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Doctoral Thesis

Higher-dimensional Supergravity, Polytopic SUSY Representation Theory, and SUSY Holography Conjecture

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Abstract of

Higher-dimensional Supergravity, Polytopic SUSY Representation Theory, and SUSY Holography Conjecture

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Doctor of Philosophy

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In this thesis, we present studies of off-shell supergravity theories in ten- and eleven-dimensional superspaces, polytopic supersymmetry representation theory, and the SUSY holography conjecture.

First, inspired by the history of how Einstein constructed General Relativity, we study the linearized Nordström supergravity in ten- and eleven-dimensional superspaces. We found no obstacles to applying the lessons we learned in 4D to higher dimensions. We also derive infinitesimal 10D superspace Weyl transformation laws. The identification of all off-shell ten-dimensional supergeometrical Weyl field strength tensors, constructed from respective torsions, is presented.

Second, since the traditional approach requires a high computational price to be paid to elucidate the θ-expansion for component fields residing within the superfields, we establish a novel approach founded by the polytopic SUSY representation theory. We realize that Lie Algebra techniques, in particular branching rules, Plethysm, and tensor product, provide the key to deciphering the *complete* list of independent fields that describe a supersymmetric multiplet in *arbitrary* spacetime dimensions efficiently. We show the explicit one-to-one correspondence between Lorentz irreps and field variables, leading to an "adynkrafield" formalism in which the traditional θ-monomials are replaced by Young Tableaux.

Further elaborations on the "scan" machinery and the consequent prepotential candidates for 10D, $\mathcal{N}$ =1, 2A, and 2B superconformal supergravity theories are presented as well.

Third, the SUSY holography conjecture proposes a way to study higher-dimensional supersymmetric models from their 1D counterparts. The problem of classifying off-shell representations of the N-extended 1D super Poincaré algebra is closely related to the classification of its graphical representations – adinkras. We define the adinkra "height yielding matrix number" (HYMN) and study HYMN equivalence classes for all $\mathcal{GR}(4,4)$ valise adinkras. Another puzzle is that considering the projection from 4D, $\mathcal{N} = 1$ SUSY to 1D, $N = 4$ SUSY, a dissection of $\mathbb{S}_4$ is required to be consistent with SUSY. We show that this dissection can be seen from the symmetries of the permutahedron and the use of weak Bruhat ordering.

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[1] "Four-point correlators of light-ray operators in CCFT," **Yangrui Hu**, Luke Lippstreu, Marcus Spradlin, Akshay Yelleshpur Srikant, and Anastasia Volovich, e-Print: 2203.04255 [hep-th]

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Chapter 1

Introduction

1.1 Ten- and Eleven-Dimensional Nordström Supergravity Theories

In 1907, Einstein described his "happiest thought" [1] which marked the commencement of the race to create the Theory of General Relativity. Unrealized, he was already decidedly at a disadvantage. As early 1900, the astronomer Karl Schwarzschild (1873-1916) [2] had written about Riemann's geometrical concepts to describe curved space - but not curved space-time. The latter would not emerge until Hermann Minkowski introduced the concept of "four-geometry" into physics [3, 4].

By 1914, there were a number of competitors. At a minimum, these included Max Abraham (1875-1922), Gustav Mie (1868-1957), and Gunnar Nordström (1881-1923). Even the accomplished mathematician David Hilbert (1862-1943) became involved but at the conclusion made the comment in his own work, "The differential equations of gravitation that results are, as it seems to me, in agreement with the magnificent theory of general relativity established by Einstein [5]."

Along the pathway to the end of the race, the idea of scalar gravitational theories was explored. It is of note that Nordström created two such sets of equations [6, 7] and even Einstein took this idea before discarding it. In the scalar approach, the usual metric of the space-time manifold is replaced by a single scalar field. A way to do this is to begin with the Minkowski metric and simply multiply it by a scalar field. This implies that, geometrically, scalar theories of gravitation are all members of the same conformal class as the usual flat Minkowski metric. Mathematically, scalar gravitation theories are perfectly consistent... they simply do not describe the physical laws observed in our universe.

In the nineteen eighties, superspace geometrical descriptions of supergravity in eleven and ten dimensions were presented in the physics literature. To the best of our knowledge a list of these inaugural publications looks as:

(a.) 11D, $\mathcal{N} = 1$ supergravity [8, 9],

(b.) 10D, $\mathcal{N} = 2$A supergravity, [10],

(c.) 10D, $\mathcal{N} = 2$B supergravity [11, 12], and

(d.) 10D, $\mathcal{N} = 1$ supergravity [13].

Of course, these theories had been obtained in other descriptions even earlier. Interested parties are directed to these works for such references.

If we think of the references [8–13] as the eleven and ten-dimensional analogs of Einstein's "happiest thought," then by analogy all developments since these works are analogs of the race to find the Theory of General Relativity. This also reveals a glaring disappointment. Since the "bell was rung" in this new race, all the competition is still at the starting line.

How would one know the race has been successfully ended? As indicated by the title of the work [9] ("Eleven-Dimensional Supergravity on the Mass-Shell in Superspace"), these descriptions possess sets of Bianchi identities that are consistent *only* when the equations of motion for the component fields in the theory satisfy their mass-shell conditions. This holds true for all of the works in [8–13]. So we may take as the sign of the successful completion of the race, if a set of superspace geometries were explicitly found such that their Bianchi identities do *not* require a mass shell condition. By way of comparison, for 4D, $\mathcal{N} = 1$ superspace supergravity, the analogs of the "happiest thought" and the conclusion of the race occurred within one year as is seen via the works completed by Wess and Zumino [14, 15].

A new urgency has recently appeared and which drives a need to improve the current situation. Recent progress [16, 17] occurred in the derivation of M-Theory corrections to 11D Supergravity. A series of procedures connecting the corrections to a 3D, $\mathcal{N} = 2$

Chern-Simons theory [18–21] (used in a role roughly analogous to world-sheet σ-model β-function calculations for string corrections) has been successfully demonstrated.

Though the works [16, 17] have presented a method of deriving these corrections beyond the supergravity limit, these *solely* treat purely bosonic M-Theory corrections, with no equivalent results describing fermionic corrections. One traditional way of accomplishing this is to embed the purely bosonic results into a superspace formulation. This impels us to a renewed interest in 11D supergravity in superspace. The goal we are pursuing is to find a Salam-Strathdee superspace [22], as modified by Wess & Zumino [14, 15], such that superspace Bianchi identities in the Nordström limit allow for the appearance of the M-Theory corrections.

This distinguishes our work from efforts taken by others. For example, there is a substantial literature that uses the concept of "pure spinors" [23–29] where the endpoint of action principles (see especially [28, 29]) has been presented. While classically such approaches appear to work, there are troubling signs [30–33] that more needs to be done to justify complete acceptance at the level of quantum theory.

Perhaps an intuitive way to argue is in order to achieve quantization, it should be implemented in terms of variables that are basically free. The non-minimal pure spinor goes some way to giving a free resolution, but the resulting space of fields does not have a well-defined trace. The work [32] offered a "fix" but *if* in the end the prescription still is not an integral over free variables with no other qualifications, then a proper quantization is still likely to be impeded.

Therefore, we adopt the rather more cautious approach by raising the query of whether it is possible to follow the pathway established by Wess & Zumino [14, 15] wherein a "simple" Salam-Strathdee superspace is used as a starting point to build, in our case, Nordström SG in which Bianchi identities do not force equations of motion.

In the earliest days of formulating the superspace understanding of supergravity, an important discovery was shown in a work by Grimm, Wess, and Zumino [34] when it was shown the dynamical equations of motion of the component fields contained within the supergravity supermultiplet could be derived with *no reference at all to a superspace* *action*. It is important to *note* all the superspace formulations in [8–13] exhibit this feature. In fact, for the case of [12], there exists no universally accepted action principle to describe the theory... even at the component level at all. Thus, using superspace formulations, the dynamics of the component fields can be obtained without complete knowledge of the structure of the superfields that contain them. This allows for a "clean separation" of issues of representations from the dynamics of the component fields so being described. Our investigation will follow this minimalist approach.

As noted by Misner and Watt [35], though scalar gravitational theories are not realistic, they have value as computational tools in numerical relativity. This raises a very intriguing question. The quote, "History doesn't repeat itself, but it often rhymes,[1]" has been stated about many situations. Since scalar gravitation models have value as computational tools for General Relativity, might extending them to eleven- and ten-dimensional supergeometries offer new ways to replicate Einstein's path from the "happiest thought" to the higher level of understanding as indicated by his lectures at Göttingen?

It is the purpose of [36] (presented in chapter 2 in the thesis) to lay a new foundation for such an exploration. We take as a guiding principle the procedure used to provide the first example of a four-dimensional supergravity supermultiplet where the closure of the local simple supersymmetry algebra was not predicated on the use of field equations. This was accomplished by Breitenlohner [37] who took his starting point as an off-shell supermultiplet, the so-called "non-abelian vector supermultiplet." The final form of Breitenlohner's initial presentation realized a reducible representation of supersymmetry. Finally, this first work also did not consider any issue surrounding the construction of actions for the supermultiplet.

