The Domino Shuffling Height Process and Its Hydrodynamic Limit

by

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A.B., Princeton University; Princeton, NJ, 2014

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Department of Mathematics at Brown University

Providence, Rhode Island

May 2018

This dissertation by Xufan Zhang is accepted in its present form by the Department of Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

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Acknowledgements

I would like to express gratitude to many people for the support and help during the five years of my graduate career.

First, I want to thank my advisor, Richard Kenyon, for introducing a diversity of intriguing mathematical topics to me and for providing support in every aspect of my studies. More importantly, through all the conversations and interactions with you, I was able to develop a better taste and understanding about mathematical research.

To my committee members, Kavita Ramanan and Benoit Pausader, I am grateful for the precious time you spent on reading my thesis and attending my defense. In addition, I would like to thank Kavita for the amazing course on stochastic processes on graphs, the conversations we had on various topics, as well as proofreading and correcting my thesis and the previous draft.

Related to the thesis, I also would like to thank Fabio Toninelli and Sanjay Ramassamy for giving very helpful advice and pointing to references. I learned a lot from Fabio through various conversations and your great summer course. I also want to thank Vincent Lerouvillois for pointing out errors in the draft.

To the energetic probability community, I would like to thank Omer Angel, Christophe Bahadoran, Alex Borodin, Patrik Ferrari and Jim Propp for inspiring conversations. I would like particularly to thank David Wilson for the amazing semester-long course on your researches and for many stimulating conversations.

To my fellow current and former Rick's students and mentees, Ian Alevy, Terrence George, Wai Yeung Lam, Xiang Li, Zhongyang Li, Olya Mandelshtam, Sanjay Ramassamy, Martin Tassy and Samuel Watson, I would like to thank all of you for the conversations and presentations throughout the years. I learned from you not only wonderful mathematics, but also curious and rigorous attitude towards researches.

I would like to thank Justin Holmer for serving on my topic exam committee, in addition to Rick and Kavita.

I want to thank Dan Katz for the teacher training and all the wonderful guidance on my teaching later.

To Audrey Aguiar, Larry Larrivee, Lori Nascimento, Carol Oliveira and Doreen Pappas, my graduate life would be impossible without your help and support. Thank you all for making the Brown math department a wonderful place to work and study.

Contents

List of Tables

List of Figures

CHAPTER 1

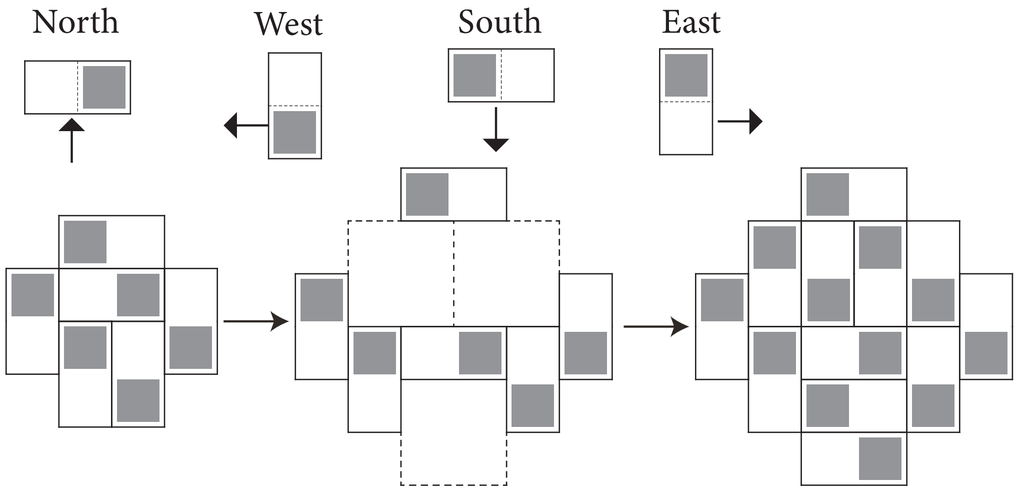
Introduction

The dimer model, which consists of the weighted perfect matchings on graphs, is a well-studied model in mathematical physics. The domino model is the dimer model on a (possibly infinite) region of the $\mathbb{Z}^2$ lattice. Alternatively, it can be thought of as tiling a region of the square grid exactly by 1×2 and 2×1 dominoes. A comprehensive survey on the subject of dimer models can be found in [18].

Domino shuffling is a discrete-time random dynamics operating on the domino model, introduced originally by Elkies et al. in [7] to compute the generating function of domino tilings of a specific family of graphs called Aztec Diamonds with periodic weights. We shall illustrate an example of shuffling by Figure 1.1 and define it rigorously later.

This example starts with an Aztec Diamond of order 2. Notice that the dominoes assume a checkerboard coloring. This coloring induces four types of dominoes, and each type moves in a specific direction by a unit distance. This example is somewhat degenerate in that there are no collisions. In cases where two dominoes move towards each other and collide, we destroy both of them. In any case, we end up with a partial tiling of an Aztec Diamond of

Figure 1.1: One example of shuffling on the Aztec Diamond



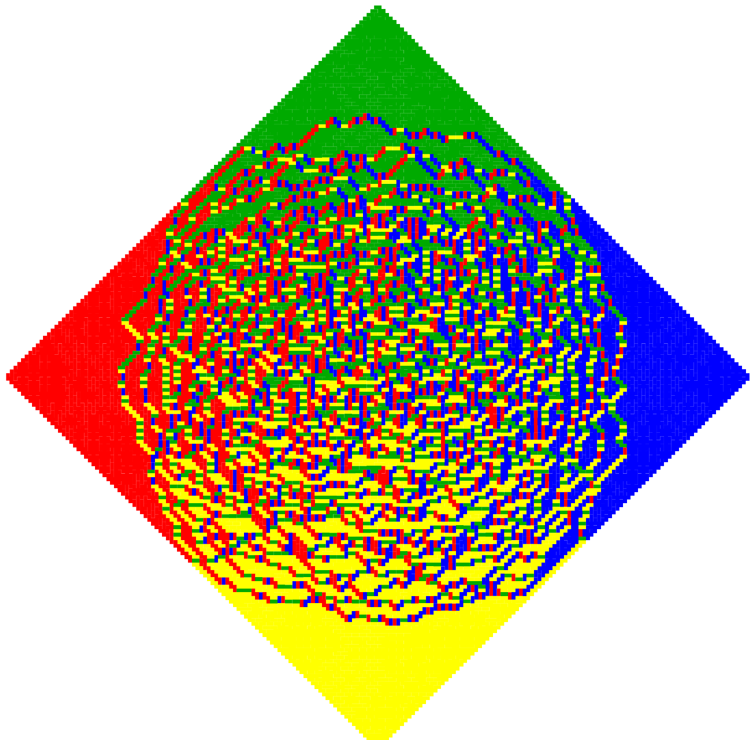
order 3. The remaining empty region is made of 2×2 blocks, and we fill each block with vertical or horizontal dominoes randomly (each with half probability in the case of uniform edge weights) and independently. A more precise description is deferred to Section 3.1.

It turns out that domino shuffling provided a simple algorithm to generate the uniform measure of tilings of Aztec Diamonds and led to the first rigorous proof of the famous Artic Circle Theorem by Jockusch et al.([11]). Roughly speaking, the theorem says that asymptotically as $n \to \infty$, with probability going to 1, a random tiling of the Aztec Diamond of order n under rescaling looks random inside the circle inscribed in the square and is deterministic in the frozen regions outside the circle. See Figure 1.2.

Domino shuffling seemed quite mysterious the way it was presented in [7] initially, in a way similar to the description above. Propp([25]) later gave a much clearer explanation using a procedure called urban renewal, or spider move, which also allows natural generalizations to different graphs. This is the approach we will take to define the shuffling dynamics in Chapter 2.

We can introduce heights to a tiling by assigning a number to each vertex in such a way that, as we walk for a unit distance along the boundary of a domino in clockwise direction,

Figure 1.2: A random tiling of a large Aztec Diamond with uniform weights. The four colors correspond to the four types of dominoes.

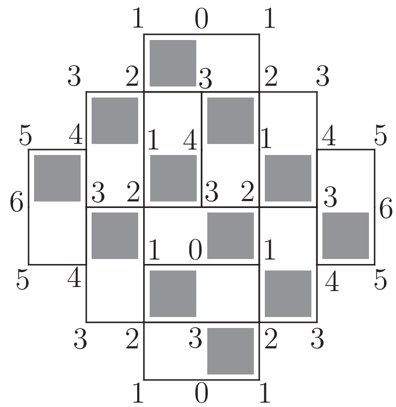


the height increases by 1 if our right hand side is white and decreases by 1 if our right hand side is black. See Figure 1.3 for an example.

As it turns out, the urban renewal perspective also naturally converts the shuffling dynamics into a random height process, which we shall call the *shuffling height process*. The height process is a $(2+1)$-dimensional evolution which is discrete in both space (2-d) and time (1-d). The purpose of the thesis is to prove that, when rescaling both space and time parameters by n, and starting nearby an arbitrary continuous profile (subject to a certain gradient requirement) on the infinite plane, as $n \to \infty$, the height process evolves according to a first-order, nonlinear Hamilton-Jacobi equation

$$u_t + H(u_x) = 0 \tag{1.1}$$

Figure 1.3: A domino tiling of an Aztec Diamond of order 3 with heights labeled



where, in the case of uniform weights, the Hamiltonian is given by

$$H(\rho_1, \rho_2) = \frac{4}{\pi} \cos^{-1}\left(\frac{1}{2}\cos\left(\frac{\pi\rho_2}{2}\right) - \frac{1}{2}\cos\left(\frac{\pi\rho_1}{2}\right)\right).$$

The convergence is uniform in any compact subset of the spacetime. A subtlety is that the PDE develops shocks even starting from a smooth profile, so we need to consider a specific weak solution called the viscosity solution.

Before describing the strategy of the proof, we would like to mention some other works on domino shuffling. Johansson([13]) and Nordenstam([24]) related domino shuffling on the Aztec Diamond to a determinantal point process. As a result, they were able to prove that, under appropriate rescaling, the boundary of the arctic circle converges to the Airy process, and the turning point converges to the GUE minor process. In 2014, Borodin and Ferrari([2]) pushed this idea further, where several different $(2+1)$-dimensional interacting particle systems and random tiling models were connected through systems of non-intersecting lines. These models are believed to belong to the Anisotropic Kardar–Parisi–Zhang (AKPZ) universality class, which means that the speed function H in the hydrodynamic limit (1.1) has the property that the determinant of its Hessian is negative. The domino shuffling dynamics is in particular one of them, with the connection explained by the same authors([3]) later in greater detail. Recently, Chhita and Toninelli([5]) analyzed the speed and fluctuation of domino shuffling on the 2-periodic $\mathbb{Z}^2$ lattice, and demonstrated a "rigid" stationary state where the fluctuation is $O(1)$.

Many of the works above focus on a specific type of region or initial condition. In terms of hydrodynamic limits starting from a more general profile, the first rigorous result in the context above was obtained in 2017 by Legras and Toninelli([34],[21]). They analyzed another stochastic interface growth model from [2], which can be viewed as a continuous-time dynamics on lozenge tilings (the dimer model on the hexagonal lattice). In this case, different from the domino shuffling dynamics, updates at a point can depend on information arbitrarily far away, and the speed function is unbounded. As a result, the hydrodynamic limit is proved either up to the first shock time or when the initial profile is convex.

We also want to mention some background in hydrodynamic limit theory. One general approach to obtaining the hydrodynamic limit of discrete systems is to first make an educated guess about the form of the limit based on a local-equilibrium heuristic, that is, assuming the system is locally at equilibrium almost everywhere for all time. This often leads to an explicit PDE, which serves as a guide. When the PDE theory provides a characterization of a unique (weak) solution, we can try to find a discrete analogue of that characterization. For example, when we are dealing with the symmetric nearest neighbor simple exclusion process, as discussed in Chapter 4 of the classical reference [20], the hydrodynamic limit is expected to be the heat equation, whose unique solution can be characterized by an integral equation. Therefore, one wants to show that, starting from a particular initial profile, the Riemann sum based on the empirical measure of the discrete system converges to an integral. This then becomes a classical problem in probability theory, showing convergence of measures. First, one invokes Prokhorov's theorem to show that every sequence of measures has a subsequential limit. Then, one utilizes knowledge about the specific discrete system to prove that any subsequential limit must agree with the desired integral equation, in this case using martingale techniques.

In the case when the expected PDE is a Hamilton-Jacobi equation, the PDE theory is more complicated. When the speed function H is convex or the initial profile is convex, the unique viscosity solution can be written down in a variational form, using either the Hopf-Lax formula or Hopf formula ([9]). Certain exclusion processes do have such a PDE as the limit, and one proof strategy consists of finding a corresponding microscopic variational formula for the discrete system and showing the convergence of this formula to the continuous one. See, for example, the works by Seppäläinen([29]) and Rezakhanlou([27]). Also the Hopf formula is used in [21] to prove the limit starting from a convex profile.

However, since the AKPZ property exactly means that the speed function H is neither convex nor concave, an explicit variational formula for the unique solution is not available. (Evans([9]) gave a general representation formula, but it is not clear how to relate it to these discrete systems.) The seminal work [26] of Rezakhanlou in 2001 provided an approach in the context of certain continuous-time exclusion processes. Just as in the case of simple exclusion processes, one would like to prove the convergence of empirical measures. However, the unique viscosity solution of the Hamilton-Jacobi equation has a peculiar characterization (among a few equivalent characterizations). It involves comparing the current solution to a family of arbitrary smooth functions at all spacetime locations. Therefore, the empirical measures which we want to demonstrate convergence of need to encode the evolution from not just one initial condition, but all possible initial conditions starting from an arbitrary time, as an approximate "semigroup". The idea is that although the encoding does not strictly satisfy the semigroup property, it will become a semigroup in the limit. A priori, to encode this much information, the space of probability measures on the encodings of evolutions would be too large, i.e. inseparable, to apply Prokhorov's theorem. The key observation of Rezakhanlou is that, if the encoding satisfies certain properties, the space of probability measures can be made separable. In [26], the full hydrodynamic limit of a family of exclusion processes in $d = 1$ was established, with a nonexplicit Hamiltonian. Incidentally, under a certain projection, domino shuffling can also be connected to a 1-d exclusion process, which is explained in [11] and is crucial for their proof. It seems that *our result is the first example in* $d > 1$ *where such a full hydrodynamic limit with a nonconvex Hamiltonian has been obtained for a discrete system.*

Another issue is that the evolution of the discrete system needs to be properly interpolated to be comparable to the evolution of the PDE, and more importantly, to keep the space of probability measures separable. We carry out the interpolation of domino shuffling in Chapter 4. The interpolation in [26] is straightforward, but it takes extra work in our case due to the differences of the models. A convenience for us, however, is that the topology can be taken to be the uniform topology, instead of the Skorohod topology. This makes the argument more transparent.

In Chapter 6, utilizing dimer theory, we identify the Gibbs measures of domino tilings as equilibrium measures of the shuffling process, and deduce the hydrodynamic limit starting from a flat initial condition. (Notice that we do not need the uniqueness of the dimer Gibbs measures.) This allows us to determine the full hydrodynamic limit in Chapter 7. While using the general theory of viscosity solutions, we have to take care of the boundedness of the spatial gradient, imposed by our model. In addition, we prove the scaling limit of shuffling on the Aztec Diamond, which in particular implies a deterministic limit shape for a random domino tiling.

The rest of the thesis is outlined as follows. In Chapter 2, we will provide some background information about the dimer model, and the dimer shuffling height process is defined in a general manner. We also establish a list of lemmas that are useful later. The specific set-up and the precise statement of the main result are presented in Chapter 3. In Chapter 5, we apply Prokhorov's theorem and the generalized Arzelà-Ascoli theorem to deduce the precompactness of the sequence of empirical measures on the approximate "semigroups" and prove some important properties of the subsequential limits necessary for the proof. In particular, the limits are bona fide semigroups.

Finally in Chapter 8, we shall discuss the 1-d exclusion process related to domino shuffling, which is interesting in its own right. We establish a family of extremal equilibrium measures, and in a certain regime prove the uniqueness of such a measure. This is an extension of the result in [11], and can be used to derive the full hydrodynamic limit of the 1-d process through variational methods, which leads to the Arctic Ellipse Theorem for random domino tilings of Aztec Diamonds weighted according to Section 3.1.

A large portion of the thesis is based on an earlier preprint [36], with some important modifications.

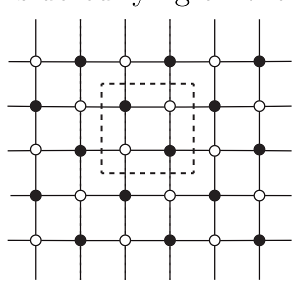
CHAPTER 2

Preliminaries

2.1 Dimer model on a periodic bipartite graph

To start with, consider a $\mathbb{Z}^2$-periodic bipartite graph $G = (V, E)$ embedded in the plane, where the vertices in each fundamental domain are colored black and white in a particular way, such that the whole graph is invariant under the natural $\mathbb{Z}^2$-translation action T matching fundamental domains. One primary example will be the graph shown in Figure 2.1. Also define $G_n := G/(n\mathbb{Z})^2$ as the quotient graph embedded on a torus.

Figure 2.1: The bipartite graph with vertices $\mathbb{Z}^2$ and one fundamental domain drawn. The vertices in this figure should be considered lying on the dual lattice of Figure 1.3.



A *dimer covering* on a bipartite graph is a subset of the edges E that form a perfect matching among the vertices V. A chosen edge is called a *dimer*. We assign a nonnegative weight $w(e)$ to each edge e invariant under T. Then on finite graphs G_1 and G_n, we can define a Boltzmann probability measure on the set of dimer coverings $\mathcal{M}$:

$$\mu[M \in \mathcal{M}] = \frac{1}{Z} \prod_{e \in M} w(e)$$

where $Z = \sum_{M \in \mathcal{M}} \prod_{e \in M} w(e)$ is the partition function. Notice that the measure is invariant under a *gauge transformation*, which means multiplying all edge weights incident to a vertex by a positive constant.

This definition of course does not make sense on G, but we can always take a sequence of Boltzmann measures on G_n with $n \to \infty$, and any weakly convergent subsequence (which is guaranteed to exist by Prokhorov's theorem) will yield a limiting Gibbs measure on G, whose finite dimensional distributions are the limit along the convergent subsequence.

2.2 Height function

A *flow* on a graph is an assignment of real numbers to the directed edges, such that edges with opposite directions are assigned opposite numbers. Given a dimer covering M on G, we can think of it as a white-to-black flow $[M]$, where each white-to-black edge is assigned either 1 or 0. If we fix some reference covering M_0, then $[M] - [M_0]$ is a divergence-free flow (the net flow into each vertex is equal to 0), which induces a gradient flow on the dual graph. In other words, we can attribute a *height function* h_M defined on the faces to the covering M, by first stipulating that one base face has height 0, and assigning neighboring faces their heights as follows. When we cross an edge, the height will increase by the net amount of flow on that edge from left to right.

This height function h_M is well defined on a planar graph up to an additive constant, and it is clear that, given the reference covering, we can recover the dimer covering M from its height function h_M. If the graph is embedded on a torus, such as G_n defined above, h_M is still well-defined locally, but is treated as a multi-valued function globally. Suppose h_M increases by x as we move towards the right once around the torus back to the same face, and increases by y as we move up once around the torus, we say that h_M or the covering M itself has *height change* (x, y). Since h_M is well defined locally, (x, y) does not depend on the choice of cycles, as long as they have homology $(1, 0)$ and $(0, 1)$ respectively.

The set of all possible height changes on $G_1 := G/\mathbb{Z}^2$ plays an important role in dimer theory. Their convex hull is called the *Newton polygon* associated to G, which, roughly speaking, contains exactly all possible "slopes" on G ([19]).

A different reference covering M_0' will define a different height function for M. The difference is determined by $[M_0] - [M_0']$, which is independent of the covering M of interest. Therefore, given two coverings M and M', the function $h_M - h_{M'}$ does not depend on the choice of reference covering.

A nice property about these height functions is that they have a lattice structure by taking pointwise maximum or minimum. This fact was briefly mentioned in [6] in the context of domino tilings. Here we give a proof in the above setting.

Lemma 2.1 (Lattice property)**.** *Fixing a reference covering M_0 on G, if h_M and $h_{M'}$ are both height functions with integer values, then both $h_M \wedge h_{M'}$ and $h_M \vee h_{M'}$ are dimer height functions, where $\wedge$ denotes pointwise minimum and $\vee$ denotes pointwise maximum.*

Proof. WLOG, suppose $g = h_M \vee h_{M'}$. We first prove by contradiction that the following scenario can never happen: there is be an edge e separating two faces f_1 and f_2 such that $g(f_1) = h_M(f_1) > h_{M'}(f_1)$ and $g(f_2) = h_{M'}(f_2) > h_M(f_2)$. Let us assume that when we cross e from f_1 to f_2, the white vertex is on the left. If $e \in M_0$, a dimer height function either stays the same or decreases by 1 going from f_1 to f_2. Then we must have $h_M(f_2) \geq h_M(f_1) - 1 \geq h_{M'}(f_1) \geq h_{M'}(f_2)$, a contradiction. Similarly, if $e \notin M_0$, a dimer height function either stays the same or increases by 1 going from f_1 to f_2. Then $h_{M'}(f_1) \geq h_{M'}(f_2) - 1 \geq h_M(f_2) \geq h_M(f_1)$, again a contradiction.

Now suppose g is not a dimer height function. One possible obstacle is that for some edge e separating two faces f_1 and f_2, $g(f_1) - g(f_2)$ is not one of the allowed dimer height changes across e. Then the scenario above must happen, which is impossible.

Another possible obstacle is when the value of g changes at least four times circling around a vertex v. (Obviously g has to change even number of times circling around any vertex.) On the other hand, any dimer height function must change either zero or two times circling around v. Also if e_0 is the unique edge in M_0 that connects v, a dimer height function that changes two times circling around v must change its value across e_0. This implies that out of the four value changes for g around v, at least one change corresponds to the scenario mentioned before, which is again impossible.

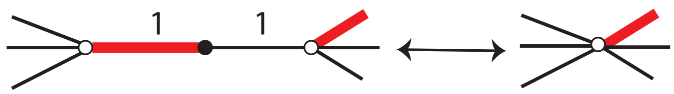
The last possible obstacle is when the value of g changes exactly two times circling around a vertex v, but neither of them happens at e_0, borrowing the notations from above. Since the scenario mentioned at the beginning is impossible, those two changes must have one of them coinciding with h_M and the other coinciding with $h_{M'}$. Therefore both h_M and $h_{M'}$ must both change across e_0, with the same net increase. But since $g = h_M \vee h_{M'}$, g must also change across e_0, a contradiction. □

2.3 Local moves

Now we define two types of local moves for the dimer model, *vertex contraction/expansion* and the *spider move*. These first appeared in [25] and were also studied in [10, 31]. These local moves happen at two different levels. One level is a local modification of the weighted bipartite graph, and another level is a possibly random mapping of dimer coverings on the graph.

Vertex contraction/expansion, on the graph level, involves either shrinking a 2-valent vertex and its two incident edges into a single vertex, or its reversal. See Figure 2.2. Assume that *the three vertices involved are all distinct*. Before shrinking, we first perform a gauge transformation to make both edge weights equal to 1. During expansion, on the other hand, simply assign both edges weight 1.

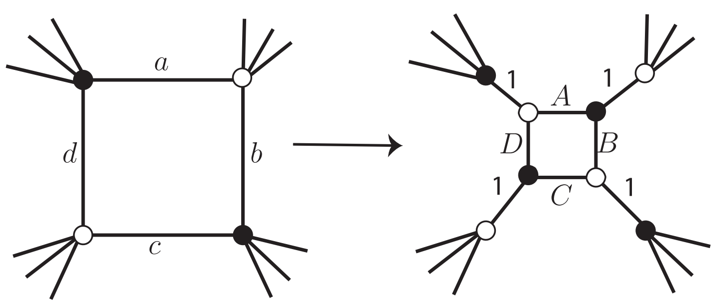
Figure 2.2: Vertex contraction/expansion move



On the dimer level, given a dimer covering before contraction, we simply delete the dimer incident to the deleted middle vertex, and keep the rest of the dimer covering after contraction. For expansion, we keep the covering, and match the added middle vertex with the unmatched side.

The spider move is the more interesting case. See Figure 2.3 for an illustration.

