A Computational Commutative Algebra Approach to Tilings

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Abstract

In 1990, Thurston proved that for any two-dimensional simply connected region, the space of tilings is flip-connected. Later, Yamzon reproved this result using a commutative algebra approach developed by Gross. In the first portion of this thesis, we further explore the algebraic structures from Gross and Yamzon's work. Specifically, we prove a primary decomposition of the flip ideal for small ($3 \times n$) regions, and give a conjecture for the primary decomposition in larger regions.

In the second portion of the thesis, we give preliminary computational results on the flip-connected components of three-dimensional tilings.

1 Introduction

Domino tiling problems have received much attention in the literature, especially due to their connection to physical models and to perfect matchings in graphs. Given a region, it is natural to ask whether this region can be perfect covered by 2×1 tiles, and if so in how many ways. Additionally, we might ask whether different tilings of a region can be related to one another in some way.

In the case of domino tilings, it is easy to determine whether a tiling exists using Hall's matching condition [1]. Additionally, in two dimensions, Kasteleyn, Temperley and Fisher answered the question of how many tilings exist in a region [7, 12].

In this thesis, we focus on the connectivity of the space of tilings by local moves. Given a region R that admits at least one tiling, consider the space of tilings for R as a graph with a vertex for each tiling of R. Given some set of local moves, we place an edge between two vertices of this graph if one tiling is transformed into the other by one of these moves.

In 1990, Thurston used height functions to prove that for any simply connected region R, the tiling space is connected by flip moves [13]. More recently, Yamzon reproved this result using Gross's commutative algebra techniques [15, 6]. In the first portion of this thesis, we expand on Gross and Yamzon's work and further investigate the algebraic structure of two-dimensional tilings.

However, in the case of three dimensional domino tilings, the picture becomes murkier. For instance, Pak and Yang showed that counting how many tilings a region admits is #P-complete [11, 14], and the space of tilings is not flip connected. In [9], Milet and Saldanha proposed a new three-dimensional local move, called the trit. Freire et al. showed that the tiling space of a three-dimensional box is connected by flips and trits up to refinement [5].

However, the connectedness of the space without refinement remains an open question. Since combinatorial proofs have so far been unsuccessful, we turn toward algebraic techniques in search of a new way to tackle the three dimensional problem. In the first portion of the thesis, we continue to develop our understanding of flip connectivity from an algebraic perspective to continue to lay the groundwork in pursuit of this goal. In the second portion of this thesis, we show computational results on small boxes to help elucidate the structure of the flip-connected components of the tiling space of a box.

2 Preliminaries

2.1 Domino Tilings

In this thesis, we consider domino tilings of 2 and 3-dimensional cubical regions. In two dimensions, each domino is a 2×1 rectangle, placed so it covers exactly two facets of the cubical region. In three dimensions, each domino is a $2 \times 1 \times 1$ rectangular prism occupying exactly two cubes of the region. A tiling in 2 or 3 dimensions is a placement of dominos such that each square or cube is occupied by exactly one domino.

2.1.1 Connections to Perfect Matchings

For R a 2-d cubical region, consider the dual graph G, where each vertex of G corresponds to a square of R, and there is an edge $v_i v_j \in E(G)$ if the respective squares share an edge.

Similarly, in three dimensions, define a vertex for each cube and place edges between cubes if they share a face.

A tiling of a region is precisely a perfect matching in this dual graph, where an edge between two vertices in the matching corresponds to placing a domino across the respective squares or cubes. An example of this correspondence in a two-dimensional region is depicted in Fig. 1.

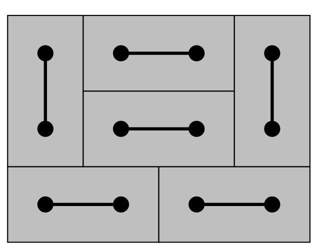


Figure 1: A tiling of a 3×4 region and its corresponding perfect matching

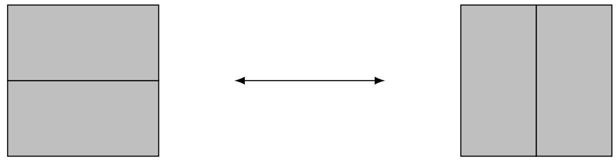


Figure 2: A flip move swaps the orientation of two dominoes covering a 2×2 square.

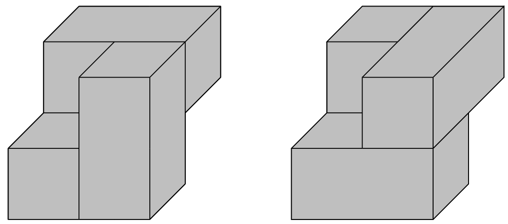


Figure 3: A trit move takes the picture on the left to the one on the right or vice versa.

2.2 Connectivity of Tilings

We consider two different local moves in the space of tilings.

In two dimensions, we only consider the flip move.

Definition (Flip Move)**.** A *flip move* in a domino tiling consists of swapping the orientation of two dominos covering a 2×2 square, as in Fig. 2.

Theorem ([13])**.** *For any simply connected two-dimensional region R, any tiling of R can be reached from any other tiling of R via flip moves.*

In three dimensions, the tiling space is not connected if limited to flip moves. Thus, we also allow *trit* moves.

Definition (Trit Move)**.** A trit move removes and replaces 3 dominos inside a $2 \times 2 \times 2$ box, one parallel to each axis, as depicted in Fig. 3.

It remains an open question whether certain spaces of three dimensional tilings are connected under flip and trit moves. There are some regions known not to be flip and trit-connected, but in others, such as boxes, the question remains unresolved.

2.3 Commutative Algebra Approach

2.3.1 Toric Ideal of a Graph

Let G be a graph with vertices $V(G)$ and edges $E(G)$.

Define the edge and vertex polynomial rings of G respectively as

$$K[E] = K[x_e : e \in E(G)]$$
$$K[V] = K[y_v : v \in V(G)]$$

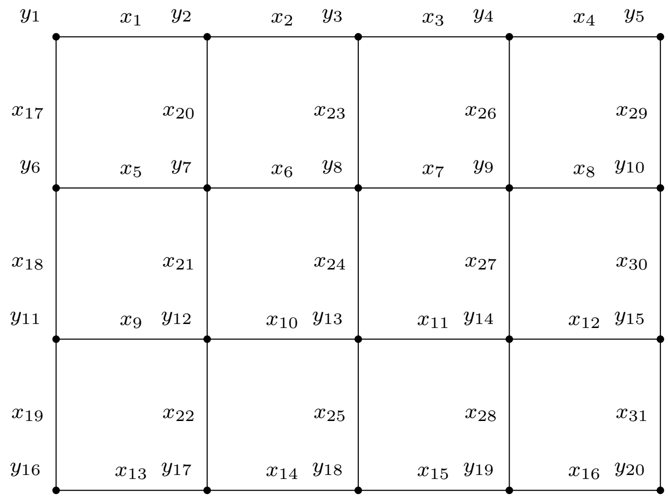
Consider the homomorphism $\phi_G : K[E] \to K[V]$ where

$$\phi_G(x_{ij}) = y_i y_j.$$

That is, ϕ_G maps each edge of G to the product of its endpoints.

Definition (Toric Ideal)**.** The *toric ideal* of G is $I_G \coloneqq \ker(\phi_G) \subset K[E]$.

Example 1. *Let G be the following graph:*



Then, for example

$$\begin{aligned}
&\phi_G(x_1x_2) = y_1y_2^2y_3\\
&\phi_G(x_1x_3 - x_{23}) = y_1y_2y_3y_4 - y_3y_8\\
&\phi_G(x_1x_5 - x_{17}x_{20}) = y_1y_2y_6y_7 - y_1y_6y_2y_7 = 0
\end{aligned}$$

so in particular $x_1x_5 - x_{17}x_{20} \in I_G$.

It is a well known result that the toric ideal I_G is generated by the primitive closed even walks in G [10]. In particular, in bipartite graphs (such as those associated to our cubical regions), the toric ideal is generated by the cycles of the graph.

2.3.2 The Tiling Ideal and the Flip Ideal

Let G be the dual graph associated to a cubical region R.

Let $T_1, T_2 \subseteq E(G)$ be tilings of R. For ease of notation, we define

$$x^{T_1} - x^{T_2} = \prod_{x_e \in T_1} x_e - \prod_{x'_e \in T_2} x'_e$$

Definition. (Tiling Ideal) The *tiling ideal* of R is

$$I_{tiling} = \langle x^{T_1} - x^{T_2} : T_1, T_2 \text{ tilings of } R\rangle$$

Note that for any tiling T, $\phi_G(x^T) = \prod y_v$, so

$$I_{tiling} \subseteq \ker(\phi_G) = I_G.$$

A flip move corresponds to a 4-cycle in G, and can be expressed as $x_ax_c - x_bx_d$, where x_a, x_c are the horizontal edges of the cycle, and x_b, x_d are the vertical edges. In this case, we say that (ac, bd) is a flip move.