His approach is equivalent to starting with the component fields of the Wess-Zumino gauge 4D, $\mathcal{N} = 1$ vector supermultiplet $(v_{\underline{a}}, \lambda_b, \mathrm{d})$ together with their familiar SUSY

[1] Though often attributed to Mark Twain, there is little evidence of this being accurate.

transformation laws,

$$
\begin{aligned}
\mathrm{D}_a\, v_{\underline{b}} &= (\gamma_{\underline{b}})_a{}^c\, \lambda_c \quad , \\
\mathrm{D}_a \lambda_b &= -\, i\, \tfrac{1}{4}([\, \gamma^{\underline{c}} \, , \, \gamma^{\underline{d}}\,])_{ab}\, (\, \partial_{\underline{c}}\, v_{\underline{d}} \; - \; \partial_{\underline{d}}\, v_{\underline{c}}\,) \; + \; (\gamma^5)_{a\, b}\, \mathrm{d} \quad , \\
\mathrm{D}_a\, \mathrm{d} &= i\, (\gamma^5 \gamma^{\underline{c}})_a{}^b\, \partial_{\underline{c}} \lambda_b \quad ,
\end{aligned}
\tag{1.1.1}
$$

followed by choosing the gauge group as the spacetime translations, SUSY generators, and the spin angular momentum generators as well as allowing additional internal symmetries. For the spacetime translations, this requires a series of replacements of the fields according to:

$$
v_{\underline{b}} \;\to\; h_{\underline{b}\,\underline{c}} \quad , \quad \lambda_b \;\to\; \psi_{\underline{c}\, b} \quad , \quad \mathrm{d} \;\to\; A_{\underline{c}} \quad , \tag{1.1.2}
$$

(in the notation in [37] $A_{\underline{a}} = B^5{}_{\underline{a}}$) while for the SUSY generators, the replacements occur according to:

$$
v_{\underline{b}} \;\to\; \chi_{\underline{b}\, c} \quad , \quad \lambda_b \;\to\; \phi_{b\, c} \quad , \quad \mathrm{d} \;\to\; \chi_c{}^5 \quad , \tag{1.1.3}
$$

and finally for the spin angular momentum generator, a replacement of

$$
v_{\underline{b}} \;\to\; \omega_{\underline{b}\,\underline{c}\,\underline{d}} \quad , \quad \lambda_b \;\to\; \chi_{b\,\underline{c}\,\underline{d}} \quad , \quad \mathrm{d} \;\to\; D_{\underline{c}\,\underline{d}} \quad , \tag{1.1.4}
$$

was used. However, to be more exact, Breitenlohner also allowed for more symmetries like chirality to be included. Because the vector supermultiplet was off-shell (up to WZ gauge transformations) the resulting supergravity theory was off-shell and included a redundant set of auxiliary component fields, i. e. this is not an irreducible description of supergravity. But as seen from (1.1.2) the supergravity fields were all present and together with the remaining component fields a complete superspace geometry can be constructed.

With the Breitenlohner approach as a guiding principle to the study of a class of curved supermanifolds containing eleven and ten dimensions, it is thus to be expected the extensions will manifest the same structure of being off-shell but reducible, and not address the issue of the construction of actions. No off-shell gauge vector supermultiplet is known beyond six dimensions, thus one is forced to deviate from completely following the Breitenlohner approach. Since a scalar superfield in any dimension greater than three is guaranteed to be off-shell, but reducible, the expectation is to be led to the study of Nordström supergravity theories in eleven and ten dimensions in this approach.

In our approach to Nordström SG, the analog of the Wess-Zumino gauge 4D, $\mathcal{N} = 1$ vector supermultiplet is played by a scalar superfield in any of the 11D or 10D superspaces to be studied. This scalar superfield guarantees off-shell supersymmetry. However, like the approach taken by Breitenlohner, the resulting theory is expected to be reducible. Also like this earlier approach, the question of an action principle is not addressed.

1.2 Weyl Covariance

The first explicit discussions in the literature on the topic of Weyl symmetry in superspace were initiated among the works [38, 39] by Howe et al.. The subject of the interplay between conformal symmetry and the constraints of superspace descriptions of Poincaré supergravity, has long been of fascination to Gates, who is one of the authors of [40]), [41–45]. Of course, numbers of other authors have also pursued this subject. A special area of these considerations involves the context of 11D, $\mathcal{N} = 1$ superspace [8, 9] related to M-Theory [46]. After its beginning, a literature (e.g. [26, 47–54]) has been built up including discussions in superspace and also at the level of component fields. These are but a small selection and the interested readers should look at the references in these works for a more complete listing of such works.

Near the end of a 1996 investigation [55], the following paragraph can be found.

> *For although we believe our observation is important, we know of at least two arguments that suggest that there must exist at least one other tensor superfield that will be required to have a completely off-shell formalism. This is to be expected even from the structure of the non-minimal 4D, $\mathcal{N} = 1$ supergravity. There it is known that there are three algebraically independent tensors $W_{\alpha\beta\gamma}$, $G_{\underline{a}}$ and T_{α}, so apparently it remains to find the eleven-dimensional analog of the $G_{\underline{a}}$. In future works, these aspects of the eleven-dimensional theory will require further study.*

From our present perspective, the "strong form" of the constraints given in a 2000 work by Cederwall, Gran, Nielsen, and Nilsson [26], appears to provide the solution of our 1996 dilemma. These authors introduced a dimension zero tensor $X_{[5]}{}^{\underline{b}}$ that, from our present understanding, is the missing analog to the 4D, $\mathcal{N}=1$ $G_{\underline{a}}$-tensor.

In the work of [56], an analysis based on prepotential superfields was undertaken regarding the scale compensating superfield Ψ and conformal semi-prepotential $\widehat{\mathrm{H}}_{\alpha}{}^{\underline{a}}$. It was shown these play an interesting role with respect to the emergence of Weyl scaling covariance in 11D, $\mathcal{N}=1$ superspace. Three *key* points were noted in this context:

(a.) *When a sufficiency of conventional constraints are imposed upon the 11D, $\mathcal{N}=1$ Poincaré superspace supergravity covariant derivative operators $(\nabla_{\alpha}, \nabla_{\underline{a}})$ so that Ψ and $\widehat{\mathrm{H}}_{\alpha}{}^{\underline{a}}$ are the only independent superfields within them, a spinorial connection field $\mathcal{J}_{\alpha}{}^{(+)}$ (for Weyl scaling in superspace) emerges among the dimension one-half Poincaré supergravity supertensor components.*

(b.) *The sufficiency of conventional constraints that leads to the existence of $\mathcal{J}_{\alpha}{}^{(+)}$ is also sufficient to lead to the emergence of a vectorial connection field $\mathcal{J}_{\underline{a}}{}^{(+)}$ for Weyl scaling in superspace.*

(c.) *The existence of $\mathcal{J}_{\alpha}{}^{(+)}$ and $\mathcal{J}_{\underline{a}}{}^{(+)}$ together with the existence of the superspace supergravity covariant derivative operators $(\nabla_{\alpha}, \nabla_{\underline{a}})$ imply the existence of modified supergravity covariant derivative operators $(\widehat{\nabla}_{\alpha}, \widehat{\nabla}_{\underline{a}})$ that transform covariantly with respect to Weyl scaling.*

In fact, precisely these three points are at the foundation of a paper written in 1991 within the context of 10D, $\mathcal{N}=1$ superspace [45].

As the focus of this analysis is the modified supergravity covariant derivative operators (and their related field strengths, Bianchi identities, etc.), this approach is mute on implications for the superfields needed to construct the superframe superfields. This is true for most of the citations among [38] - [54] as they are not focused upon the superfield variables that are "inside" of $(\nabla_{\alpha},\ \nabla_{\underline{a}})$, i. e. the prepotentials.

We will make an observation about the relation of the 11D, $\mathcal{N} = 1$ Nordström theory and the infinitesimal super Weyl transformation laws of the complete non-linear Poincaré superspace supergravity derivatives (∇_α, $\nabla_{\underline{a}}$) as motivating a pathway for the derivation of similar results in all ten dimensional superspaces. Inspired by the success of relating the emergence of Weyl symmetry in 11D, $\mathcal{N} = 1$ superspace, together with our recent studies of supergravity in eleven and ten dimensions [36, 57–59], we are motivated to extend the discussion and results of [56] into the domain of *all* ten-dimensional supergravity theories in superspace. In chapter 7, we will present the extension in ten-dimensional superspaces.

1.3 Polytopic SUSY Representation Theory

The concept of the Salam-Strathdee "superfield", for space-time supersymmetry, was introduced in 1974 [60, 61]. The use of superfields makes the presence of space-time supersymmetry (SUSY) manifest in such discussions, just as the use of 4-vectors makes the symmetries of special relativity manifest within their domains of consideration.

Superfields have proven vital in the identification of collections of component fields of various spins that provide representations of space-time supersymmetry. Their utility is most important to identify such collections that realize this symmetry independent of dynamical conditions (i.e. *equations of motion*) imposed on component fields in these collections. Collections of fields that satisfy this condition are said to describe an "off-shell realization" of space-time supersymmetry.

On the other hand, there exist collections of component fields of differing spins with the property that space-time supersymmetry is *only* realized in the presence of dynamical conditions (i.e. *equations of motion*) imposed on all of the component fields in these collections. Collections of fields that satisfy this condition are said to describe an "on-shell realization" of space-time supersymmetry.

This subtle distinction between "off-shell realization" and "on-shell realization" has an important implication. Only formulations of the foremost category can provide manifestly space-time supersymmetric realizations in the path integral approach to quantization. Thus, the area of supersymmetrical relativistic quantum field theory (SQFT) has for decades existed in the peculiar situation that the vast majority of these theories cannot be investigated via path integrals in a manner where space-time SUSY is manifest. This is a statement of the notorious, and often overlooked, "off-shell SUSY" problem. It is, perhaps, the oldest major problem in the conceptual foundation of the field involving SUSY.

Although there is still no apparent solution to this problem on the horizon, we have in a recent series of work [57–59, 62] succeeded in answering a question that can provide a step in solving this half-century-old puzzle. We will refer to this as the "Salam-Strathdee Riddle[2]" as it is implicitly contained in the Salam-Strathdee proposition of the notion of superfields and can be posed as a question, "In an arbitrary dimension, what is the explicit collection of component fields that occur in a superfield which is an arbitrary representation of the space-time Lorentz symmetry group and provides an off-shell SUSY representation?"

By creating a new conceptual network-enabled framework, we have presented a concept that is a logically and mathematically independent alternative to the concept of the Salam-Strathdee superfield. Using this to create efficient code, there now exists a way to efficiently answer the puzzle.

The network-enabled framework mentioned above is the idea that the topology of certain types of graphs, "*adynkras*," can be used to replace the 'theta-expansions' required in the Salam-Strathdee approach. The overlap of the topics of SUSY and topology has been the domain of exploration of Jim Gates in many guises. Some of these range from local superspace integration theory, Chern-Simons models, Wess-Zumino-Novikov-Witten (WZNW) models, and finally adynkras.