Figure 2.3: Spider move on graph level



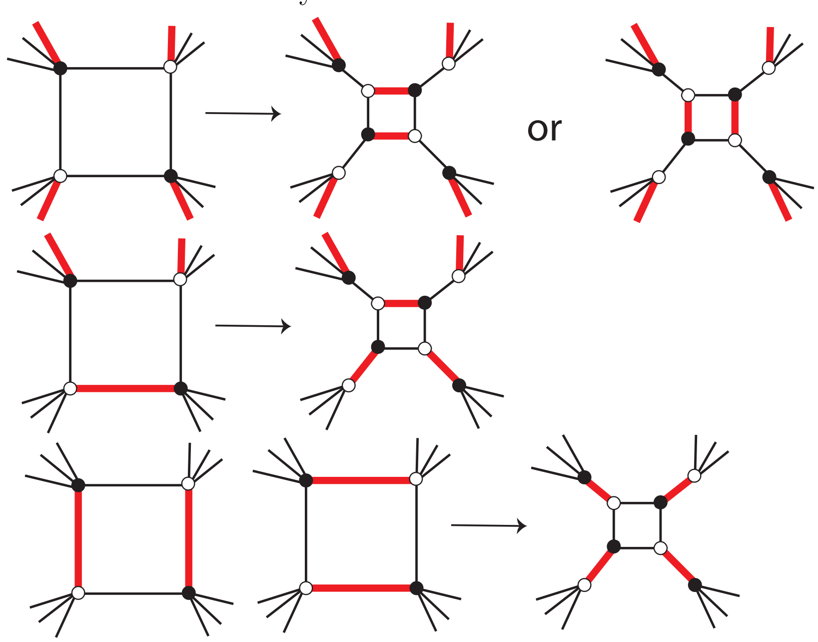
Start with a quadrilateral face with a top-left black vertex such that *all four vertices are distinct*. On the graph level, first insert four tentacles at the corners. The four tentacles are assigned weight 1. The other four weights are assigned as follows:

$$A = \frac{c}{ac+bd}, B = \frac{d}{ac+bd}, C = \frac{a}{ac+bd}, D = \frac{b}{ac+bd}. \tag{2.1}$$

For the spider move with the opposite coloring, first perform vertex expansion at four corners and then implement the spider move on the internal face. For the reversal with the original coloring, perform the opposite-coloring spider move on the internal face and shrink the four 2-valent edges. Hence we only call the move in Figure 2.3 the spider move.

On the dimer level, we keep all the dimers that are not one of the four internal edges. Then depending on whether each of four original vertices were matched externally or internally, we make some choices in order to complete a dimer covering. See Figure 2.4 for some of the cases.

Figure 2.4: Spider moves on the dimer level. Only the first row has randomness. In the second row, we omitted three other symmetric cases.



In the first row of Figure 2.4, a choice is made between the two possibilities according to the ratio between their total weights. The two horizontal dimers are chosen with probability

$\frac{AC}{AC+BD}$ and the two vertical dimers are chosen with probability $\frac{BD}{AC+BD}$. Many variants of the following statement appeared in the literature([10, 25, 31]).

Proposition 2.2. *For any finite bipartite graph H embedded on a torus, the local moves preserve the Boltzmann measure of dimer coverings on H, in the sense that applying a local move to the Boltzmann measure on H results in the Boltzmann measure on the new graph H'.*

Proof. The statement for vertex contraction/expansion move is obvious, because the mapping does not change the total weight of the dimer covering.

To prove the statement for the spider move, we refer to Figure 2.3 and 2.4. By an abuse of notation, we use a, b, c, d, A, B, C, D to denote both the edges and their weights. If S is a set of edges in the inner square on the LHS, then let w_S^T be the weight of the dimer covering where exactly S among the four edges of the inner square are included and T is the rest of the edges in the covering. Define $\widetilde{w}_S^T$ similarly for the weight of the dimer covering on the RHS formed exactly by S, T and a subset of the four new tentacles, where S is a set of edges in the inner square and T is a set of edges in the complement of the inner square and the four tentacles. Now consider the three rows in Figure 2.4 and the three omitted rows. We shall first verify that the ratio between the LHS and RHS of every row in Figure 2.4 is the same. By (2.1),

$$\begin{aligned}
&\frac{\widetilde{w}^T_{\{AC\}} + \widetilde{w}^T_{\{BD\}}}{w^T_{\emptyset}} = AC + BD = \frac{ac+bd}{(ac+bd)^2} = \frac{1}{ac+bd},\\
&\frac{\widetilde{w}^T_{\{A\}}}{w^T_{\{c\}}} = \frac{\widetilde{w}^T_{\{B\}}}{w^T_{\{d\}}} = \frac{\widetilde{w}^T_{\{C\}}}{w^T_{\{a\}}} = \frac{\widetilde{w}^T_{\{D\}}}{w^T_{\{b\}}} = \frac{1}{ac+bd},\\
&\frac{\widetilde{w}^T_{\emptyset}}{w^T_{\{ac\}} + w^T_{\{bd\}}} = \frac{1}{ac+bd}.
\end{aligned}$$

Therefore, if the dimer coverings were distributed according to their weights before the spider move, the probabilities after the spider move are also proportional to their weights, except in the first row of Figure 2.4, the two results on the RHS are grouped together. It only remains to check that their probabilities are also proportional to their weights. Indeed,

$$\frac{\widetilde{w}^T_{\{AC\}}}{\widetilde{w}^T_{\{BD\}}} = \frac{AC}{BD},$$

and the spider move selects A and C with probability $\frac{AC}{AC+BD}$ and selects B and D with probability $\frac{BD}{AC+BD}$. □

2.4 Shuffling height process

The following is not a precise definition, but rather a general description of the type of process we are considering.

First, we consider a *global operation* where we choose a local move on G or G_n, and perform it at all T-periodic counterparts simultaneously, requiring that *the edges involved do not overlap with each other*. The spider moves at different locations are independent in terms of their randomness. See Figure 3.1 for an example.

By Proposition 2.2, such a global operation still preserves the Boltzmann measure, since we can consider it as performing the local moves sequentially. It is also well defined on G, as we can perform the local moves in the increasing order of their distance from the origin, so that every finite region of G will be determined after some finite number of local moves.

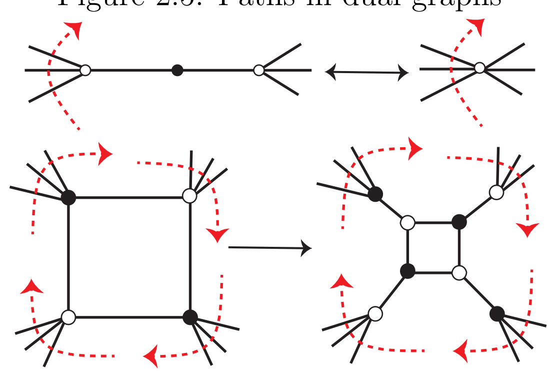
Second, we want to compare the height functions before and after a local move. Given the reference covering M_0 and a height function h_M before a local move, we may choose (in most cases there is a natural choice) the reference covering after the local move to be deterministically one of the possible outcomes of the local move applied to M_0. This defines a new height function $h_{M'}$ for the random outcome M'.

Lemma 2.3. *With the choice above of the reference covering on G, $h_{M'}$ agrees with h_M at* *every face except the inner face of the spider move (up to a global additive constant which is made zero), where M' is an outcome of M after a local move.*

For a moment let us forget about the precise embedding. When we say $h_{M'}$ agrees with h_M at a face, we mean that the heights defined at the *combinatorially corresponding faces* agree, since no face is created or destroyed.

Proof. We can relate the heights at different faces as in Figure 2.5. Since we assume that all vertices involved in the local moves are distinct, the dimer configuration along the dotted paths drawn remains the same. Therefore, the height on the surrounding faces can be made invariant before and after the local move. Then the height at every other face also stays unchanged except the middle one in the spider move. □

Figure 2.5: Paths in dual graphs



A consequence of Lemma 2.3 is that during a global operation, the heights of all the faces except the inner faces of spider moves can be kept unchanged, since such is true after every local move. This is the reason why we will be able to discuss the evolution of height functions. Also in this case, we will choose the same new reference dimer for each local move. That is, we can assume that *the reference covering remains T-periodic* after a global operation.

When there are two dimer coverings M_1 and M_2 on G, we can perform a local move or a global operation on them simultaneously. The only requirement is that, they are *coupled* so that each pair of corresponding faces during a spider move share the same randomness. Then we have the following "monotonicity" lemma.

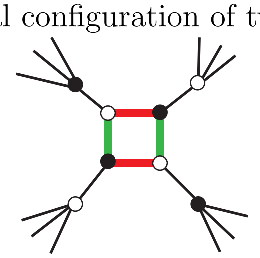
Lemma 2.4 (Monotonicity)**.** *With the same choice of reference covering on G as in Lemma 2.3, if $h_{M_1} \geq h_{M_2}$ at each corresponding face before a local move, then $h_{M_1'} \geq h_{M_2'}$ still holds afterwards, where M_1' and M_2' are the coupled results of the local move performed on M_1 and M_2 respectively.*

Proof. The heights of the faces do not change in vertex contraction/expansion, so there is nothing to prove there.

It suffices to consider $h_{M_1} - h_{M_2}$ and $h_{M_1'} - h_{M_2'}$, since they do not depend on the choice of reference covering. By assumption $h_{M_1} - h_{M_2} \geq 0$. By Lemma 2.3, we can assume that, for both h_{M_1} and h_{M_2}, the heights at all other faces stay the same except the internal face of the spider move, denoted as f.

Therefore, we have $h_{M_1'} - h_{M_2'} \geq 0$ at all faces except f. Suppose the statement is wrong, then $h_{M_1'} - h_{M_2'} < 0$ at f. By considering the flow $[M_1'] - [M_2']$, the only case this can happen is shown in Figure 2.6, where the green vertical dimers are in M_2' and the red horizontal dimers are in M_1'. This cannot happen since we assumed that they are coupled at f during the spider move. $\square$

Figure 2.6: A local configuration of two dimer coverings



Now to obtain a tractable height process, we make a strong assumption.

Assumption: After a particular choice of sequence of global operations being performed on G_n (resp. G), the resulting graph coincides with G_n (resp. G) in terms of the actual embedding, with combinatorially corresponding faces matching each other, and the edge weights are also the same up to gauge transformations.

Another assumption is that the reference dimer covering also remains the same. This is not necessary because there are only finitely many T-periodic coverings, so we can make this true by running longer time if necessary, given the assumption above.

If the assumption above is true, we call one iteration of such a sequence of global operations a *shuffle*, and the corresponding height evolution a *shuffling height process*.

The assumption might seem very strong. The specific example that we will analyze is the one in Figure 2.1. But already in $\mathbb{Z}^2$ lattice, by increasing the size of the fundamental domain, some special phenomena in the steady state fluctuation are discovered in [5].

The reason why we introduced this process in this more general manner is, first, the existence of the hydrodynamic limit of any such process can be obtained by a similar approach, even though the specific PDE might be hard to compute; and second, the necessary lemmas listed above and their proofs do not assume much about the specific graph structure, so it seems more natural to state them independently.

CHAPTER 3

The main result

3.1 The specific example

Now we turn to the simplest example where a shuffling height process can be defined, the 1-periodic domino tiling model in Figure 2.1.

We assume that the vertical edges have weight $\sqrt{a}$ and the horizontal edges have weight 1, as any other choice of positive weights with the same fundamental domain is gauge equivalent to this one. By convention in the literature, we choose the reference flow $[M_0]$ in the initial graph to be 1/4 on each edge. We define the height function h on the faces, which are labeled by coordinates in $\mathbb{Z}^2$. With this coordinate, the graph is periodic under the action T, generated by translations $(2, 0)$ and $(0, 2)$.

We assume that initially the face at $(0, 0)$ has a top-left black vertex. We call faces (i, j) with $i + j$ even *even faces*, and otherwise *odd faces*. Also, we multiply the heights by 4, so that all of them are integers. This is the height function of domino tilings defined in [32], as shown in Figure 1.3, where heights are labeled on vertices because that picture is dual to the graph we are working with.

To be clear, when we speak of an "edge", we always refer to an edge on the primal graph as in Figure 2.1, etc. By definition of the height function, when we cross an edge with a white vertex on the left, the height either increases by 3 or decreases by 1. Therefore once we fix the height of face $(0,0)$ modulo 4, all other faces are determined modulo 4.

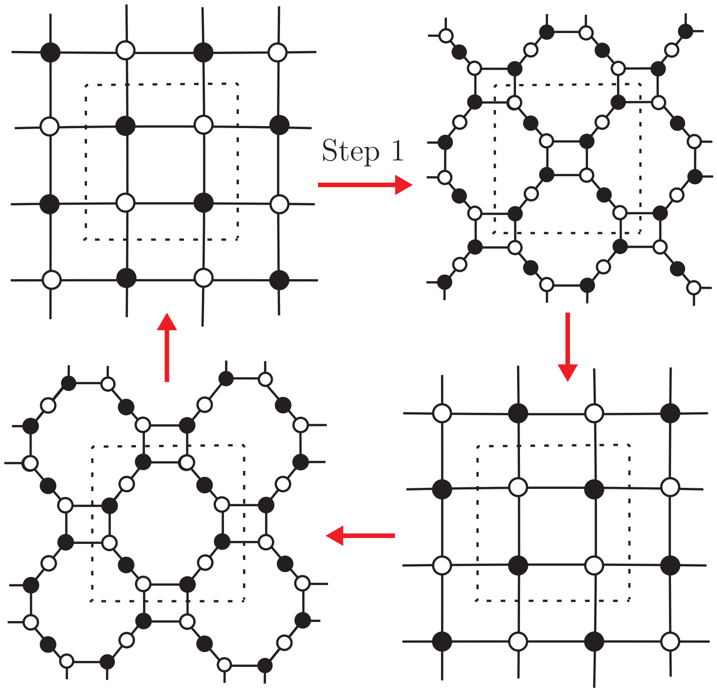
The shuffling procedure is essentially the same as the domino shuffling algorithm from [7]. Propp [25] rephrased this shuffling procedure in terms of the local moves described in Section 2.3. By our definition, a shuffle consists of the following steps.

1. Perform a spider move at all even faces (i,j) (these are two T-periodic families of local moves);
2. Perform vertex contraction at all 2-valent vertices;
3. Perform a spider move at all odd faces (i,j);
4. Perform vertex contraction at all 2-valent vertices;

See Figure 3.1 for one iteration. By formula (2.1), after Step 1, the horizontal edge weights become $\frac{1}{1+a}$, and the vertical edge weights become $\frac{\sqrt{a}}{1+a}$. So up to gauge transformations, the edge weights remain the same after Step 2, and after Step 4 as well. By viewing the reference flow $[M_0]$ as a convex combination of four different integer coverings, each consisting of a single type of edges, $[M_0]$ also remains $1/4$ on all horizontal and vertical edges. Therefore, the assumption for a shuffling height process is satisfied.

Consider the height at certain even face (i,j). By Lemma 2.3, it only gets modified at Step 1 of a shuffle, since that is the only time when that face undergoes a spider move. A small inconvenience is that the height module 4 changes after a shuffle. To compensate for that, from now on, *we subtract all heights by* 2 *after every shuffle.* This amounts to adding a constant drift to the hydrodynamic limit, which does no harm. Since the only relevant

Figure 3.1: One domino shuffle with a fixed fundamental domain labeled. Each step consists of several T-periodic families of local moves.



faces during a spider move are the neighboring ones, we can list all the possible outcomes at (i, j) after a shuffle in Table 3.1. We see that the height at (i, j) is nonincreasing, and stays the same modulo 4. The same table also describes the height evolution at an odd face (i, j), where the left column represents the height after Step 2. In other words, the entire height process can be defined using the local rules described in Table 3.1, without mentioning dimers.

3.2 Hydrodynamic limit

To state a hydrodynamic limit result on the height evolution, we need to introduce a time parameter t. Suppose the initial condition at $t = 0$ is the height function of certain dimer covering on G, which is a function $\mathbb{Z}^2 \to \mathbb{Z}$, as defined previously, the dynamics is that a shuffle happens at $t = 1, 2, 3, \ldots$. This defines a random process $h : \mathbb{Z}^2 \times \mathbb{N} \to \mathbb{Z}$ where

Table 3.1: The height evolution at an even face (i, j) during a shuffle

Local heights centered at (i, j)	The height at (i, j) after the shuffle
$\begin{array}{ccc} & h+1 & \\ h-1 & h & h-1 \\ & h-3 & \end{array}$	$h-4$
$\begin{array}{ccc} & h-3 & \\ h-1 & h & h-1 \\ & h+1 & \end{array}$	$h-4$
$\begin{array}{ccc} & h+1 & \\ h-1 & h & h-1 \\ & h+1 & \end{array}$	h with probability $\frac{a}{1+a}$ $h-4$ with probability $\frac{1}{1+a}$
$\begin{array}{ccc} & h+1 & \\ h-1 & h & h+3 \\ & h+1 & \end{array}$	h
$\begin{array}{ccc} & h+1 & \\ h+3 & h & h-1 \\ & h+1 & \end{array}$	h
$\begin{array}{ccc} & h-3 & \\ h-1 & h & h-1 \\ & h-3 & \end{array}$	$h-4$
$\begin{array}{ccc} & h+1 & \\ h+3 & h & h+3 \\ & h+1 & \end{array}$	h

$h(x, t)$ denotes the height at face x and time t. A priori, the spatial function $h(\cdot, t)$ for any t is a domino height function. As mentioned before, $h(x, \cdot)$ is nonincreasing. Furthermore, we require that $h(0, 0) \equiv 0 \pmod 4$. This immediately determines all $h(x, t) \pmod 4$. We will call such height function $h(\cdot, t)$ *admissible*.

The underlying probability space Ω consists of a collection of iid Bernoulli random variables at each $(x, t) \in \mathbb{Z}^2 \times \mathbb{N}$, which take value 1 with probability $\frac{1}{1+a}$ and value 0 with probability $\frac{a}{1+a}$. In particular, given $\omega \in \Omega$, $\omega(x, t)$ dictates the randomness of a spider move that happens at face x and time t.

Define the space of *asymptotic height functions* Γ to be the set of all 2-*spatially-Lipschitz functions* from $\mathbb{R}^2$ to $\mathbb{R}$, which in this thesis means

$$|f(x) - f(y)| \leq 2\,|x - y|_\infty\,,$$

where $|\cdot|_p$ is the ℓ_p norm.

The choice of Γ comes from the Newton polygon, as defined in Section 2.2. In this case, the (rescaled) Newton polygon bounds the region $U := \{x : |x|_1 \leq 2\}$, and a differentiable function is in Γ iff its gradient lies in U. See Lemma 7.3 for a similar statement.

Now suppose we are given some $g \in \Gamma$ and a sequence of initial conditions $(h_n(\cdot, 0))_{n\in\mathbb{N}}$ approximating g. What we mean exactly is that $(h_n(\cdot, 0))$ is a sequence of admissible height functions, random or not, independent or not, such that

$$\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1 \leq R} \left| \frac{1}{n} h_n(\lfloor nx \rfloor, 0) - g(x) \right| = 0 \tag{3.1}$$

for every finite $R > 0$, and the expectation $\mathbb{E}$ is taken over the probability space Ω_0 of the initial condition $(h_n(\cdot, 0))_{n\in\mathbb{Z}_{>0}}$.

The height evolution of (h_n) is governed by the single probability space Ω. This means implicitly that the randomness of a spider move at (x, t) is coupled for all h_n.

Now we are ready to state the hydrodynamic limit.

Theorem 3.1. *We have for every $R > 0$,*

$$\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1 \leq R, t \leq R} \left| \frac{1}{n} h_n(\lfloor nx \rfloor, \lfloor nt \rfloor) - u(x, t) \right| = 0 \tag{3.2}$$

where $u : \mathbb{R}^2 \times \mathbb{R}_{\geq 0} \to \mathbb{R}$ *is the unique viscosity solution of*

$$\begin{cases} u_t + H(u_x) &= 0 \\ u(x,0) &= g(x), \end{cases} \tag{3.3}$$

H *is defined on* U *by*

$$H(\rho_1, \rho_2) = \frac{4}{\pi} \cos^{-1}\left(\frac{a}{1+a}\cos\left(\frac{\pi\rho_2}{2}\right) - \frac{1}{1+a}\cos\left(\frac{\pi\rho_1}{2}\right)\right), \tag{3.4}$$

$u_t := \frac{\partial u}{\partial t}$ *and* u_x *denotes the spatial gradient of* u*.*

The precise definition of viscosity solutions is delayed to Section 7.1.

Remark. Notice that H is continuous on U. One can compute the determinant of the Hessian of H to be

$$-\left(\frac{a\pi\left(\cos\left(\frac{\pi\rho_1}{2}\right) + \cos\left(\frac{\pi\rho_2}{2}\right)\right)}{(a+1)^2 - \left(\cos\left(\frac{\pi\rho_1}{2}\right) - a\cos\left(\frac{\pi\rho_2}{2}\right)\right)^2}\right)^2$$

which is negative in the interior of U.

If the initial condition has all the gradients lying in some convex subset of U, then the future gradients remain in that subset.

The result here is for the shuffling in the plane. It can be replaced by a torus or cylinder, and the formula remains the same. For the shuffling on an Aztec Diamond, as how it was defined in the first place, one has to impose some boundary condition on the PDE.

CHAPTER 4

Smoothing out the height process

The goal of this section is to embed the height process $h(s,t)$ in a suitable space, which, as shown later, also contains the semigroup solving the PDE. Since h is discrete in both space and time, we need to extend it to a continuous process.

4.1 Useful properties of the height process

We first take a closer look at our height process $h(x,t)$. Let Φ denote the space of all admissible height functions, which are domino height functions whose value at $(0,0)$ is 0 (mod 4). The following lemma is rephrasing Lemma 2.1.

Lemma 4.1 (Lattice property). *If $\varphi_1, \varphi_2 \in \Phi$, then $\varphi_1 \wedge \varphi_2 \in \Phi$ and $\varphi_1 \vee \varphi_2 \in \Phi$.*

Define $h(x,t;\varphi,\omega)$ as the height at position x and time t of the deterministic height process with an initial configuration $h(\cdot,0) = \varphi \in \Phi$ and Bernoulli mark $\omega \in \Omega$.

Since $\varphi \in \Phi$ is well defined up to a global additive constant that is a multiple of 4, for all $k \in \mathbb{Z}$,

$$h(x,t;\varphi+4k,\omega) = h(x,t;\varphi,\omega) + 4k. \tag{4.1}$$

The following "monotonicity" lemma is the deterministic version of Lemma 2.4.

Lemma 4.2 (Monotonicity)**.** *Given $\varphi_1, \varphi_2 \in \Phi$, if $\varphi_1 \leq \varphi_2$, then $h(\cdot,t;\varphi_1,\omega) \leq h(\cdot,t;\varphi_2,\omega)$ for all $t \in \mathbb{N}$.*

Now we state a simple but crucial lemma, which states that information propagates at linear speed. This can also be easily generalized to other shuffling height processes.

Lemma 4.3 (Linear propagation)**.** *Given $\varphi_1, \varphi_2 \in \Phi$ and $x \in \mathbb{Z}^2$, if $\varphi_1(y) = \varphi_2(y)$ for all y such that $|x-y|_1 \leq R$, then $h(y,t;\varphi_1,\omega) = h(y,t;\varphi_2,\omega)$ for all y such that $|x-y|_1 \leq R-2t$.*

Proof. During each shuffle, there are two rounds of height updates. In the first step, all even faces (i,j) get modified. But since ω is given, the new height at (i,j) is just a function of the heights of its four neighbors. Similarly, in the third step, heights at odd faces (i,j) are updated according to its four neighbors. Therefore, the new height at any face x after a shuffle is just a function of original heights at y where $|y-x|_1 \leq 2$. Now the statement follows by an induction on t. $\square$

Combing the few statements, we deduce a "localization" property of shuffling height processes.

Proposition 4.4 (Localization)**.** *Assume $k \in \mathbb{N}$.*

1. *Given $\varphi_1, \varphi_2 \in \Phi$ and $x \in \mathbb{Z}^2$, if $|\varphi_1(y) - \varphi_2(y)| \leq 4k$ for all y such that $|x-y|_1 \leq R$, then $|h(y,t;\varphi_1,\omega) - h(y,t;\varphi_2,\omega)| \leq 4k$ for all y such that $|x-y|_1 \leq R-2t$;*

2. *Given $\varphi_1, \varphi_2 \in \Phi$, if $|\varphi_1(y) - \varphi_2(y)| \leq 4k$ for all y, then $|h(y,t;\varphi_1,\omega) - h(y,t;\varphi_2,\omega)| \leq 4k$ for all y.*

Proof. Let $\varphi_3 = \varphi_1 \vee (\varphi_2 + 4k)$, which is admissible by Lemma 4.1. Since $\varphi_1(y) \leq \varphi_2(y) + 4k$ for all y such that $|x-y|_1 \leq R$, we have $\varphi_3(y) = \varphi_2(y) + 4k$ for all such y. By Lemma 4.3, $h(y,t;\varphi_3,\omega) = h(y,t;\varphi_2+4k,\omega)$ for all y such that $|x-y|_1 \leq R-2t$. Therefore, for all such y,

$$\begin{aligned} h(y,t;\varphi_1,\omega) &\leq h(y,t;\varphi_3,\omega) \\ &= h(y,t;\varphi_2+4k,\omega) \\ &= h(y,t;\varphi_2,\omega) + 4k. \qquad \text{(by (4.1))} \end{aligned}$$

The other inequality can be proved similarly, so the first statement holds. The second statement follows by taking $R = \infty$. □

Another property of the height process is that the vertical drift speed is linearly bounded.

Lemma 4.5 (Vertical speed bound)**.** *For any $\varphi \in \Phi$, $x \in \mathbb{Z}^2$ and $t \in \mathbb{N}$,*

$$\varphi(x) - 4t \leq h(x,t;\varphi,\omega) \leq \varphi(x).$$

Proof. This follows easily from Table 3.1 and the same table at odd faces (i,j). □

Define the space translation operator τ_y for $y \in \mathbb{Z}^2$ on both height functions h and $\omega \in \Omega$ by

$$\tau_y h(x) = h(x-y), \tau_y \omega(x,t) = \omega(x-y,t)$$

for every $x \in \mathbb{Z}^2$ and $t \in \mathbb{Z}$. Then we have

$$h(x-2y,t;\varphi,\omega) = h(x,t;\tau_{2y}\varphi,\tau_{2y}\omega) \tag{4.2}$$

for every $x, y \in \mathbb{Z}^2$, $\varphi \in \Phi$, $\omega \in \Omega$. The factor 2 is present so that $\tau_{2y}\varphi \in \Phi$.