Definition. (Flip Ideal) The *flip ideal* of R is

$$I_{flip} = \langle x_a x_c - x_b x_d : (ac, bd) \text{ is a flip move}\rangle$$

Flip moves are cycles and thus generators of I_G, so

$$I_{flip} \subseteq I_G.$$

Theorem 2.1. *[15] In two dimensions, a region R is flip-connected if and only if $I_{tiling} \subseteq I_{flip}$.*

Yamzon showed that this is the case for any cubical region R, thus reproving Thurston's result using commutative algebra [15]. In this thesis, we study the primary decomposition of I_{flip} for cubical regions. The eventual goal is to extend this technique to three dimensional regions.

2.4 Groebner Bases

Groebner bases are a useful tool when studying ideals of a polynomial ring.

Definition (Groebner Basis)**.** Fix a monomial order. For any polynomial f, let $\mathrm{LT}(f)$ denote the leading term of f with respect to this order. A finite subset $G = \{g_1, \ldots, g_n\}$ of an ideal I is a *Groebner basis* of I if

$$\langle \mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_n)\rangle = \langle \mathrm{LT}(f) : f \in I\rangle.$$

Groebner bases have many useful properties. In this thesis, we use the following two.

Theorem 2.2. *[3] Fix a monomial order. Then every ideal $I \subset k[x_1, \ldots, x_n]$ other than $\{0\}$ has a Groebner basis with respect to this order. Furthermore, any Groebner basis for an ideal I is a basis of I.*

Theorem 2.3. *[3] Fix a monomial order, and let $G = \{g_1, \ldots, g_n\}$ be a Groebner basis for an ideal $I \subset k[x_1, \ldots, x_n]$. Then for any $f \in k[x_1, \ldots, x_n]$, there is a unique $r \in k[x_1, \ldots, x_n]$ such that*

(i) No term of r is divisible by any of $\mathrm{LT}(g_1), \ldots, \mathrm{LT}(g_n)$, and

(ii) There is a $g \in I$ such that $f = g + r$.

In particular, this means that $f \in I$ if and only if the remainder of division of f by G is zero.

3 Primary Decomposition of the Flip Ideal

In this section, we will study the primary decomposition of I_{flip} for particular rectangular regions. Recall the definitions of primary ideals and primary decompositions.

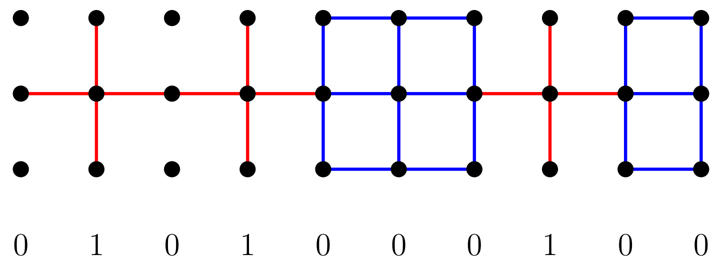


Figure 4: Ideal $I_{G_{10}}^{(0101000100)}$. Here, the blue grids represent the toric ideal on that subgraph, while each red edge is a monomial generator.

Definition (Primary Ideal)**.** An ideal $I \subseteq R$ is *primary* if $I \neq R$ and whenever $pq \in I$, either $p \in I$ or $q^n \in I$ for some n.

Definition (Primary Decomposition)**.** A *primary decomposition* of an ideal $I \subseteq R$ is an expression of I as a finite intersection of primary ideals, say

$$I = \bigcap_{i=1}^{n} \mathfrak{q}_i$$

where each $\mathfrak{q}_i \subset R$ is a primary ideal.

In this section, we show a primary decomposition for I_{flip} on grid graphs of size $3 \times n$ vertices.

Let S be the set of 0/1 strings of length n with no consecutive 1s and 0s in the first and last spots. Then, for each $s \in S$, we construct a primary ideal $I_{G_n}^{(s)}$ as follows.

Consider all i such that $s_i = 1$. For each such i, add 4 monomial generators: both vertical edges in the ith column, and the middle horizontal edges on either side. Then, any time there are consecutive 0s from s_j to s_k, add the generators of the toric ideal on columns j through k.

To illustrate this construction, Fig. 4 shows the ideal corresponding to $s = 0101000100$ in a 3×10 grid.

Theorem 3.1. *For any $s \in S$, $I_G^{(s)}$ is a prime ideal.*

Proof. Recall that a prime ideal is an ideal I such that whenever $fg \in I$, either $f \in I$ or $g \in I$.

Let $f, g \in K[E]$ such that $fg \in I_G^{(s)}$. Note that $I_G^{(s)} = I_{G'} + \sum(x_i)$, where G' is the subgraph of G not covered by the monomial crosses, and x_i are the monomial generators. Further, since $I_{G'}$ is a toric ideal, it is prime.

Write $f = f_1 + f_2 + f', g = g_1 + g_2 + g'$, where $f_1, g_1 \in I_{G'}$, $f_2, g_2 \in \sum(x_i)$, and f', g' are either 0, or are not in $I_{G'}$ and do not have any terms divisible by a monomial generator.

Then,

$$fg = f_1(g_1 + g_2 + g') + g_1 f' + f_2(g_1 + g_2 + g') + g_2 f' + f'g'$$

The first two terms are in $I_{G'}$, and the second two terms are in $\sum(x_i)$. Since $fg \in I_{G'} + \sum(x_i)$, this means that $f'g' \in I_{G'} + \sum(x_i)$.

Since no terms of f' or g' contain any monomial generators, no terms of the product are in $\sum(x_i)$ so $f'g' \in I_{G'}$. Since $I_{G'}$ is prime, and since we constructed f', g' to either be zero or not in $I_{G'}$, we have $f' = 0$ or $g' = 0$ and thus either f or g is in $I_G^{(s)}$. $\square$

We are now ready to give the primary decomposition of I_{flip} for $3 \times n$ grids.

Theorem 3.2. *Let G be a grid graph on $3 \times n$ vertices, and let I_{flip} be its flip ideal. Let S be the set of 0/1 strings of length n with no consecutive 1s and with 0s in the first and last positions. Using the same construction as above,*

$$I_{flip} = \bigcap_{s \in S} I_{G_n}^{(s)}.$$

Proof. ($\Rightarrow$) The first inclusion $I_{flip} \subseteq \bigcap_{s \in S} I_G^{(s)}$ follows directly from the constructions of each $I_G^{(s)}$.

I_{flip} is generated by binomials of the form $f = x_1x_3 - x_2x_4$, where x_1, x_2, x_3, x_4 form a box in G, with x_1 opposite x_3 and x_2 opposite x_4. In any $I_G^{(s)}$, this box either falls in a quadrant of a monomial cross, or within a toric ideal portion of the graph. In the first case, one of the edges in each term is a generator of the ideal, so f is in the ideal. In the second case, the binomial itself is a generator of the ideal.

Thus, each generator of I_{flip} is contained in each ideal in the intersection, so

$$I_{flip} \subseteq \bigcap_{s \in S} I_G^{(s)}.$$

$\square$

It remains to show that $I_{flip} \supseteq \bigcap_{s \in S} I_G^{(s)}$. This direction is considerably more difficult.

Remark 1. *Theorem 3.2 and Theorem 3.1 imply that I_{flip} has a primary decomposition of all primes, and is thus radical.*

3.1 Characterization of cycles in the flip ideal

Throughout this section, we consider only *single cycles* from I_G. In the next section, we leverage these results to prove Theorem 3.2 in full generality.

Definition (Tail of a Cycle)**.** Consider a cycle $C = \prod_{x_e \in T_1} x_e - \prod_{x_e \in T_2} x_e$. We say that edges x_i, x_j, x_k are a *tail* of C if the following hold:

(1) $x_i, x_k \in T_1$,

(2) $x_j \in T_2$, and

(3) there exists a 4th edge x_e such that $x_ix_k - x_jx_e$ is a flip move.

An example of a tail can be see in Fig. 5.

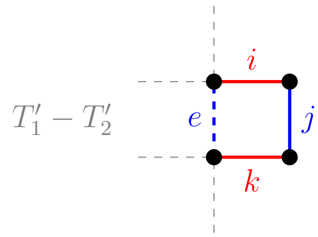


Figure 5: Tail of a cycle. Here, T_1' and T_2' are the edges of the cycle other than x_i, x_j, x_k.

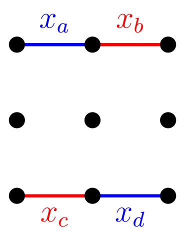


Figure 6: Parallel edge pattern in a cycle. Here blue edges are in one term of the cycle and red edges in the other.

Lemma 3.3. *Let $C = T_1' x_i x_k - T_2' x_j$ be a cycle in our graph, and let x_i, x_j, x_k be a tail of C. Then $C \in I_{flip}$ if the cycle formed by truncating the tail is in I_{flip}.*

Proof. Let x_e be the edge that completes the flip $x_i x_k - x_j x_e$. Then, rewrite C as

$$\begin{aligned} C &= T_1' x_i x_k - T_2' x_j \\ &= T_1'(x_i x_k - x_j x_e) + T_1' x_j x_e - T_2' x_j \\ &= T_1'(x_i x_k - x_j x_e) + x_j(T_1' x_e - T_2') \end{aligned}$$

The first term of the sum is a multiple of a flip and thus in I_{flip}.