In chapter 5, we will show the newly developed techniques, both algorithmic and analytical, that allow the *first complete and explicit SO(1,9) Lorentz descriptions of all component fields contained in 10D*, $\mathcal{N} = 1$, $\mathcal{N} = 2A$, and $\mathcal{N} = 2B$ *unconstrained scalar superfields.* They form the maximal reducible and relevant supermultiplets. For the case of the $\mathcal{N} = 1$ theory, our results agree with the first report on this topic [50] given

[2] Thus, mathematical/theoretical physics sometimes can resemble the challenges similar to the ones the Sphinx gave to unwary wanderers.

by Bergshoeff and de Roo. These authors took as their starting point an analysis of the "supercurrent corresponding to the $\mathcal{N} = 1$ supersymmetric Maxwell theory in ten dimensions." An approach of this type is tied to making *an initial assumption about the dynamics of some supersymmetrical system*, i.e. the choice of the energy-momentum tensor.

The previous sentence illustrates a peculiarity of the study of spacetime supersymmetry in comparison to other symmetries. Namely, almost all studies of the representations of spacetime supersymmetry take as a starting a set of assumptions related to some dynamic system. This is shared by no other symmetry known to these authors. Thus, there is a possibility by exploring avenues that do not depart from dynamical assumptions, a resolution to the puzzles may be found.

We will present the maximal reducible relevant representations of supergravity in 10D related to unconstrained scalar superfields by "tensoring" them with different representations of the SO(1,9) Lorentz group and search for graviton(s) and gravitino(s). To achieve this goal, we borrow Breitenlohner's approach [37], which gave the first off-shell 4D, $\mathcal{N} = 1$ supergravity description. It utilized the known off-shell structure of the 4D, $\mathcal{N} = 1$ vector supermultiplet $(v_{\underline{a}}, \lambda_\beta, d)$. Since the structure of off-shell supermultiplets in ten dimensions is poorly understood, we implement the approach of Breitenlohner by use of a 10D scalar superfield in place of the vector supermultiplet. This will ensure an off-shell supersymmetry realization.

There are three guiding principles implied by the Breitenlohner appoaoch that are implemented in our discussion, use:

(a.) no *dynamical* assumptions in the derivation of representation

(b.) *only* conventional Wess-Zumino superspaces, and

(c.) superfields that admit an *unconstrained prepotential* formulation.

These are the principles that allowed for the complete off-shell superspace descriptions of supergravity in 4D, $\mathcal{N} = 1$ theories. Our current investigation looks to extend these beyond the lower dimensional context and into the domain of the 10D theories. The operational steps to implement the paradigm of the approach include

(a.) identify an appropriate scalar supermultiplet free of *dynamical* assumptions with regards to the closure of the superalgebra,

(b.) tensor the scalar supermultiplet with representations of the Lorentz algebra, and

(c.) among the results found by the tensoring operation, search for components that align with the Lorentz representations of conformal graviton and gravitino.

As this approach is independent of any assumptions about dynamics of the component fields contained in supermultiplets, there should be an expectation of differences in comparison with any past results derived from an analysis where dynamics are built into a conceptual foundation.

It is known that when a supersymmetrical theory is expressed in terms of prepotentials, the reasons for many unexpected enhancements of vanishing quantum corrections become obvious. So we restrict our considerations to formulations that are consistent with prepotentials. This further restricts our considerations to solely theories that demonstrate an absence of "off-shell central charges" [63–67].

The keys to our progress in this realm are higher dimensional adynkras (built upon the well-established foundation of one-dimensional ones) and advanced computational tools. These techniques are derived via properties of representation theory and thus are *independent of assumptions about dynamics.* Our work is also a beneficiary of a report undertaken by N. Yamatsu [68] who presented results for grand unified model-building based on finite-dimensional Lie algebras. In this thesis, the reader can find that projection matrices, branching rules between Lie algebras and their subalgebras up to high ranks, as well as Dynkin labels and Weyl dimensional formulae of irreducible representations, Plethysm, and much more have proven to be particularly useful when applied to the representation theory of Lorentz groups we consider. It is of historical note, that the work by Yamatsu falls among the path of earlier explorations [69–71].

The final enabling tool for our work is based on algorithmic efficiency and here the work by Feger and Kephart [72] played an important role. These authors developed and presented a Mathematica application - Lie Algebras and Representation Theory (LieART) which carries out computations typically occurring in the context of Lie algebras and representation theory. There is another powerful Mathematica application - Susyno [73] developed by Renato Fonseca. They both provide a robust algorithm that enables the study of weight systems.

For the first time in the physics literature, the Lorentz representations of all 2,147,483,648 bosonic degrees of freedom and 2,147,483,648 fermionic degrees of freedom in an unconstrained eleven-dimensional scalar superfield were presented in [58]. Comparisons of the conceptual bases for this advance in terms of component field, superfield, and adinkra arguments, respectively, were also made. These highlight the computational efficiency of the adynkra-based approach over the others. It is noted at level sixteen in the 11D, $\mathcal{N} = 1$ scalar superfield, the $\{65\}$ representation of SO(1,10), the conformal graviton, is present. Thus, adinkra-based arguments suggest the surprising possibility that the 11D, $\mathcal{N} = 1$ scalar superfield alone might describe a Poincaré supergravity prepotential or semi-prepotential in analogy to one of the off-shell versions of 4D, $\mathcal{N} = 1$ superfield supergravity.

1.4 SUSY Holography

The SUSY Holography conjecture was first proposed in [74] as

SUSY Holography Conjecture: reduce higher-dimensional supersymmetric models to one dimension, and one-dimensional models encode the structure of their higher-dimensional counterparts.

In this program, the key object to study is the one-dimensional models. In what follows, we will see that the graphical representations of 1D, N-extended supersymmetry algebra, called “adinkras”, is turned out to be powerful and useful in this research regime. The ultimate goal for this program is to find out what mathematical structures in the one-dimensional objects secretly encode the higher dimensional information. Namely, the real hologram is the goal, so that one-dimensional statements can be promoted to higher-dimensional statements. As we know, higher-dimensional systems, in particular ten-dimensional and eleven-dimensional models which are of interest, are usually very complicated and difficult to study. If we can construct their one-dimensional holograms, the computational cost will be dramatically deduced.

Graphs have demonstrated an unexpected power to "clear out the mathematical underbrush" encountered by theoretical physicists. Feynman Diagrams are a spectacular example of this. While it is difficult to recall, there was a time when Feynman Diagrams were not held in high regard. This all changed rapidly when a sufficient number of physicists employed their power to efficiently calculate quantum corrections to physical processes.

While we make no similar claims about the breadth of possible impacts from the developments which began with the recognition of the existence of the $\mathcal{GR}(\mathrm{d}, \mathcal{N})$ ("Garden Algebras") [75, 76]) as a foundation of supersymmetric representation theory and their evolution into the introduction of adinkras [77], we do hold that adinkras provide similarly important tools within the domain of the representation theory of supersymmetrical theories. One hint about this involves the pathways [78–80] that adinkras have opened from supersymmetrical theories, including field theories, to error-correction codes.

There is a substantial and rapidly growing literature on the relation of quantum error-correction codes [81] to the very structure of space-time (e. g. [82]) itself. Over and above this particular focus, there is the related similar discussion underway regarding quantum entanglement and space-time. Literally, it is bona fide to ask, "Is space-time a quantum error-correction code?" However, from the perspective of field theory, the fields themselves are primary dynamical entities. This suggests the similar question, "Are there relations between fields that describe physical reality and quantum error-correction codes?"

To our knowledge, the works in [78–80] are the singular ones that hint at such a linkage. To be clear, there have been *no claims* that these works provide for a role for *quantum* error-correction codes. The implication of the works in [78–80] is *all irreducible supersymmetric field theory representations in four dimensions involve classical error-corrections codes*. However, it is also true that the **L**-matrices and their inverses the

R-matrices that arise from adinkras are generalizations of Pauli matrices. Looking at many discussions of quantum error-correction codes, the Pauli matrices play an important role.

In section 9.1, a conjecture is made that the weight space for 4D, $\mathcal{N}$-extended supersymmetrical representations is embedded within the permutohedra associated with permutation groups $\mathbb{S}_d$. Adinkras and Coxeter Groups associated with minimal representations of 4D, $\mathcal{N} = 1$ supersymmetry provide evidence supporting this conjecture. It is shown that the appearance of the mathematics of 4D, $\mathcal{N} = 1$ minimal off-shell supersymmetry representations is equivalent to solving a four-color problem on the truncated octahedron. This observation suggests an entirely new way to approach the off-shell SUSY auxiliary field problem based on IT algorithms probing the properties of $\mathbb{S}_d$.

In the study of SUSY holography, there is a puzzle that considering the projection from 4D, $\mathcal{N} = 1$ SUSY to 1D, $N = 4$ SUSY, a dissection of $\mathbb{S}_4$ is required to be consistent with SUSY. We would like to find out how this can be uncovered without the use of supersymmetry-based arguments, relying solely on properties of $\mathbb{S}_4$. This would lead to a better understanding of which mathematical structures inside adinkras secretly contain information from higher dimensions, which provides a possibility to obtain insights for higher-dimensional theories from 1D statements. In the work of [83], we found that this dissection can be seen from the symmetries of the permutohedron and the use of weak Bruhat ordering.

Among the most primitive of adinkras are those which have the structure of nodes *only* occurring at two distinct height. These are called "valise adinkras" with all bosonic nodes at the same value of height and all fermion nodes at the same (but different from the bosonic one) height. The fact that the nodes representing functions can be differentiated or integrated is reflected in a change of nodal height in adinkras [84]. Various prescriptions have been presented for ways to map the graphical representations into numerical data. While some of these involve the use of eigenvalues for this purpose, we recently used a modified formulation [85] (called Height Yielding Matrix Numbers - HYMNs) also using eigenvalues. The "HYMNs" approach encodes the heights of various nodes into diagonal matrices. This approach called, "dressing," was introduced in work by Toppan et. al. [86–90]. In section 9.2, we present the definition and applications of the HYMNs.