Another observation is that h satisfies a semigroup-like property. Define the time translation operator γ_s for $s \in \mathbb{N}$ on ω, by $\gamma_s\omega(x,t) = \omega(x, t+s)$. Then for all $s, t \in \mathbb{N}$, $s \leq t$,

$$h(x,t;\varphi,\omega) = h(x, t-s; h(\cdot, s; \varphi, \omega), \gamma_s\omega). \tag{4.3}$$

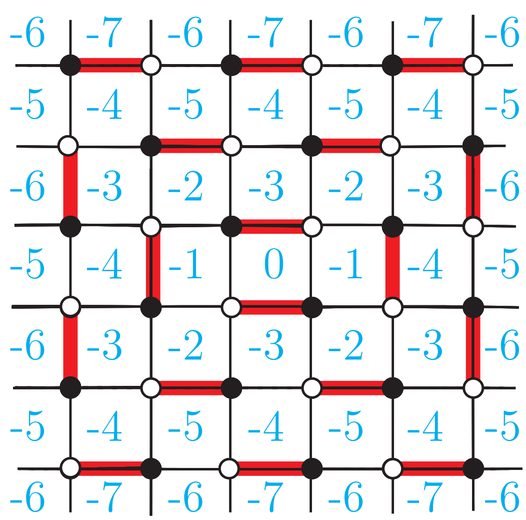
4.2 Smoothing out the height process spatially

Define the pyramid height function $v : \mathbb{Z}^2 \to \mathbb{Z}$ to be

$$v\,(x) = \min\left\{\varphi(x) : \varphi \in \Phi, \varphi(0) = 0\right\}.$$

Here we can take pointwise minimum because the height difference between 0 and x is a bounded integer. This is an admissible height function due to Lemma 4.1. To help visualize, the corresponding dimer covering is shown in Figure 4.1.

Figure 4.1: The pyramid height function v



To convince ourselves that this is exactly v, notice that from the origin to any other face, there exists a face path such that the height decreases by 1 every step. Since the height can either decrease by 1 or increase by 3 at those steps, we have the correct height at every face. Observe that $v(x) = v(-x)$.

A more constructive way to describe v that also works for different graphs G is the following. Consider the boundary vertices of the Newton polygon $\{x : |x|_1 = 2\}$. Each of them corresponds to a covering on G_1. Lift them to periodic coverings on G, and find their height functions that equal to 0 at the origin. Now take the pointwise minimum.

Fix $n \in \mathbb{Z}_{>0}$. Let us start with an input function $g \in \Gamma$. We will define a height function $\varphi_g^n \in \Phi$ close to g, and use it as the initial condition for the height process. Set

$$\begin{aligned} \Phi_g &:= \left\{ \varphi \in \Phi, \varphi = \tau_y v + k \text{ for some } y \in \mathbb{Z}^2, k \in \mathbb{Z} \text{ such that } k \leq ng\left(\frac{y}{n}\right) \right\}, \\ \varphi_g^n(x) &:= \max_{\varphi \in \Phi_g} \{\varphi(x)\}. \end{aligned}$$

To see that φ_g^n is well defined, take any $x, y \in \mathbb{Z}^2$, and $k \in \mathbb{Z}$ such that $k \leq ng\left(\frac{y}{n}\right)$. Since g is 2-spatially-Lipschitz,

$$\begin{aligned} \left| ng\left(\frac{y}{n}\right) - ng\left(\frac{x}{n}\right) \right| &\leq 2n \left| \frac{y}{n} - \frac{x}{n} \right|_\infty = 2\,|y - x|_\infty, \\ \Rightarrow ng\left(\frac{y}{n}\right) &\leq ng\left(\frac{x}{n}\right) + 2\,|y - x|_\infty. \end{aligned} \tag{4.4}$$

Then observe that $v(x) \leq -2|x|_\infty + 1$, so we have

$$\begin{aligned} (\tau_y v)(x) + k &\leq -2|x - y|_\infty + 1 + ng\left(\frac{y}{n}\right) \\ &\leq -2|x - y|_\infty + 1 + ng\left(\frac{x}{n}\right) + 2\,|y - x|_\infty = 1 + ng\left(\frac{x}{n}\right) \end{aligned} \tag{4.5}$$

where we used (4.4) for the second inequality.

On the other hand, there must exist $k \in \mathbb{Z}$ such that $ng\left(\frac{x}{n}\right) - 4 < k \leq ng\left(\frac{x}{n}\right)$ and

$\tau_x v + k \in \Phi$, and with such k,

$$(\tau_x v)(x) + k = k > ng\left(\frac{x}{n}\right) - 4. \tag{4.6}$$

Combining (4.5) and (4.6), we conclude that

$$ng\left(\frac{x}{n}\right) - 4 < \varphi_g^n(x) \le ng\left(\frac{x}{n}\right) + 1. \tag{4.7}$$

Due to Lemma 4.1, φ_g^n is an admissible height function. Furthermore, we can show that φ_g^n is determined locally by g at every x. This property will be important in Section 4.4.

Proposition 4.6. *Given $g \in \Gamma$, $\forall x \in \mathbb{Z}^2$,*

$$\varphi_g^n(x) = \varphi(x)$$

for some $\varphi \in \Phi$ such that $\varphi = \tau_y v + k$, $|y - x|_1 \le 1$, $k \le ng\left(\frac{y}{n}\right)$.

Proof. Consider some $x \in \mathbb{Z}^2$. By definition, there exists $y \in \mathbb{Z}^2$, $k \in \mathbb{Z}$ such that $\varphi_1 = \tau_y v + k \in \Phi_g$, $\varphi_g^n(x) = \varphi_1(x) = v(x-y) + k$, and

$$k \le ng\left(\frac{y}{n}\right). \tag{4.8}$$

Assume $|y - x|_1 \ge 2$, otherwise there is nothing to prove.

Again, since g is 2-spatially-Lipschitz,

$$ng\left(\frac{x}{n}\right) \ge ng\left(\frac{y}{n}\right) - 2\left|y - x\right|_\infty. \tag{4.9}$$

If $v(x-y) = -2\left|y-x\right|_\infty$, then we claim that $\varphi_2 = \tau_x v + k + v(x-y) \in \Phi_g$ and $\varphi_2(x) = \varphi_1(x)$.

Indeed, we have $\varphi_2(x) = v(x-x) + k + v(x-y) = k + v(x-y)$. Since $\varphi_1 \in \Phi$ and

$\varphi_1(x) = v(x-y)+k$, in particular φ_2 is also in Φ. Furthermore,

$$k + v(x-y) = k - 2\,|y-x|_\infty \leq ng\left(\frac{y}{n}\right) - 2\,|y-x|_\infty \leq ng\left(\frac{x}{n}\right), \tag{4.10}$$

by (4.8) and (4.9), so the claim is true. The statement then follows with candidate φ_2.

We are left with the case when $v(x-y) \neq -2\,|y-x|_\infty$. Observe that $v(x-y) = -2\,|y-x|_\infty$ iff $y-x=(i,j)$ with $i+j$ even. Therefore, if $v(x-y) = -2\,|y-x|_\infty$ does not hold for some x, there is a neighboring face x' of x, such that $|x'-y|_\infty = |x-y|_\infty - 1$, and $v(x'-y) = -2\,|y-x'|_\infty$. Let $\varphi_3 = \tau_{x'}v+k+v(x'-y)$. Since $\varphi_3(x') = v(x'-x')+k+v(x'-y) = k+v(x'-y) = \varphi_1(x')$, by the same argument as above, $\varphi_3 \in \Phi_g$.

We have $\varphi_3(x) = \tau_{x'}v(x)+k+v(x'-y) = v(x-x')+k+v(x'-y)$. With the help of Figure 4.1, it is easy to verify that $v(x-x')+v(x'-y) = v(x-y)$, knowing that $|y-x|_1 \geq 2$, $y-x=(i,j)$ with $i+j$ odd, $|x'-x|_1 = 1$, and $|x'-y|_\infty = |x-y|_\infty - 1$. Therefore, the statement holds with candidate φ_3. $\square$

With some fixed $s,t \in \mathbb{N}$, $s<t$, $g \in \Gamma$, $\omega \in \Omega$, consider the height process

$$h\left(x, t-s; \varphi_g^n, \gamma_s\omega\right)$$

as a function of $x \in \mathbb{Z}^2$ only. Its direct linear interpolation is not in Γ, because when x changes by 1, the function might change by 3. Instead, we define a new function $\psi_{s,t}$ such that for all $x \in \mathbb{Z}^2$,

$$\psi_{s,t}(2x) := h\left(2x, t-s; \varphi_g^n, \gamma_s\omega\right). \tag{4.11}$$

The function $\psi_{s,t}$, for now, is only defined on $2\mathbb{Z}^2$, where $k\mathbb{Z}^2$ for some constant k denotes the set $\{(ki,kj) : (i,j) \in \mathbb{Z}^2\}$. In other words, $\psi_{s,t}$ agrees with the height process at all the T-translations of the origin. Then $\psi_{s,t}$ is 2-spatially-Lipschitz on $2\mathbb{Z}^2$, by checking the pyramid height function.

We want to further interpolate $\psi_{s,t}$ to a 2-spatially-Lipschitz function $\mathbb{R}^2 \to \mathbb{R}$. More specifically, given the heights at the four faces, listed in counterclockwise order,

$$(2i, 2j), (2i + 2, 2j), (2i + 2, 2j + 2), (2i, 2j + 2),$$

we first interpolate $\psi_{s,t}$ along the four sides linearly. Inside the square $[2i, 2i+2] \times [2j, 2j+2]$, we need to be careful about interpolating $\psi_{s,t}$ to keep it 2-spatially-Lipschitz. We specify the interpolation in one such square. Suppose we know the values of a function f at, for example, the four points $(0, 0)$, $(2, 0)$, $(2, 2)$, $(0, 2)$, and want to interpolate f inside the square formed by these four points. Up to symmetry and a vertical shift, we shall assume that $f(0, 0) = 0$, f is nonnegative at the other three points, and $f(2, 0) \leq f(0, 2)$. We consider all possible cases:

Case 1: If $f(2, 0) = f(0, 2) = f(2, 2) = 0$, we let $f = 0$ inside the square.

Case 2: If $f(2, 0) = f(0, 2) = 0$, and $f(2, 2) = 4$, for $(x, y) \in [0, 2]^2$, we let $f(x, y) = 2y$ if $x \geq y$, and $f(x, y) = 2x$ if $x < y$.

Case 3: If $f(2, 0) = 0$, $f(0, 2) = 4$ and $f(2, 2) = 0$, this is same as Case 2 under rotation.

Case 4: If $f(2, 0) = 0$, $f(0, 2) = 4$ and $f(2, 2) = 4$, define $f(x, y) = 2y$.

Case 5: If $f(2, 0) = f(0, 2) = 4$ and $f(2, 2) = 0$, for $(x, y) \in [0, 2]^2$, we let $f(x, y) = 2x$ for $y \leq \min\{2-x, x\}$, $f(x, y) = 4-2y$ for $2-x \leq y \leq x$, $f(x, y) = 4-2x$ for $y \geq \max\{2-x, x\}$, and $f(x, y) = 2y$ for $x \leq y \leq 2 - x$.

Case 6: If $f(2, 0) = f(0, 2) = f(2, 2) = 4$, this is an inverted version of Case 2.

Notice that in all the cases, the interpolation is linear on the four edges of the 2×2 square. Therefore, we can glue together the interpolations on all the 2×2 squares to obtain a global 2-spatially Lipschitz interpolation. Also the interpolation is monotone in the four corner values.

Due to the finiteness of the fundamental domain,

$$\left|\psi_{s,t}(x) - h\left(x, t-s; \varphi_g^n, \gamma_s\omega\right)\right| < C_0, \forall x \in \mathbb{Z}^2 \tag{4.12}$$

for some global constant $C_0 > 0$ that comes from the interpolation.

Now given $s, t \in \frac{1}{n}\mathbb{N}$, $s < t$, we define

$$S_n(s,t;g,\omega)(x) := \frac{1}{n}\psi_{ns,nt}(nx).$$

Since $\psi_{ns,st}$ is in Γ, $S_n(s,t;g,\omega)$ is in Γ as well. From (4.12), we deduce that

$$\left|S_n(s,t;g,\omega)(x) - \frac{1}{n}h\left(nx, n(t-s); \varphi_g^n, \gamma_{ns}\omega\right)\right| < \frac{C_0}{n}, \forall x \in \frac{1}{n}\mathbb{Z}^2. \tag{4.13}$$

Whenever $s, t \in \frac{1}{n}\mathbb{N}$, $s \geq t$, simply define

$$S_n(s,t;g,\omega)(x) := g.$$

Next, we wish to interpolate S_n with respect to the time variables s and t. We want the interpolation to be continuous (even Lipschitz) in s and t, but for this to make sense, we have to specify the image space and the topology on it.

4.3 The space of continuous evolutions

Following [26], we first define a general function space where we will embed the fixed-time evolutions of both the interpolated shuffling height process and the PDE.

Given $g_1, g_2 \in \Gamma$ and $k \in \mathbb{N} \cup \{\infty\}$, let

$$\|g_1 - g_2\|_k := \sup_{|x|_1 \leq k} |g_1(x) - g_2(x)|, \tag{4.14}$$

$$d(g_1, g_2) := \sum_{i=1}^{\infty} 2^{-i} \|g_1 - g_2\|_i. \tag{4.15}$$

By definition of Γ,

$$\begin{aligned}
\|g_1 - g_2\|_k &\leq \sup_{|x|_\infty \leq k} |g_1(x) - g_2(x)| \\
&\leq |g_1(0) - g_2(0)| + \sup_{|x|_\infty \leq k} |g_1(x) - g_1(0)| + \sup_{|x|_\infty \leq k} |g_2(x) - g_2(0)| \\
&\leq |g_1(0) - g_2(0)| + 4k.
\end{aligned} \tag{4.16}$$

So $\|g_1 - g_2\|_k$ grows at most linearly with respect to k, and the sum in (4.15) converges. It is clear that $d(g_1, g_2) = 0$ iff $g_1 = g_2$, and the triangle inequality is easy to check, so (4.15) defines a metric on Γ.

Let $\mathscr{E}_q^r$ $(r, q \geq 0)$ denote the space of functions $F : \Gamma \to \Gamma$ with the following properties:

1. $F(g + m) = F(g) + m$ for every constant m. (4.17)
2. $F(g_1) \leq F(g_2)$ whenever $g_1 \leq g_2$. (4.18)
3. $\sup_{g \in \Gamma} \|F(g) - g\|_0 \leq r < \infty$. (4.19)
4. If $g_1(x) = g_2(x)$ for all x with $|x|_1 \leq R,$ where $R \geq q,$ then $F(g_1)(x) = F(g_2)(x)$ for all x with $|x|_1 \leq R - q$. (4.20)

Remark 4.7. Notice the similarities between these properties and (4.1), Lemma 4.2, Lemma 4.5, Lemma 4.3.

We can define a metric D on $\mathscr{E}_q^r$. Given $F_1, F_2 \in \mathscr{E}_q^r$, let

$$\|F_1 - F_2\|_k := \sup_{g\in\Gamma} \|F_1(g) - F_2(g)\|_k,$$
$$D(F_1, F_2) := \sum_{i=1}^{\infty} 2^{-i} \frac{\|F_1 - F_2\|_i}{1 + \|F_1 - F_2\|_i}. \tag{4.21}$$

By (4.16),

$$\begin{aligned}\|F_1(g) - F_2(g)\|_k &\leq |F_1(g)(0) - F_2(g)(0)| + 4k \\ &= |(F_1(g)(0) - g(0)) - (F_2(g)(0) - g(0))| + 4k \\ &\leq 2r + 4k\end{aligned}$$

by Property (4.19). So $\|F_1 - F_2\|_k$ is finite, and $D(F_1, F_2)$ is well defined. The triangle inequality is easy to check, and $D(F_1, F_2) = 0$ iff $F_1(g) = F_2(g)$ for every $g \in \Gamma$.

There is a "localization" lemma for functions in $\mathscr{E}$.

Lemma 4.8 (Localization)**.** *Given any $F \in \mathscr{E}_q^r$, $g_1, g_2 \in \Gamma$, $R \geq q$,*

$$\|F(g_1) - F(g_2)\|_{R-q} \leq \|g_1 - g_2\|_R.$$

In particular, $\|F(g_1) - F(g_2)\|_\infty \leq \|g_1 - g_2\|_\infty$.

Proof. We first show that Γ is closed under $\vee$ and $\wedge$ operations. Given $g_1, g_2 \in \Gamma$, $x, y \in \mathbb{R}^2$, if $g_1(x) \geq g_2(x)$, $g_1(y) \geq g_2(y)$, then

$$|(g_1 \vee g_2)(x) - (g_1 \vee g_2)(y)| = |g_1(x) - g_1(y)| \leq 2|x - y|_\infty.$$

If $g_1(x) \geq g_2(x)$, $g_1(y) \leq g_2(y)$, then

$$|(g_1 \vee g_2)(x) - (g_1 \vee g_2)(y)| = |g_1(x) - g_2(y)|.$$

Since

$$-2|x-y|_\infty \leq g_1(y) - g_1(x) \leq g_2(y) - g_1(x) \leq g_2(y) - g_2(x) \leq 2|x-y|_\infty,$$

we duduce that

$$|(g_1 \vee g_2)(x) - (g_1 \vee g_2)(y)| \leq 2|x-y|_\infty.$$

The other two cases are similar, so we conclude that Γ is closed under $\vee$. Taking negative shows closedness under $\wedge$. By Remark 4.7, the rest of the proof proceeds in exactly the same way as the proof of Proposition 4.4, using Properties (4.17), (4.18) and (4.20). □

Lemma 4.9. *Functions F in $\mathscr{E}_q^r$ are 2^q-Lipschitz continuous.*

Proof. Suppose $g_1, g_2 \in \Gamma$ satisfy that $d(g_1, g_2) \leq \delta$. By Lemme 4.8,

$$\begin{aligned} d(F(g_1), F(g_2)) &= \sum_{i=1}^{\infty} 2^{-i} \|F(g_1) - F(g_2)\|_i \\ &\leq \sum_{i=1}^{\infty} 2^{-i} \|g_1 - g_2\|_{i+q} = 2^q \sum_{i=1}^{\infty} 2^{-i-q} \|g_1 - g_2\|_{i+q} \\ &\leq 2^q d(g_1, g_2). \end{aligned}$$

□

Lemma 4.10. *The space $\mathscr{E}_q^r$ is compact.*

Although we are working with a different functional space, the proof of this lemma is essentially the same as that of Lemma 3.2 in [26]. We include a sketch of the proof here for completeness.

Proof. We need to show that $\mathscr{E}_q^r$ is closed and totally bounded. To show that it is closed, suppose F_n is a Cauchy sequence in $\mathscr{E}_q^r$. Then by definition of the metric (4.21), $F_n(g)(x)$ is also a Cauchy sequence for every $g \in \Gamma$ and $x \in \mathbb{R}^2$. Therefore, we can define a limiting functional F so that $F(g)(x) = \lim_{n\to\infty} F_n(g)(x)$. It is easy to check that F lies in $\mathscr{E}_q^r$, so $\mathscr{E}_q^r$ is indeed closed.

To show totally-boundedness, we would like to choose a finite set of functions in $\mathscr{E}_q^r$, so that every $F \in \mathscr{E}_q^r$ can be approximated by at least one of them up to a required precision. Due to Property (4.17), a function $F \in \mathscr{E}_q^r$ is completely characterized by its image of $g \in \Gamma_0$, where Γ_0 denotes the subset of Γ that satisfies $g(0) = 0$. Given a positive integer k, due to the 2-spatially-Lipschitz-continuity, there exists a finite subset A_k of Γ_0 such for every $g \in \Gamma_0$, there exists $a \in A_k$ such that $\|g - a\|_k \le k^{-1}$. Let Σ_k denote the finite set of all tuples $\sigma = (\sigma^1, \sigma^2)$ where

$$\sigma^1 : A_{k+q} \to A_k, \quad \sigma^2 : A_{k+q} \to \{-\lfloor rk \rfloor, -\lfloor rk \rfloor + 1, \cdots, \lceil rk \rceil\}.$$

Given $\sigma \in \Sigma_k$, we define $\mathscr{A}_\sigma$ to be the subset of $F \in \mathscr{E}_q^r$ such that

$$\|F(a) - F(a)(0) - \sigma^1(a)\|_k \le k^{-1}, \quad F(a)(0) \in \left[\frac{\sigma^2(a)}{k}, \frac{\sigma^2(a)+1}{k}\right)$$

for every $a \in A_{k+q}$. For every nonempty $\mathscr{A}_\sigma$, we pick a representative F_σ. These form the finite set that we desired. To see this, for any $F \in \mathscr{E}_q^r$, we can find $\sigma \in \Sigma_k$ such that $F \in \mathscr{A}_\sigma$. For any $g \in \Gamma_0$, there exists $a \in A_{k+q}$ such that $\|g - a\|_{k+q} \le (k+q)^{-1}$. By Lemma 4.8, $\|F(g) - F(a)\|_k \le (k+q)^{-1}$, and similarly for F_σ. On the other hand, $\|F(a) - F_\sigma(a)\|_k = O(k^{-1})$ because both F and F_σ belong to $\mathscr{A}_\sigma$. Therefore by the triangle inequality, $D(F, F_\sigma) = O\left(k^{-1}\right)$. $\square$

Now let $\mathscr{C}_T := C\left([0,T] \times [0,T]; \mathscr{E}_q^r\right) = C\left([0,T]; C\left([0,T]; \mathscr{E}_q^r\right)\right)$ be the space of continuous functions $S : [0,T] \times [0,T] \to \mathscr{E}_q^r$, with the uniform topology, where the distance between two functions $R_1, R_2 \in \mathscr{C}_T$ is given by

$$\rho(R_1, R_2) := \sup_{s,t \in [0,T]^2} D\left(R_1(s,t), R_2(s,t)\right). \tag{4.22}$$

It is well known that if Y is a Polish space (separable complete metric space), then $C([0,T]; Y)$ is also a Polish space (see for example [14, Theorem 4.19]). Then since $\mathscr{E}_q^r$ is compact, it is in particular Polish. Thus $\mathscr{C}_T$ is also Polish.

4.4 Smoothing out the height process temporally

Having the abstract space set up, we return to the function $S_n(s,t;g,\omega)(x)$ defined previously. When $s < t$, the function $S_n(s,t;\cdot,\omega)$ does not belong to $\mathscr{E}_q^r$ since the value of $S_n(s,t;g,\omega)(2x)$ at $x \in \frac{1}{n}\mathbb{Z}^2$ belongs to $\frac{1}{n}\mathbb{Z}$, while the constant m in Property (4.17) of $\mathscr{E}_q^r$ is arbitrary, which is used in the proof of Lemma 4.8 and Lemma 4.10. To force Property (4.17), we could consider a modified function $g \mapsto S_n(s,t;g - g(0),\omega) + g(0)$, but then one can check that Property (4.18) no longer holds. To guarantee both Property (4.17) and 4.18, we consider the following modification:

$$\widehat{S}_n(s,t;g,\omega) := \frac{1}{4}\int_{u=0}^{4} \left[S_n\left(s,t;g - g(0) - \frac{u}{n},\omega\right) + \frac{u}{n}\right] du + g(0).$$

Roughly speaking, we are averaging the result of S_n across a vertical range of $\frac{4}{n}$, so it is easy to see that $\widehat{S}_n(s,t;\cdot,\omega)$ still maps Γ to Γ.

First, we prove a couple of lemmas, which will be used to show that $\widehat{S}_n$ is in $\mathscr{E}_q^r$, and also later in the thesis.

Lemma 4.11. *For any $s, t \in \frac{1}{n}\mathbb{N}$ such that $s < t$, and $g \in \Gamma$, we have*

$$\|S_n(s, t; g, \omega) - g\|_\infty \leq \frac{6 + C_0}{n} + 4(t - s).$$

Proof. From (4.7), it follows that $|ng(x) - \varphi_g^n(x)| \leq 4$ for all $x \in \mathbb{Z}^2$. And by Lemma 4.5, for all $x \in \frac{1}{n}\mathbb{Z}^2$,

$$\left|\varphi_g^n(x) - h(nx, n(t - s); \varphi_g^n, \gamma_{ns}\omega)\right| \leq 4n(t - s).$$

Combined with (4.13), we get for all $x \in \frac{1}{n}\mathbb{Z}^2$,

$$|S_n(s, t; g, \omega)(x) - g(x)| \leq \frac{4 + C_0}{n} + 4(t - s).$$

Finally, since S_n and g are both in Γ, we deduce that for all $x \in \mathbb{R}^2$,

$$|S_n(s, t; g, \omega)(x) - g(x)| \leq \frac{4 + C_0 + 2}{n} + 4(t - s).$$

□

Lemma 4.12. *Given $x \in \mathbb{R}^2$ and $g_1, g_2 \in \Gamma$, if $g_1(y) = g_2(y)$ for all y such that $|y - x|_1 \leq R$, then $S_n(s, t; g_1, \omega)(y) = S_n(s, t; g_2, \omega)(y)$ for all y such that $|y - x|_1 \leq R - 2(t - s) - \frac{9}{n}$.*

Proof. Assuming that $g_1(y) = g_2(y)$ for all $y \in \mathbb{R}^2$ such that $|y - x|_1 \leq R$, by Proposition 4.6, we have $\varphi_{g_1}^n(z) = \varphi_{g_2}^n(z)$ for all $z \in \mathbb{Z}^2$ with $|\frac{z}{n} - x|_1 \leq R - \frac{1}{n}$. This is equivalent to

$$|z - nx|_1 = |(z - \lfloor nx \rfloor) + (\lfloor nx \rfloor - nx)|_1 \leq nR - 1.$$

Since $|\lfloor nx \rfloor - nx|_1 \leq 2$, we know that $\varphi_{g_1}^n(z) = \varphi_{g_2}^n(z)$ for all $z \in \mathbb{Z}^2$ with $|z - \lfloor nx \rfloor|_1 \leq nR - 3$.