We note that $T_1 x_e - T_2$ is exactly the binomial arising from the cycle formed by C with the tail truncated. If $T_1 x_e - T_2 \in I_{flip}$, then both terms of the sum are in I_{flip}, so $C \in I_{flip}$. $\square$

Lemma 3.4. *Let C be a cycle in G such that the parallel edge pattern in Fig. 6 is not found in C. Then C is in the flip ideal.*

Proof. We proceed by induction on the number of squares in the interior of C.

As a base case, if C is a cycle around a single square, then C is exactly a flip move and thus in I_{flip}.

Now, suppose that for any cycle containing $k-1$ squares, if the cycle does not contain the parallel edge pattern, then it is in the flip idea.

Consider a cycle C containing k squares that does not have the parallel edge pattern. Construct a new graph G' from C, with a vertex for every box in the interior of C, and an edge between two adjacent boxes if their shared side is not in C.

If C has no parallel edge patterns, then G' is a tree. Indeed, since C is a single cycle, it has a contiguous interior, so G' is connected. Suppose that G' has a cycle. Then C must

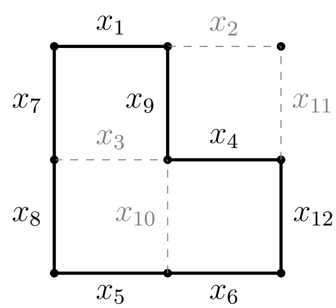


Figure 7: Example of a cycle with no parallel edge pattern

have the parallel edge pattern (namely around the boxes whose vertices have a cycle in G'). Since C lacks this pattern, we conclude that G' is a connected, acyclic graph (i.e. a tree).

Then, since G' is a tree, it must have a leaf. For a vertex v to be a leaf in G', three of the edges around its corresponding box must be in C. In other words, v is the box inside a tail of C. Thus, we can trim v from our tree by executing a flip move on C.

This new cycle still does not have a parallel edge structure, so by our inductive hypothesis it will be in I_{flip}. Moreover, by Lemma 3.3, if this new cycle is in the flip ideal, then C is in the flip ideal. Thus, $C \in I_{flip}$, as desired. □

Example 2. *To illustrate the construction in the proof of Lemma 3.4, consider the cycle in Fig. 7, which has no parallel edges.*

We begin with the x_4, x_6, x_{12} tail.

$$\begin{aligned}x_1x_4x_6x_8 - x_5x_7x_9x_{12} &= x_1x_8(x_4x_6 - x_{10}x_{12}) + x_{12}(x_1x_8x_{10} - x_5x_7x_9)\\ &= x_1x_8(x_4x_6 - x_{10}x_{12}) + x_1x_{12}(x_8x_{10} - x_3x_5) + x_5x_{12}(x_1x_3 - x_7x_9)\end{aligned}$$

Taking the contrapositive, Lemma 3.4 tells us that if $C \notin I_{flip}$, then the parallel edge pattern appears somewhere in C.

Next we show that if the parallel edge pattern appears, then $C \notin \bigcap_{s\in S} I_G^{(s)}$.

Lemma 3.5. *Let C be a cycle in G that contains the parallel edge pattern. Then there exists an $s \in S$ such that $C \notin I_G^{(s)}$. In particular, this means that $C \notin \bigcap_{s\in S} I_G^{(s)}$.*

Proof. Suppose C is such a cycle, suppose it has a parallel edge pattern centered around column i. Let s be the string with all 0s except for a 1 in the ith position, as depicted in Fig. 8. Then $C \notin I_G^{(s)}$.

Since C contains a parallel edge pattern, it can be written $C = C_1'x_ax_d - C_2'x_bx_c$, where C_1' and C_2' are the monomials containing the edges outside of the parallel edge pattern that complete the cycle.

We show that C cannot be written as a combination of the generators of $I_G^{(s)}$.

Suppose it could, so $C = \sum_i \alpha_i f_i$, where f_i are the generators of $I_G^{(s)}$ and α_i are edge polynomials. Rewrite this as $C = C_1 - C_2 = \sum_i \beta_i g_i$, where each β_i is a monomial and each g_i is some generator of $I_G^{(s)}$.

Consider the term $\beta_1 g_1$ that produces the leading term of C. Since x_a, x_d are in the first term of C but do not appear in any generators of $I_G^{(s)}$, they must appear in β_1. None of the

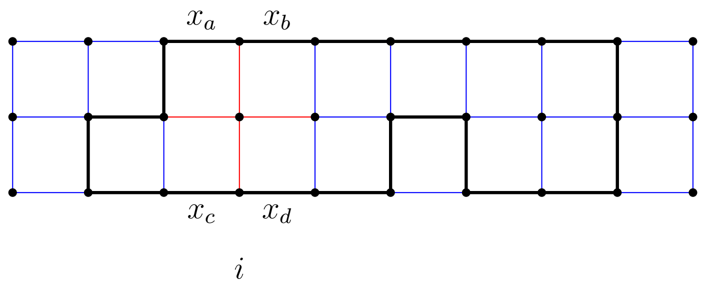


Figure 8: A cycle with parallel edges around column $i = 4$, and its corresponding $I_G^{(s)}$. For this cycle, we could have picked $i = 4, 5$, or 8.

monomial generators appear in C, so g_1 cannot be one of the monomial generators and no monomial generator appears in β_1. Letting $g_1 = T_1 - U_1$ and $\beta_1 = x_a x_d \beta_1'$,

$$C = C_1 - C_2 = \sum_{i=1}^{N} \beta_i g_i = x_a x_d \beta_1' T_1 - x_a x_d \beta_1' U_1 + \sum_{i=2}^{N} \beta_i g_i$$

Then $C_2 = x_a x_d \beta_1' U_1 - \sum_{i=2}^{N} \beta_i g_i$. We know that $C_2 \neq x_a x_d \beta_1' U_1$, since x_a, x_d do not appear in C_2. Thus, we must have some term $\beta_i g_i$ in our sum that cancels out the $x_a x_d \beta_1' T_2$.

Without loss of generality, reorder the sum such that $\beta_1 U_1 = \beta_2 T_2$. Since x_a and x_d do not appear in any of our generators, $x_a x_d$ must be in β_2. Moreover, since no monomial generators appear in $\beta_1 U_1$, no monomial generators appear in $\beta_2 T_2$. Thus g_2 is a binomial, so $\beta_2 g_2$ gives us another term which contains $x_a x_d$ and does not contain any monomial generators of $I_G^{(s)}$. By the same argument, we continue this process, each time cancelling one term that contains $x_a x_d$ and no monomial generators and getting another such term.

Therefore, at any point in this process, the leading term in the sum contains $x_a x_d$. Since the sum is finite, we are left with $C_2 = x_a x_d \beta_n' U_n$. However, $x_a x_d$ does not appear in C_2, so this is a contradiction.

Thus, $C \notin I_G^{(s)}$, so in particular $C \notin \bigcap_{s \in S} I_G^{(s)}$. □

Now, combine Lemma 3.4 and Lemma 3.5. Let C be a cycle in $I_G \setminus I_{flip}$. Then, C must have a parallel edge pattern, so C is not in the intersection.

Therefore, for any single cycle C, $C \in I_{flip}$ if and only if $C \in \bigcap_{s \in S} I_G^{(s)}$.

3.2 Characterization of Arbitrary Elements

We now prove Theorem 3.2 in full generality.

Proposition 3.6. *If $C \in I_G \setminus I_{flip}$, then $C \notin \bigcap_{s \in S} I_G^{(s)}$.*

To prove Theorem 3.2, we first show that the Proposition 3.6 holds when C is a product of cycles. Then, we consider $C \in I_G \setminus I_{flip}$ in full generality by considering sums of products of cycles.

Before proving this proposition, we must first prove several lemmas and theorems. The first states that if a product of tail-less cycles is not in I_{flip}, then there is some s such that

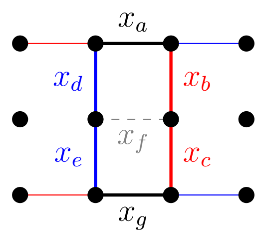


Figure 9: Cycles have parallel edges overlapping by one column.

$C \notin I_G^{(s)}$. The second expands this result to $\beta \prod C_i$, where β is a polynomial where no term is divisible by the leading term of any cycle.

Lemma 3.7. *If $\{C_\alpha\}$ is a collection of tail-less cycles such that $\prod C_\alpha \notin I_{flip}$, then $\prod C_\alpha \notin I_G^{(s)}$ for some $s \in S$.*

Proof. It is sufficient to show it is possible to pick columns such that,

- We pick a column in at least one parallel edge pattern of each C_α
- We do not pick any columns whose monomial crosses overlap with edges in a C_α

Then, each $C_\alpha \notin I_G^{(s)}$, and so $\prod C_\alpha \notin I_G^{(s)}$ since $I_G^{(s)}$ is prime.

Since $\prod C_\alpha \notin I_{flip}$, each cycle is not in I_{flip} and hence by Lemma 3.4, each has some parallel edge pattern.

If the parallel edges all occupy disjoint columns, then it is easy to pick our columns.

If two cycles share a parallel edge pattern (i.e. overlap by two or more columns), then pick the column in the middle of a shared pattern to place a cross for both of them.