Works by mathematicians [91,92] have connected adinkras to algebraic geometry and in particular Riemann surfaces. The concept of discrete Morse functions for oriented triangular meshes on Riemann surfaces was introduced into the mathematical literature some time ago [37]. Thus, nodal heights in adinkras are associated with values of these Morse functions. The latter are piecewise-linear over the meshes constructed by linear extension across edges and faces of the plaquettes associated with the meshes. The bottom line is the height assignment of any node in an adinkra corresponds to the integer value of the Morse function and the height assignment can be "Banchoff index" of the node. The fact that the adinkra contains many nodes leads to matrices of these indices that we call "Banchoff Matrices." The Height Yielding Matrix Numbers - HYMNs - are the eigenvalues of these matrices.

HYMNs provide an intrinsic definition of the shape of an adinkra. In a vaguely reminiscent way of how the Riemann curvature tensor provides a definition of intrinsic curvature for hyper-surfaces, HYMNs provide an intrinsic way to define the shape of any adinkra.

Chapter 2

Linearized Nordström Supergravity in Eleven- and Ten-dimensional Superspaces

This chapter is based on [36].

As the full off-shell theories of supergravity in the important dimensions of eleven and ten are currently unknown, in this chapter, we introduce a superfield formalism as a foundation and experimental laboratory to explore the possibility that the scalar versions of the higher dimensional supergravitation theory can be constructed. We also present aspects of the component description of linearized Nordström Supergravity in eleven and ten dimensions. The presentation includes low order component fields in the supermultiplet, the supersymmetry variations of the scalar graviton and gravitino trace, their supercovariantized field strengths, and the supersymmetry commutator algebra of these theories.

We organize this chapter in the manner described below.

Section 2.1 provides a self-consistent introduction to the field-theory and gauge-theory based formulation of gravitation described solely by a metric in D dimensions. We use a frame field/spin-connection formulation from the beginning point of our discussion. This eases the transition to the case of superspace as for these latter theories it is an impossibility [93] to introduce a metric/Christoffel-Riemannian formulation in the context of a superspace geometry appropriate to supersymmetry. The restriction of the full frame field to retain only the degree of freedom associated with its determinant is presented along with:

(a.) the well-known vanishing of the Weyl tensor, and

(b.) the residual form of the Einstein-Hilbert action under this restriction.

Section 2.2 is a transitional one where we review 4D, $\mathcal{N}$ = 1 supergravity as a paradigm setting arena. We show how the structure of this superspace in this well-studied theory suggests pathways that can be pursued for how to carry out constructions of scalar supergravitation in all higher-dimensions including ten- and eleven-dimensional theories.

In section 2.3 through 2.6, we deploy the lessons found in the 2.1 to work in making respective proposals for linearized theories of scalar supergravitation in the 11D, $\mathcal{N}$ = 1, 10D, $\mathcal{N}$ = 2A, 10D, $\mathcal{N}$ = 2B, and 10D, $\mathcal{N}$ = 1 superspaces.

Section 2.7 is devoted to presentations of all the component level results implied by the previous four sections. Each respective higher-dimensional theory is treated as a subsection of this section.

Section 2.8 is a short section in comparison to the two that precede it. In 4D, $\mathcal{N}$ = 1 supergravity [94, 95], the concept of the "conformal compensator" was introduced some time ago. The first of these references is distinguished from the second in that the compensator is described by a chiral superfield, while in the second the alternative case where the compensator that is a complex linear superfield was introduced. It is the latter case which is relevant to our exploration. However, we will demonstrate evidence that chiral superfields exist for the 10D, $\mathcal{N}$ = 2B superspace. This is unique among superspaces of eleven and ten dimensions. However, we also present evidence that though such chiral superfields appear to consistently exist, the linearized Nordström superspace is such that a chiral superfield of this type cannot be used as a compensator.

2.1 Gauge Theory Perspective On Ordinary Gravity

The traditional geometrical approach to describing gravity can be regarded as having driven an apparent wedge between general relativity and theory of elementary particles.

Instead, a gauge theory and field theory based point of view provides a logical foundation for gravity which permits an alternative to geometry.

For gravitational theories in D dimensions, the gauge group can be taken as the Poincaré group, and the Lie algebra generators are momentum $P_{\underline{m}} = -i\partial_{\underline{m}}$ and spin angular momentum generator $\mathcal{M}_{\underline{ab}}$. These are taken to satisfy the following commutation relations,

$$[\, P_{\underline{m}} \, , P_{\underline{n}} \,] \;=\; 0 \quad , \quad [\, \mathcal{M}_{\underline{ab}} \, , P_{\underline{m}} \,] \;=\; 0 \quad , \quad [\, \mathcal{M}_{\underline{ab}} \, , \partial_{\underline{c}} \,] \;=\; \eta_{\underline{ca}}\partial_{\underline{b}} \;-\; \eta_{\underline{cb}}\partial_{\underline{a}} \quad , \tag{2.1.1}$$

$$[\, \mathcal{M}_{\underline{ab}} \, , \mathcal{M}_{\underline{cd}} \,] \;=\; \eta_{\underline{ca}}\mathcal{M}_{\underline{bd}} \;-\; \eta_{\underline{cb}}\mathcal{M}_{\underline{ad}} \;-\; \eta_{\underline{da}}\mathcal{M}_{\underline{bc}} \;+\; \eta_{\underline{db}}\mathcal{M}_{\underline{ac}} \quad , \tag{2.1.2}$$

and it might appear that the definition $P_{\underline{m}} = -i\partial_{\underline{m}}$ together with the second and third equations among (2.1) are in contradiction. The resolution to this conundrum is to note

$$\partial_{\underline{a}} \;\equiv\; \delta_{\underline{a}}{}^{\underline{m}}\, \partial_{\underline{m}} \quad , \tag{2.1.3}$$

and the factor of $\delta_{\underline{a}}{}^{\underline{m}}$ actually corresponds to the vacuum value of the inverse frame field $e_{\underline{a}}{}^{\underline{m}}$ whose first index transforms under the action of $\mathcal{M}_{\underline{ab}}$ and whose second index is inert under the action of the spin angular momentum generator. To distinguish between these two types of quantities, we use the "early" latin letters, $\underline{a}$, $\underline{b}$, etc. to denote indices that transform under the action of $\mathcal{M}_{\underline{ab}}$. Similarly, we use the "late" latin letters, $\underline{m}$, $\underline{n}$, etc. to denote indices that do *not* transform under the action of $\mathcal{M}_{\underline{ab}}$.

The covariant derivative with respect to this gauge group is

$$\nabla_{\underline{a}} \;\equiv\; e_{\underline{a}}{}^{\underline{m}}\partial_{\underline{m}} \;+\; \tfrac{1}{2}\omega_{\underline{ac}}{}^{\underline{d}}\mathcal{M}_{\underline{d}}{}^{\underline{c}} \quad , \tag{2.1.4}$$

where $e_{\underline{a}}{}^{\underline{m}}$ is related to the metric through its inverse $e_{\underline{m}}{}^{\underline{a}}$ via $g_{\underline{mn}} = e_{\underline{m}}{}^{\underline{a}}\eta_{\underline{ab}}e_{\underline{n}}{}^{\underline{b}}$. The commutator of $\nabla_{\underline{a}}$ generates field strengths torsions $T_{\underline{abc}}$ and curvatures $R_{\underline{abcd}}$

$$[\, \nabla_{\underline{a}} \;,\; \nabla_{\underline{b}} \,] \;=\; T_{\underline{ab}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; \tfrac{1}{2}R_{\underline{abc}}{}^{\underline{d}}\mathcal{M}_{\underline{d}}{}^{\underline{c}} \quad . \tag{2.1.5}$$

Scalar gravitation can be defined by restricting the form of the inverse frame field to

$$e_{\underline{a}}{}^{m} \;=\; \psi\, \delta_{\underline{a}}{}^{m} \quad , \tag{2.1.6}$$

where ψ is a finite scalar field. By definition, this defines a class of geometries that is conformally flat in the context of strictly Riemannian spaces. To see this we begin by setting $T_{\underline{abc}} = 0$, which implies

$$\nabla_{\underline{a}} \;=\; \psi\,[\, \partial_{\underline{a}} \;-\; (\partial_{\underline{b}} \ln \psi)\mathcal{M}_{\underline{a}}{}^{\underline{b}}\,] \quad . \tag{2.1.7}$$

and allows the full Riemann curvature tensor to be solely expressed in terms of the ψ field as

$$R_{\underline{ab}}{}^{\underline{cd}} \;=\; -\psi(\partial_{[\underline{a}}\partial^{[\underline{c}}\psi)\delta_{\underline{b}]}{}^{\underline{d}]} \;+\; (\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi)\delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}} \quad , \tag{2.1.8}$$

similarly for the Ricci curvature we find

$$R_{\underline{a}}{}^{\underline{c}} \;=\; R_{\underline{ab}}{}^{\underline{cb}} \;=\; -(D-2)\psi(\partial_{\underline{a}}\partial^{\underline{c}}\psi) - \psi(\Box\psi)\delta_{\underline{a}}{}^{\underline{c}} \;+\; (D-1)(\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi)\delta_{\underline{a}}{}^{\underline{c}} \quad , \tag{2.1.9}$$

and finally for the curvature scalar by $R = \delta_{\underline{c}}{}^{\underline{a}} R_{\underline{a}}{}^{\underline{c}}$,

$$R \;=\; -2(D-1)\psi(\Box\psi) \;+\; D(D-1)(\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi) \quad . \tag{2.1.10}$$

The Weyl tensor $\mathcal{C}_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}}$ is defined by the equation

$$\mathcal{C}_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \;=\; R_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \;-\; \left[\frac{1}{D-2}\right] R_{[\underline{a}}{}^{[\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}]} \;+\; \left[\frac{1}{(D-2)(D-1)}\right]\delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}}\, R \quad , \tag{2.1.11}$$

and when the results in (2.1.8) - (2.1.10) are used, this is found to vanish.