Applying Lemma 4.3, we have

$$h(ny, nt-ns; \varphi^n_{g_1}, \gamma_{ns}\omega) = h(ny, nt-ns; \varphi^n_{g_2}, \gamma_{ns}\omega)$$

for all $y \in \frac{1}{n}\mathbb{Z}^2$ such that $|ny - \lfloor nx \rfloor|_1 \leq nR - 3 - 2n(t-s)$. By construction, the value of S_n at a point $y \in \mathbb{R}^2$ is determined by the value of h at $(2i, 2j), (2i+2, 2j), (2i+2, 2j+2), (2i, 2j+2)$ such that $i, j \in \mathbb{Z}$, and the square formed by these four points contains ny. Therefore, $S_n(s, t; g_1, \omega)(y) = S_n(s, t; g_2, \omega)(y)$ for all $y \in \mathbb{R}^2$ such that

$$\begin{aligned} |ny - \lfloor nx \rfloor|_1 &\leq nR - 3 - 2n(t-s) - 4 \\ |n(y-x) + (nx - \lfloor nx \rfloor)|_1 &\leq nR - 7 - 2n(t-s). \end{aligned}$$

Since $|nx - \lfloor nx \rfloor|_1 \leq 2$, we conclude that $S_n(s, t; g_1, \omega)(y) = S_n(s, t; g_2, \omega)(y)$ for all y such that

$$\begin{aligned} |n(y-x)|_1 &\leq nR - 7 - 2n(t-s) - 2 \\ |y-x|_1 &\leq R - 2(t-s) - \frac{9}{n}. \end{aligned}$$

$\square$

Proposition 4.13. *Given $T > 0$, there exist universal constants $r, q > 0$ such that $\widehat{S}_n(s, t; \cdot, \omega) \in \mathscr{E}^r_q$, where r, q will be specified below, for all $n \in \mathbb{Z}_{>0}$ and $s, t \in [0, T] \cap \frac{1}{n}\mathbb{N}$.*

Proof. We assume that $s < t$, for otherwise $S_n(s, t; \cdot, \omega)$ and thus $\widehat{S}_n(s, t; \cdot, \omega)$ is simply identity.

Property (4.17) holds because for any constant m,

$$\begin{aligned}\widehat{S}_n(s,t;g+m,\omega) &= \frac{1}{4}\int_{u=0}^{4}\left[S_n\left(s,t;g+m-g(0)-m-\frac{u}{n},\omega\right)+\frac{u}{n}\right]du+g(0)+m\\ &= \widehat{S}_n(s,t;g,\omega)+m.\end{aligned}$$

From Lemma 4.11, we know that $|S_n(s,t;g,\omega)(0)-g(0)-u| \le (6+C_0+4)/n+4(t-s) \le 10+C_0+4T$ for all $u \in [0,\frac{n}{4})$, so by taking the integral over u, Property (4.19) holds for $\widehat{S}_n(s,t;\cdot,\omega)$ with $r = 10+C_0+4T$.

We confirm that Property (4.20) holds with $q = 2T+9$ by specializing Lemma 4.12 to $x = 0$.

Finally, to verify Property (4.18), suppose $g_1, g_2 \in \Gamma$ satisfy that $g_1 \le g_2$. We will construct a measure-preserving bijection $\eta : [0,4) \to [0,4)$. We set up some notations for convenience:

$$\begin{aligned}\delta_0 &:= g_2(0)-g_1(0), \quad \delta := \inf_{x\in\mathbb{R}^2}\{g_2(x)-g_1(x)\} \ge 0 \in [0,+\infty),\\ f_1 &:= g_1-g_1(0), \quad f_2 := g_2-g_2(0).\end{aligned}$$

Let us consider

$$\begin{aligned}&\left[S_n\left(s,t;g_2-g_2(0)-\frac{u}{n},\omega\right)+\frac{u}{n}+g_2(0)\right]-\left[S_n\left(s,t;g_1-g_1(0)-\frac{\eta(u)}{n},\omega\right)+\frac{\eta(u)}{n}+g_1(0)\right]\\ =&S_n\left(s,t;f_2-\frac{u}{n},\omega\right)-S_n\left(s,t;f_1-\frac{\eta(u)}{n},\omega\right)+\frac{u-\eta(u)}{n}+\delta_0. \qquad (4.23)\end{aligned}$$

We define $\eta(u)$ to be the unique element in $[0,4)$ such that $\delta-\delta_0+\frac{\eta(u)-u}{n}$ is an integer multiple of $\frac{4}{n}$, say $\frac{4k}{n}$ for some $k \in \mathbb{Z}$.

By construction of S_n,

$$S_n\left(s,t;f_2-\frac{u}{n},\omega\right)=S_n\left(s,t;f_2-\frac{u}{n}-\frac{4k}{n},\omega\right)+\frac{4k}{n}, \tag{4.24}$$

and

$$\begin{aligned}\left(f_2-\frac{u}{n}-\frac{4k}{n}\right)-\left(f_1-\frac{\eta(u)}{n}\right)&=g_2-g_1-(g_2(0)-g_1(0))+\frac{\eta(u)-u}{n}-\frac{4k}{n}\\&\geq\delta-\delta_0+\frac{\eta(u)-u}{n}-\frac{4k}{n}=0.\end{aligned}$$

Therefore again by construction of S_n and Lemma 4.2,

$$S_n\left(s,t;f_2-\frac{u}{n}-\frac{4k}{n},\omega\right)-S_n\left(s,t;f_1-\frac{\eta(u)}{n},\omega\right)\geq 0. \tag{4.25}$$

Putting (4.23), (4.24),(4.25) together, we get that

$$\text{LHS of (4.23)}\geq\frac{4k}{n}+\frac{u-\eta(u)}{n}+\delta_0=\delta\geq 0.$$

Now integrating LHS of (4.23) over $u\in[0,4)$, we conclude that $\widehat{S}_n(s,t;g_1,\omega)\leq\widehat{S}_n(s,t;g_2,\omega)$.

□

So far $\widehat{S}_n(s,t;\cdot,\omega)\in\mathscr{E}_q^r$ is only defined for $s,t\in\frac{1}{n}\mathbb{N}$. We would like to interpolate it to a Lipschitz continuous function in $(s,t)\in[0,T]^2$ with a universal Lipschitz constant (taking ℓ_1 metric on $[0,T]^2$), and treat it as an element of $\mathscr{C}_T$. We shall first check the Lipschitz continuity of $\widehat{S}_n(s,t;\cdot,\omega)$ on $(s,t)\in\frac{1}{n}\mathbb{N}^2$, and then interpolate bilinearly to the entire $[0,T]^2$. It suffices to consider S_n since $\widehat{S}_n$ is just an average of S_n.

Proposition 4.14. *For all $n \in \mathbb{Z}_{>0}$, $(s,t) \in \frac{1}{n}\mathbb{N}^2$, $g \in \Gamma$,*

$$\left\| S_n(s,t;g,\omega) - S_n\left(s, t+\frac{1}{n}; g,\omega\right)\right\|_\infty \leq \frac{10+2C_0}{n},$$
$$\left\| S_n(s,t;g,\omega) - S_n\left(s+\frac{1}{n}, t; g,\omega\right)\right\|_\infty \leq \frac{10+2C_0}{n}.$$

Proof. To check Lipschitz continuity with respect to t, take $F_1 = S_n(s,t;\cdot,\omega)$ amd $F_2 = S_n\left(s,t+\frac{1}{n};\cdot,\omega\right)$ where (s,t) and $\left(s,t+\frac{1}{n}\right)$ are both in $\frac{1}{n}\mathbb{Z}^2_{\geq 0}$. When $s=t$, $F_1(g)=g$. Then Lemma 4.11 implies that

$$\|F_1(g) - F_2(g)\|_\infty = \|g - F_2(g)\|_\infty \leq \frac{6+C_0}{n} + \frac{4}{n} = \frac{10+C_0}{n}, \quad \forall g \in \Gamma.$$

When $s<t$, given an input g, F_1 and F_2 are computed from the same φ^n_g. The outputs were interpolated from $h(nx, n(t-s); \varphi^n_g, \gamma_{ns}\omega)$ and $h(nx, n(t-s)+1; \varphi^n_g, \gamma_{ns}\omega)$ respectively. Applying (4.3) and Lemma 4.5, these two height functions differ by at most 4 everywhere. Thus by (4.13), $\|F_1(g) - F_2(g)\| \leq (4+2C_0+2)\frac{1}{n}$. When $s>t$ both functions are identity. Combining the bounds, we obtain the first inequality.

Checking Lipschitz continuity with respect to s is somewhat different. This time, take $F_1 = S_n(s,t;\cdot,\omega)$ amd $F_2 = S_n\left(s+\frac{1}{n},t;\cdot,\omega\right)$ where (s,t) and $\left(s+\frac{1}{n},t\right)$ both lie in $\frac{1}{n}\mathbb{N}^2$. If $s+\frac{1}{n} = t$, $F_2(g) = g$, so again Lemma 4.11 implies that $\|F_1(g) - F_2(g)\|_\infty \leq (10 + C_0)\frac{1}{n}$. When $s+\frac{1}{n} < t$, $F_1(g)$ and $F_2(g)$ are interpolated from $h(nx, n(t-s); \varphi^n_g, \gamma_{ns}\omega)$ and $h(nx, n(t-s)-1; \varphi^n_g, \gamma_{ns+1}\omega)$ respectively. By (4.3),

$$h(nx, n(t-s); \varphi^n_g, \gamma_{ns}\omega) = h(nx, n(t-s)-1; h(\cdot, 1; \varphi^n_g, \gamma_{ns}\omega), \gamma_{ns+1}\omega),$$

and furthermore by Lemma 4.5,

$$|h(nx, 1; \varphi_g^n, \gamma_{ns}\omega) - \varphi_g^n(nx)| \le 4, \qquad \forall x \in \frac{1}{n}\mathbb{Z}^2.$$

Combined with Proposition 4.4, we obtain that

$$|h(nx, n(t-s); \varphi_g^n, \gamma_{ns}\omega) - h(nx, n(t-s)-1; \varphi_g^n, \gamma_{ns+1}\omega)| \le 4, \qquad \forall x \in \frac{1}{n}\mathbb{Z}^2.$$

Finally using (4.13), we conclude that $\|F_1(g) - F_2(g)\|_\infty \le (4 + 2C_0 + 2)\frac{1}{n}$. When $s \ge t$, both functions are identity. As a result, we get the second inequality. $\square$

The last step is to interpolate $\widehat{S}_n$ to continuous time bilinearly. To be more specific, for each $(s,t) \in \frac{1}{n}\mathbb{N}^2$ such that $0 \le s, t \le T - \frac{1}{n}$, and each $g \in \Gamma$, let

$$\begin{aligned} f_{00} &= \widehat{S}_n(s,t;g,\omega), & f_{10} &= \widehat{S}_n\left(s + \tfrac{1}{n}, t; g, \omega\right), \\ f_{01} &= \widehat{S}_n\left(s, t + \tfrac{1}{n}; g, \omega\right), & f_{11} &= \widehat{S}_n\left(s + \tfrac{1}{n}, t + \tfrac{1}{n}; g, \omega\right). \end{aligned}$$

Then for each $(x,y) \in [0,1]^2$, define

$$\widehat{S}_n\left(s + \frac{x}{n}, t + \frac{y}{n}; g, \omega\right) := f_{00} + (f_{10} - f_{00})x + (f_{01} - f_{00})y + (f_{00} + f_{11} - f_{01} - f_{10})xy.$$

It is easy to see that $\widehat{S}_n(s,t;g,\omega)$ assumes the original values at $(s,t) \in \frac{1}{n}\mathbb{N}^2$, and is linear in (s,t) whenever s or t is constant, with slopes bounded by the constant $10 + 2C_0$ from Proposition 4.14. In particular, $\forall (s_1, t_1), (s_2, t_2) \in \mathbb{R}^2$,

$$\|\widehat{S}_n(s_1, t_1; g, \omega) - \widehat{S}_n\left(s_2, t_2; g, \omega\right)\|_\infty \le (10 + 2C_0)(|s_1 - s_2| + |t_1 - t_2|). \tag{4.26}$$

Moreover, this linear property shows that $\widehat{S}_n(s,t;\cdot,\omega)$ is in $\mathscr{E}_q^r$ for all $(s,t) \in [0,T]^2$. To summarize, we have the following statement.

Proposition 4.15. *For all* $n \in \mathbb{Z}_{>0}$, $\widehat{S}_n(s,t;\cdot,\omega)$ *as a function of* $(s,t) \in [0,T]^2$ *is an element of* $\mathscr{C}_T$ *with Lipschitz constant* $10 + 2C_0$, *where* $[0,T]^2$ *is equipped with* ℓ_1 *metric, i.e.*

$$D\left(\widehat{S}_n(s_1,t_1;\cdot,\omega), \widehat{S}_n\left(s_2,t_2;\cdot,\omega\right)\right) \le (10+2C_0)(|s_1-s_2| + |t_1-t_2|).$$

Even though we will define the limit using $\widehat{S}_n$, we are not deviating much from S_n based on the following proposition.

Proposition 4.16. *For all* $(s,t) \in \frac{1}{n}\mathbb{N}^2$,

$$\left\|\widehat{S}_n(s,t;g,\omega) - S_n\left(s,t;g,\omega\right)\right\|_\infty \le \frac{18+2C_0}{n}. \tag{4.27}$$

Proof. Suppose k is the largest integer such that $4k/n \le g(0)$, and given any $u \in [0,4)$, we have $\varphi^n_{g-g(0)-u/n} + 4k = \varphi^n_{g-g(0)-u/n+4k/n}$. By (4.1) and the construction of S_n, for all $(s,t) \in \frac{1}{n}\mathbb{N}^2$,

$$S_n\left(s,t;g-g(0)-\frac{u}{n},\omega\right) + \frac{4k}{n} = S_n\left(s,t;g-g(0)-\frac{u}{n}+\frac{4k}{n},\omega\right). \tag{4.28}$$

Since $|4k/n - g(0) - u/n| \le 4/n$, by (4.7), $|\varphi^n_{g-g(0)-u/n+4k/n} - \varphi^n_g| \le 4+4+4 = 12$. Then as before, invoking Proposition 4.4 and (4.13), we deduce that $\forall (s,t) \in \frac{1}{n}\mathbb{N}^2$

$$\left\|S_n\left(s,t;g-g(0)-\frac{u}{n}+\frac{4k}{n},\omega\right) - S_n\left(s,t;g,\omega\right)\right\|_\infty \le \frac{1}{n}\left(12+2C_0+2\right). \tag{4.29}$$

On the other hand, by definition

$$\widehat{S}_n(s,t;g,\omega) = \frac{1}{4}\int_{u=0}^{4}\left[S_n\left(s,t;g-g(0)-\frac{u}{n},\omega\right) + \frac{4k}{n} + \left(\frac{u}{n}+g(0)-\frac{4k}{n}\right)\right]du,$$

so combining (4.28) and (4.29), we have

$$\left\|\widehat{S}_n(s,t;g,\omega) - S_n\left(s,t;g,\omega\right)\right\|_\infty \leq \frac{1}{n}\left(12 + 2C_0 + 2 + 4\right).$$

$\square$

CHAPTER 5

Limit points

5.1 Precompactness

Throughout the previous section, we kept the Bernoulli mark $\omega \in \Omega$ fixed. From now on we shall work with the whole probability space Ω again. Then $\widehat{S}_n$ becomes a random variable on Ω, and induces a probability measure μ_n on $\mathscr{C}_T$. Since $\mathscr{C}_T$ is Polish (separable complete metric space), by Prokhorov's theorem, the family $\{\mu_n\}$ is tight iff $\{\mu_n\}$ is precompact. Recall the definition that the sequence $\{\mu_n\}$ is tight if for any $\varepsilon > 0$, there exists a compact set K_ε such that $\mu_n(K_\varepsilon) \geq 1 - \varepsilon$ for all n. On the other hand, $\{\mu_n\}$ is precompact if its closure is sequentially compact, that is, every subsequence of $\{\mu_n\}$ further contains a weakly convergent subsequence. See [1] for more background.

Let $\mathscr{C}'_T$ denote the subset of $\mathscr{C}_T$ which consists of $(10+2C_0)$-Lipschitz continuous functions on $[0, T]^2$ equipped with the ℓ_1 metric. By Proposition 4.15, $\mu_n(\mathscr{C}'_T) = 1$ for all n, so to obtain the tightness of $\{\mu_n\}$, it suffices to show that $\mathscr{C}'_T$ is compact. We shall use a generalized version of the Arzelà-Ascoli theorem (see for example [15, Theorem 7.17]).

Theorem 5.1. *A subset E of the space of continuous function from a compact Hausdorff space X to a metric space Y with uniform topology is compact iff the following three conditions hold:*

1. *E is closed,*

2. *$E(x)$ has a compact closure for every $x \in X$,*

3. *E is equicontinuous.*

Applied to $\mathscr{C}_T$, the subset $\mathscr{C}'_T$ is equicontinuous because it consists of $(10+2C_0)$-Lipschitz continuous functions. The second condition is given for free, because $\mathscr{E}_q^r$ is compact. To check that $\mathscr{C}'_T$ is closed, suppose a sequence $(F_n)_{n\in\mathbb{N}}$ in $\mathscr{C}'_T$ converges uniformly to F in $\mathscr{C}_T$. Let d_1 denote the ℓ_1 distance on $[0,T]^2$. Given $x, y \in [0,T]^2$, we have $D(F_n(x), F_n(y)) \leq (10 + 2C_0)d_1(x, y)$ since F_n is in $\mathscr{C}'_T$. Due to convergence, given $\varepsilon > 0$, there exists N such that for all $n > N$, $D(F_n(x), F(x)) < \varepsilon$ and $D(F_n(y), F(y)) < \varepsilon$. Then by triangle inequality, for $n > N$,

$$\begin{aligned} D(F(x), F(y)) &\leq D(F(x), F_n(x)) + D(F_n(x), F_n(y)) + D(F_n(y), F(y)) \\ &< 2\varepsilon + (10 + 2C_0)d_1(x, y). \end{aligned}$$

As $\varepsilon > 0$ is arbitrary, $D(F(x), F(y)) \leq (10 + 2C_0)d_1(x, y)$, and thus F is also in $\mathscr{C}'_T$.

5.2 Characterization of limit points

Now we know that the closure of $\{\mu_n\}$ is sequentially compact. In order to identify the subsequential limit points of $(\mu_n)_{n\in\mathbb{Z}_{>0}}$ as semigroups of Hamilton-Jacobi equations, we shall first make some characterization.

Since $\mathscr{C}'_T$ is compact, and $\mu_n(\mathscr{C}'_T) = 1$, we will restrict to $\mathscr{C}'_T$ from now on. Let $\mathscr{C}^0_T \subset \mathscr{C}'_T$ be the subset of elements S that satisfy the following additional conditions:

1. For all $g \in \Gamma, S(t_1, t_2; g) = g$ when $t_1 \geq t_2$, (5.1)
2. For all $g \in \Gamma$ and t_1, t_2, t_3 such that $0 \leq t_1 \leq t_2 \leq t_3 \leq T, S(t_2, t_3; S(t_1, t_2; g)) = S(t_1, t_3; g)$, (5.2)
3. Given $x \in \mathbb{R}^2$ and $g_1, g_2 \in \Gamma,$ if $g_1(y) = g_2(y)$ for all y such that $|y - x|_1 \leq R$, then $S(s, t; g_1)(y) = S(s, t; g_2)(y)$ for all y such that $|y - x|_1 \leq R - 2(t - s)$ and all s, t such that $0 \leq s < t \leq T$. (5.3)

Proposition 5.2. *All subsequential limits of $(\mu_n)_{n \in \mathbb{Z}_{>0}}$ lie in $\mathscr{C}^0_T$ with probability 1.*

When we say *subsequential limits*, we mean the limit of a weakly convergent subsequence $(\mu_{n_i})_{i \in \mathbb{N}}$ such that $(n_i)_{i \in \mathbb{N}}$ is a strictly increasing sequence of positive integers.

Proof. Suppose $(\mu_{n_i})_{i \in \mathbb{N}}$ is such a weakly convergent subsequence. For convenience, we will denote this subsequence by $(\mu_i)_{i \in \mathbb{N}}$ where $\mu_i := \mu_{n_i}$. Also denote the limiting measure by μ. By Portmanteau theorem (see for example [1, Theorem 2.1]), the weak convergence is equivalent to

$$\limsup_{i \to \infty} \mu_i(E) \leq \mu(E)$$

for all closed subset E of $\mathscr{C}'_T$.

Let E_1 denote the subset of $\mathscr{C}'_T$ that satisfies Condition (5.1). Then clearly E_1 is a closed set, as uniform convergence in $\mathscr{C}'_T$ implies pointwise convergence. Also by definition we have $\mu_i(E_1) = 1$, so $\mu(E_1) = 1$ as well.

Now we turn to Condition (5.2), which describes a semigroup property. If E_2 denotes the subset with such property, due to the discrete nature and interpolation, it is unlikely that any μ_i has probability 1 on E_2, and it is unclear what is $\limsup \mu_i(E_2)$. To work around this, given $\varepsilon > 0$, let E_2^ε denote the set of $S \in \mathscr{C}'_T$ that satisfies the following condition:

$$D(S(t_1, t_3; \cdot), S(t_2, t_3; S(t_1, t_2; \cdot))) \leq \varepsilon, \quad \forall t_1, t_2, t_3 \text{ such that } 0 \leq t_1 \leq t_2 \leq t_3 \leq T.$$

Here the distance $D(\cdot, \cdot)$ is defined by (4.21). The expression is well defined because $S(t_2, t_3; S(t_1, t_2; \cdot))$ is obviously in $\mathscr{E}_{2q}^{2r}$.

Lemma 5.3. *The subset E_2^ε is closed in $\mathscr{C}'_T$.*

Proof. Suppose $(F_n)_{n\in\mathbb{N}}$ in E_2^ε converges to $F \in \mathscr{C}'_T$, and take t_1, t_2, t_3 such that $0 \leq t_1 \leq t_2 \leq t_3 \leq T$. Given some large $k \in \mathbb{Z}_{>0}$ and small $\delta > 0$, define

$$a(k, \delta) = 2^{-k} \frac{\delta}{1+\delta}.$$

Then there exists N_a such that

$$D(F_n(s,t), F(s,t)) < a(k, \delta), \quad \forall n > N_a, (s,t) \in [0,T]^2.$$

For such n, in particular we have $D(F_n(t_1, t_2), F(t_1, t_2)) < a(k, \delta)$. This implies that

$$\sup_{g\in\Gamma} \|F_n(t_1, t_2; g) - F(t_1, t_2; g)\|_k \leq \delta. \tag{5.4}$$

We also have $D(F_n(t_2, t_3), F(t_2, t_3)) < a(k, \delta)$, which tells us that

$$\sup_{g\in\Gamma} \|F_n(t_2, t_3; F_n(t_1, t_2; g)) - F(t_2, t_3; F_n(t_1, t_2; g))\|_k \leq \delta. \tag{5.5}$$

By Lemma 4.8, if $k \geq q$, then $\forall g \in \Gamma$,

$$\|F(t_2, t_3; F_n(t_1, t_2; g)) - F(t_2, t_3; F(t_1, t_2; g))\|_{k-q} \leq \|F_n(t_1, t_2; g) - F(t_1, t_2; g)\|_k \leq \delta, \quad (5.6)$$

where the last inequality is due to (5.4). From (5.5) and (5.6), we deduce that

$$D(F_n(t_2, t_3; F_n(t_1, t_2; \cdot)), F(t_2, t_3; F(t_1, t_2; \cdot))) \leq 2\delta + 2^{-k+q}.$$

Combined with $D(F_n(t_1, t_3), F(t_1, t_3)) < a(k, \delta)$ and the fact that $F_n \in E_2^\varepsilon$, we have

$$\begin{aligned} &D(F(t_1, t_3; \cdot), F(t_2, t_3; F(t_1, t_2; \cdot))) \leq D(F(t_1, t_3; \cdot), F_n(t_1, t_3; \cdot)) + \\ &+ D(F_n(t_1, t_3; \cdot), F_n(t_2, t_3; F_n(t_1, t_2; \cdot))) + D(F_n(t_2, t_3; F_n(t_1, t_2; \cdot)), F(t_2, t_3; F(t_1, t_2; \cdot))) \\ &\leq a(k, \delta) + \varepsilon + 2\delta + 2^{-k+q}. \end{aligned}$$

By taking $k \to \infty$ and $\delta \to 0$ simultaneously, we conclude that

$$D(F(t_1, t_3; \cdot), F(t_2, t_3; F(t_1, t_2; \cdot))) \leq \varepsilon,$$

so E_2^ε is indeed closed. □

Now we want to show that $\limsup_{i\to\infty} \mu_i(E_2^\varepsilon) = 1$. In fact, we claim the following is true.