Finally, consider the case where two cycles overlap by exactly one column, as in Fig. 9. Since $C_1C_2 \in I_G^{(s)}$ if and only if $-C_1C_2 \in I_G^{(s)}$, it is sufficient to consider the case where both cycles have x_a in their positive term and x_g in their negative term. Write $C_1 = T_1x_ax_c - U_1x_bx_g, C_2 = T_2x_ax_e - U_2x_dx_g$. Then,

$$\begin{aligned} C_1C_2 &= (T_1x_ax_c - U_1x_bx_g)(T_2x_ax_e - U_2x_dx_g) \\ &= T_1T_2x_a^2x_cx_e - T_1U_2x_ax_cx_dx_g - T_2U_1x_ax_bx_ex_g + U_1U_2x_bx_dx_g^2 \end{aligned}$$

Expand the first and last terms as

$$\begin{aligned} T_1T_2x_a^2x_cx_e &= T_1T_2x_a^2(x_cx_e - x_fx_g) + T_1T_2x_ax_g(x_ax_f - x_dx_b) + T_1T_2x_ax_bx_dx_g \\ U_1U_2x_bx_dx_g^2 &= U_1U_2x_g^2(x_bx_d - x_ax_f) + U_1U_2x_fx_g(x_ax_g - x_bx_d) + U_1U_2x_bx_dx_fx_g \end{aligned}$$

Thus,

$$\begin{aligned} C_1C_2 = T_1T_2x_ax_bx_dx_g &- T_1U_2x_ax_cx_dx_g - T_2U_1x_ax_bx_ex_g + U_1U_2x_bx_dx_fx_g \\ &+ T_1T_2x_a^2(x_cx_e - x_fx_g) + T_1T_2x_ax_g(x_ax_f - x_dx_b) \\ &+ U_1U_2x_g^2(x_bx_d - x_ax_f) + U_1U_2x_fx_g(x_ax_g - x_bx_d) \\ = x_ax_g(T_1x_d - &U_1x_e)(T_2x_b - U_2x_c) + T_1T_2x_a^2(x_cx_e - x_fx_g) \\ &+ T_1T_2x_ax_g(x_ax_f - x_dx_b) + U_1U_2x_g^2(x_bx_d - x_ax_f) + U_1U_2x_fx_g(x_ax_g - x_bx_d) \end{aligned}$$

Note that $T_1x_d - U_1x_e, T_2x_b - U_2x_c$ are exactly the cycles that arise by truncating the shared column from each cycle, and that the remaining terms are all multiples of flip moves.

Therefore, if $C_1C_2 \notin I_{flip}$, neither is $C_1'C_2'$, where C_1' and C_2' are formed by truncating the overlapping column from the respective cycles. Thus, C_1 and C_2 must have parallel edges outside of the overlapping column, so we can place a monomial cross in each of them, as desired. $\square$

Throughout the remainder of this section, assume the edges are numbered across rows from top to bottom then down columns from left to right, as in Fig. 10. Let F be a Groebner basis for I_{flip} using this numbering scheme and lexicographic order.

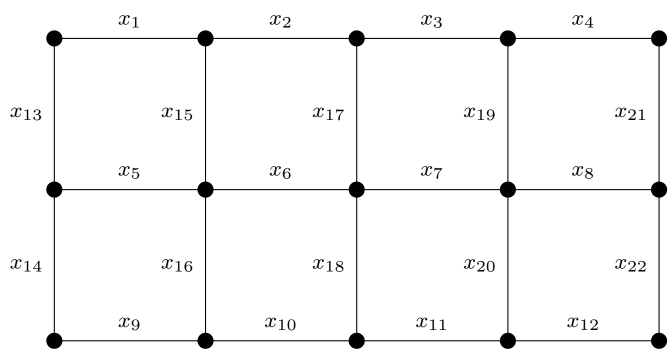


Figure 10: The labelling of the edge monomials on the graph.

Lemma 3.8. *Suppose β is a polynomial such that no leading term of a cycle in I_G divides any term of β. Let $\{C_i\}$ be a collection of cycles such that $\beta \prod C_i \notin I_{flip}$, and suppose that no leading term of F divides any term of $\beta \prod C_i$. Then there exists $s \in S$ such that $C = \beta \prod C_i \notin I_G^{(s)}$.*

Proof. Since $\beta \prod C_i \notin I_{flip}$, we know that each C_i has a parallel edge pattern. Moreover, no leading term of F divides any term of $\beta \prod C_i$, so each C_i must be tail-less. Then, by Lemma 3.7, we can pick columns such that $\prod C_i \notin I_G^{(s)}$.

Since $I_G^{(s)}$ is prime, it then remains to show that there is an s such that $\beta \notin I_G^{(s)}$ and $C_i \notin I_G^{(s)}$ for all i. Since no term of β is divisible by the leading term of a cycle, the only way for this to fail to happen is if every s such that $\prod C_i \notin I_G^{(s)}$ has monomial generators such that at least one of the generators is found in each term of β.

By assumption, no term of $\beta \prod C_i$ is divisible by the leading term of a flip move, so no term of β contains a horizontal edge in the middle of parallel edges in some C_i. Thus, we only need to concern ourselves with vertical edges in β.

Consider some term β_k of β, and suppose that every s such that $\prod C_i \notin I_G^{(s)}$ has a monomial generator that appears in β_k. If this is the case, then $\beta_k \prod C_{i_j} \in I_{flip}$.

Suppose not, and pick a counterexample $\beta_k \prod C_i \notin I_{flip}$ so that the sum of the areas of the C_i is minimal.

We can pick our columns for $I_G^{(s)}$ in the proof of Lemma 3.7 so that we take the leftmost column whenever we have a choice. Then some selected column is in the left-most parallel edges of some C_i, so β_k must have a vertical edge directly to the right of the left edge of some C_i. Then, we can execute a flip move on $\beta_k C_i$ and another to remove the resulting tail, thus shrinking C_i.

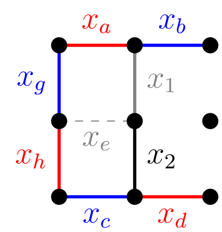


Figure 11: A cycle C_i (shown in red and blue) where β_k has a vertical edge (x_2) in the next column over.

More formally, in the situation of Fig. 11,

$$\begin{aligned}\beta_k C_i &= x_2 \beta_k'(x_a x_d x_h T_j - x_b x_c x_g U_j)\\ &= \beta_k'(x_2 x_h - x_c x_e) x_a x_d T_j + \beta_k'(x_a x_e - x_1 x_g) x_c x_d T_j + x_c x_g \beta_k'(x_1 x_d T_j - x_2 x_b U_j)\end{aligned}$$

In the case where $x_1 | \beta_k$ instead, the argument goes through similarly. In either case, $\beta_k C_i$ can be written as a combination of flip moves plus a monomial times a smaller cycle.

Let x_i be the vertical edge that divides β_k, and redefine $\beta_k' = x_c x_g \beta_k / x_i$. Write $\beta_k C_i = \beta_k' C_i' + f$, where $f \in I_{flip}$ and C_i' is the new, smaller cycle.

Then,

$$\begin{aligned}\beta_k \prod_i C_i &= \beta_k C_i \prod_{i' \neq i} C_{i'}\\ &= (f + \beta_k' C_i') \prod_{i' \neq i} C_{i'}\\ &= f \prod_{i' \neq i} C_{i'} + \beta_k' C_i' \prod_{i' \neq i} C_{i'}\end{aligned}$$

By assumption $\beta_k \prod C_i \notin I_{flip}$, so

$$\beta_k' C_i' \prod_{i' \neq i} C_{i'} \notin I_{flip}.$$

However, we have moved the left edge of some C_i right by one column, so the total area of the C_i is smaller, which contradicts our assumption of minimal area.

Therefore, if we cannot pick columns to avoid β, then $\beta_k \prod C_i \in I_{flip}$ for some term β_k.

By assumption, no term of $\beta \prod C_i$ is divisible by a leading term of F. By the properties of Groebner bases, this implies that no subset of the terms will sum to a nonzero element of I_{flip}, so this is a contradiction. Therefore, we will always be able to pick columns to avoid β, as desired. □

Now, we consider sums of these $\beta \prod C_i$. To prove this result, we will rely on the following technical lemmas.

Lemma 3.9. *Let $C_1, \ldots, C_n$ be cycles such that each $C_i = T_i - U_i \in I_{flip}$. Then their union $C = \bigcup_{i=1}^{n} C_i = \prod T_i - \prod U_i$ is in I_{flip}.*

Proof. Induct on the number of cycles, n.

If $n = 1$, then $C = C_1 \in I_{flip}$, so the base case holds.