We define $e \equiv \det(e_{\underline{a}}{}^{\underline{m}}) = \psi^D$ and the Einstein-Hilbert action takes the form

$$\begin{aligned} S_{EH} &= \frac{3}{\kappa^2}\int d^Dx\, e^{-1}R(\psi) = \frac{3}{\kappa^2}\int d^Dx\, \psi^{-D}\ \big[-2(D-1)\psi(\Box\psi) + D(D-1)(\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi)\big] \\ &= \frac{3}{\kappa^2}\int d^Dx\ \big[-2(D-1)\,\psi^{1-D}\,(\Box\psi) + D(D-1)\,\psi^{-D}\,(\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi)\big] \\ &= -\frac{3}{\kappa^2}\,(D-1)\int d^Dx\,\big\{\,(D-2)\,\big[\,\psi^{-D}\,(\partial^{\underline{e}}\psi)(\partial_{\underline{e}}\psi)\big]\ +\ 2\,\partial^{\underline{e}}\big[\,\psi^{1-D}\,(\partial_{\underline{e}}\psi)\big]\,\big\}\quad . \end{aligned} \tag{2.1.12}$$

As the full off-shell description of 10D and 11D supergravities are yet unknown, we work with a toy model - scalar supergravity in the higher dimensions, which we expect gives part of the complete solutions. In the subsequent sections, we replace ψ by $1+\Psi$, where Ψ is an infinitesimal superfield, and study the corresponding linearized supergravity.

2.2 Nordström Supergravity in 4D, $\mathcal{N}=1$ Supergeometry

As a preparatory step for our eventual goals, it is important that we re-visit four-dimensional $\mathcal{N}=1$ linearized supergravity as there are important lessons to be gained from asking questions solely in this domain prior to making the leap to eleven and ten dimensions. The formulation of linearized 4D, $\mathcal{N}=1$ supergravity in term of the usual supergravity pre-potential $H^{\underline{a}}$ was identified long ago [96]. It is perhaps of importance to note that supergravity pre-potentials bare some resemblance to other better-known concepts important for the mathematical description of theories describing ordinary gravitation.

One of the most computationally enabling formulations of the dynamics of ordinary gravitation is based on the Arnowitt-Deser-Meisner (ADM) formulation [97] wherein the quadratic form involving the metric is expressed in the form

$$\begin{aligned} dx^{\underline{m}}\, g_{\underline{m}\,\underline{n}}\, dx^{\underline{n}} \ &=\ \Big(-N^2\ +\ N_{\underline{i}}\delta^{\underline{i}\,\underline{j}}N_{\underline{j}}\Big)\,dt\otimes dt\ +\ N_{\underline{i}}\,\big(\,dt\otimes dx^{\underline{i}}\ +\ dx^{\underline{i}}\otimes dt\,\big) \\ &\quad +\ \mathrm{G}_{\underline{i}\,\underline{j}}\, dx^{\underline{i}}dx^{\underline{j}}\quad , \end{aligned} \tag{2.2.1}$$

in terms of the "lapse" function N, "shift" vector $N_{\underline{i}}$, and induced 3-metric $\mathrm{G}_{\underline{i}\,\underline{j}}$. For the equation above to be valid, we can write $dx^{\underline{m}} = \big(c\,dt,\ dx^{\underline{1}},\ dx^{\underline{2}},\ dx^{\underline{3}} \big)$ and

$$g_{\underline{m}\,\underline{n}} = \begin{bmatrix} \frac{1}{c^2}\,[\,-N^2 \,+\, N_{\underline{i}}\delta^{\underline{i}\,\underline{j}}N_{\underline{j}}\,] & \frac{1}{c}\,N_{\underline{1}} & \frac{1}{c}\,N_{\underline{2}} & \frac{1}{c}\,N_{\underline{3}} \\ \frac{1}{c}\,N_{\underline{1}} & \mathrm{G}_{\underline{1}\,\underline{1}} & \mathrm{G}_{\underline{1}\,\underline{2}} & \mathrm{G}_{\underline{13}} \\ \frac{1}{c}\,N_{\underline{2}} & \mathrm{G}_{\underline{2}\,\underline{1}} & \mathrm{G}_{\underline{2}\,\underline{2}} & \mathrm{G}_{\underline{2}\,\underline{3}} \\ \frac{1}{c}\,N_{\underline{3}} & \mathrm{G}_{\underline{3}\,\underline{1}} & \mathrm{G}_{\underline{3}\,\underline{2}} & \mathrm{G}_{\underline{3}\,\underline{3}} \end{bmatrix} \quad . \tag{2.2.2}$$

The introduction of frame fields can be accomplished by observing that the quadratic form in (2.2.1) may also be written as

$$dx^{\underline{m}}\, g_{\underline{m}\,\underline{n}}\, dx^{\underline{n}} \;\equiv\; dx^{\underline{m}}\, e_{\underline{m}}{}^{\underline{a}}\, \eta_{\underline{a}\underline{b}}\, e_{\underline{n}}{}^{\underline{b}}\, dx^{\underline{n}} \quad , \tag{2.2.3}$$

by "factorizing" the metric into the product of two frame fields $e_{\underline{m}}{}^{\underline{a}}(x)$ and $e_{\underline{n}}{}^{\underline{b}}(x)$ multiplied by the constant Minkowski metric, $\eta_{\underline{a}\underline{b}}$, of flat spacetime. Thus, there exist relations between the ADM variables and the frame fields [98, 99].

The point of the above discussion is to note that the inverse frame fields $e_{\underline{a}}{}^{\underline{m}}$ may be regarded as functions of the ADM variables, i. e.

$$e_{\underline{a}}{}^{\underline{m}} \;=\; e_{\underline{a}}{}^{\underline{m}}\big(N,\, N_{\underline{i}},\, \mathrm{G}_{\underline{i}\,\underline{j}}\big) \quad , \tag{2.2.4}$$

and that for numerical relativity calculations, the latter are far more useful than the frame fields $e_{\underline{m}}{}^{\underline{a}}$, or even the metric $g_{\underline{m}\,\underline{n}}$ itself. As we will see later, it is the form of 4D, $\mathcal{N} = 1$ supergravity often called the "Breitenlohner auxiliary field set" that is relevant to our work. For this formulation, it was first shown in the work of [95] the super-frame superfields $E_A{}^M$ are expressed in terms of a more fundamental set of superfields, i.e. the prepotentials $H^{\underline{m}}$ and Ψ (with the "conformal compensator" explicitly dependent upon a complex linear superfield). In an "echo" of the utility of the ADM variables, the prepotentials are far more useful when component calculations, or quantum calculations are undertaken, with the latter able to utilize the technology of super Feynman graphs.

As in the discussion of section 7.5 in [100], we write (with X being the superfield linearization of Ψ)

$$
\begin{aligned}
\mathrm{E}_\alpha &= \mathrm{D}_\alpha + \overline{X}\mathrm{D}_\alpha + i\frac{1}{2}(\mathrm{D}_\alpha H^{\underline{b}})\partial_{\underline{b}} \quad , \\
\mathrm{E}_{\dot\alpha} &= \overline{\mathrm{D}}_{\dot\alpha} + X\overline{\mathrm{D}}_{\dot\alpha} - i\frac{1}{2}(\overline{\mathrm{D}}_{\dot\alpha} H^{\underline{b}})\partial_{\underline{b}} \quad ,
\end{aligned}
\tag{2.2.5}
$$

$$
\begin{aligned}
\mathrm{E}_{\underline{a}} &= \partial_{\underline{a}} + i\Big[\frac{1}{2}\overline{\mathrm{D}}^2\mathrm{D}_{(\alpha}H^{\gamma)}{}_{\dot\alpha} - (\overline{\mathrm{D}}_{\dot\alpha}\overline{X})\delta_\alpha{}^\gamma\Big]\mathrm{D}_\gamma + i\Big[-\frac{1}{2}\mathrm{D}^2\overline{\mathrm{D}}_{(\dot\alpha}H_\alpha{}^{\dot\gamma)} - (\mathrm{D}_\alpha X)\delta_{\dot\alpha}{}^{\dot\gamma}\Big]\overline{\mathrm{D}}_{\dot\gamma} \\
&\quad + \Big[-\frac{1}{2}(\,[\mathrm{D}_\alpha\,,\,\overline{\mathrm{D}}_{\dot\alpha}]H^{\underline{b}}) + (X+\overline{X})\delta_{\underline{a}}{}^{\underline{b}}\Big]\partial_{\underline{b}} \quad .
\end{aligned}
\tag{2.2.6}
$$

for the linearized superframe superfields. Similar to the ADM formulation of ordinary gravity, the superframe is expressed in terms of two independent superfields, $H^{\underline{a}}$, and X. The remaining structures needed to specify the supergravity supercovariant derivatives are the spin-connections which here take the forms

$$
\begin{aligned}
\Phi_{\alpha\beta\gamma} &= -\,C_{\alpha(\beta}\mathrm{D}_{\gamma)}\overline{X} \quad , \\
\Phi_{\alpha\dot\beta\dot\gamma} &= \frac{1}{2}\mathrm{D}^2\overline{\mathrm{D}}_{(\dot\beta|}H_{\alpha|\dot\gamma)} \quad , \\
\Phi_{\dot\alpha\beta\gamma} &= -\,\frac{1}{2}\overline{\mathrm{D}}^2\mathrm{D}_{(\beta}H_{\gamma)\dot\alpha} \quad , \\
\Phi_{\underline{a}\beta\gamma} &= i\frac{1}{2}\mathrm{D}_\alpha\overline{\mathrm{D}}^2\mathrm{D}_{(\beta}H_{\gamma)\dot\alpha} + iC_{\alpha(\beta|}\overline{\mathrm{D}}_{\dot\alpha}\mathrm{D}_{|\gamma)}\overline{X} \quad .
\end{aligned}
\tag{2.2.7}
$$