Lemma 5.4. *There exists $N > 0$ such that for all $n > N$, $\widehat{S}_n \in E_2^\varepsilon$ with probability 1.*

Proof. By definition (4.21), it suffices to show that, when n is large enough, for all t_1, t_2, t_3 such that $0 \leq t_1 \leq t_2 \leq t_3 \leq T$, $\omega \in \Omega$, $g \in \Gamma$,

$$\|\widehat{S}_n(t_1, t_3; g, \omega) - \widehat{S}_n(t_2, t_3; \widehat{S}_n(t_1, t_2; g, \omega), \omega)\|_\infty \leq \varepsilon. \quad (5.7)$$

Let $t_i' = \lfloor nt_i \rfloor \frac{1}{n}$ for $i = 1, 2, 3$. Then by (4.26), for any $g' \in \Gamma$, $i, j \in \{1, 2, 3\}$ such that $i < j$,

$$\begin{aligned}
&\|\widehat{S}_n(t_1, t_2; g', \omega) - S_n(t_i', t_j'; g', \omega)\|_\infty \\
\le &\|\widehat{S}_n(t_i, t_j; g', \omega) - \widehat{S}_n(t_i', t_j'; g', \omega)\|_\infty + \|\widehat{S}_n(t_i', t_j'; g', \omega) - S_n(t_i', t_j'; g', \omega)\|_\infty \\
\le &(10 + 2C_0)\frac{2}{n} + \frac{18 + 2C_0}{n} = \frac{38 + 6C_0}{n}.
\end{aligned} \tag{5.8}$$

where the second inequality uses (4.26) and Proposition 4.16.

Therefore, applying Lemma 4.8,

$$\begin{aligned}
&\|\widehat{S}_n(t_2, t_3; \widehat{S}_n(t_1, t_2; g, \omega), \omega) - \widehat{S}_n(t_2, t_3; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty \\
\le &\|\widehat{S}_n(t_1, t_2; g, \omega) - S_n(t_1', t_2'; g, \omega)\|_\infty \le \frac{38 + 6C_0}{n}.
\end{aligned} \tag{5.9}$$

Combining (5.8) and (5.9), we can bound

$$\begin{aligned}
&\|\widehat{S}_n(t_1, t_3; g, \omega) - \widehat{S}_n(t_2, t_3; \widehat{S}_n(t_1, t_2; g, \omega), \omega)\|_\infty \le \|\widehat{S}_n(t_1, t_3; g, \omega) - S_n(t_1', t_3'; g, \omega)\|_\infty + \\
&+ \|\widehat{S}_n(t_2, t_3; \widehat{S}_n(t_1, t_2; g, \omega), \omega) - \widehat{S}_n(t_2, t_3; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty + \\
&+ \|\widehat{S}_n(t_2, t_3; S_n(t_1', t_2'; g, \omega), \omega) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty + \\
&+ \|S_n(t_1', t_3'; g, \omega) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty \\
&\le 3\frac{38 + 6C_0}{n} + \|S_n(t_1', t_3'; g, \omega) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty.
\end{aligned}$$

Suppose n is large enough so that $3(38 + 6C_0)/n < \varepsilon/2$, then we are left to show that

$$\|S_n(t_1', t_3'; g, \omega) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty \le \varepsilon/2. \tag{5.10}$$

By (4.13), for all $g' \in \Gamma$ and $x \in \frac{1}{n}\mathbb{Z}^2$, $i, j \in \{1,2,3\}$ such that $i < j$,

$$\left| S_n(t_i', t_j'; g', \omega)(x) - \frac{1}{n} h\left(nx, n(t_j' - t_i'); \varphi_{g'}^n, \gamma_{nt_i'}\omega\right) \right| < \frac{C_0}{n}. \tag{5.11}$$

Let

$$\lambda_g := \varphi^n_{S_n(t_1', t_2'; g, \omega)}.$$

Then by (4.7), for all $x \in \frac{1}{n}\mathbb{Z}^2$,

$$\left| \frac{1}{n}\lambda_g(nx) - S_n(t_1', t_2'; g, \omega)(x) \right| \leq \frac{4}{n}. \tag{5.12}$$

Combining (5.11) and (5.12) yields

$$\left| \lambda_g(nx) - h\left(nx, n(t_2' - t_1'); \varphi_g^n, \gamma_{nt_1'}\omega\right) \right| \leq 4 + C_0, \quad \forall x \in \frac{1}{n}\mathbb{Z}^2. \tag{5.13}$$

For all $x \in \frac{1}{n}\mathbb{Z}^2$,

$$\begin{aligned}
&|S_n(t_1', t_3'; g, \omega)(x) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)(x)| \\
\leq &\left| S_n(t_1', t_3'; g', \omega)(x) - \frac{1}{n} h\left(nx, n(t_3' - t_1'); \varphi_g^n, \gamma_{nt_1'}\omega\right)(x) \right| + \\
&+ \left| S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)(x) - \frac{1}{n} h\left(nx, n(t_3' - t_2'); \lambda_g, \gamma_{nt_2'}\omega\right)(x) \right| + \\
&+ \left| \frac{1}{n} h\left(nx, n(t_3' - t_2'); \lambda_g, \gamma_{nt_2'}\omega\right)(x) - \frac{1}{n} h\left(nx, n(t_3' - t_2'); h\left(nx, n(t_2' - t_1'); \varphi_g^n, \gamma_{nt_1'}\omega\right), \gamma_{nt_2'}\omega\right)(x) \right| + \\
&+ \left| \frac{1}{n} h\left(nx, n(t_3' - t_1'); \varphi_g^n, \gamma_{nt_1'}\omega\right)(x) - \frac{1}{n} h\left(nx, n(t_3' - t_2'); h\left(nx, n(t_2' - t_1'); \varphi_g^n, \gamma_{nt_1'}\omega\right), \gamma_{nt_2'}\omega\right)(x) \right| \\
&\leq \frac{2C_0}{n} + \frac{4 + C_0}{n} + 0 = \frac{4 + 3C_0}{n},
\end{aligned}$$

where in the second inequality we used (5.11) on the first two terms, (5.13) and Proposition 4.4 on the third term, and (4.3) on the fourth term. We also know that $S_n(s, t; g', \omega) \in \Gamma$ for any $s, t \in \frac{1}{n}\mathbb{N}$ and $g' \in \Gamma$, so

$$\|S_n(t_1', t_3'; g, \omega) - S_n(t_2', t_3'; S_n(t_1', t_2'; g, \omega), \omega)\|_\infty \leq \frac{4 + 3C_0 + 2}{n} = \frac{6 + 3C_0}{n}.$$

Again choosing n large enough, we can make sure $(6 + 3C_0)/n < \varepsilon/2$. □

Lemma 5.3 and Lemma 5.4 together imply that $\mu(E_2^\varepsilon) = 1$. Since $\varepsilon > 0$ can be arbitrarily small, we conclude that μ-almost surely, $\forall t_1, t_2, t_3$ such that $0 \leq t_1 \leq t_2 \leq t_3 \leq T$,

$$D(S(t_1, t_3; \cdot), S(t_2, t_3; S(t_1, t_2; \cdot))) = 0,$$

which implies Condition (5.2)

$$S(t_1, t_3; g) = S(t_2, t_3; S(t_1, t_2; g)), \quad \forall g \in \Gamma.$$

For Condition (5.3), define E_3^ε to be the set of $S \in \mathscr{C}_T'$ with the following property: given $x \in \mathbb{R}^2$ and $g_1, g_2 \in \Gamma$, if $g_1(y) = g_2(y)$ for all y such that $|y - x|_1 \leq R$, then $S(s, t; g_1)(y) = S(s, t; g_2)(y)$ for all y such that $|y - x|_1 \leq R - 2(t - s) - \varepsilon$ and all s, t such that $0 \leq s < t \leq T$. Then again since uniform convergence in $\mathscr{C}_T'$ implies pointwise convergence, the set E_3^ε is closed in $\mathscr{C}_T'$.

To show that $\widehat{S}_n \in E_3^\varepsilon$ for all large enough n, first we notice that the function $u \mapsto S_n(s, t; g - u/n, \omega) + u/n : \mathbb{R} \to \Gamma$ has a period of 4 by the construction of S_n. Therefore

$$\widehat{S}_n(s, t; g, \omega) = \frac{1}{4} \int_{u=0}^{4} \left[S_n\left(s, t; g - g(x) - \frac{u}{n}, \omega\right) + \frac{u}{n} \right] du + g(x).$$

Now applying Lemma 4.12, we deduce that $\widehat{S}_n \in E_3^\varepsilon$ as long as $\frac{9}{n} < \varepsilon$, so $\mu(E_3^\varepsilon) = 1$. Since

$\varepsilon > 0$ is arbitrary, we conclude that Condition (5.3) holds μ-almost surely. $\square$

Following the same proof procedure as Proposition 4.4 and Lemma 4.8, Condition (5.3) implies the following localization property about the limit points.

Lemma 5.5. *Suppose $S \in \mathscr{C}'_T$ satisfies Condition* (5.3)*, then given $g_1, g_2 \in \Gamma$ and $x \in \mathbb{R}^2$,*

$$\sup_{y:|y-x|_1 \le R-2(t-s)} |S(s,t;g_1)(y) - S(s,t;g_2)(y)| \le \sup_{y:|y-x|_1 \le R} |g_1(y) - g_2(y)| \, .$$

CHAPTER 6

Equilibrium measures

6.1 Construction of Gibbs measures

We will make use of a particular family of Gibbs measures of dimer coverings in the plane. Specifically, for each slope vector $\rho := (\rho_1, \rho_2) \in U^o$, the interior of the Newton polygon, we would like a Gibbs measure whose height function in large scale is concentrated on the slope ρ. One such choice is to restrict the Boltzmann measure on the toroidal graph G_n to those with height change $(\lfloor n\rho_1 \rfloor, \lfloor n\rho_2 \rfloor)$, and take any weak limit as $n \to \infty$. However, it is unclear how to compute the limiting local probabilities.

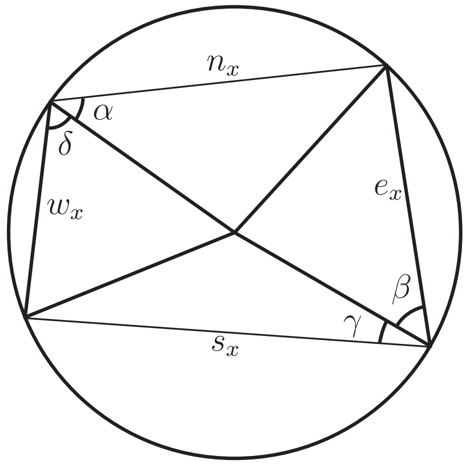
A different approach, following [16, 6, 19], is to consider the full Boltzmann measure on G'_n, modified from G_n by gauge transformations of the edge weights while keeping the periodicity. Then conditioned on the dimer configuration outside a finite region, the measure inside the finite region is unchanged, so a limiting Gibbs measure is still a Gibbs measure of the original graph G. However, the absolute probability of a dimer covering on G_n is modified according to its height change, with certain height changes preferred to others. Then by saddle point analysis, the limiting Gibbs measure is concentrated on a preferred slope. The advantage of this approach is that the local dimer probabilities of the Boltzmann measure on tori are computable from the relevant entries of the inverse Kasteleyn matrices, where a Kasteleyn matrix is a weighted adjacency matrix with certain choice of signs. The Kasteleyn matrices on G'_n can be inverted via explicit diagonalization, and it can be shown that, as $n \to \infty$, the inverse matrix converges along a common subsequence to a limiting matrix, which is called the infinite inverse Kasteleyn matrix. (In fact, the whole sequence converges due to the uniqueness result of Gibbs measures by Sheffield([30]), but this uniqueness is not needed in this thesis.) We choose our Gibbs measure to be the weak limit along this subsequence, where the local dimer probabilities of the Gibbs measure are computed from the relevant entries of the infinite inverse Kasteleyn matrix in the same way as on tori.

When the dimer graph and the edge weights satisfy an "isoradiality" condition, as in our case, it is shown in [33] based on [17] that the entries of the infinite inverse Kasteleyn matrix have simple expressions based on local geometry. As a result, explicit formula for the local dimer probabilities in the Gibbs measure can be derived.

We shall state the relevant results for our specific example. For each even face x, denote the four edges of the face x on the north, east, south and west side by n_x, e_x, s_x, w_x respectively. The gauge transformation on G_n and G is such that the dimer weights are the same in every even face x (this is true on the original graphs). The isoradiality condition is that the four edges can be represented by the four sides of a quadilateral with unit circumcircle as in Figure 6.1, such that the dimer weights of n_x, e_x, s_x, w_x are given by $\sin\alpha, \sin\beta, \sin\gamma, \sin\delta$ respectively.

Then the Gibbs measure has the following local dimer probabilities, where $P(\text{some edges})$

Figure 6.1: Isoradial weights



is shorthand for the probability that the set of edges are included simultaneously,

$$P(n_x) = \frac{\alpha}{\pi}, \quad P(e_x) = \frac{\beta}{\pi}, \quad P(s_x) = \frac{\gamma}{\pi}, \quad P(w_x) = \frac{\delta}{\pi}, \tag{6.1}$$

$$P(n_x, s_x) = \frac{1}{\pi^2}\left(\alpha\gamma + \frac{\sin\alpha\sin\gamma}{\sin\beta\sin\delta}\beta\delta\right), \tag{6.2}$$

$$P(e_x, w_x) = \frac{1}{\pi^2}\left(\beta\delta + \frac{\sin\beta\sin\delta}{\sin\alpha\sin\gamma}\alpha\gamma\right) = aP(n_x, s_x). \tag{6.3}$$

The slope $\rho = (\rho_1, \rho_2)$ is given by the expected height change along x and y directions, so

$$\rho_1 = 2\left(P(e_x) - P(w_x)\right) = 2\frac{\beta - \delta}{\pi} \tag{6.4}$$

$$\rho_2 = 2\left(P(s_x) - P(n_x)\right) = 2\frac{\gamma - \alpha}{\pi}. \tag{6.5}$$

Also, it can be easily checked that, this new set of weights being a gauge transformation of the original graph is equivalent to that

$$\frac{\sin\beta\sin\delta}{\sin\alpha\sin\gamma} = a, \tag{6.6}$$

which also guarantees that the shuffling dynamics is identical with the new weights. Then we can compute $\alpha, \beta, \gamma, \delta$ as functions of ρ, similar to [6]. Using $\alpha+\beta+\gamma+\delta=\pi$, we can rewrite (6.6) as

$$\cos(\beta-\delta)-\cos(\beta+\delta)=a\left(\cos(\alpha-\gamma)+\cos(\beta+\delta)\right),$$

and plugging in (6.4) and (6.5) to get

$$\frac{1}{1+a}\cos\left(\frac{\pi\rho_1}{2}\right)-\frac{a}{1+a}\cos\left(\frac{\pi\rho_2}{2}\right)=\cos\left(\beta+\delta\right).$$

Denoting the LHS by M and combining this with (6.4), we get

$$\beta=\frac{\pi\rho_1}{4}+\frac{1}{2}\cos^{-1}\left(M\right),\quad \delta=-\frac{\pi\rho_1}{4}+\frac{1}{2}\cos^{-1}\left(M\right).$$

Similarly, we obtain that

$$\alpha=-\frac{\pi\rho_2}{4}+\frac{1}{2}\cos^{-1}\left(-M\right),\quad \gamma=\frac{\pi\rho_2}{4}+\frac{1}{2}\cos^{-1}\left(-M\right).$$

We shall verify that the slope of the Gibbs measure does concentrate on ρ. Let Ω_ρ, where $\rho\in U^o$, denote the probability space of height functions $h(\cdot)$ distributed according to the Gibbs measure π_ρ constructed above with slope ρ, and shifted vertically so that $h(0)=0$ always holds.

Lemma 6.1. *Suppose $h(\cdot)$ is given by Ω_ρ. For all $R>0$,*

$$\lim_{n\to\infty}\mathbb{E}\sup_{|x|_1\le R}\left|\frac{1}{n}h(\lfloor nx\rfloor)-x\cdot\rho\right|=0.$$

Proof. As a sanity check, notice that $\mathbb{E}(h(\lfloor nx\rfloor))-\lfloor nx\rfloor\cdot\rho$ is bounded. By [19], the variance of $h(x)$ is $O(\log|x|)$. Therefore, given some small $\varepsilon > 0$, by Chebyshev's inequality,

$$P\left(|h(\lfloor nx \rfloor) - \lfloor nx \rfloor \cdot \rho| > n\varepsilon\right) \leq \frac{C_1 \log^2 |\lfloor nx \rfloor|}{n^2 \varepsilon^2} \tag{6.7}$$

for all large enough n and some constant C_1 that depends on ρ only.

Pick a set of points $A_\varepsilon \subset \{x : |x|_1 \leq R\}$ with cardinality $O(\varepsilon^{-2} R^2)$ such that every point x with $|x|_1 \leq R$ is within ε ℓ_1-distance from at least one point in A_ε. Let M_ε^n denote the event such that $|h(\lfloor nx \rfloor) - \lfloor nx \rfloor \cdot \rho| \leq n\varepsilon$ for all $x \in A_\varepsilon$. Then by (6.7),

$$P(M_\varepsilon^n) \geq 1 - \sum_{x \in A_\varepsilon} \frac{C_1 \log^2 |\lfloor nx \rfloor|}{n^2 \varepsilon^2}. \tag{6.8}$$

This is a finite sum, and when $n \to \infty$, n^2 clearly outgrows $\log^2 |\lfloor nx \rfloor|$, so $P(M_\varepsilon^n) \to 1$.

Since heights at neighboring faces differ by at most 3, if $h \in M_\varepsilon^n$,

$$\begin{aligned} &\sup_{|x|_1 \leq R} |h(\lfloor nx \rfloor) - nx \cdot \rho| \leq C_2 n\varepsilon + \rho \\ \Rightarrow &\sup_{|x|_1 \leq R} \left| \frac{1}{n} h(\lfloor nx \rfloor) - x \cdot \rho \right| \leq C_2 \varepsilon + \frac{\rho}{n} \end{aligned} \tag{6.9}$$

for some constant C_2 that depends on ρ only.

On the other hand, $\sup_{|x|_1 \leq R} \left| \frac{1}{n} h(\lfloor nx \rfloor) - x \cdot \rho \right|$ is at most $|3 + \rho| R$. Combining this with (6.8) and (6.9), we see that

$$\limsup_{n \to \infty} \mathbb{E} \sup_{|x|_1 \leq R} \left| \frac{1}{n} h(\lfloor nx \rfloor) - x \cdot \rho \right| \leq C_2 \varepsilon.$$

Now taking $\varepsilon \to 0$, we obtain the statement. $\square$

6.2 Evolution at equilibrium

Now we shall relate the Gibbs measures in the previous section to shuffling dynamics. The first observation is the following.

Proposition 6.2. *The Gibbs measure π_ρ, $\rho \in U^o$, is invariant under shuffling.*

Proof. This is the direct consequence of Proposition 2.2. The Gibbs measure was constructed as the weak limit of a sequence of Boltzmann measures on tori with increasing sizes. Since the local moves preserve the Boltzmann measures on tori, the shuffling procedure, viewed as a sequence of local moves, also preserves the Boltzmann measures on tori. Also since the edge weights of the new graph are the same as the original graph up to gauge transformation, the shuffling procedure in fact results in the same exact Boltzmann measure as before the shuffling.

By Lemma 4.3 (and its version on tori), after a shuffle, the resulting configuration on a finite region V is a random function of the original configuration on the region

$$V' = \{x : x \text{ is within a constant } \ell_1\text{-distance of } V\},$$

where the randomness only comes from the Bernoulli random variables that govern the outcomes of the spider moves on V'. For each finite region, such function and the Bernoulli random variables are the same for the shuffles on all large enough tori and the infinite plane. Due to weak convergence, the measures of cylindrical sets on V' and V on tori converge to those in the Gibbs measure. Therefore when we perform the shuffling to the Gibbs measure in the plane, the resulting distribution on V also remains the same. This is true for all V, and since π_ρ is characterized by distribution on all finite regions, we conclude that π_ρ is invariant under shuffling. $\square$

The invariance of Gibbs measures will let us find the hydrodynamic limit when the height is initially distributed according to a Gibbs measure. To do so, we first investigate how the height evolution at the origin depends on the local dimer configuration. From Table 3.1, it is easy to see the following rules about the height at an even face x after a shuffle:

1. The height remains the same when either s_x or e_x is present,
2. The height decreases by 4 when either n_x or s_x is present,
3. When none of n_x, e_x, s_x, w_x is present, with probability $\frac{a}{1+a}$ the height remains the same, and with probability $\frac{1}{1+a}$ the height decreases by 4.

Consider the shuffling height process with initial configuration given by Ω_ρ. The whole process lives on the product probability space $\Omega_\rho \times \Omega$. Define the event $Q(x,t)$ for $t \in \mathbb{N}$ as

$$\begin{aligned} Q(x,t) :=&\{\text{At time } t \text{, the even face } x \text{ has either } n_x \text{ or } s_x \text{ present, or,} \\ &\text{has none of } n_x, e_x, s_x, w_x \text{ present and } \omega(x,t) = 1\}. \end{aligned}$$

Recall that $\omega(x,t) = 1$ with probability $\frac{1}{1+a}$. We can use the information gathered in Section 6.1 and the invariance of the Gibbs measure under shuffling to compute explictly $P(Q(x,t))$:

$$\begin{aligned} P(Q(x,t)) =& P_{\pi_\rho}(n_x) + P_{\pi_\rho}(s_x) - P_{\pi_\rho}(n_x, s_x) + \frac{1}{1+a}(1 - P_{\pi_\rho}(e_x) - P_{\pi_\rho}(w_x) + P_{\pi_\rho}(e_x, w_x) \\ & - P_{\pi_\rho}(n_x) - P_{\pi_\rho}(s_x) + P_{\pi_\rho}(n_x, s_x)) \\ =& P_{\pi_\rho}(n_x) + P_{\pi_\rho}(s_x) - \frac{a}{1+a} P_{\pi_\rho}(n_x, s_x) + \frac{1}{1+a} P_{\pi_\rho}(e_x, w_x) \\ =& P_{\pi_\rho}(n_x) + P_{\pi_\rho}(s_x) \\ =& \frac{1}{\pi} \cos^{-1}\left(\frac{a}{1+a}\cos\left(\frac{\pi\rho_2}{2}\right) - \frac{1}{1+a}\cos\left(\frac{\pi\rho_1}{2}\right)\right). \end{aligned}$$

The result is a function of ρ only. We define $H(\rho) := 4P(Q(x,t))$.

Now we are ready to prove a law of large numbers for the height evolution at the origin. Finer fluctuation results could be obtained as in [5].

Lemma 6.3. *Suppose the height process $h(x,t)$ has an initial condition $h(\cdot,0)$ given by Ω_ρ, $\rho \in U^o$, then*

$$\lim_{t\to\infty} \mathbb{E}\left|\frac{1}{t}h(0,t) + H(\rho)\right| = 0.$$

Proof. Since $\frac{1}{t}h(0,t)$ is bounded between 0 and 4, to prove the lemma, it suffices to show that for every $\varepsilon > 0$,

$$\lim_{t\to\infty} P\left(\left|\frac{1}{t}h(0,t) + H(\rho)\right| > \varepsilon\right) = 0. \tag{6.10}$$

By the local rules above, we know that for every even face x, and $t \in \mathbb{N}$,

$$\mathbb{E}(h(x,t) - h(x,0)) = \mathbb{E}\left(-4\sum_{i=1}^{t} \mathbf{1}_{Q(x,t)}(h)\right) = -tH(\rho). \tag{6.11}$$

In particular, $\mathbb{E}(h(0,t)) = -tH(\rho)$. The intuition is that, if $\left|\frac{1}{t}h(0,t) + H(\rho)\right| > \varepsilon$, then faces near origin also have large deviation simultaneously. Since at each t, the correlation of $Q(x,t)$ for different x decays at least quadratically in distance by [19], this event is unlikely to happen.

For convenience, we will work on faces $2x$ where $x \in \mathbb{Z}^2$, i.e. T-translations of the origin. They have the extra benefit that $\mathbb{E}_{\Omega_\rho}(h(2x)) = 2x \cdot \rho$ exactly (otherwise there is a bounded error). For $R \in \mathbb{N}$, let

$$V_R(t) := \mathrm{Var}\left(\sum_{|x|_1=R} (h(2x,t) - h(2x,0))\right), \widetilde{V}_R(t) := \mathrm{Var}\left(\sum_{|x|_1=R} \mathbf{1}_{Q(2x,t)}(h)\right),$$

then by the local rules

$$\begin{aligned} V_R(t+1) =& \operatorname{Var}\left(\sum_{|x|_1=R} (h(2x,t)-h(2x,0)) - 4\sum_{|x|_1=R} \mathbf{1}_{Q(2x,t)}(h)\right) \\ =& V_R(t) - 8\operatorname{Cov}\left(\sum_{|x|_1=R} (h(2x,t)-h(2x,0)), \sum_{|x|_1=R} \mathbf{1}_{Q(2x,t)}(h)\right) + 16\widetilde{V}_R(t) \end{aligned}$$

Using Cauchy-Schwarz inequality, the covariance is bounded in absolute value by $\sqrt{V_R(t)\widetilde{V}_R(t)}$. As mentioned above, $|\operatorname{Cov}(Q(x,t),Q(y,t))| = O(|x-y|_1^2)$, so $\widetilde{V}_R(t) = O(R)$. Therefore $V_R(t)$ satisfies the recurrence inequality

$$|V_R(t+1) - V_R(t)| \leq a\sqrt{R}\sqrt{V_R(t)} + bR$$

for some constants a, b, which implies $V_R(t) = O(t^2 R)$.