Suppose that a union of $k-1$ cycles from I_{flip} is in I_{flip}, and let C be a union of k cycles in I_{flip}. Then,

$$\begin{aligned} C &= \prod_{i=1}^{k} T_i - \prod_{i=1}^{k} U_i \\ &= (T_k - U_k)\left(\prod_{i=1}^{k-1} T_i\right) + U_k\left(\prod_{i=1}^{k-1} T_i - \prod_{i=1}^{k-1} U_i\right) \end{aligned}$$

The first term is in I_{flip} since $C_k = T_k - U_k \in I_{flip}$, and the second is in I_{flip} since

$$\bigcup_{i=1}^{k-1} C_i = \prod_{i=1}^{k-1} T_i - \prod_{i=1}^{k-1} U_i \in I_{flip}$$

by the inductive hypothesis, so $C \in I_{flip}$. □

Lemma 3.10. *Let $m\left(\prod X_i - \prod Y_i\right) \in I_{G_s}$ be a union of tail-less cycles, and suppose that for every column in the interior of any $X_i - Y_i$, one of the vertical edges in that column divides either $m \prod X_i$ or $m \prod Y_i$. Then $m\left(\prod X_i - \prod Y_i\right) \in I_{flip}$.*

Proof. Without loss of generality, order the $X_i - Y_i$ so that cycles whose leftmost edges are further to the left come before those whose leftmost edges are further right.

If the leftmost edges of $X_1 - Y_1$ are one column over from the rightmost column, then all the cycles have width 1 and are thus in I_{flip}, so $\prod X_i - \prod Y_i \in I_{flip}$ by Lemma 3.9.

Assume for the sake of contradiction that the proposition does not hold. Then there is some leftmost column c such that if $X_1 - Y_1$ has its leftmost edges to the right of c, then $m\left(\prod X_i - \prod Y_i\right) \in I_{flip}$. Pick $m(\prod X_1 - \prod Y_1) \notin I_{flip}$ such that $X_1 - Y_1$ has its leftmost edges at column c, $\prod X_i - \prod Y_i$ has the minimal number of cycles with their leftmost edges at column c, and among such cycles the sum of the areas of the cycles is minimized.

If any cycle is a single column wide, then it is in I_{flip} so by the argument in the proof of Lemma 3.9 we only need show that the union of the remaining cycles is in I_{flip}. Thus, consider only the cycles that span at least two columns.

By hypothesis, either x_1 or x_2 divides either $m \prod X_i$ or $m \prod Y_i$, so we are in the case depicted in Fig. 12. There are several cases:

(1) $x_1 | m$ or $x_2 | m$

(2) $x_1 | X_1$ or $x_2 | Y_1$ (We do not need to consider the reverse, since $X_1 - Y_1$ is a cycle and therefore does not have adjacent edges in the same term.)

(3) For some $j > 1$, $x_1 | X_j$ or $x_1 | Y_j$ or $x_2 | X_j$ or $x_2 | Y_j$

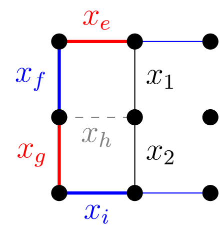


Figure 12: The left end of a cycle $X_1 - Y_1 = x_f x_i X_1' - x_e x_g Y_1'$. Either x_1 or x_2 must be in the term of C_k we are considering.

Case 1: Suppose $x_1|m$ or $x_2|m$. Without loss of generality, assume $x_1|m$. Otherwise, multiply by -1 to reverse the roles of X_1 and Y_1. Then, by reversing the roles of x_1 and x_2, the argument follows identically.

Let $m' = m/x_1, X_1 = x_f x_i X_1', Y_1 = x_e x_g Y_1'$. Then

$$\begin{aligned} m\left(\prod X_k - \prod Y_k\right) &= m' x_1 x_f x_i X_1' \prod_{k\neq 1} X_k - m' x_1 x_e x_g Y_1' \prod_{k\neq 1} Y_k \\ &= m' x_i X_1' \prod_{k\neq 1} X_k (x_1 x_f - x_e x_h) + m' x_e X_1' \prod_{k\neq 1} X_k (x_h x_i - x_2 x_g) \\ &\quad + m' x_e x_g \left(x_2 X_1' \prod_{k\neq 1} X_k - x_1 Y_1' \prod_{k\neq 1} Y_k \right) \end{aligned}$$

Thus, since our original union of cycles was not in I_{flip}, neither is the last term of this sum. However, $m'\left(x_2 X_1' \prod_{k\neq 1} X_k - x_1 Y_1' \prod_{k\neq 1} Y_k\right)$ is a union of cycles that still has the same vertical edges as our original binomial, but now has one fewer cycle touching column c. Then by our minimality assumptions, $m'\left(x_2 X_1' \prod_{k\neq 1} X_k - x_1 Y_1' \prod_{k\neq 1} Y_k\right) \in I_{flip}$, which is a contradiction.

Case 2: Suppose $x_1|X_1$ or $x_2|Y_1$. Assume $x_1|X_1$. The argument for $x_2|Y_1$ follows identically by reversing the roles of x_1 and x_2, and multiplying by -1 to swap the roles of X_1 and Y_1.

Let $X_1 = x_1 x_f x_i X_1'$, $Y_1 = x_e x_g Y_1'$. Then

$$\begin{aligned} \prod_k X_k - \prod_k Y_k &= x_1 x_f x_i X_1' \prod_{k\neq 1} X_k - x_e x_g Y_1' \prod_{k\neq 1} Y_k \\ &= (x_1 x_f - x_e x_h) x_i X_1' \prod_{k\neq 1} X_k + (x_h x_i - x_2 x_g) x_e X_1' \prod_{k\neq 1} X_k \\ &\quad + x_e x_g \left(x_2 X_1' \prod_{k\neq 1} X_k - Y_1' \prod_{k\neq 1} Y_k \right) \end{aligned}$$

The last term again has the same vertical edges as the original binomial and one fewer cycles touching column c, so it is in I_{flip}. However, this implies $\prod_k X_k - \prod_k Y_k \in I_{flip}$, which is a contradiction.

Case 3: Suppose there is some $j \neq 1$ such that $x_1 | X_j$ or $x_1 | Y_j$ or $x_2 | X_j$ or $x_2 | Y_j$. Since $X_j - Y_j$ is a cycle with no tails, if $x_1 | X_j$ then $x_2 | Y_j$ and if $x_2 | X_j$ then $x_1 | Y_j$. Thus, there are actually only two cases: either $x_1 | X_j$ and $x_2 | Y_j$, or $x_2 | X_j$ and $x_1 | Y_j$.

Consider the second of these cases, and write $X_j = x_2 X_j'$, $Y_j = x_1 Y_j'$. Then

$$\begin{aligned}\prod X_k - \prod Y_k &= x_2 X_1 X_j' \prod_{k \neq 1,j} X_k - x_1 Y_1 Y_j' \prod_{k \neq 1,j} Y_k \\ &= (x_1 + x_2)(X_1 X_j' \prod_{k \neq 1,j} X_k - Y_1 Y_j' \prod_{k \neq 1,j} Y_k) \\ &\quad - \left(x_1 X_1 X_j' \prod_{k \neq 1,j} X_k - x_2 Y_1 Y_j' \prod_{k \neq 1,j} Y_k \right)\end{aligned}$$

Note that $X_1 X_j' \prod_{k \neq 1,j} X_k - Y_1 Y_j' \prod_{k \neq 1,j} Y_k$ is the union of cycles with the same multiplicity at column c as our original binomial, but with strictly smaller total area. Multiplying it by either x_1 or x_2 ensures that each column where our original binomial had a vertical edge still has at least one vertical edge in one of the terms, so by the minimality of our original binomial, the first term is in I_{flip}. Additionally, the second two terms are exactly the binomial that arises if $x_1 | X_j$ and $x_2 | Y_j$, so it is sufficient to consider that case.

Let $X_1 = x_f x_i X_1'$, $Y_1 = x_e x_g Y_1'$, $X_j = x_1 X_j'$, $Y_j = x_2 Y_j'$. Then

$$\begin{aligned}\prod X_k - \prod Y_k &= x_1 x_f x_i X_1' X_j' \prod_{k \neq 1,j} X_k - x_2 x_e x_g Y_1' Y_j' \prod_{k \neq 1,j} Y_k \\ &= (x_1 x_f - x_e x_h) x_i X_1' X_j' \prod_{k \neq 1,j} X_k + (x_h x_i - x_2 x_g) x_e X_1' X_j' \prod_{k \neq 1,j} X_k \\ &\quad + x_2 x_e x_g \left(X_1' X_j' \prod_{k \neq 1,j} X_k - Y_1' Y_j' \prod_{k \neq 1,j} Y_k \right)\end{aligned}$$

The first two terms are multiples of flip moves and thus in I_{flip}. The third term is a union of cycles with lower multiplicity in column c than our original. Since we multiply through by x_2, we also guarantee that this new element of I_{G_s} also has a vertical edge in each column. Thus, the third term is also in I_{flip}, so our original binomial is in I_{flip}, which is a contradiction.

Since we reach a contradiction in all cases, we conclude that $m \left(\prod X_i - \prod Y_i \right) \in I_{flip}$, as desired. $\square$

Now, we use these results to show that sums of products of cycles not in I_{flip} are not in some $I_G^{(s)}$.