The superfield X introduced above is a general scalar superfield. This implies that the linearized formulation described above *is* reducible. There are two widely familiar choices that lead to irreducibility. One choice is implemented by picking X to depend on $H^{\underline{a}}$, and a chiral superfield ϕ (i. e. $\overline{\mathrm{D}}_{\dot\alpha}\phi = 0$). This is the path that leads to the minimal off-shell formulation of 4D, $\mathcal{N} = 1$ supergravity. For this choice, the commutator algebra of the superspace supergravity covariant derivatives takes the forms

$$
\begin{aligned}
[\nabla_\alpha, \nabla_\beta\} &= -2\overline{R}\,\mathcal{M}_{\alpha\beta} \quad , \qquad [\nabla_\alpha, \overline{\nabla}_{\dot\alpha}\} \;=\; i\nabla_{\underline{a}} \quad , \\
[\nabla_\alpha, \nabla_{\underline{b}}\} &= -i\,C_{\alpha\beta}\Big[\,\overline{R}\,\overline{\nabla}_{\dot\beta} \;-\; G^{\gamma}{}_{\dot\beta}\nabla_\gamma\,\Big] \;-\; i\,(\overline{\nabla}_{\dot\beta}\overline{R})\mathcal{M}_{\alpha\beta} \\
&\quad +\; i\,C_{\alpha\beta}\Big[\,\overline{W}_{\dot\beta\dot\gamma}{}^{\dot\delta}\,\overline{\mathcal{M}}_{\dot\delta}{}^{\dot\gamma} \;-\; (\nabla^{\delta}G_{\gamma\dot\beta})\mathcal{M}_{\delta}{}^{\gamma}\,\Big] \quad , \\
[\nabla_{\underline{a}}, \nabla_{\underline{b}}\} &= \Big\{\;\Big[\;C_{\dot\alpha\dot\beta}W_{\alpha\beta}{}^{\gamma} \;+\; \tfrac{1}{2}C_{\alpha\beta}(\overline{\nabla}_{(\dot\alpha}G^{\gamma}{}_{\dot\beta)}) \;-\; \tfrac{1}{2}C_{\dot\alpha\dot\beta}\,(\nabla_{(\alpha}R)\delta_{\beta)}{}^{\gamma}\;\Big]\nabla_\gamma \\
&\quad +\; i\tfrac{1}{2}C_{\alpha\beta}G^{\gamma}{}_{(\dot\alpha|}\nabla_{\gamma|\dot\beta)} \\
&\quad -\; \Big[\;C_{\dot\alpha\dot\beta}\Big(\nabla_\alpha W_{\beta\delta\gamma} \;+\; \tfrac{1}{2}C_{\delta(\alpha}C_{\beta)\gamma}\,(\overline{\nabla}^2\overline{R} \;+\; 2R\overline{R})\;\Big) \\
&\quad -\; \tfrac{1}{2}C_{\alpha\beta}\,(\overline{\nabla}_{(\dot\alpha|}\nabla_\gamma G_{\delta|\dot\beta)})\;\Big]\mathcal{M}^{\gamma\delta}\;\Big\} \;+\; \text{h.c.} \quad .
\end{aligned}
\tag{2.2.8}
$$

The other widely known choice "the Breitenlohner auxiliary field set" is implemented by picking X to depend on $H^{\underline{a}}$, and a complex linear superfield Σ (i. e. $\overline{\mathrm{D}}^2\Sigma = 0$). This is the path that leads to the non-minimal off-shell formulation of 4D, $\mathcal{N} = 1$ supergravity. For this choice, the commutator algebra of the superspace supergravity covariant derivatives takes the forms

$$
\begin{aligned}
[\nabla_\alpha \;,\; \nabla_\beta\} \;&=\; \frac{1}{2}T_{(\alpha}\nabla_{\beta)} \;-\; 2\,\overline{R}\mathcal{M}_{\alpha\beta} \quad , \\
[\nabla_\alpha \;,\; \overline{\nabla}_{\dot\beta}\} \;&=\; i\,\nabla_{\alpha\dot\beta} \quad , \\
[\nabla_\alpha \;,\; \nabla_{\underline{b}}\} \;&=\; \frac{1}{2}\,T_\beta\nabla_{\alpha\dot\beta} \;-\; iC_{\alpha\beta}\Big[\overline{R} \;+\; \frac{1}{4}(\nabla^{\gamma}T_\gamma)\Big]\overline{\nabla}_{\dot\beta} \\
&\quad +\; i\,\Big[\;C_{\alpha\beta}\,G^{\gamma}{}_{\dot\beta} \;-\; \frac{1}{2}C_{\alpha\beta}\big((\nabla^{\gamma} \;+\; \tfrac{1}{2}T^{\gamma})\overline{T}_{\dot\beta}\big) \;+\; \frac{1}{2}(\,\overline{\nabla}_{\dot\beta}T_\beta\,)\delta_\alpha{}^{\gamma}\;\Big]\nabla_\gamma \\
&\quad -\; i\,\Big[\;C_{\alpha\beta}\,(\nabla^{\gamma}G_{\delta\dot\beta})\mathcal{M}_{\gamma}{}^{\delta} \;+\; \big((\overline{\nabla}_{\dot\beta} - \overline{T}_{\dot\beta})\overline{R}\,\big)\mathcal{M}_{\alpha\beta}\;\Big] \\
&\quad +\; i\,C_{\alpha\beta}\,\Big[\;\overline{W}_{\dot\beta\dot\gamma}{}^{\dot\delta}\,\overline{\mathcal{M}}_{\dot\delta}{}^{\dot\gamma} \;+\; \frac{1}{6}\,\big(\nabla^{\delta}\,(\nabla_\delta \;+\; \tfrac{1}{2}T_\delta\,)\,\overline{T}_{\dot\gamma}\,\big)\,\overline{\mathcal{M}}_{\dot\beta}{}^{\dot\gamma} \;+\; \frac{1}{3}\,\overline{R}\,\overline{T}_{\dot\gamma}\,\overline{\mathcal{M}}_{\dot\beta}{}^{\dot\gamma}\;\Big] \quad .
\end{aligned}
\tag{2.2.9}
$$

The final commutator $[\nabla_{\underline{a}}\ ,\ \nabla_{\underline{b}}\}$ is found from equation (2.2.9) to be explicitly

$$
\begin{aligned}
[\nabla_{\underline{a}}\ ,\ \nabla_{\underline{b}}\} = \Big\{ & i\frac{1}{2}(\nabla_\beta \overline{T}_{\dot\beta})\nabla_{\underline{a}} \;-\; i\frac{1}{2}(\overline{\nabla}_{\dot\alpha} T_\beta)\nabla_{\alpha\dot\beta} \;-\; i\,C_{\dot\alpha\dot\beta}\Big[\, G_\beta{}^{\dot\gamma} \;+\; \frac{1}{2}\,(\,(\overline{\nabla}^{\dot\gamma} \;+\; \tfrac{1}{2}\overline{T}^{\dot\gamma})\,T_\beta\,)\,\Big]\nabla_{\alpha\dot\gamma} \\
& +\; \frac{1}{2}\Big[\,(\overline{\nabla}_{\dot\alpha}\overline{\nabla}_{\dot\beta}T_\beta) \;-\; \frac{1}{2}\overline{T}_{\dot\beta}\,(\overline{\nabla}_{\dot\alpha}T_\beta)\,\Big]\,\nabla_\alpha \\
& -\; C_{\dot\alpha\dot\beta}\Big[\,(\nabla_\alpha R) \;+\; \frac{1}{4}(\nabla_\alpha\overline{\nabla}^{\dot\gamma}\overline{T}_{\dot\gamma}) \;+\; \frac{1}{3}R\,T_\alpha \;-\; \frac{1}{12}\,(\,\overline{\nabla}^{\dot\gamma}\,(\overline{\nabla}_{\dot\gamma} \;-\; \overline{T}_{\dot\gamma})\,T_\alpha\,)\,\Big]\nabla_\beta \\
& +\; C_{\dot\alpha\dot\beta}\,W_{\beta\alpha}{}^{\gamma}\,\nabla_\gamma \;+\; C_{\alpha\beta}\Big[\,(\overline{\nabla}_{\dot\alpha}G^\gamma{}_{\dot\beta}) \;-\; \frac{1}{2}\overline{T}_{\dot\beta}\,G^\gamma{}_{\dot\alpha}\,\Big]\nabla_\gamma \;+\; \frac{1}{2}C_{\alpha\beta}\Big[\,\frac{1}{2}(\nabla^\gamma\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta} \\
& \qquad -\;(\,\overline{\nabla}_{\dot\alpha}\,(\nabla^\gamma \;+\; \tfrac{1}{2}T^\gamma)\,\overline{T}_{\dot\beta}\,) \;+\; \frac{1}{2}\,(\,(\overline{\nabla}_{\dot\alpha} \;+\; \tfrac{1}{2}\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta}\,)\,T^\gamma\,\Big]\nabla_\gamma \\
& +\; \Big[\,-\;(\overline{\nabla}_{\dot\alpha}\overline{\nabla}_{\dot\beta}\overline{R}) \;+\; \frac{1}{2}(\overline{\nabla}_{\dot\alpha}\overline{R})\,\overline{T}_{\dot\beta} \;+\; \overline{R}\,(\overline{\nabla}_{\dot\alpha} + \tfrac{1}{2}\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta}\,\Big]\mathcal{M}_{\alpha\beta} \\
& \qquad +\; 2\,C_{\dot\alpha\dot\beta}\,\overline{R}\Big[\,R \;+\; \frac{1}{4}(\overline{\nabla}^{\dot\gamma}\overline{T}_{\dot\gamma})\,\Big]\mathcal{M}_{\alpha\beta} \\
& -\; \frac{1}{6}C_{\dot\alpha\dot\beta}\Big[\,R\,T_\beta\,T_\gamma \;+\; \frac{1}{2}T_\beta\,(\overline{\nabla}^{\dot\delta}\overline{\nabla}_{\dot\delta}T_\gamma) \;+\; \frac{1}{4}T_\beta\,T_\gamma\,(\overline{\nabla}^{\dot\delta}\overline{T}_{\dot\delta})\,\Big]\mathcal{M}_\alpha{}^\gamma \\
& +\; \frac{1}{6}C_{\dot\alpha\dot\beta}\Big[\,2\,(\nabla_\alpha R)\,T_\gamma \;+\; 2\,R\,(\nabla_\alpha T_\gamma) \;+\; (\,\nabla_\alpha\overline{\nabla}^{\dot\delta}\,(\overline{\nabla}_{\dot\delta} \;+\; \tfrac{1}{2}\overline{T}_{\dot\delta})\,T_\gamma\,) \\
& \qquad +\; \tfrac{1}{2}(\overline{\nabla}^{\dot\delta}\overline{T}_{\dot\delta})\,(\nabla_\alpha T_\gamma)\,\Big]\mathcal{M}_\beta{}^\gamma \\
& -\; C_{\alpha\beta}\Big[\,(\overline{\nabla}_{\dot\alpha}\nabla^\gamma G_{\delta\dot\beta}) \;-\; \frac{1}{2}\overline{T}_{\dot\beta}\,(\nabla^\gamma G_{\delta\dot\alpha})\,\Big]\mathcal{M}_\gamma{}^\delta \\
& \qquad +\; C_{\dot\alpha\dot\beta}\Big[\,(\nabla_\alpha W_{\beta\delta}{}^\gamma) \;-\; \frac{1}{2}T_\beta\,W_{\alpha\delta}{}^\gamma\,\Big]\mathcal{M}_\gamma{}^\delta\Big\} \\
& +\; \text{h. c.} \quad .
\end{aligned}
\tag{2.2.10}
$$