From now on we let $R = R(\varepsilon, t) := \lfloor \varepsilon t/20 \rfloor$, then $V_R(t) = O(\varepsilon t^3)$. By Chebyshev's inequaility,

$$P\left(\left|\sum_{|x|_1=R} (h(2x,t)-h(2x,0)) - \mathbb{E}\sum_{|x|_1=R} (h(2x,t)-h(2x,0))\right| \geq C\varepsilon^2 t^2\right) = \frac{C'}{\varepsilon^3 t} \tag{6.12}$$

which goes to 0 as $t \to \infty$.

By Lemma 6.1, given $\delta > 0$, for all large enough t,

$$P\left(\sup_{|x|_1=R}\left|\frac{1}{t}h(2x,0) - \mathbb{E}\left(\frac{1}{t}h(2x,0)\right)\right| \leq \frac{\varepsilon}{10}\right) > 1-\delta. \tag{6.13}$$

If $\frac{1}{t}h(0,t) + H(\rho) > \varepsilon$, since height functions restricted to even faces are 2-spatially-Lipschitz, we will have, for all x such that $|x|_1 = R$,

$$\frac{1}{t}h(2x,t) - \mathbb{E}\left(\frac{1}{t}h(2x,t)\right) \geq \frac{1}{t}h(0,t) + H(\rho) - \left|\frac{1}{t}h(2x,t) - \frac{1}{t}h(0,t)\right| - \left|\mathbb{E}\left(\frac{1}{t}h(2x,t) - \frac{1}{t}h(0,t)\right)\right|$$
$$\geq \varepsilon - 2\frac{\varepsilon}{10} - 2\frac{\varepsilon}{10} = \frac{3}{5}\varepsilon.$$

If this event along with the event in (6.13) both happen, then the event in (6.12) happens, because in this case for all x such that $|x|_1 = R$ (which has cardinality $\Omega(\varepsilon t)$),

$$(h(2x,t) - h(2x,0)) - \mathbb{E}(h(2x,t) - h(2x,0)) \geq t\left(\frac{3}{5}\varepsilon - \frac{1}{10}\varepsilon\right) = \frac{\varepsilon t}{2}.$$

Therefore

$$\lim_{t\to\infty} P\left(\frac{1}{t}h(0,t) + H(\rho) > \varepsilon\right) \leq \delta.$$

Similarly, we can deduce that

$$\lim_{t\to\infty} P\left(\frac{1}{t}h(0,t) + H(\rho) < -\varepsilon\right) \leq \delta.$$

Taking $\delta \to 0$, we proved (6.10). □

Proposition 6.4. *With the same assumption as Lemma 6.3, for all $R > 0$ and $t > 0$,*

$$\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1 \leq R} \left|\frac{1}{n}h(\lfloor nx \rfloor, \lfloor nt \rfloor) - x\cdot\rho + tH(\rho)\right| = 0.$$

Proof. On one hand, by Lemma 6.3,

$$\lim_{n\to\infty} \mathbb{E}\left|\frac{1}{n}h(0,\lfloor nt \rfloor) + tH(\rho)\right| = t \lim_{n\to\infty} \mathbb{E}\left|\frac{1}{nt}h(0,\lfloor nt \rfloor) + H(\rho)\right| = 0. \tag{6.14}$$

On the other hand, since the shuffling process preserves the Gibbs measure of dimer coverings, the random function $h(\cdot, \lfloor nt \rfloor) - h(0, \lfloor nt \rfloor)$ is also distributed according to Ω_ρ. Therefore, by Lemma 6.1, we know that

$$\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1 \le R} \left| \frac{1}{n} \left(h(\lfloor nx \rfloor, \lfloor nt \rfloor) - h(0, \lfloor nt \rfloor) \right) - x \cdot \rho \right| = 0. \tag{6.15}$$

Now we combine (6.14) and (6.15) to deduce that

$$\begin{aligned}
&\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1 \le R} \left| \frac{1}{n} h(\lfloor nx \rfloor, \lfloor nt \rfloor) - x \cdot \rho + tH(\rho) \right| \\
\le &\lim_{n\to\infty} \mathbb{E} \left(\sup_{|x|_1 \le R} \left| \frac{1}{n} \left(h(\lfloor nx \rfloor, \lfloor nt \rfloor) - h(0, \lfloor nt \rfloor) \right) - x \cdot \rho \right| + \left| \frac{1}{n} h(0, \lfloor nt \rfloor) + tH(\rho) \right| \right) \\
= &0 + 0 = 0.
\end{aligned}$$

□

6.3 More on limit points

It turns out that the information we obtained from the Gibbs measures tells us more about the limit points in Section 5.2. First, we need a lemma that relates the initial conditions in Section 4.2 and Section 6.2.

Recall from Section 3.2 that the sequence of height processes $(h_n(x,t))_{n \in \mathbb{Z}_{>0}}$ has its initial conditions $h_n(\cdot, 0)$ given by the probability space Ω_0.

Lemma 6.5. *Given $g \in \Gamma$, if*

$$\lim_{n\to 0} \mathbb{E} \sup_{|x|_1 \le R} \left| \frac{1}{n} h_n(\lfloor nx \rfloor, 0) - g(x) \right| = 0 \tag{6.16}$$

for all $R > 0$, then

$$\lim_{n\to 0} \mathbb{E} \sup_{|x|_1 \le R, t\le T} \left| \frac{1}{n} h_n(\lfloor nx \rfloor, \lfloor nt \rfloor) - \widehat{S}_n(0,t;g)(x) \right| = 0 \tag{6.17}$$

for all $R > 0$.

Proof. By (4.7), we might as well replace the assumption (6.16) by

$$\lim_{n\to 0} \mathbb{E} \sup_{|x|_1 \le R} \frac{1}{n} \left| h_n(\lfloor nx \rfloor, 0) - \varphi_g^n(\lfloor nx \rfloor) \right| = 0, \quad \forall R > 0. \tag{6.18}$$

To be precise about the source of the randomness, we let $\omega_0 \in \Omega_0$ refer to the initial condition, and $\omega \in \Omega$ refer to the Bernoulli mark as usual that dictates the shuffling. Using (4.13) and Proposition 4.16, we know that, for a given $t \geq 0$,

$$\begin{aligned} &\sup_{|x|_1 \le R} \left| \frac{1}{n} h_n(\lfloor nx \rfloor, \lfloor nt \rfloor; \omega_0; \omega) - \widehat{S}_n(0,t;g,\omega) \right| \\ \le &\sup_{|x|_1 \le R} \frac{1}{n} \left| h_n(\lfloor nx \rfloor, \lfloor nt \rfloor; \omega_0; \omega) - h\left(\lfloor nx \rfloor, \lfloor nt \rfloor; \varphi_g^n, \omega\right) \right| + \frac{C_3}{n} \end{aligned} \tag{6.19}$$

for some global constant C_3. And by Proposition 4.4,

$$\begin{aligned} &\sup_{|x|_1 \le R} \frac{1}{n} \left| h_n(\lfloor nx \rfloor, \lfloor nt \rfloor; \omega_0; \omega) - h\left(\lfloor nx \rfloor, \lfloor nt \rfloor; \varphi_g^n, \omega\right) \right| \\ \le &\sup_{|x|_1 \le R+2t+1} \frac{1}{n} \left| h_n(\lfloor nx \rfloor, 0; \omega_0) - \varphi_g^n(\lfloor nx \rfloor) \right| \end{aligned} \tag{6.20}$$

for all n large enough.

Therefore, combining (6.19) and (6.20),

$$
\begin{aligned}
&\limsup_{n\to\infty} \mathbb{E} \sup_{|x|_1\le R, t\le T} \left|\frac{1}{n} h_n(\lfloor nx\rfloor, \lfloor nt\rfloor) - \widehat{S}_n(0,t;g)(x)\right| \\
=&\limsup_{n\to\infty} \int \sup_{|x|_1\le R, t\le T} \left|\frac{1}{n} h_n(\lfloor nx\rfloor, \lfloor nt\rfloor;\omega_0;\omega) - \widehat{S}_n(0,t;g,\omega)(x)\right| d\omega_0 d\omega \\
\le&\limsup_{n\to\infty} \int \sup_{|x|_1\le R+2T+1} \left(\frac{1}{n}\left|h_n(\lfloor nx\rfloor, 0;\omega_0) - \varphi_g^n(\lfloor nx\rfloor)\right| + \frac{C_3}{n}\right) d\omega_0 \\
=&\lim_{n\to\infty} \mathbb{E} \sup_{|x|_1\le R+2T+1} \frac{1}{n}\left|h_n(\lfloor nx\rfloor, 0) - \varphi_g^n(\lfloor nx\rfloor)\right| + \lim_{n\to 0}\frac{C_3}{n} = 0
\end{aligned}
$$

by our assumption (6.18). □

Now we can make the following additional characterization about the limit points of $(\mu_n)_{n\in\mathbb{Z}_{>0}}$.

Proposition 6.6. *All subsequential limits of $(\mu_n)_{n\in\mathbb{Z}_{>0}}$ satisfy the following property almost surely:*

$$S(s,t;g_\rho) = g_\rho - (t-s)H(\rho), \quad \forall \rho\in U, 0\le s\le t\le T$$

where $g_\rho(x) = x\cdot\rho$.

Proof. Suppose a subsequence $(\mu_i)_{i\in\mathbb{N}} := (\mu_{n_i})_{i\in\mathbb{N}}$ converges weakly to μ. First let us fix some $R>0$, $\rho\in U^o$ and $t\in(0,T]$. Combining Lemma 6.1, Proposition 6.4 and Lemma 6.5, we obtain that

$$\lim_{i\to\infty} \mathbb{E} \sup_{|x|_1\le R} \left|\widehat{S}_{n_i}(0,t;g_\rho)(x) - g_\rho(x) + tH(\rho)\right| = 0. \tag{6.21}$$

Viewing $\sup_{|x|_1\le R} |S(0,t;g_\rho)(x) - g_\rho(x) + tH(\rho)|$ as a function of $S\in\mathscr{C}'_T$, it is clearly continuous, tracing through the definitions (4.22) (4.21) of the metric for the uniform topology.

It is also bounded because $S(0,t;\cdot) \in \mathscr{E}_q^{cr}$. Therefore, by the definition of weak convergence,

$$\mathbb{E}_\mu \sup_{|x|_1 \le R} |S(0,t;g_\rho)(x) - g_\rho(x) + tH(\rho)| = \lim_{i\to\infty} \mathbb{E} \sup_{|x|_1 \le R} \left| \widehat{S}_{n_i}(0,t;g_\rho)(x) - g_\rho(x) + tH(\rho) \right| = 0. \tag{6.22}$$

Since $R > 0$ is arbitrary, we deduce that $S(0,t;g_\rho) = g_\rho - tH(\rho)$ μ-almost surely.

We can choose a dense subset of $t \in (0,T]$ and $\rho \in U^o$ so that $S(0,t;g_\rho) = g_\rho - tH(\rho)$ holds μ-almost surely for all such t and ρ.

Now because $H(\rho)$ is in fact continuous on the entire U, by definition (4.15), $d(g_\rho - tH(\rho), g_{\rho'} - tH(\rho')) \to 0$ as $\rho' \to \rho$. Then by Lemma 4.9, we know that, fixing t, $S(0,t;g_\rho) = g_\rho - tH(\rho)$ for all $\rho \in U$ μ-almost surely.

Recall that we are working on $\mathscr{C}_T'$, so $S(0,t;\cdot)$ is $(10 + 2C_0)$-Lipschitz continuous in t μ-almost surely. Again due to uniform topology, we conclude that $S(0,t;g_\rho) = g_\rho - tH(\rho)$ holds for all $t \in [0,T]$ and $\rho \in U$ μ-almost surely.

We already know from Proposition 5.2 that Condition (5.2) holds μ-almost surely. Then μ-almost surely, for all s,t such that $0 \le s \le t \le T$, using Property (4.17),

$$\begin{aligned} S(s,t;g_\rho) &= S(s,t;g_\rho - sH(\rho)) + sH(\rho) = S(s,t;S(0,s;g_\rho)) + sH(\rho) \\ &= S(0,t;g_\rho) + sH(\rho) = g_\rho - (t-s)H(\rho), \end{aligned}$$

so the proposition is proved. $\square$

CHAPTER 7

Viscosity solution

7.1 Proof of the main result

We recall some PDE theory about Hamilton-Jacobi equations. For more details, see [8]. Given $g, H \in C^0(\mathbb{R}^2)$, consider the following first-order partial differential equation with initial condition g

$$\begin{cases} u_t + H(u_x) &= 0 \\ u(x,0) &= g(x) \end{cases} \tag{7.1}$$

where the solution $u(x,t)$ is a function on $\mathbb{R}^2 \times [0,T]$, and u_x is supposed to be its gradient with respect to the two spatial coordinates. It is possible to apply method of characteristics to obtain short-time solution, but even if g and H are smooth, shocks can form at finite time and the solution $u(x,t)$ becomes nondifferentiable. In order to describe the long-time evolution of the PDE and to deal with nonsmooth initial conditions, we have to consider weak solutions, which are not differentiable but still solve the PDE in some sense. A priori, the uniqueness and existence of weak solutions are not guaranteed. The viscosity solution is a particular choice that guarantee both, even though we do not need the existence result from PDE theory.

Definition 7.1. *A function $u(x,t)$ is called a viscosity solution to* (7.1) *if the following conditions hold,*

1. *u is continuous on $\mathbb{R}^2 \times [0,T]$,*

2. *$u(\cdot, 0) = g$,*

3. *If $\phi \in C^\infty(\mathbb{R}^2 \times [0,T])$ and $(x_0,t_0) \in \mathbb{R}^2 \times (0,T)$ satisfy that $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi \geq u$ in a neighborhood of (x_0,t_0), then*

$$\phi_t(x_0,t_0) + H(\phi_x(x_0,t_0)) \leq 0, \tag{7.2}$$

4. *If $\phi \in C^\infty(\mathbb{R}^2 \times [0,T])$ and $(x_0,t_0) \in \mathbb{R}^2 \times (0,T)$ satisfy that $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi \leq u$ in a neighborhood of (x_0,t_0), then*

$$\phi_t(x_0,t_0) + H(\phi_x(x_0,t_0)) \geq 0. \tag{7.3}$$

To prove the main result, we show that the semigroup of the shuffling height process satisfies the definition of a viscosity solution.

Proposition 7.2. *All subsequential limits of $(\mu_n)_{n\in\mathbb{Z}>0}$ are concentrated on a single $S \in \mathscr{C}'_T$, such that $u(x,t) := S(0,t;g)(x)$ is the unique viscosity solution to* (3.3).

Proof. We know that all subsequential limits of $(\mu_n)_{n\in\mathbb{Z}>0}$ have probability 1 on $S \in \mathscr{C}'_T$ satisfying Condition (5.1), 5.2 and 5.3 as well as the property in Proposition 6.6.

We will check the four criteria for the viscosity solution. The continuity of u is guaranteed by the fact that $S(0,t;g) \in \Gamma$ and the Lipschitz continuity of S in t. Also Condition (5.1) implies that $u(\cdot,0) = g$. We shall verify criterion (7.2). The last one (7.3) is similar.

Let ϕ and (x_0, t_0) be as described in the assumption for (7.2). Suppose $B \subset \mathbb{R}^2 \times (0,T)$ is an open ball centered at (x_0, t_0) in which $\phi \geq u$. An issue here is that $\phi(\cdot, t)$ *is not necessarily in* Γ, so we cannot directly apply S on $\phi(\cdot,t)$. First, we prove the following lemma.

Lemma 7.3. *$\phi_x(x_0,t_0) \in U$.*

Proof. Let $\mathbf{v} := \phi_x(x_0,t_0)$, and define vector $\mathbf{w} := (w_1, w_2)$. By definition, for all small $k \in \mathbb{R}$,

$$\phi\left(x_0 + k\mathbf{w}, t_0\right) = \phi\left(x_0, t_0\right) + k\mathbf{v}\cdot\mathbf{w} + o(k). \tag{7.4}$$

Since $\phi(x_0,t_0) = u(x_0,t_0)$ and $\phi \geq u$ in B, for small k,

$$\phi\left(x_0 + k\mathbf{w}, t_0\right) - \phi\left(x_0, t_0\right) \geq u\left(x_0 + k\mathbf{w}, t_0\right) - u\left(x_0, t_0\right) \geq -2|k||\mathbf{w}|_\infty. \tag{7.5}$$

Combining (7.4) and (7.5), we get for all small k,

$$k\mathbf{v}\cdot\mathbf{w} + o(k) \geq -2|k||\mathbf{w}|_\infty.$$

Divided by k, this implies

$$-2|\mathbf{w}|_\infty \leq \mathbf{v}\cdot\mathbf{w} \leq 2|\mathbf{w}|_\infty.$$

Taking $\mathbf{w} = (\pm 1, \pm 1)$, we obtain a set of inequalities of the form $\pm v_1 \pm v_2 \leq 2$. Therefore $\mathbf{v}$ must satisfy that $|\mathbf{v}|_1 \leq 2$, or $\mathbf{v} \in U$. $\square$

Now we define a new function affine in space

$$\widetilde{\phi}(x,t) := \phi(x_0,t) + (x-x_0)\cdot\phi_x(x_0,t_0).$$

We have $\widetilde{\phi}(x_0,t_0) = \phi(x_0,t_0) = u(x_0,t_0)$. Since ϕ is smooth, we can assume that for $(x,t) \in B$,

$$\begin{aligned}\phi(x,t) - \widetilde{\phi}(x,t) &= \phi(x,t) - \phi(x_0,t) - (x-x_0)\cdot\phi_x(x_0,t_0)\\ &= (x-x_0)\cdot\phi_x(x_0,t) + O((x-x_0)^2) - (x-x_0)\cdot\phi_x(x_0,t_0)\\ &= O((x-x_0)^2 + |x-x_0||t-t_0|).\end{aligned}$$

As $\phi \geq u$ in B, we have

$$\widetilde{\phi}(x,t) - u(x,t) \geq -C((x-x_0)^2 + |x-x_0||t-t_0|) \tag{7.6}$$

for some constant $C > 0$ and all $(x,t) \in B$. Furthermore $\widetilde{\phi}_x(x,t) = \phi_x(x_0,t_0)$, and $\widetilde{\phi}_t(x,t) = \phi_t(x_0,t)$. In particular, $\widetilde{\phi}(\cdot,t) \in \Gamma$ for all t.

For all small enough δ, we have

$$\{(x,t) : |x-x_0|_1 \leq 2\delta, |t-t_0|_1 \leq \delta\} \subset B.$$

By Condition (5.2),

$$\widetilde{\phi}(x_0,t_0) = u(x_0,t_0) = S(t_0-\delta,t_0;S(0,t_0-\delta;g))(x_0) = S(t_0-\delta,t_0;u(\cdot,t_0-\delta))(x_0).$$

Define $g_1 = u(\cdot,t_0-\delta)$ and $g_2 = u(\cdot,t_0-\delta) \wedge \widetilde{\phi}(\cdot,t_0-\delta)$ ($g_2 \in \Gamma$ as shown in the proof of

Lemma 4.8). From (7.6), we have

$$\sup_{y:|y-x_0|_1\le 2\delta} |g_1(y)-g_2(y)| \le C\delta^2,$$

Now we apply Lemma 5.5 to g_1 and g_2 to deduce

$$|S(t_0-\delta,t_0;g_1)(x_0)-S(t_0-\delta,t_0;g_2)(x_0)| \le C\delta^2.$$

Since $\widetilde{\phi}(\cdot,t_0-\delta)\ge g_2$, by Property (4.18),

$$\begin{aligned} S(t_0-\delta,t_0;\widetilde{\phi}(\cdot,t_0-\delta))(x_0) &\ge S(t_0-\delta,t_0;g_2)(x_0) \\ &\ge S(t_0-\delta,t_0;g_1)(x_0)-C\delta^2=\widetilde{\phi}(x_0,t_0)-C\delta^2. \end{aligned}$$

On the other hand, using Property (4.17) and the property in Proposition 6.6,

$$\begin{aligned} S(t_0-\delta,t_0;\widetilde{\phi}(\cdot,t_0-\delta))(x_0) &= S(t_0-\delta,t_0;\widetilde{\phi}(\cdot,t_0-\delta)-\widetilde{\phi}(0,t_0-\delta))(x_0)+\widetilde{\phi}(0,t_0-\delta) \\ &= S(t_0-\delta,t_0;g_{\phi_x(x_0,t_0)})(x_0)+\widetilde{\phi}(0,t_0-\delta) \\ &= g_{\phi_x(x_0,t_0)}(x_0)-\delta H(\phi_x(x_0,t_0))+\widetilde{\phi}(0,t_0-\delta) \\ &= \widetilde{\phi}(x_0,t_0-\delta)-\widetilde{\phi}(0,t_0-\delta)-\delta H(\phi_x(x_0,t_0))+\widetilde{\phi}(0,t_0-\delta) \\ &= \widetilde{\phi}(x_0,t_0-\delta)-\delta H(\phi_x(x_0,t_0)) \end{aligned}$$

where we recall $g_\rho := x\cdot\rho$ for $\rho\in\mathbb{R}^2$. Along with the inequality above, we get

$$\begin{aligned} &\widetilde{\phi}(x_0,t_0-\delta)-\delta H(\phi_x(x_0,t_0)) \ge \widetilde{\phi}(x_0,t_0)-C\delta^2 \\ \Rightarrow\ &\frac{\widetilde{\phi}(x_0,t_0)-\widetilde{\phi}(x_0,t_0-\delta)}{\delta}+H(\phi_x(x_0,t_0)) \le C\delta. \end{aligned}$$

Since $\widetilde{\phi}_t(x_0, t_0) = \phi_t(x_0, t_0)$, by taking $\delta \to 0$, we obtain that

$$\phi_t(x_0, t_0) + H(\phi_x(x_0, t_0)) \leq 0,$$

exactly as desired.

Even though H is not defined on the whole $\mathbb{R}^2$, notice that we only used the values of H on U, so we still have the uniqueness of $u(x, t)$ with initial condition $g \in \Gamma$. $\square$

The same argument shows that in fact $S(s, t; g)$ is determined for all $(s, t) \in [0, T]^2$ and $g \in \Gamma$.

Proof of Theorem 3.1. Let u be the unique viscosity solution to (3.3) with initial condition g, whose existence is provided by Proposition 7.2. From the precompactness of $\{\mu_n\}$, we deduce that in fact the entire sequence of random variables $\widehat{S}_n$ converges weakly to the deterministic $\widehat{S} \in \mathscr{C}'_T$ characterized by $\widehat{S}(s, t; g) = u(0, t - s)$.

Tracing through the definition of the metric (4.22), (4.21) and (4.14), it is easy to see that the following function is continuous on $\mathscr{C}'_T$,

$$f(S) := \sup_{|x|_1 \leq T, t \leq T} |S(0, t; g)(x) - u(x, t)| .$$

It is also bounded because $S(0, t; g)(x)$ is bounded when $|x|_1 \leq T, t \leq T$, due to Property (4.19) and the fact that $g \in \Gamma$. Therefore

$$\begin{aligned} \lim_{n \to \infty} \mathbb{E} \sup_{|x|_1 \leq T, t \leq T} \left| \widehat{S}_n(0, t; g)(x) - u(x, t) \right| &= \lim_{n \to \infty} \mathbb{E} f(\widehat{S}_n) = \mathbb{E} f(\widehat{S}) \\ &= \mathbb{E} \sup_{|x|_1 \leq T, t \leq T} \left| \widehat{S}(0, t; g)(x) - u(x, t) \right| = 0. \end{aligned} \tag{7.7}$$

From (4.13) and Proposition 4.16, we know that

$$\lim_{n\to\infty} \mathbb{E} \sup_{x\in\mathbb{R}^2, t\le T} \left| \widehat{S}_n(0,t;g)(x) - \frac{1}{n} h\left(\lfloor nx \rfloor, \lfloor nt \rfloor; \varphi_g^n\right) \right| = 0. \tag{7.8}$$

Also by (4.7),

$$\lim_{n\to\infty} \sup_{x\in\mathbb{R}^2} \left| \frac{1}{n} \varphi_g^n(\lfloor nx \rfloor) - g(x) \right| = 0. \tag{7.9}$$

Now consider a sequence of shuffling height processes $(h_n(x,t))_{n\in\mathbb{Z}_{>0}}$ with initial conditions given by a probability space Ω_0, satisfying (3.1) for every $R > 0$. Then (3.2) follows with $R = T$, by putting together (7.7), (7.8), (7.9) and Lemma 6.5. Finally we can choose $T > 0$ arbitrarily, so Theorem 3.1 is established. □

7.2 Some variants

One byproduct of the main result is that we get for free that the viscosity solution to (3.3) belongs to $\mathscr{C}'_T$ and satisfies Properties (4.17)-(4.20) and Conditions (5.1)-(5.3).

We then alter the domain of the shuffling process. Instead of the $\mathbb{Z}^2$ lattice, we could also perform the shuffling on a cylinder $\mathbb{Z}/n\mathbb{Z} \times \mathbb{Z}$ or a torus $\mathbb{Z}/n \times \mathbb{Z}/m$ where at least one of m and n is even. Intuitively, due to Proposition 4.4 and Condition 5.3 of the viscosity solution, locally these cases are indistinguishable from the infinite plane case on both the discrete and continuous levels. To be rigorous, we go through the procedure in Chapter 4 and 5 for the shuffling on a cylinder or a torus instead. For any subsequential limit, we extend it in the spatial coordinates periodically to the universal cover $\mathbb{R}^2$. We claim that the extension corresponds to the unique viscosity solution on $\mathbb{R}^2$ with a periodic initial condition. This is because the argument proving Proposition 7.2 can be made purely local using Property (4.20) and Condition (5.3).