Theorem 3.11. *Suppose $C \in I_G \setminus I_{flip}$ and can be written*

$$C = \sum_{i=1}^{n} \beta_i \prod_{\alpha} C_{i,\alpha}$$

for $\beta_i \in K[E]$ and $C_{i,\alpha}$ cycles, such that

- *If $i \neq j$, then $\prod C_{i,\alpha} \neq \pm \prod C_{j,\alpha}$*
- *Each β_i has fully combined like-terms*
- *No nonempty subset of the $\beta_i \prod C_{i,\alpha}$ sum to an element of I_{flip}*
- *No term of β_i is divisible by the leading term of a cycle in I_G*
- *No term of any $\beta_i \prod_\alpha C_{i,\alpha}$ is divisible by the leading term of an element of F*
- *For each $i = 1, \ldots, n$, there exists an $s_i \in S$ such that $\beta_i \prod C_{i,\alpha} \notin I_G^{(s_i)}$*

Then there exists s such that $C \notin I_G^{(s)}$.

Proof. We show that the sum of the first k terms is not in some $I_G^{(s)}$ for all $k = 1, \ldots, n$, and thus by taking $k = n$, show that $C \notin I_G^{(s)}$ for some s. Let S_k denote the sum of the first k terms.

As a base case, our hypothesis tells us that the first term is not in some $I_G^{(s_1)}$, so $S_1 \notin I_G^{(s_1)}$.

Now, suppose that $S_{k-1} = \sum_{i=1}^{k-1} \beta_i \prod_\alpha C_{i_\alpha} \notin I_G^{(s_{k-1})}$ for some s_{k-1}. By hypothesis, there is some s_k such that $\beta_k \prod_\alpha C_{k_\alpha} \notin I_G^{(s_k)}$. We wish to find an s such that $\sum_{i=1}^{k} \beta_i \prod_\alpha C_{i_\alpha} \notin I_G^{(s)}$.

First, if there is any s such that $S_{k-1} \in I_G^{(s)}$ and $C_k \notin I_G^{(s)}$ or vice versa, then $S_k = S_{k-1} + C_k \notin I_G^{(s)}$, and we are done.

Thus, we need only consider the case where $S_{k-1} \in I_G^{(s)} \Leftrightarrow C_k \in I_G^{(s)}$. Pick $s \in S$ with the maximum number of 1s such that $S_{k-1}, C_k \notin I_G^{(s)}$. Then, $S_k \notin I_G^{(s)}$.

Indeed, since $C_k \notin I_G^{(s)}$, each $C_{k,\alpha} \notin I_G^{(s)}$ and thus has parallel edges around some monomial cross of $I_G^{(s)}$. Thus, we can write $C_{k_\alpha} = x_{a,\alpha} x_{d,\alpha} T_\alpha - x_{b,\alpha} x_{c,\alpha} U_\alpha$, where $x_{a,\alpha}, x_{b,\alpha}, x_{c,\alpha}, x_{d,\alpha}$ are the parallel edges around some monomial cross of $I_G^{(s)}$.

Expanding out the product, we see

$$\begin{aligned}
C_k &= \beta_k \prod_{\alpha=1}^{r} C_{k_\alpha} \\
&= \beta_k \prod_{\alpha=1}^{n} (x_{a,\alpha} x_{d,\alpha} T_\alpha - x_{b,\alpha} x_{c,\alpha} U_\alpha) \\
&= \beta_k \sum_{A \subseteq \{1,\ldots,r\}} \left(\prod_{\alpha \in A} (x_{a,\alpha} x_{d,\alpha} T_\alpha) \prod_{\alpha \notin A} (-x_{b,\alpha} x_{c,\alpha} U_\alpha) \right)
\end{aligned}$$

To complete the proof, we will use the following, which respectively appear as the first part of Corollary 1.7 and as Proposition 1.10 in [4].

Theorem 3.12 (Corollary 1.7 from [4]). *Let $I \subset k[x_1, \dots, x_n]$ be a binomial ideal, $m_1, \dots, m_t$ monomials and $f_1, \dots, f_t$ polynomials such that $\sum f_i m_i \in I$, and let $f_{i,j}$ denote the terms of f_i. Then for each term $f_{i,j}m_i$, either $f_{i,j}m_i \in I$, or there is a term $f_{i',j'}$ distinct from $f_{i,j}$ and a scalar $a \in k$ such that $f_{i,j}m_i + af_{i',j'}m_{i'} \in I$.*

Theorem 3.13 (Proposition 1.10 from [4]). *Let B be a binomial ideal and M a monomial ideal in $k[x_1, \dots, x_n]$. If $f \in B + M$ and f' is the sum of those terms in f that are not individually in $B + M$, then $f' \in B$.*

Suppose for the sake of contradiction that $S_{k-1} + C_k \in I_G^{(s)}$. Note that $I_G^{(s)} = I_{G_s} + M_s$, where I_{G_s} is the toric ideal of a subgraph and M_s is the ideal generated by the monomial crosses. Thus, $S_{k-1} + C_k = B + m$, where $B \in I_{G_s}$ is the sum of the terms containing no monomial generators and $m \in M_s$.

Let $C_k' = (\beta_{k,1} + \cdots + \beta_{k,m}) \prod C_{k,\alpha}$, where each $\beta_{k,j} \notin M_s$. Since $C_k' \notin I_G^{(s)}$, $C_k' \neq 0$, and none of its terms of have any monomial generators. Thus $B = C_k' + S_{k-1}' \in I_{G_s}$, where S_{k-1}' is the sum of some subset of the terms of S_{k-1}.

We can now fully expand out

$$B = S_k' + \sum_{A \subseteq \{1,\dots,r\}} (\beta_{k,1} + \cdots + \beta_{k,m}) \left(\prod_{\alpha \in A} (x_{a,\alpha} x_{d,\alpha} T_\alpha) \prod_{\alpha \notin A} (-x_{b,\alpha} x_{c,\alpha} U_\alpha) \right) \in I_{G_s}$$

I_{G_s} is a toric ideal of a graph and thus does not have any monomial elements, so by Theorem 3.12, for each $A \subseteq \{1, \dots, r\}$ and $\beta_{k,j}$, there exists another term $f_{i',j'}m_{i'}$ and a scalar $a \in k$ such that

$$\beta_{k,j} \prod_{\alpha \in A} (x_{a,\alpha} x_{d,\alpha} T_\alpha) \prod_{\alpha \notin A} (-x_{b,\alpha} x_{c,\alpha} U_\alpha) + af_{i',j'}m_{i'} \in I_{G_s}.$$

Using the fact that $f_{i,j'}m_i'$ is a term of some C_i, rewrite this expression as

$$\beta_{k,j} \prod_{\alpha \in A} (x_{a,\alpha} x_{d,\alpha} T_\alpha) \prod_{\alpha \notin A} (-x_{b,\alpha} x_{c,\alpha} U_\alpha) + a\beta_{i,j'} \prod_{\alpha' \in A'} (x_{a,\alpha'} x_{d,\alpha'} T_{i,\alpha'}) \prod_{\alpha' \notin A'} (-x_{b,\alpha'} x_{c,\alpha'} U_{i,\alpha'}) \in I_{G_s}$$

Since the parallel edges do not appear in any generator of I_{G_s}, they appear with the same multiplicity in both terms. Thus, since I_{G_s} is prime,

$$\beta_{k,j} \prod_{\alpha \in A} T_\alpha \prod_{\alpha \notin A} U_\alpha + a\beta_{i,j'} \prod_{\alpha' \in A'} T_{i,\alpha'} \prod_{\alpha' \notin A'} U_{i,\alpha'} \in I_{G_s}$$

Since I_{G_s} is the toric ideal of a subgraph, the edges of the two terms must be incident to the same vertices with the same multiplicity, so the sum is a monomial times some union of cycles in I_{G_s}. To arrive at our contradiction, we use Lemmas 3.9 and 3.10 to show that the sum is in fact in I_{flip}.

Given that $\beta_{k,j} \prod_{\alpha \in A} T_\alpha \prod_{\alpha \notin A} U_\alpha + \beta_{i,j'} \prod_{\alpha' \in A'} T_{i,\alpha'} \prod_{\alpha' \notin A'} U_{i,\alpha'} \in I_{G_s}$, it must be a monomial times a union of cycles in I_{G_s}.

Then, for some monomial m and cycles $X_i - Y_i$,

$$\beta_{k,j} \prod_{\alpha \in A} T_\alpha \prod_{\alpha \notin A} U_\alpha + a\beta_{i,j'} \prod_{\alpha' \in A'} T_{i,\alpha'} \prod_{\alpha' \notin A'} U_{i,\alpha'} = m \left(\prod X_i - \prod Y_i \right) \tag{1}$$

where $\prod X_i$ and $\prod Y_i$ share no common variables. Without loss of generality,

$$m\prod X_i = \beta_{k,j}\prod_{\alpha\in A} T_\alpha \prod_{\alpha\notin A} U_\alpha$$
$$-m\prod Y_i = a\beta_{i,j'}\prod_{\alpha'\in A'} T_{i,\alpha'}\prod_{\alpha'\notin A'} U_{i,\alpha'}$$

Since neither $\beta_{k,j}\prod T_\alpha \prod U_\alpha$ nor $\beta_{i,j'}\prod_{\alpha'\in A'} T_{i,\alpha'}\prod_{\alpha'\notin A'} U_{i,\alpha'}$ is divisible by the leading term of any flip move, none of the cycles have a tail.