Under either choice, one can use the definitions of the superframe superfields in (2.2.5) - (2.2.7) together with the set of equations of *either* (2.2.8) or (2.2.9) and (2.2.10) to find the dependence of $W_{\alpha\beta\gamma}$, $G_{\underline{a}}$, and $\overline{R}$ (for the minimal theory) on $H^{\underline{a}}$, and ϕ, or the dependence of $W_{\alpha\beta\gamma}$, $G_{\underline{a}}$, $\overline{R}$, and T_α on $H^{\underline{a}}$, and Σ (for the non-minimal theory). These are the standard and well discussed theories of off-shell 4D, $\mathcal{N} = 1$ supergravity, i. e. the consistency of the Bianchi identities associated with (2.2.8) or (2.2.9) and (2.2.10) for the algebra of the superspace supergravity covariant derivatives do *not* require on-shell conditions to be imposed on the component fields contained within the superfields.

The process of imposing the Einstein Field Equations in the non-supersymmetrical case in the absence of matter amounts to the condition

$$R_{\underline{a}\,\underline{b}} \;-\; \tfrac{1}{2}\eta_{\underline{a}\underline{b}}\,R \;=\; 0 \quad , \tag{2.2.11}$$

excluding the cosmological constant[1]. The equivalence in the case of superspace supergravity is accomplished by setting $G_{\underline{a}}$, and $\overline{R}$ (for the minimal theory) to zero or by setting $G_{\underline{a}}$, and T_α (for the non-minimal theory) to zero. The condition $T_\alpha = 0$ also forces $\overline{R} = 0$ in the non-minimal theory. Under these conditions, the algebra of superspace supergravity covariant derivatives takes the universal form

$$\begin{aligned}
[\nabla_\alpha, \nabla_\beta\} &= \;0 \quad , \quad [\nabla_\alpha, \overline{\nabla}_{\dot{\alpha}}\} \;=\; i\nabla_{\underline{a}} \quad , \\
[\nabla_\alpha, \nabla_{\underline{b}}\} &= \; i\,C_{\alpha\beta}[\,\overline{W}_{\dot{\beta}\dot{\gamma}}{}^{\dot{\delta}}\,\overline{\mathcal{M}}_{\dot{\delta}}{}^{\dot{\gamma}}\,] \quad , \\
[\nabla_{\underline{a}}\ ,\ \nabla_{\underline{b}}\} &= \; C_{\dot{\alpha}\dot{\beta}}\,W_{\beta\alpha}{}^{\gamma}\,\nabla_\gamma + \; C_{\dot{\alpha}\dot{\beta}}\,(\nabla_\alpha W_{\beta\gamma}{}^{\delta})\,\mathcal{M}_\delta{}^{\gamma} \;+\; \text{h. c.} \quad .
\end{aligned} \tag{2.2.12}$$

At this point, we can take a largely unexplored path as it is possible to consider the limit of these equations wherein $H^{\underline{a}} = 0$. This is the route to the 4D, $\mathcal{N} = 1$ superspace version of scalar supergravitation theory à la Nordström in the eleven and ten dimensions that are the targets of our study.

The curious reader may wonder from where does the condition $H^{\underline{a}} = 0$ arise? On page 335 of [100] there appears the following text.

> *Nonsupersymmetric deSitter covariant derivatives can be obtained from gravitational covariant derivatives by eliminating all field components except the (density) compensating field (i.e., the determinant of the metric or vierbein). This follows from the fact that in deSitter space the Weyl tensor vanishes, which says that there is no conformal (spin 2) part to the metric: It is conformally flat.*

[1] The reason the cosmological constant can be ignored in our considerations is because unlike in four dimensions, there exists no evidence currently available in the literature that it is possible to construct spaces of constant curvature in ten or eleven dimensions with their respective superspaces.

From the discussion given in section 2.1, we saw that Nordström geometries in all dimensions are necessarily such as to describe Weyl tensors that vanish and are hence conformally flat.

Also in the work of [100] it is explained that the conformal part of the metric arises solely from $H^{\underline{a}}$. Since the Nordström limit is a conformally flat bosonic space, it must corresponds to setting $H^{\underline{a}} = 0$. Thus, to our knowledge, the passage above from "Superspace" marked the first indication of this. Of course, other authors such as in the work of [101] later reaffirmed this point about the structure of superspace of supergravity.

In the limit of our interest, we find

$$\begin{aligned} \mathrm{E}_\alpha &= \mathrm{D}_\alpha + \overline{X}\mathrm{D}_\alpha \quad , \quad \mathrm{E}_{\dot{\alpha}} = \overline{\mathrm{D}}_{\dot{\alpha}} + X\overline{\mathrm{D}}_{\dot{\alpha}} \quad , \\ \mathrm{E}_{\underline{a}} &= \partial_{\underline{a}} - i\Big[(\overline{\mathrm{D}}_{\dot{\alpha}}\overline{X})\delta_\alpha{}^\gamma\Big]\mathrm{D}_\gamma - i\Big[(\mathrm{D}_\alpha X)\delta_{\dot{\alpha}}{}^{\dot{\gamma}}\Big]\overline{\mathrm{D}}_{\dot{\gamma}} + \Big[(X+\overline{X})\delta_{\underline{a}}{}^{\underline{b}}\Big]\partial_{\underline{b}} \quad , \\ \Phi_{\alpha\beta\gamma} &= -\, C_{\alpha(\beta}\mathrm{D}_{\gamma)}\overline{X} \quad , \quad \Phi_{\alpha\dot{\beta}\dot{\gamma}} = 0 \quad , \quad \Phi_{\dot{\alpha}\beta\gamma} = 0 \quad , \\ \Phi_{\underline{a}\beta\gamma} &= iC_{\alpha(\beta|}\overline{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}_{|\gamma)}\overline{X} \quad . \end{aligned} \tag{2.2.13}$$

In response to this restriction, the forms of the algebras in (2.2.8) or (2.2.9) and (2.2.10) also change. In particular, the superfield $W_{\alpha\beta\gamma}$ (and consequently $W_{\alpha\beta\gamma\delta}$) is identically zero. The latter condition is consistent with the component level description of scalar gravitation in the previous section as the Weyl tensor of (2.1.10) is the leading component field that occurs in $W_{\alpha\beta\gamma\delta}$ and occurs at first order in the θ-expansion of $W_{\alpha\beta\gamma}$. The third result in (2.2.13) also contains two useful bits of information:

(a.) The final term of the equation informs us that the leading term in the θ-expansion of $X + \overline{X}$ corresponds to the linearization of ψ seen in equation (2.1.6).

(b.) The second term of the equation informs us that the leading term in the θ-expansion of $(\overline{\mathrm{D}}_{\dot{\alpha}}\overline{X})\delta_\alpha{}^\gamma$ corresponds to the spin-1/2 remnant of the gravitino!

Another point to discuss is the dependence of the field strength superfields $G_{\underline{a}}$, and $\overline{R}$ (for the minimal theory) and $G_{\underline{a}}$, $\overline{R}$, and T_α (for the non-minimal theory) on the superfield X. Direct calculation shows that the reality of $G_{\underline{a}}$ in both cases implies that it only depends on the difference $i(X - \overline{X})$. The superfield T_α is found to depend on the first spinor derivative (i. e. D_α) of X. Finally, the superfield $\overline{R}$ is found to depend on the second spinorial derivative of an expression linear in X and $\overline{X}$.