Furthermore, we can consider the original shuffling algorithm on Aztec Diamonds. This is different from the previous cases as the evolution starts from a single point after rescaling and has an explicit boundary condition with a region varying with time. Therefore we require a different uniqueness result from PDE theory. For that we need to be precise about the domain and the boundary condition.

The rescaled evolution is defined on an inverted square pyramid. Due to the way we are rescaling space and time, the terminal time T is no longer relevant in the proof, as the sequence of discrete evolutions remain the same regardless of the terminal time. In other words, the scaling limits for different terminal times are related simply by a scaling factor. Therefore we shall set the terminal time $T = 1$. The open domain of the rescaled evolution is

$$V = \left\{(x,t) : x \in \mathbb{R}^2, t \in \mathbb{R}, |x|_1 < t, 0 < t < 1\right\}.$$

There is no longer an initial condition, but there is a boundary condition on the sides of the pyramid

$$\partial V^1 = \left\{(x,t) : x \in \mathbb{R}^2, t \in \mathbb{R}, |x|_1 = t, 0 \leq t \leq 1\right\},$$

where we put a superscript 1 to distinguish it from the entire boundary. In fact, the boundary condition is deterministic at the discrete level and thus also in any scaling limit. It is easier to see this from the domino picture Figure 1.3. We see that the height differences are deterministic on the four sides. Starting from either one of the two possible 2×2 starting configurations, the two initial dominoes are never destroyed and always stay on the boundary, so using their heights as references, the entire boundary heights evolve deterministically. We try to repeat the semigroup construction with this boundary condition. There is a slight issue in Section 4.2 that φ_g^n might have wrong boundary heights. We can fix this by taking $(\varphi_g^n \vee \varphi_{\min}) \wedge \varphi_{\max}$, where $\varphi_{\min}$ (resp. $\varphi_{\max}$) is the minimum (resp. maximum) height function on the Aztec Diamond with the correct boundary values. A bigger issue is that $S_n(s,t;\cdot,\omega)$ lies in different functional spaces for different (s,t). Even worse, if $s=0$, S_n does not even take inputs. To deal with this, we start from a small positive time $\varepsilon > 0$ and consider instead the domain and boundary

$$V_\varepsilon = \left\{(x,t) : x \in \mathbb{R}^2, t \in \mathbb{R}, |x|_1 < t, \varepsilon < t < 1\right\},$$
$$\partial V_\varepsilon^1 = \left\{(x,t) : x \in \mathbb{R}^2, t \in \mathbb{R}, |x|_1 = t, \varepsilon \le t \le 1\right\} \cup \left\{(x,t) : x \in \mathbb{R}^2, |x|_1 < \varepsilon, t = \varepsilon\right\},$$

where ∂V_ε^1 consists of the four sides and the bottom small face. For any $\widehat{S}_n(s,t;\cdot,\omega)$, for the purpose of extracting subsequential limit only, we rescale its input and output so that they are both functions defined on $\{x : x \in \mathbb{R}^2, |x|_1 \le 1\}$. By similar argument, the conditions of Theorem 5.1 are still satisfied. Any subsequential limit must have a deterministic boundary condition f defined on ∂V_ε^1, as argued above.

Now we need a uniqueness result for the viscosity solution with our specific boundary condition. It is difficult to find a result in PDE theory that addresses our situation directly, so for completeness we provide a proof here.

We need the following theorem, restated for our situation.

Theorem 7.4 ([4], Theorem 6.4)**.** *Consider the first-order equation*

$$u_t + H(u_x) = 0$$

with a Lipschitz-continuous H. Let V be an open subset of the joint spacetime

$$\{(x,t) : x \in \mathbb{R}^n, t \in \mathbb{R}\} \cong \mathbb{R}^{n+1},$$

with the boundary component $\partial V = \partial V^1 \cup \partial V^2$ such that ∂V^2 is relatively open in ∂V, $\partial V^1 \cap \partial V^2 = \emptyset$. If u and v are continuous on $\overline{V}$, u is a bounded viscosity solution of $u_t + H(u_x) \leq -\delta < 0$ on $V \cup \partial V^2$ and v is a bounded viscosity solution of $u_t + H(u_x) \geq 0$ on $V \cup \partial V^2$, and $u \leq v$ on ∂V^1, then $u \leq v$ on V.

Here "a viscosity solution of $u_t + H(u_x) \geq 0$ on $V \cup \partial V^2$" means a function v that satisfies Condition 4 of Definition 7.1 of the viscosity solution for all $(x_0, t_0) \in V \cup \partial V^2$. Similarly "a viscosity solution of $u_t + H(u_x) \leq -\delta$ on $V \cup \partial V^2$" means a function u that satisfies Condition 3 of the viscosity solution to equation $u_t + H(u_x) + \delta = 0$ for all $(x_0, t_0) \in V \cup \partial V^2$.

We would like to apply the theorem above to prove the following comparison principle, which provides a uniqueness result that is sufficient for our purpose. Recall that any subsequential limit is uniformly bounded by Property (4.19).

Proposition 7.5. *Consider the equation along with a boundary condition*

$$\begin{cases} u_t + H(u_x) &= 0 \qquad (x,t) \in V_\varepsilon \\ \qquad\qquad u &= f \quad (x,t) \in \partial V_\varepsilon^1 \end{cases} \tag{7.10}$$

where $f, V_\varepsilon, \partial V_\varepsilon^1$ are described previously and H is given by (3.4). Also define

$$\partial V_\varepsilon^2 = \left\{ (x,t) : x \in \mathbb{R}^2, |x|_1 < 1, t = 1 \right\}.$$

If u and v are continuous on $\overline{V_\varepsilon}$, u is a bounded viscosity solution of $u_t + H(u_x) \leq 0$ on $V_\varepsilon \cup \partial V_\varepsilon^2$, $u \leq f$ on ∂V_ε^1, v is a bounded viscosity solution of $u_t + H(u_x) \geq 0$ on $V_\varepsilon \cup \partial V_\varepsilon^2$, $v \geq f$ on ∂V_ε^1, then $u \leq v$ on $\overline{V_\varepsilon}$.

Proof. Notice that we have $u \leq f \leq v$ on ∂V_ε^1. In order to apply Theorem 7.4 with $V = V_\varepsilon$, $\partial V^1 = \partial V_\varepsilon^1$ and $\partial V^2 = \partial V_\varepsilon^2$, the only obstacle is the δ term in Theorem 7.4. Instead of u, we consider $u_k(x,t) := u(x,t) - t/k$ for a positive integer k. We claim that u_k is a bounded viscosity solution of $u_t+H(u_x) \leq -\frac{1}{k} < 0$ on $V_\varepsilon \cup \partial V_\varepsilon^2$. To see this, take any $(x_0,t_0) \in V_\varepsilon \cup \partial V_\varepsilon^2$ and $\phi \in C^\infty(V_\varepsilon)$ such that $\phi(x_0,t_0) = u_k(x_0,t_0)$ and $\phi \geq u_k$ in a neighborhood of (x_0,t_0). Define $\widetilde{\phi} := \phi + t/k$, then $\widetilde{\phi}(x_0,t_0) = u(x_0,t_0)$ and $\widetilde{\phi} \geq u$ in a neighborhood of (x_0,t_0). By the assumption on u, we have

$$\begin{aligned} &\widetilde{\phi}_t(x_0,t_0) + H(\widetilde{\phi}_x(x_0,t_0)) \leq 0, \\ \Rightarrow\ &\phi_t(x_0,t_0) + \frac{1}{k} + H(\phi_x(x_0,t_0)) \leq 0, \end{aligned}$$

which proves the claim. Now we can apply Theorem 7.4 to u_k and v instead and deduce that $u_k \leq v$ on $\overline{V_\varepsilon}$. Taking $k \to \infty$, we get $u \leq v$ on $\overline{V_\varepsilon}$. □

We again proceed as in Proposition 7.2 to show that any subsequential limit is a continuous bounded viscosity solution on $V_\varepsilon \cup \partial V_\varepsilon^2$ and equals to f on ∂V_ε^1. By Proposition 7.5, we obtain a deterministic limit on V_ε, since the argument can be made purely local by Proposition 4.4 and Condition 5.3. Because we have a predetermined boundary condition, similar to Lemma 5.5, we can deduce that

$$\sup_y |S(\varepsilon,t;g_1)(y) - S(\varepsilon,t;g_2)(y)| \leq \sup_y |g_1(y) - g_2(y)|, \tag{7.11}$$

where g_1, g_2 are defined on $\{x : x \in \mathbb{R}^2, |x|_1 \leq \varepsilon\}$. Since g_1, g_2 are 2-spatially-Lipschitz continuous with a common boundary condition, the right hand side of 7.11 is $O(\varepsilon)$. Therefore by taking the limit $\varepsilon \to 0$ and using the continuity of $S(\cdot,t)$, we conclude that we have a deterministic scaling limit if we start shuffling from time 0.

CHAPTER 8

Equilibrium measures of a related 1-D exclusion process

This section is mostly independent of the previous sections. We consider the discrete-time TASEP (Totally Asymmetric Simple Exclusion Process) model defined as following:

On the $\mathbb{Z}$ (or $\mathbb{Z}_N$) lattice, each site can be occupied by either 0 or 1 particle. We denote the configuration at time $t \in \mathbb{Z}$ by σ_t, where $\sigma_t(x) = 1$ if site $x \in \mathbb{Z}$ (or $\mathbb{Z}_N$) is occupied and $\sigma_t(x) = 0$ otherwise. Given an initial configuration σ_0, we perform a global update at each time $t \in \mathbb{Z}^+$: a particle at site x independently jumps to $x+1$ with probability $\beta \in (0,1)$ if $\sigma_{t-1}(x+1) = 0$.

This can be phrased using the graphical construction. We take a Bernoulli point process on space-time $\mathbb{Z} \times \mathbb{Z}$ (or $\mathbb{Z} \times \mathbb{Z}_N$), such that each point (t,x) is included with probability β independently. Let $\omega \in \Omega$ be a realization (also called Bernoulli mark), where $\Omega = \{0,1\}^{\mathbb{Z}\times\mathbb{Z}}$ (or $\{0,1\}^{\mathbb{Z}\times\mathbb{Z}_N}$) is the configuration space for the Bernoulli point process. Then the TASEP configuration σ_{t+1} is a deterministic function of σ_t and $\omega(t,\cdot)$, where $\omega(t,x)$ indicates whether a particle at site x and time t, if exists, attempts to jump at time $t+1$.

This model is referred to as the discrete-time TASEP model "with fully parallel updates" in [23], because we may update all particles simultaneously at each time t. For convenience, we will refer to this specific model simply as TASEP, unless specified otherwise. This model (in the case of $\beta = \frac{1}{2}$) was first introduced in [11] to prove the Arctic Circle Theorem due to its connection to the domino shuffling algorithm. For instance, the evolution of the boundary arc of the bottom yellow frozen region in Figure 1.2 during the domino shuffling algorithm can be mapped to the TASEP. For the details of the correspondence, one can refer to that paper. Another seminal paper on this exact same model is [12], which demonstrates that the fluctuation of this process under certain scaling converges to the Tracy-Widom distribution.

It is also worth mentioning that there are several other versions of discrete-time TASEP with different update rules, studied extensively in [23]. However, the model considered here is only briefly mentioned in [23]. The main distinction from other discrete-time TASEPs or the TASEP in continuous time (see [22]) is that the invariant measures here are not product measures.

We shall extend the existence and uniqueness results on the invariant measures of TASEP in [11] via somewhat different proofs. These can be useful for various scaling limit calculations. For example, one can proceed exactly as in Chapter 9 of [28] to derive the hydrodynamic limit of the TASEP based on a variational method, and hence deduce the Arctic Ellipse Theorem.

It is also worth mentioning that if we let $\beta \to 0$, and rescale the time by β^{-1}, this TASEP converges to the model in continuous time, but we will assume that β is fixed in the following discussion.

8.1 Invariant measures on $\mathbb{Z}_N$ and $\mathbb{Z}$

First we describe the invariant measures of TASEP on a cycle of length N. The state space is then $\{0,1\}^{\mathbb{Z}_N}$. This is a finite-state aperiodic and irreducible Markov chain, so there exists a unique stationary measure once we fix β.

Theorem 8.1. *For any integers $0 \le M \le N$, the unique stationary measure $\mu_{N,M}$ for a TASEP on $\mathbb{Z}_N$ with M particles is given by*

$$\mu_{N,M}(\sigma) = \frac{1}{Z}(1-\beta)^{-\#Peak(\sigma)}$$

where $\#Peak(\sigma)$ counts the number of $\overline{01}$ neighboring pairs in state σ (or equivalently the number of $\overline{10}$ pairs since we are on $\mathbb{Z}_N$), and Z is the normalizing constant.

Proof. We just need to show that, for any state $\sigma \in \{0,1\}^{\mathbb{Z}_N}$ starting from $\mu_{N,M}$, the probability of σ remains the same after one step. There are exactly $2^{\#Peak(\sigma)}$ many different states which can jump to σ in one step, corresponding to whether the 1 in each $\overline{01}$ pair in σ just moved or not. We denote the set of these $2^{\#Peak(\sigma)}$ possible predecessors by A. Then after one step, the probability of σ is

$$\sum_{\sigma' \in A} \frac{1}{Z}(1-\beta)^{-\#Peak(\sigma')}\beta^{\#Jump(\sigma',\sigma)}(1-\beta)^{\#Peak(\sigma')-\#Jump(\sigma',\sigma)}$$

where $\#Jump(\sigma',\sigma)$ is the number of 1's in σ' that jumped to turn into σ. This becomes

$$\begin{aligned} &\sum_{\sigma' \in A} \frac{1}{Z}\beta^{\#Jump(\sigma',\sigma)}(1-\beta)^{-\#Jump(\sigma',\sigma)} \\ =&\sum_{\sigma' \in A} \frac{1}{Z}(1-\beta)^{-\#Peak(\sigma)}\beta^{\#Jump(\sigma',\sigma)}(1-\beta)^{\#Peak(\sigma)-\#Jump(\sigma',\sigma)} \end{aligned}$$

$$
\begin{aligned}
=&\mu_{N,M}(\sigma)\sum_{\sigma'\in A}\beta^{\#Jump(\sigma',\sigma)}(1-\beta)^{\#Peak(\sigma)-\#Jump(\sigma',\sigma)}\\
=&\mu_{N,M}(\sigma)(\beta+1-\beta)^{\#Peak(\sigma)}=\mu_{N,M}(\sigma)
\end{aligned}
$$

□

Now we turn to TASEP on $\mathbb{Z}$. Denote the state space $\{0,1\}^{\mathbb{Z}}$ by $\mathcal{X}$. It is given the product topology, with a metric, e.g.,

$$d(\sigma,\sigma')=\sum_{x\in\mathbb{Z}}2^{-|x|}\left|\sigma(x)-\sigma'(x)\right|.$$

We set up some notations for convenience. Let $\mathcal{T}$ be the space of translation invariant measures on $\mathcal{X}$, and let $\mathcal{I}$ be the space of stationary measures on $\mathcal{X}$ under the TASEP dynamics. By Prokhorov's theorem, both these spaces are weakly compact, and they are obviously convex. We define $\mathcal{T}_e$ and $\mathcal{I}_e$ as their extreme points (points which cannot be written as a nontrivial linear combination of other points in the space). We will also consider the intersection $\mathcal{T}\cap\mathcal{I}$, the translation invariant and stationary measures of TASEP, and its extreme points $(\mathcal{T}\cap\mathcal{I})_e$. For more detailed background, see [22].

Theorem 8.2. *The space $(\mathcal{T}\cap\mathcal{I})_e$ contains a 1-parameter family of Markov measures μ_p characterized by*

$$p=\mathbb{E}_{\mu_p}[\sigma(0)]$$

and

$$q_{00}q_{11} = (1-\beta)q_{10}q_{01} \tag{8.1}$$

where

$$q_{ij} = \mathbb{P}_{\mu_p}[\sigma(x+1) = j | \sigma(x) = i].$$

If $\beta \in (0, \frac{1}{2}]$*, these are all the measures in* $(\mathcal{T} \cap \mathcal{I})_e$*.*

The specific case when $\beta = \frac{1}{2}$ is proven in [11].

First of all, let us compute q_{ij}'s explicitly in terms of p. By definition, $q_{01} = 1 - q_{00}$ and $q_{10} = 1 - q_{11}$, so (8.1) yields

$$\begin{aligned} q_{00}q_{11} &= (1-\beta)(1-q_{00})(1-q_{11}) \\ q_{11} &= \frac{1-q_{00}}{1+\gamma q_{00}}, \text{ writing } \gamma = \frac{\beta}{1-\beta} \text{ for convenience.} \end{aligned} \tag{8.2}$$

Also by the translation invariance,

$$p = pq_{11} + (1-p)q_{01}$$

which can be solved

$$p = \frac{(1-q_{00})(1+\gamma q_{00})}{1+2\gamma q_{00} - \gamma q_{00}^2} \tag{8.3}$$

By taking derivative, one can check that p as a function of q_{00} is strictly decreasing, which goes from 0 to 1 when q_{00} goes from 1 to 0. So the equation (8.3) can be inverted:

$$q_{00} = \frac{2\beta(1-p) - 1 + \sqrt{4\beta p(p-1)+1}}{2\beta(1-p)} \tag{8.4}$$

From above one can find

$$pq_{10} = (1-p)q_{01} = \frac{1 - \sqrt{4\beta p(p-1)+1}}{2\beta} \tag{8.5}$$

which says that $(0,1)$ and $(1,0)$ neighboring pairs are equally likely, as expected.

Remark. The extremality of μ_p is easy to see. The cases $p = 0$ and $p = 1$ are trivial. When $0 < p < 1$, μ_p is a Markov process with a 2×2 transtion matrix, and correlation decays exponentially, so we can show that μ_p is in fact mixing under translation by approximating events in the σ-algebra with cylindrical sets. Therefore μ_p is spatially ergodic, meaning it is an extreme point of $\mathcal{I}$, which strictly contains $\mathcal{I} \cap \mathcal{T}$. As a result, μ_p belongs to $(\mathcal{I} \cap \mathcal{T})_e$.

The theorem immediately implies the following very similar but weaker statement:

Corollary 8.3. *When $\beta \in (0, \frac{1}{2}]$, the unique TASEP-stationary measures among the spatially ergodic translation invariant measures on $\mathcal{X}$ are the ones described in Theorem 8.2.*

Proof. As mentioned in the previous remark, if a measure μ is spatially ergodic, and also stationary, it must belong to $(\mathcal{I} \cap \mathcal{T})_e$, so it is μ_p for some p as described in Theorem 8.2. $\square$

The rest of the proof is split into two parts. Again we assume $0 < p < 1$.

8.2 Stationarity of μ_p

One can prove the stationarity directly as in [11], but we choose to derive it from the results on $\mathbb{Z}_N$, because this tells us also about the relationship between TASEPs on $\mathbb{Z}_N$ and $\mathbb{Z}$.

Given any $p \in (0,1)$, we take the sequence of stationary measures $\{\mu_{N_i,M_i}\}$ on $\mathbb{Z}_{N_i}$, such that $N_i = i$ and $M_i = \lceil pN_i \rceil$. We view μ_{N_i,M_i} as a measure with state space $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ by extending every configuration periodically to $\mathbb{Z}$.

Proposition 8.4. *$\{\mu_{N_i,M_i}\}$ converges weakly to μ_p.*

Recall that we made $\mathcal{X}$ a compact metric space. It is then equivalent to show that the marginal measures of $\{\mu_{N_i,M_i}\}$ on all cylindrical sets converge to μ_p. Due to rotational symmetry, the absolute position of the cylindrical set does not matter. Also by our choice of M_i, the marginal measure at a single site, say $x = 0$, converges to that of μ_p. In the next two lemmas, we will verify the marginal measure on two neighboring sites, and then extend to any finite consecutive sites by establishing the Markov property.

Lemma 8.5. *Denoting*

$$q_{jk}^{(i)} = \mathbb{P}_{\mu_{N_i,M_i}}[\sigma(1) = k | \sigma(0) = j]$$

we have $\lim_{i\to\infty} q_{jk}^{(i)} = q_{jk}$ where q_{jk} is defined in Theorem 8.2.

Proof. We know that $q_{11}^{(i)} = 1 - q_{10}^{(i)}$, $q_{01}^{(i)} = 1 - q_{00}^{(i)}$, $(M_i/N_i)q_{11}^{(i)} + (1 - M_i/N_i)q_{01}^{(i)} = M_i/N_i$ where $\lim_{i\to\infty} M_i/N_i = p$. So to prove the lemma, it suffices to show that

$$\lim_{i\to\infty} \frac{q_{00}^{(i)} q_{11}^{(i)}}{q_{01}^{(i)} q_{10}^{(i)}} = 1 - \beta \tag{8.6}$$

We can rewrite this as

$$\begin{aligned}
\lim_{i\to\infty} \frac{q_{00}^{(i)} q_{11}^{(i)}}{q_{10}^{(i)} q_{01}^{(i)}} &= \lim_{i\to\infty} \frac{(1 - M_i/N_i)q_{00}^{(i)}(M_i/N_i)q_{11}^{(i)}}{(1 - M_i/N_i)q_{01}^{(i)}(M_i/N_i)q_{10}^{(i)}} \\
&= \lim_{i\to\infty} \frac{\mathbb{P}_{\mu_{N_i,M_i}}[\sigma(0) = 0, \sigma(1) = 0]\,\mathbb{P}_{\mu_{N_i,M_i}}[\sigma(0) = 1, \sigma(1) = 1]}{\mathbb{P}_{\mu_{N_i,M_i}}[\sigma(0) = 0, \sigma(1) = 1]\,\mathbb{P}_{\mu_{N_i,M_i}}[\sigma(0) = 1, \sigma(1) = 0]} = 1 - \beta
\end{aligned}$$

The ratio of these probabilities can be computed explicitly from Theorem 8.1. For example,

$$\begin{aligned}&\mathbb{P}_{\mu_{N_i,M_i}}\left[\sigma(0)=0,\sigma(1)=0\right]\\ =&\frac{1}{Z}\sum_k \#\left\{\sigma\in\{0,1\}^{\mathbb{Z}_{N_i}}|\sigma \text{ has } M_i \text{ particles and } k \text{ peaks}, \sigma(0)=0,\sigma(1)=0\right\}(1-\beta)^{-k}\end{aligned}$$

The coefficient in the sum is counting the sequence $0^{a_0}1^{b_1}0^{a_1}\dots 1^{b_k}0^{a_k}$ where $a_0+\cdots+a_k=N_i-M_i$, $b_1+\cdots+b_k=M_i$ and all a's and b's are positive integers. By counting partitions, this is just $\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k}$. Computing other probabilities similarly, the goal (8.6) becomes

$$\frac{\sum_k\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\sum_k\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}}\Bigg/\frac{\sum_k\binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\sum_k\binom{M_i-1}{k}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}}\to 1 \tag{8.7}$$

where the $1-\beta$ in (8.6) is absorbed by the sum. So it amounts to estimating the ratio of certain sums. Notice that the summands are product of binomial coefficients and exponentials, so in fact each sum should be concentrated on a small window of $o(N_i)$ terms near the maximal summand.

Take $\sum_k\binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}$ as an example. The quotient of the k-th term divided by the $(k-1)$-th term is

$$\frac{\binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k+1}}=\frac{(M_i-k)(N_i-M_i-k)}{(1-\beta)k^2}$$

The maximal term occurs when the ratio crosses 1, or

$$(M_i-\widetilde{k})(N_i-M_i-\widetilde{k})=(1-\beta)\widetilde{k}^2\Rightarrow \widetilde{k}=N_i\frac{1-\sqrt{1-4\beta(M_i/N_i)(1-M_i/N_i)}}{2\beta} \tag{8.8}$$

approximately. Some elementary calculus shows two properties about $\widetilde{k}(i)$:

I. The summand is increasing on the left of $\widetilde{k}$ and decreasing on the right;

II. The limit of $\widetilde{k}/N_i$, which depends on p and β, is strictly between 0 and $\min\{p,1-p\}$.

For now let us restrict the sum to indices k between ϵN_i and $(\min\{M_i, N_i - M_i\} - \epsilon N_i)$ for some small $\epsilon > 0$ that only depends on the limit of $\widetilde{k}/N_i$. Then $k, M_i, M_i - k, N_i - M_i - k$ are all of linear order in N_i, so we can apply Sterling's formula to the summand:

$$\begin{aligned}
&\binom{M_i - 1}{k}\binom{N_i - M_i - 1}{k}(1-\beta)^{-k} \\
=&\frac{(M_i-1)!(N_i-M_i-1)!}{k!(M_i-1-k)!k!(N_i-M_i-1-k)!}(1-\beta)^{-k} \\
=&\left(\frac{1}{2\pi}+o(1)\right)\sqrt{\frac{(M_i-1)(N_i-M_i-1)}{k^2(M_i-1-k)(N_i-M_i-1-k)}}\left(\frac{M_i-1}{k}\right)^k\left(\frac{M_i-1}{M_i-1-k}\right)^{M_i-1-k} \\
&\left(\frac{N_i-M_i-1}{k}\right)^k\left(\frac{N_i-M_i-1}{N_i-M_i-1-k}\right)^{N_i-M_i-1-k}(1-\beta)^{-k}
\end{aligned}$$

If we take logarithm of this, and let $\kappa = k/N_i$, $\alpha = (M_i - 1)/N_i$, we get

$$\begin{aligned}
&\log\left(\frac{1}{2\pi}+o(1)\right)+\frac{1}{2}\log\left(\frac{\alpha(1-\alpha)}{N_i^2\kappa^2(\alpha-\kappa)(1-\alpha-\kappa)}\right)+N_i\kappa\log\frac{\alpha}{\kappa} \\
&+N_i(\alpha-\kappa)\log\frac{\alpha}{\alpha-\kappa}+N_i\kappa\log\frac{1-\alpha}{\kappa}+N_i(1-\alpha-\kappa)\log\frac{1-\alpha}{1-\alpha-\kappa}-N_i\kappa\log(1-\beta)
\end{aligned}$$

where the last five terms of order N_i dominate the first two terms, so we can further estimate this as $(1 + o(1))N_i f(\alpha, \kappa)$ with

$$\begin{aligned}
f(\alpha,\kappa)=\kappa\log\frac{\alpha}{\kappa}+(\alpha-\kappa)\log\frac{\alpha}{\alpha-\kappa}+\kappa\log\frac{1-\alpha}{\kappa} \\
+(1-\alpha-\kappa)\log\frac{1-\alpha}{1-\alpha-\kappa}-\kappa\log(1-\beta).
\end{aligned}$$

Unsurprisingly, the maximum of f with respect to κ agrees with the discrete case computed in (8.8).