Consider the $m(\prod X_i - Y_k)$ from Eq. (1). Since we picked a maximal s to construct I_{G_s}, wherever G_s is more than a single column wide, every internal column of that portion of G_s either has an edge that appears in $\beta_{k,j}$, or else contains edges from some $C_{k,\alpha}$. Since $m\prod X_i = \beta_{k,j}\prod_{\alpha\in A} T_{k,\alpha}\prod_{\alpha\notin A} U_{k,\alpha}$, it is divisible by an edge from each column of G_s (and hence an edge from every column contained in any $X_i - Y_i$). Therefore, we can apply Lemma 3.10, and so

$$\beta_{k,j}\prod_{\alpha\in A} T_\alpha \prod_{\alpha\notin A} U_\alpha + a\beta_{i,j'}\prod_{\alpha'\in A'} T_{i,\alpha'}\prod_{\alpha'\notin A'} U_{i,\alpha'} = m\left(\prod X_i - \prod Y_i\right) \in I_{flip} \tag{2}$$

Further, this is a non-zero element of I_{flip}. Consider two cases: either $i = k$ or $i \neq k$.

First, suppose $i = k$. Since the parallel edges need to match, $A = A'$. The conclusion of Theorem 3.12 states that the terms in the binomial are distinct, so $j \neq j'$ and thus $\beta_{k,j} \neq \pm a\beta_{k,j'}$. Therefore, the terms cannot cancel.

Alternatively, if $i \neq k$, then C_i multiplies different cycles than C_k. Since $\beta_{k,j}$ and $\beta_{i,j'}$ are not divisible by the leading term of any cycles, the terms again cannot cancel.

Thus, $\beta_{k,j}\prod_{\alpha\in A} T_\alpha \prod_{\alpha\notin A} U_\alpha + a\beta_{i,j'}\prod_{\alpha'\in A'} T_{i,\alpha'}\prod_{\alpha'\notin A'} U_{i,\alpha'} \in I_{flip} - \{0\}$.

However, neither of the terms is divisible by the leading term of a binomial in our Groebner basis F. Thus, dividing by F will give a non-zero remainder (namely the binomial itself). Given a Groebner basis for an ideal, a polynomial is in the ideal if and only if dividing by the Groebner basis gives a remainder of zero. Thus, $\beta_{k,j}\prod_{\alpha\in A} T_\alpha \prod_{\alpha\notin A} U_\alpha + a\beta_{i,j'}\prod_{\alpha'\in A'} T_{i,\alpha'}\prod_{\alpha'\notin A'} U_{i,\alpha'} \notin I_{flip}$, contradicting Eq. (2).

Therefore, $S'_{k-1} + C_k \notin I_{G_s}$, and so $S_k \notin I_G^{(s)}$, as desired. □

We can now complete the proof of Proposition 3.6 that $I_{flip} \supseteq \bigcap_{s\in S} I_G^{(s)}$.

Proof. Note that when s is all 0s, $I_G^{(s)}$ is precisely the toric ideal of the graph, so

$$\bigcap_{s\in S} I_G^{(s)} \subseteq I_G.$$

Moreover, I_{flip} is generated by a subset of the cycles of the graph, so

$$I_{flip} \subseteq I_G.$$

Therefore, it is sufficient to show that for all $C \in I_G$, if $C \notin I_{flip}$, then $C \notin \bigcap_{s\in S} I_G^{(s)}$.

Let $C \in I_G \setminus I_{flip}$. Dividing C by F yields $C = f + C'$, where $f \in I_{flip}$ and no term of C' is divisible by a leading term in F. Since $C \notin I_{flip}$, we know $C' \notin I_{flip}$.

Since $C' \in I_G$, we can write it as a combination of cycles of G,

$$C = f + \sum_{i=1}^{n} \alpha_i C_i$$

Pulling out any cycles whose leading terms divide a term of α_i, reducing each by F, and pulling out any subset of $\beta_i \prod C_{i\alpha}$ that sums to an element of I_{flip} yields

$$C = f + \sum \beta_i \prod_{\alpha} C_{i,\alpha}$$

Let $C' = \sum_{i=1}^{n} \beta_i \prod_{\alpha} C_{i,\alpha} \in I_G \setminus I_{flip}$. Then, after combining like terms,

- If $i \neq j$, then $\prod C_{i,\alpha} \neq \pm \prod C_{j,\alpha}$, since we combined like terms
- Also since we combined like terms, each β_i has fully combined like terms.
- No nonempty subset of the $\beta_i \prod C_{i,\alpha}$ sums to an element of I_{flip}, since we pulled all such subsets into f
- No term of β_i is divisible by the leading term of a cycle in I_G, since we pulled all possible divisors out from α_i
- No term of $\beta_i \prod C_{i,\alpha}$ is divisible by a leading term of F, since we reduced each by F
- Each $\beta_i \prod C_{i,\alpha} \notin I_{flip}$, and no leading term of I_G divides β_i, so by Lemma 3.8 there is some s_i such that $\beta_i \prod C_{i,\alpha} \notin I_G^{(s)}$

Thus, C' satisfies the hypotheses of Theorem 3.11, so $C' \notin I_G^{(s)}$ for some s. Since $I_{flip} \subseteq \bigcap I_G^{(s)}$ for all s, we know that $f \in I_{flip} \subseteq I_G^{(s)}$, and so $C \notin I_G^{(s)}$, as desired.

Therefore, if $C \in I_G \setminus I_{flip}$, then $C \notin \bigcap_{s \in S} I_G^{(s)}$, so $I_{flip} \supseteq \bigcap_{s \in S} I_G^{(s)}$. □

This completes the second direction of the inclusion, so $I_{flip} = \bigcap_{s \in S} I_G^{(s)}$, completing the proof of Theorem 3.2.

4 Computational Results

The computational aspect of this thesis has two distinct components: symbolic computations in Macaulay2, and exploring three-dimensional flip connected components using Python.

4.1 Macaulay2 Computations

We used Macaulay2 to compute primary decompositions for some small grids, providing the motivation for the main theorem of the previous section. To aid in analyzing the output, we created a Python script to draw pictures highlighting which edges appeared in which primary components. An example output from this script is shown in Fig. 13.

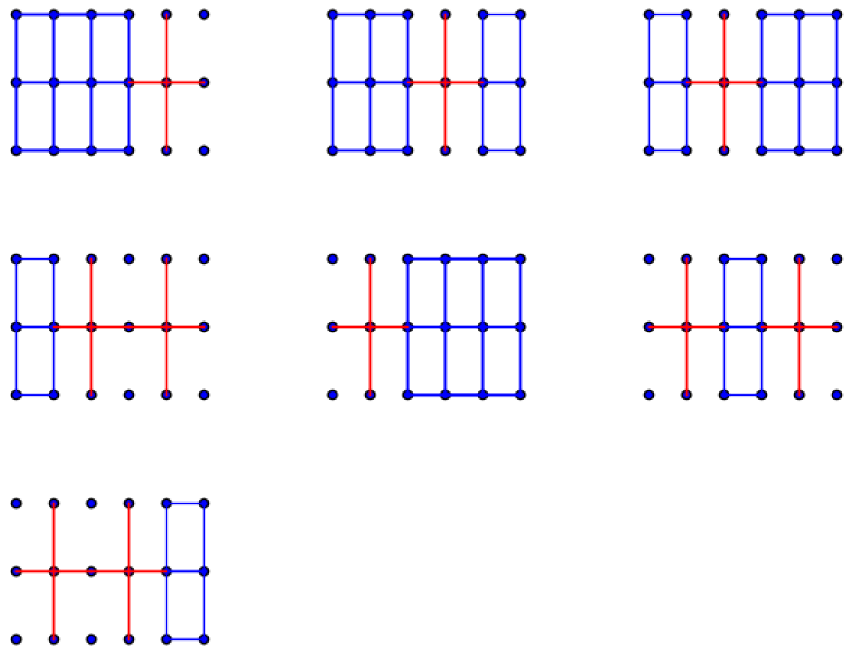


Figure 13: A visualization of the primary components of I_{flip} in a 3×6 grid. Red edges represent monomial generators, while blue grids represent the toric ideal on that portion of the graph.

We also ran primary decompositions for flip ideals of some slightly larger grid graphs. The output for a 4×6 grid is shown in Fig. 14.

These examples, along with our results from $3 \times n$ grids and the symmetry of the problem, lead us to the following conjecture about the algebraic structure of I_{flip} in the general case.

Conjecture 4.1. *Let G be the a $m \times n$ grid graph, and let S be the set of $m \times n$ 0/1 arrays such that the border of the array is all 0s and each 1 is surrounded by 8 zeros.*

For each $s \in S$, construct $I_G^{(s)}$ by adding a monomial generator for each edge having an endpoint labelled with a 1 in s, and adding the generators for the toric ideal on the remainder of the graph, analagously to Theorem 3.2.

Then,

$$I_{flip} = \bigcap_{s \in S} I_G^{(s)}$$

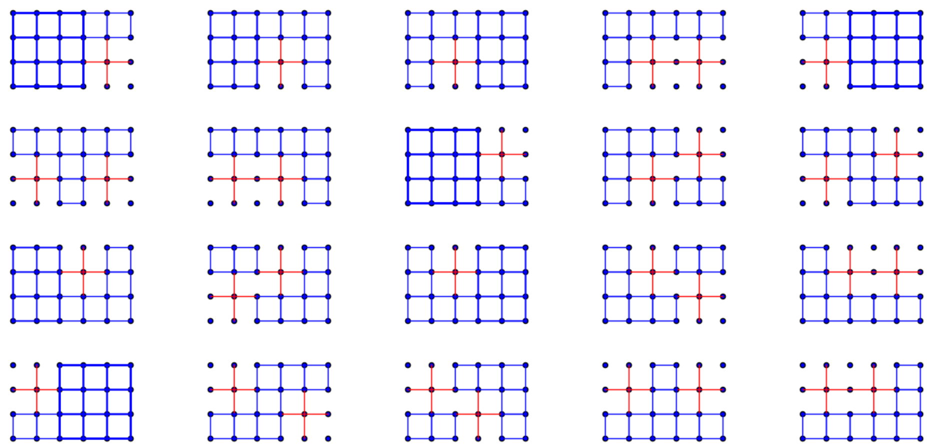


Figure 14: Primary components for I_{flip} on a 4×6 grid.

4.2 Flip Component Exploration

The second computational aspect of this thesis was writing code to explore the flip-connected components of three dimensional regions.

To start, we created a program to detect and execute flips and trits on a given tiling. To aid in this and to build intuition, we also added functionality to create and display an interactive visualization of a matching, a snapshot of which can be seen in Fig. 15.

The goal was to use our code to explore flip-connected components, as was done in [8]. However, computational limitations prevented us from fully validating the results of Milet and Saldanha from [8]. Even after optimizing our breadth-first search to save memory by only storing the total number of nodes and the most recent layer, or by writing all the data to disk, the queue still grew too large to fit in memory after finding approximately 152,320,000 tilings in the giant component.

However, we were able to confirm Milet and Saldanha's results on the smaller components. In addition to validating the counts, we were able to produce the full graphs for these smaller flip-connected components.

We hypothesized that the flip component graphs should be approximately cubical, since we can execute two non-overlapping flip moves in either order and arrive at the same result. However, the graphs for the $4 \times 4 \times 4$ box were too crowded to be visually inspected for this structure, even on the smaller components, so we instead focus on the visualizations of flip components on smaller boxes. In each of these figures, red, green, or blue vertices represent tilings where all dominoes are parallel to the x, y, or z axis, respectively.

The $2 \times 2 \times 2$ box clearly demonstrates a cubical structure, as shown in Fig. 16.

The cubical structure can also be easily seen in a slightly less trivial context in the single flip component of a $2 \times 2 \times 3$ box, as in Fig. 17.

In order to see how tilings with trit moves fit into a flip component, we must look at a still larger box. Namely, the smallest box with more than one flip-connected component is the $3 \times 3 \times 2$. In this box, we have two isolated tilings with no flip moves, each of which has trit moves into the giant component. The giant component is pictured in Fig. 18. In this component, there are too many tilings to adequately visually inspect for symmetry

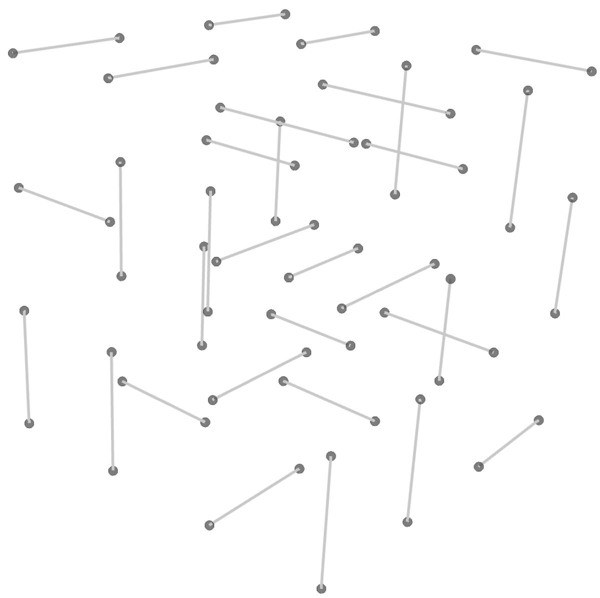


Figure 15: A snapshot from our interactive visualization of a matching. The user can explore and rotate the visualization in three-dimensions to better explore the structure.

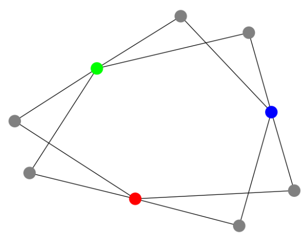


Figure 16: The single flip component of a $2 \times 2 \times 2$ box. The red, green, and blue vertices represent the tilings where all dominoes are aligned with the x, y, and z axes, respectively.

or cubical structures. However, in the sparser areas around the edges, the structure and symmetry appear to hold. In the future, we would like to be able to give a more rigorous characterization and proof of this apparent symmetry.

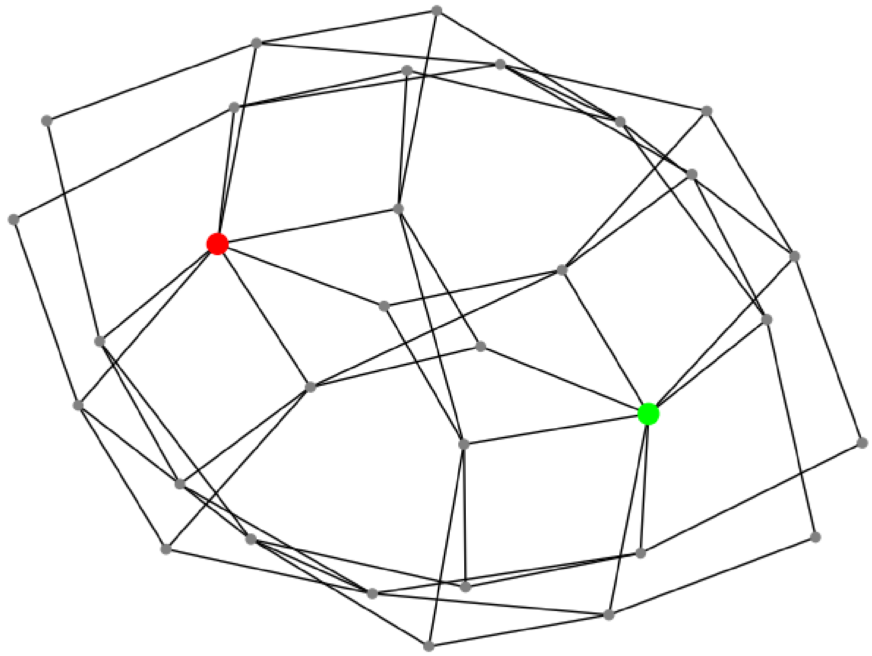


Figure 17: The cubical structure of the single flip component of a $2 \times 2 \times 3$ box.

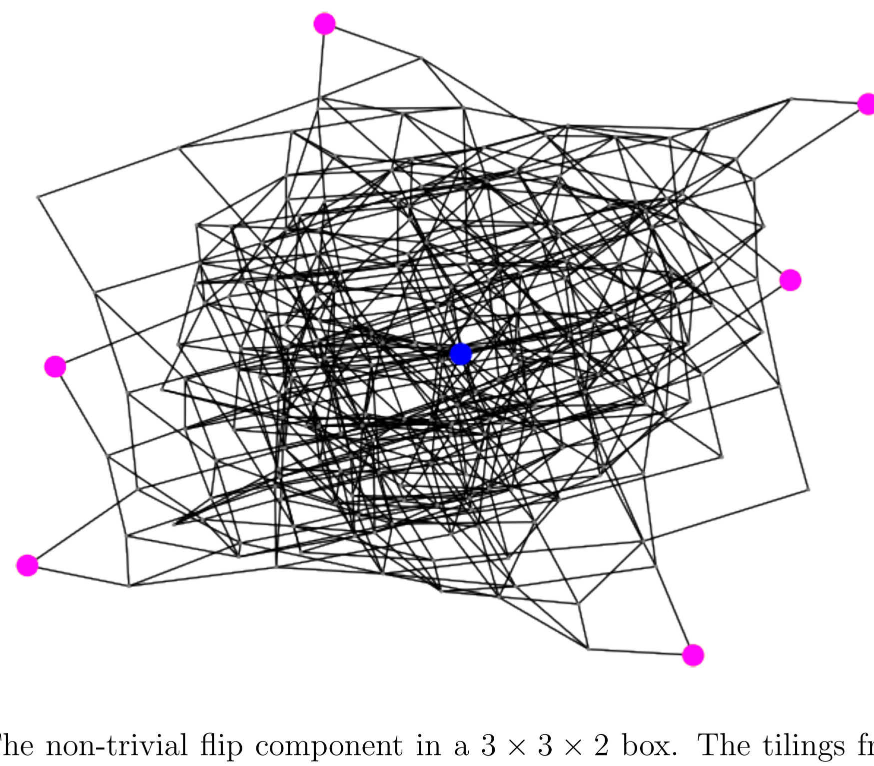


Figure 18: The non-trivial flip component in a $3 \times 3 \times 2$ box. The tilings from which you can take a trit move to one of the isolated tilings are drawn in pink.

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