Due to property I above, we can estimate the sum $\sum_k \binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}$ by an integral $\int e^{(1+o(1))N_i f(\alpha,\kappa)}d\kappa$ with appropriate boundaries, plus those $2\epsilon N_i$ omitted terms at the edges. Due to property II, if we choose ϵ small enough independent of N_i, by Laplace's method (see for example [35]), the integral is completely dominated by a small window of size, say, $N_i^{-1/3}$ near the maximum of f. This implies that the sum, with the edge terms omitted, is dominated by terms within $N_i^{2/3}$ of index $\widetilde{k}$ from (8.8). Again thanks to property I and II, those edge terms are also dominated if ϵ is chosen small enough independent of N_i.

Arguing similarly, we conclude that for each of the four sums that appear in (8.7), which we write in shorthand as $\sum_k a_k^{(i)}$, the following holds:

$$\lim_{i\to\infty} \frac{\sum_{k=\widetilde{k}-N_i^{2/3}}^{\widetilde{k}+N_i^{2/3}} a_k^{(i)}}{\sum_k a_k^{(i)}} = 1 \tag{8.9}$$

where $\widetilde{k} = \widetilde{k}(i)$ is given by (8.8). (For the other three sums, the maximal terms have different indices, but they are only $O(1)$ away from $\widetilde{k}$, so it does not matter.) Therefore, we can restrict all the indices in (8.7) to the small window. That is, it suffices to show

$$\frac{\sum_{k=\widetilde{k}-N_i^{2/3}}^{\widetilde{k}+N_i^{2/3}} \binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\sum_{k=\widetilde{k}-N_i^{2/3}}^{\widetilde{k}+N_i^{2/3}} \binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}} \Bigg/ \frac{\sum_{k=\widetilde{k}-N_i^{2/3}}^{\widetilde{k}+N_i^{2/3}} \binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\sum_{k=\widetilde{k}-N_i^{2/3}}^{\widetilde{k}+N_i^{2/3}} \binom{M_i-1}{k}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}} \to 1 \tag{8.10}$$

Let us compute the ratio between individual summands with the same index. For the first quotient in (8.10), we have

$$\frac{\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\binom{M_i-1}{k-1}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}} = \frac{N_i-M_i-k}{k}$$

and for the second quotient in (8.10)

$$\frac{\binom{M_i-1}{k}\binom{N_i-M_i-1}{k}(1-\beta)^{-k}}{\binom{M_i-1}{k}\binom{N_i-M_i-1}{k-1}(1-\beta)^{-k}} = \frac{N_i-M_i-k}{k}$$

By property II about $\widetilde{k}$, there exists $\epsilon > 0$ such that for large enough i,

$$\epsilon N_i < \widetilde{k}(i) < N_i - M_i - \epsilon N_i$$

and since $k \in [\widetilde{k} - N_i^{2/3}, \widetilde{k} + N_i^{2/3}]$, both ratios above converge to

$$\begin{aligned} \lim_{i\to\infty} \frac{N_i - M_i - \widetilde{k}}{\widetilde{k}} &= \lim_{i\to\infty} \frac{1 - M_i/N_i - \widetilde{k}/N_i}{\widetilde{k}/N_i} \\ &= \frac{2\beta(1-p) - 1 + \sqrt{1 - 4\beta p(1-p)}}{1 - \sqrt{1 - 4\beta p(1-p)}}. \end{aligned} \tag{8.11}$$

More importantly, this convergence is "uniform in k", in the sense that the worst case happens at $k = \widetilde{k}(i) \pm N_i^{2/3}$, and those two both converge to the same limit above.

Therefore, the two big quotients in (8.10) both converge to the same limit, and (8.10) is established. $\square$

Lemma 8.6 (Markov property in the limit)**.**

$$\lim_{i\to\infty} \frac{q_{jk}^{(i)}}{q_{y;k}^{(i)}} = 1$$

where $q_{y;k}^{(i)}$ *denotes*

$$\mathbb{P}_{\mu_{N_i,M_i}} [\sigma(1) = k | \sigma(0) = j = y_1, \sigma(-1) = y_2, \sigma(-2) = y_3, \dots, \sigma(-n) = y_n]$$

for any fixed $n, k, y_1, y_2, \dots, y_n$.

Proof. The argument is similar to the proof of Lemma 8.5. We will assume that $y_n = 1$ and $j = y_1 = 0$. Other three cases are essentially the same.

Since we are dealing with conditional probabilities, to establish the lemma, it suffices to show that

$$\lim_{i\to\infty} \frac{q_{00}^{(i)}}{q_{01}^{(i)}} \Big/ \frac{q_{y;0}^{(i)}}{q_{y;1}^{(i)}} = 1.$$

The first quotient $q_{00}^{(i)}/q_{01}^{(i)}$ was shown to be equal to the first quotient in (8.7). Suppose that there are H occurrences of 0's and Q occurrences of $\overline{10}$'s in $\overline{y_n y_{n-1} \dots y_1}$. Again we use Theorem 8.1, similar to the reasoning behind (8.7), to get

$$\frac{q_{y;0}^{(i)}}{q_{y;1}^{(i)}} = \frac{\sum_k \binom{M_i-n+H}{k}\binom{N_i-M_i-H-1}{k}(1-\beta)^{-k-Q}}{\sum_k \binom{M_i-n+H}{k}\binom{N_i-M_i-H-1}{k-1}(1-\beta)^{-k-Q}}$$

Since n, H, Q are constants independent of i, this in fact shares the same limit as the second quotient in (8.7) due to essentially the same computation. Therefore this lemma follows from (8.7).

$\square$

Proof of Proposition 8.4. Continuing with the remark after the proposition, we combine Lemma 8.5 and Lemma 8.6 to conclude that $\{\mu_{N_i,M_i}\}$ converges to μ_p on all cylindrical sets with consecutive sites, so $\{\mu_{N_i,M_i}\}$ converges weakly to μ_p. $\square$

This convergence in particular implies the following.

Proposition 8.7. *μ_p is stationary under the discrete updates.*

Proof. Suppose a measure μ on $\mathcal{X} = \{0,1\}^{\mathbb{Z}}$ becomes μ' after one global update. It suffices to show that, for any given cylindrical set with form $B = \{\sigma \in \mathcal{X} : \sigma(1) = b_1, \dots, \sigma(n) = b_n\}$, we have stationarity $\mu_p(B) = \mu_p'(B)$.

Again we consider the sequence of measures $\{\mu_{N_i,M_i}\}$ (viewed as measures on $\mathcal{X}$ by periodic extension) that converges weakly to μ_p according to Proposition 8.4. Note that the configuration σ_{t+1} on sites $1, \dots, n$ is a deterministic function of $\sigma_t(0), \sigma_t(1), \dots, \sigma_t(n+1)$

and iid Bernoulli marks $\omega(t,0), \omega(t,1), \ldots, \omega(t,n)$. When $N_i > n+2$, sites $0, \ldots, n+1$ are within a single period, so one cannot distinguish between the single-step evolution restricted to these sites on $\mathbb{Z}$ and $\mathbb{Z}_{N_i}$. Then by Theorem 8.1, we have $\mu'_{N_i,M_i}(B) = \mu_{N_i,M_i}(B)$.

If we break down this equality, it just means

$$\begin{aligned} &\mu_{N_i,M_i}(B) \\ = &\sum_{\overline{a}=(a_0,\ldots,a_{n+1})\in\{0,1\}^{n+2}} \mu_{N_i,M_i}\left(\{\sigma \in \mathcal{X} : \sigma(0) = a_0, \ldots, \sigma(n+1) = a_{n+1}\}\right) \mathbb{P}\left[\overline{a} \to \overline{b}\right] \end{aligned}$$

where $\mathbb{P}\left[\overline{a} \to \overline{b}\right]$ is the transition probability

$$\mathbb{P}\left[\sigma'(1) = b_1, \ldots, \sigma'(n) = b_n | \sigma(0) = a_0, \ldots, \sigma(n+1) = a_{n+1}\right].$$

And since the measures $\{\mu_{N_i,M_i}\}$ converge to μ_p on each cylindrical set as $i \to \infty$, the equality above still holds after we replace all μ_{N_i,M_i} by μ_p, which means $\mu'_p(B) = \mu_p(B)$ as desired. $\square$

8.3 Uniqueness of μ_p

Throughout this section, we assume that $\beta \in (0, \frac{1}{2}]$. Given any $\mu \in \mathcal{I} \cap \mathcal{T}$, we can find a number $p = \mathbb{E}_\mu[\sigma(0)]$. Then our goal is to show that μ is the same as μ_p in distribution.

We define a partial ordering on $\mathcal{X}$ by saying that $\sigma^1 \geq \sigma^2$ iff $\sigma^1(x) \geq \sigma^2(x)$ for all $x \in \mathbb{Z}$. The key ingredient of the proof is a coupling between two evolutions that preserves the partial ordering. Particle systems with such coupling are "attractive" [22]. For the continuous-time TASEP or other versions of discrete-time TASEP, the basic coupling works, i.e. imposing that particles at the same position share the same random outcome. However, this no longer works in the current situation, and [11] invented a clever coupling that solves the issue. We will modify their coupling, but proceed as [22] does, because there is some gap in the proof of [11].

To define a coupling, we describe an update rule for $\mathcal{X} \times \mathcal{X}$ so that the marginal rule on each copy coincides with the TASEP. Given $(\sigma^1, \sigma^2) \in \mathcal{X} \times \mathcal{X}$ at some time moment, we say that two neighboring positions x and $x+1$ are linked if either $\sigma^1(x) = 1$, $\sigma^1(x+1) = 0$ or $\sigma^2(x) = 1$, $\sigma^2(x+1) = 0$. If positions $x, x+1, \ldots, y$ are all linked, and we cannot extend the sequence further on either end, then we call the corresponding segment of (σ^1, σ^2) a **block**. There exists a unique block decomposition for (σ^1, σ^2). For convenience we assume that all blocks have finite length, but one can easily generalize the following rule to blocks with infinite length. A block can have minimum length 1.

It is clear that we can update each block independently, because no particle can jump across blocks in this update step. For length-1 blocks, no jump happens. For blocks with length 2, we have no freedom of choosing coupling except the case

$$\begin{array}{ll} \sigma^1: & 1\ 0 \\ \sigma^2: & 1\ 0 \end{array}$$

where we force the two particles to behave the same.

For blocks with length 3 or more, there are two general cases (up to exchanging σ^1 and σ^2):

$$\begin{array}{lcccccc} \sigma^1: & \boxed{1} & 0 & 1 & \ldots & 0 & ? \\ \sigma^2: & ? & \boxed{1} & 0 & \ldots & 1 & 0 \end{array}$$

or

$$\begin{array}{lcccccc} \sigma^1: & \boxed{1} & 0 & 1 & \ldots & 1 & 0 \\ \sigma^2: & ? & \boxed{1} & 0 & \ldots & 0 & ? \end{array}$$

where ? can be either 0 or 1. We couple each particle at position x in σ^1 with the particle at position $x+1$ in σ^2. The leftmost coupled pairs are boxed in the diagram above. For each pair independently, we pick one random real number R uniformly between 0 and 1. Depending on the value of R, three things can happen:

$$\begin{cases} \text{1. If } R < \beta\text{, the top particle jumps to the right, and the bottom particle stays;} \\ \text{2. If } R > 1-\beta\text{, the top particle stays, and the bottom particle jumps to the right;} \\ \text{3. If } \beta \le R \le 1-\beta\text{, both particles stay.} \end{cases}$$

Here we used the condition that $\beta \le \frac{1}{2}$. Observe that the marginal rule on each copy indeed agrees with our original model, i.e. each particle with an empty right neighbor jumps with probability β, and the choices within each copy are independent. Also, the coupled dynamics commutes with translation on $\mathcal{X} \times \mathcal{X}$. More importantly,

Proposition 8.8. *The coupling preserves the partial ordering on $\mathcal{X}$ in the sense that if $\sigma^1 \ge \sigma^2$, and (σ^1, σ^2) turns into $(\widetilde{\sigma}^1, \widetilde{\sigma}^2)$ after one global update, then $\widetilde{\sigma}^1 \ge \widetilde{\sigma}^2$.*

Proof. If $\sigma^1 \ge \sigma^2$, then the block decomposition of (σ^1, σ^2) consists of only the following types of blocks:

$$\begin{matrix} 0 \\ 0 \end{matrix}\;,\; \begin{matrix} 1 \\ 0 \end{matrix}\;,\; \begin{matrix} 1 \\ 1 \end{matrix}\;,\; \begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}\;,\; \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}\;,\; \begin{matrix} 1 & 1 \\ 1 & 0 \end{matrix}\;,\; \begin{matrix} 1 & 1 & 0 \\ 1 & 0 & 0 \end{matrix}$$

and the coupling clearly preserves partial ordering on these blocks. □

When a partial ordering is not satisfied, this coupling still tends to bring the configurations closer, hence the name "attractive". This is made precise as following:

Proposition 8.9. *After one global update, the number of mismatches (positions x such that $\mu^1(x) \ne \mu^2(x)$) within a block does not increase. Most types of blocks have strictly fewer mismatches with positive probability, which we shall call* ***unstable blocks****. The exceptions* *are blocks of length 1, and the following blocks up to exchanging* σ^1 *and* σ^2*,*

$$\begin{matrix} 1 & 0 \\ 0 & 0 \end{matrix}\ ,\quad \begin{matrix} 1 & 0 \\ 1 & 0 \end{matrix}\ ,\quad \begin{matrix} 1 & 0 \\ 1 & 1 \end{matrix}\ ,\quad \begin{matrix} 1 & 0 & 0 \\ 1 & 1 & 0 \end{matrix}$$

which we shall call ***stable blocks****, because the number of mismatches remains the same with probability 1.*

Proof. Blocks with length 1 and 2 can be checked manually. We subdivide a block of length 3 or more into segments of length 2 or less, and there are four cases to consider:

I. The beginning segment always looks like

$$\begin{matrix} \boxed{1} & 0 \\ ? & \boxed{1} \end{matrix}$$

Let Δ denote the net change of number of mismatches within this segment, and again R denotes the random number that determines the movement of the particles in this segment, then

$$\Delta = \begin{cases} 0 & \beta \leq R \leq 1-\beta \\ 0 & R < \beta \text{ and } ? = 1 \\ -2 & R < \beta \text{ and } ? = 0 \\ -1 & R > 1-\beta \end{cases}$$

II. For an intermediate segment

$$\begin{matrix} \boxed{1} & 0 \\ 0 & \boxed{1} \end{matrix}$$

$$\Delta \leq \begin{cases} 0 & \beta \leq R \leq 1-\beta \\ -1 & R < \beta \text{ or } R > 1-\beta \end{cases}$$

III. For an ending segment that looks like

$$\begin{matrix} 1 & 0 \\ 0 & ? \end{matrix}$$

only the top particle can jump.

$$\Delta \leq \begin{cases} 0 & R \geq \beta \\ -1 & R < \beta \text{ and } ? = 1 \\ 1 & R < \beta \text{ and } ? = 0 \end{cases}$$

and $\Delta = 1$ iff in the second last segment which has form I or II, the bottom particle jumps.

IV. For an ending segment with form

$$\begin{matrix} ? \\ 0 \end{matrix}$$

which happens when the block length is odd, no jump happens.

$$\Delta \leq \begin{cases} 0 & ? = 1 \\ 1 & ? = 0 \end{cases}$$

and $\Delta = 1$ iff in the second last segment which has form I or II, the bottom particle jumps.

Now we sum up these segments to obtain the result for the entire block of length 3 or more. Case I and II both are nonincreasing and have positive probability to strictly reduce mismatches. Case III and IV can increase mismatches by 1, but those happen only if the bottom particle on the left jumps, which guarantees a strict decrease of mismatches in the second last segment, so two effects cancel out. Such consideration also shows that there is only one stable block of length 3 or more, as mentioned in the proposition. □

From here we will follow the treatment in [22]. Following previous notations, define $\mathcal{T}^*$ as the space of translation invariant coupled measures on $\mathcal{X} \times \mathcal{X}$, and $\mathcal{I}^*$ as the space of stationary measures on $\mathcal{X} \times \mathcal{X}$ under the coupling defined above.

Since mismatches tend to decrease after updates, if a coupled measure is already stationary, its mismatches have to be in some sense minimum. This is the content of the next proposition, which is a discrete version of [22, Lemma VIII.3.2].

Proposition 8.10. *If* $\pi \in \mathcal{I}^* \cap \mathcal{T}^*$,

$$\mathbb{P}_\pi[\sigma^1(0) = 0, \sigma^2(0) = 1, \sigma^1(i) = 1, \sigma^2(i) = 0] = 0$$

for any $i \in \mathbb{Z}$.

Proof. Suppose to the contrary $\mathbb{P}_\pi[\sigma^1(0) = 0, \sigma^2(0) = 1, \sigma^1(i) = 1, \sigma^2(i) = 0] > 0$ for some i. Since we can exchange σ^1 and σ^2, WLOG assume $i > 0$, and is minimum. Then with positive probability under π, σ^1 and σ^2 have some specified values from position 0 to i:

$$\begin{array}{rccc} \textit{Position} & 0 & \dots & i \\ \sigma^1: & 0 & \dots & 1 \\ \sigma^2: & 1 & \dots & 0 \end{array}$$

There is no mismatch in the middle, otherwise we can shrink the interval and obtain a smaller i by translation invariance. Suppose there are j particles in σ^1 from position 1 to position $i-1$. Then we keep applying the coupling dynamics to π, and look at the evolution on this interval. There is a positive probability that first all the matched particles in the middle move to the right while particles on position 0 and i stay fixed, so the interval now looks like

$$\begin{array}{llllllllll} \textit{Position} & 0 & 1 & \dots & (i-j-1) & (i-j) & \dots & (i-1) & i \\ \sigma^1: & 0 & 0 & \dots & 0 & 1 & \dots & 1 & 1 \\ \sigma^2: & 1 & 0 & \dots & 0 & 1 & \dots & 1 & 0 \end{array}$$

In the second stage, all the particles in σ^2 from position $i-j$ to $i-1$ will move to position $i-j+1$ to i while everything else in the interval stay fixed:

$$\begin{array}{llllllllll} \textit{Position} & 0 & 1 & \dots & (i-j-1) & (i-j) & (i-j+1) & \dots & (i-1) & i \\ \sigma^1: & 0 & 0 & \dots & 0 & 1 & 1 & \dots & 1 & 1 \\ \sigma^2: & 1 & 0 & \dots & 0 & 0 & 1 & \dots & 1 & 1 \end{array}$$

In the third stage, the particle in σ^2 at position 0 will move to position $i-j-1$ while everything else in the interval stay fixed:

$$\begin{array}{llllllllll} \textit{Position} & 0 & \dots & (i-j-2) & (i-j-1) & (i-j) & (i-j+1) & \dots & (i-1) & i \\ \sigma^1: & 0 & \dots & 0 & 0 & 1 & 1 & \dots & 1 & 1 \\ \sigma^2: & 0 & \dots & 0 & 1 & 0 & 1 & \dots & 1 & 1 \end{array}$$

So in fact with positive probability we have a local pattern

$$\begin{array}{cc} 0 & 1 \\ 1 & 0 \end{array}$$

which must be part of an unstable block by Proposition 8.9. By translation invariance and again Proposition 8.9, this means $\mathbb{P}_\pi[\sigma^1(0) \neq \sigma^2(0)]$ strictly decreases after a global update (this is true even if the unstable block is infinite, because mismatches strictly decrease with positive probability throughout the whole unstable block uniformly). But this contradicts the assumption that $\pi \in \mathcal{I}^*$. $\square$

We also need a discrete version of [22, Theorem III.2.15].

Lemma 8.11. *If $\mu_1, \mu_2 \in (\mathcal{I} \cap \mathcal{T})_e$, there exists $\pi \in (\mathcal{I}^* \cap \mathcal{T}^*)_e$ which projects to μ_1 on the first component and projects to μ_2 on the second component.*

Proof. Take $\nu_0 = \mu_1 \times \mu_2$, which is clearly translation invariant, and consider the sequence

$$\frac{1}{n}\sum_{i=1}^{n} \nu_i, \qquad n = 1, 2, \dots$$

where $\nu_i \in \mathcal{T}^*$ is the resulting measure after i iterations of coupled dynamics started with ν_0. The space of probability measures on $\mathcal{X} \times \mathcal{X}$ is compact under the weak topology, so there exists a weakly convergent subsequence which converges to a probability measure $\widetilde{\nu}$:

$$\lim_{n\to\infty} \frac{1}{t_n}\sum_{i=1}^{t_n} \nu_i = \widetilde{\nu}, \qquad t_n \uparrow \infty.$$

It is clear that $\widetilde{\nu} \in \mathcal{T}^*$ since all ν_i are. We claim that $\widetilde{\nu} \in \mathcal{I}^*$. Suppose $\widetilde{\nu}$ turns into $\widetilde{\nu}'$ after one global update, then it suffices to show $\widetilde{\nu}(A) = \widetilde{\nu}'(A)$ for all cylindrical set A. By monotone covergence theorem,

$$\begin{aligned}
\widetilde{\nu}'(A) - \widetilde{\nu}(A) &= \lim_{n\to\infty} \frac{1}{t_n}\sum_{i=1}^{t_n} \nu_{i+1}(A) - \lim_{n\to\infty} \frac{1}{t_n}\sum_{i=1}^{t_n} \nu_i(A) \\
&= \lim_{n\to\infty} \frac{1}{t_n}\left(\sum_{i=1}^{t_n} \nu_{i+1}(A) - \sum_{i=1}^{t_n} \nu_i(A)\right) \\
&= \lim_{n\to\infty} \frac{1}{t_n}\left(\nu_{t_n+1}(A) - \nu_1(A)\right) = 0
\end{aligned}$$

So $\widetilde{\nu} \in \mathcal{I}^* \cap \mathcal{T}^*$ and has the desired marginal measures on the two components. Again, the space $\mathcal{I}^* \cap \mathcal{T}^*$ is compact and convex, so $\widetilde{\nu}$ can be written as a convex combination of its extreme points. We can restrict the convex combination to the first and second copy of

$\mathcal{X} \times \mathcal{X}$, and since $\mu_1, \mu_2 \in (\mathcal{I} \cap \mathcal{T})_e$, all the summands in the convex combinations summing up to $\widetilde{\nu}$ must have marginal measures μ_1 and μ_2 on the two components respectively. We take one of the summands to be $\pi \in (\mathcal{I}^* \cap \mathcal{T}^*)_e$. □

Proof of uniqueness in Theorem 8.2. Now assume μ is in $(\mathcal{I} \cap \mathcal{T})_e$ such that $\mathbb{E}_\mu[\sigma(0)] = p$. Our goal is to show that μ is the same as μ_p. Using Lemma 8.11, we know there exists an element $\pi \in (\mathcal{I}^* \cap \mathcal{T}^*)_e$ which projects to μ on the first component and projects to μ_p on the second component.

By Proposition 8.10, π is supported on $\mathcal{Y}_1 \cup \mathcal{Y}_2 \cup \mathcal{Y}_3$, where

$$\mathcal{Y}_1 = \{(\sigma^1, \sigma^2) : \sigma^1 \leq \sigma^2, \sigma^1 \neq \sigma^2\}$$
$$\mathcal{Y}_2 = \{(\sigma^1, \sigma^2) : \sigma^1 \geq \sigma^2, \sigma^1 \neq \sigma^2\}$$
$$\mathcal{Y}_3 = \{(\sigma^1, \sigma^2) : \sigma^1 = \sigma^2\}$$

By Proposition 8.8, all these three spaces are closed under the coupled dynamics, and they are also translation invariant. So by the extremality of π, it must have full measure on one of these sets. In particular, π is supported on either $\{(\sigma^1, \sigma^2) : \sigma^1 \geq \sigma^2\}$ or $\{(\sigma^1, \sigma^2) : \sigma^1 \leq \sigma^2\}$. But we assumed that $\mathbb{E}_\mu[\sigma(0)] = p = E_{\mu_p}[\sigma(0)]$, so in fact π is supported on $\{(\sigma^1, \sigma^2) : \sigma^1 = \sigma^2\}$, and μ equals μ_p. □

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