We have argued previously [56], the minimal supergravity theory does not extend from four dimensions to eleven dimensions since there is no concept of chirality in the higher dimensions. This implies that only the features seen in the non-minimal theory should be expected to occur in the subsequent sections of this work. As we shall see, this is indeed the case. The commutator algebra for the superspace supergravity covariant derivatives responds to the condition $H^{\underline{a}} = 0$, by the elimination of all terms proportional to $W_{\alpha\beta\gamma}$ and $W_{\alpha\beta\gamma\delta}$. Thus, we find 4D, $\mathcal{N}$ = 1 Nordström supergravity *that descends* from ten or eleven dimensions and only contains the generators associated with 4D, $\mathcal{N}$ = 1 simple supegravity is described by

$$
\begin{aligned}
[\nabla_\alpha \, , \, \nabla_\beta\} \; &= \frac{1}{2} T_{(\alpha} \nabla_{\beta)} \; - \; 2\, \overline{R} \mathcal{M}_{\alpha\beta} \quad , \\
[\nabla_\alpha \, , \, \overline{\nabla}_{\dot{\beta}}\} \; &= i \, \nabla_{\alpha\dot{\beta}} \quad , \\
[\nabla_\alpha \, , \, \nabla_{\underline{b}}\} \; &= \frac{1}{2}\, T_\beta \nabla_{\alpha\dot{\beta}} \; - \; i C_{\alpha\beta} \Big[\overline{R} \; + \; \frac{1}{4} (\nabla^\gamma T_\gamma) \Big] \overline{\nabla}_{\dot{\beta}} \\
&\quad + \; i \Big[\; C_{\alpha\beta} \, G^\gamma{}_{\dot{\beta}} \; - \; \frac{1}{2} C_{\alpha\beta} \big((\nabla^\gamma \; + \; \tfrac{1}{2} T^\gamma) \overline{T}_{\dot{\beta}} \big) \; + \; \frac{1}{2} (\, \overline{\nabla}_{\dot{\beta}} T_\beta \,) \delta_\alpha{}^\gamma \; \Big] \nabla_\gamma \\
&\quad - \; i \Big[\; C_{\alpha\beta} \, (\nabla^\gamma G_{\delta\dot{\beta}}) \mathcal{M}_\gamma{}^\delta \; + \; \big((\overline{\nabla}_{\dot{\beta}} - \overline{T}_{\dot{\beta}}) \overline{R} \big) \mathcal{M}_{\alpha\beta} \; \Big] \\
&\quad + \; i \, C_{\alpha\beta} \Big[\; \frac{1}{6} \big(\nabla^\delta \; (\nabla_\delta \; + \; \tfrac{1}{2} T_\delta \,) \, \overline{T}_{\dot{\gamma}} \, \big) \; \overline{\mathcal{M}}_{\dot{\beta}}{}^{\dot{\gamma}} \; + \; \frac{1}{3} \, \overline{R} \, \overline{T}_{\dot{\gamma}} \, \overline{\mathcal{M}}_{\dot{\beta}}{}^{\dot{\gamma}} \; \Big] \quad ,
\end{aligned}
$$

$$
\begin{aligned}
[\nabla_{\underline{a}}\,,\,\nabla_{\underline{b}}\} \;=\; & \Big\{\, i\frac{1}{2}\,(\nabla_\beta \overline{T}_{\dot\beta})\nabla_{\underline{a}} \;-\; i\frac{1}{2}\,(\overline{\nabla}_{\dot\alpha} T_\beta)\nabla_{\alpha\dot\beta} \;-\; i\,C_{\dot\alpha\dot\beta}\Big[\, G_\beta{}^{\dot\gamma} \;+\; \frac{1}{2}\,(\,(\overline{\nabla}^{\dot\gamma} \;+\; \tfrac{1}{2}\overline{T}^{\dot\gamma})\,T_\beta\,)\,\Big]\nabla_{\alpha\dot\gamma} \\
& +\; \frac{1}{2}\Big[\,(\overline{\nabla}_{\dot\alpha}\overline{\nabla}_{\dot\beta}T_\beta) \;-\; \frac{1}{2}\,\overline{T}_{\dot\beta}\,(\overline{\nabla}_{\dot\alpha}T_\beta)\,\Big]\,\nabla_\alpha \\
& -\; C_{\dot\alpha\dot\beta}\Big[\,(\nabla_\alpha R) \;+\; \frac{1}{4}\,(\nabla_\alpha\overline{\nabla}^{\dot\gamma}\overline{T}_{\dot\gamma}) \;+\; \frac{1}{3}\,R\,T_\alpha \;-\; \frac{1}{12}\,(\,\overline{\nabla}^{\dot\gamma}\,(\overline{\nabla}_{\dot\gamma} \;-\; \overline{T}_{\dot\gamma})\,T_\alpha\,)\,\Big]\,\nabla_\beta \\
& +\; C_{\alpha\beta}\Big[\,(\overline{\nabla}_{\dot\alpha}G^\gamma{}_{\dot\beta}) \;-\; \frac{1}{2}\,\overline{T}_{\dot\beta}\,G^\gamma{}_{\dot\alpha}\,\Big]\,\nabla_\gamma \;+\; \frac{1}{2}\,C_{\alpha\beta}\Big[\,\frac{1}{2}\,(\nabla^\gamma\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta} \\
& \qquad -\;(\,\overline{\nabla}_{\dot\alpha}\,(\nabla^\gamma \;+\; \tfrac{1}{2}T^\gamma)\,\overline{T}_{\dot\beta}\,) \;+\; \frac{1}{2}\,(\,(\overline{\nabla}_{\dot\alpha} \;+\; \tfrac{1}{2}\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta}\,)\,T^\gamma\,\Big]\,\nabla_\gamma \\
& +\; \Big[\,-\;(\overline{\nabla}_{\dot\alpha}\overline{\nabla}_{\dot\beta}\overline{R}) \;+\; \frac{1}{2}(\overline{\nabla}_{\dot\alpha}\overline{R})\,\overline{T}_{\dot\beta} \;+\; \overline{R}\,(\overline{\nabla}_{\dot\alpha} + \tfrac{1}{2}\overline{T}_{\dot\alpha})\,\overline{T}_{\dot\beta}\,\Big]\,\mathcal{M}_{\alpha\beta} \\
& \qquad +\; 2\,C_{\dot\alpha\dot\beta}\,\overline{R}\Big[\,R \;+\; \frac{1}{4}(\overline{\nabla}^{\dot\gamma}\overline{T}_{\dot\gamma})\,\Big]\,\mathcal{M}_{\alpha\beta} \\
& -\; \frac{1}{6}\,C_{\dot\alpha\dot\beta}\Big[\,R\,T_\beta\,T_\gamma \;+\; \frac{1}{2}\,T_\beta\,(\overline{\nabla}^{\dot\delta}\overline{\nabla}_{\dot\delta}T_\gamma) \;+\; \frac{1}{4}\,T_\beta\,T_\gamma\,(\overline{\nabla}^{\dot\delta}\overline{T}_{\dot\delta})\,\Big]\,\mathcal{M}_\alpha{}^\gamma \\
& +\; \frac{1}{6}\,C_{\dot\alpha\dot\beta}\Big[\,2\,(\nabla_\alpha R)\,T_\gamma \;+\; 2\,R\,(\nabla_\alpha T_\gamma) \;+\; (\,\nabla_\alpha\overline{\nabla}^{\dot\delta}\,(\overline{\nabla}_{\dot\delta} \;+\; \tfrac{1}{2}\overline{T}_{\dot\delta})\,T_\gamma\,) \\
& \qquad +\; \tfrac{1}{2}(\overline{\nabla}^{\dot\delta}\overline{T}_{\dot\delta})\,(\nabla_\alpha T_\gamma)\,\Big]\,\mathcal{M}_\beta{}^\gamma \\
& -\; C_{\alpha\beta}\Big[\,(\overline{\nabla}_{\dot\alpha}\nabla^\gamma G_{\delta\dot\beta}) \;-\; \frac{1}{2}\,\overline{T}_{\dot\beta}\,(\nabla^\gamma G_{\delta\dot\alpha})\,\Big]\,\mathcal{M}_\gamma{}^\delta\Big\} \\
& +\; \text{h. c.} \quad .
\end{aligned}
\tag{2.2.14}
$$

To our knowledge, the results in (2.2.14) mark the first time that a superspace description using non-minimal supergravity (one irreducible version of the Breitenlohner formulation) of 4D, $\mathcal{N} = 1$ Nordström supergravity has appeared in the literature. As we consider higher-dimensional theories, we set the cosmological constant to zero and the original non-minimal theory is sufficient for our purposes.

To summarize, the limit of off-shell 4D, $\mathcal{N} = 1$ superfield supergravity where we *only* retain the conformal compensator provides a superspace extension of the Nordström supergravitation theory. We will make a working assumption that such an approach is universally applicable to all superspaces. In particular, in the subsequent sections, we will apply this assumption to superspaces whose bosonic subspaces possess either eleven or ten dimensions.

2.3 Linearized Nordström Supergravity in 11D, $\mathcal{N} = 1$ Supergeometry

We begin our discussion by reviewing the work of [8, 9] where it was shown that the entire structure of the torsions, curvatures, and 4-form field strengths could be written in terms of a single superfield denoted by $W_{\underline{abcd}}$. Using the conventions of [56], we can write

$$
\begin{aligned}
T_{\alpha\beta}{}^{\underline{c}} &= i(\gamma^{\underline{c}})_{\alpha\beta} \quad , \quad F_{\alpha\beta\underline{cd}} = \tfrac{1}{2}(\gamma_{\underline{cd}})_{\alpha\beta} \quad , \quad F_{\underline{abc}\delta} = 0 \quad , \\
F_{\alpha\beta\gamma\delta} &= F_{\alpha\beta\gamma\underline{d}} = 0 \quad , \quad F_{\underline{cdef}} = W_{\underline{cdef}} \quad , \\
T_{\alpha\beta}{}^{\gamma} &= 0 \quad , \quad T_{\alpha\underline{b}}{}^{\underline{c}} = 0 \quad , \\
T_{\alpha\underline{b}}{}^{\gamma} &= i\frac{1}{144}(\gamma_{\underline{b}}{}^{\underline{cdef}} + 8\delta_{\underline{b}}{}^{\underline{c}}\gamma^{\underline{def}})_{\alpha}{}^{\gamma}W_{\underline{cdef}} \quad , \\
R_{\alpha\beta\underline{cd}} &= \frac{1}{3}(\gamma^{\underline{ef}})_{\alpha\beta}W_{\underline{cdef}} + \frac{1}{72}(\gamma_{\underline{cd}}{}^{\underline{efgh}})_{\alpha\beta}W_{\underline{efgh}} \quad .
\end{aligned}
\tag{2.3.1}
$$

In addition to the torsion and curvature supertensors, the formulation above includes the 4-form supertensor, F_{ABCD}. It should be noted that these equations in (2.3.1) are the eleven-dimensional analog of the equations in (2.2.12). In other words, the supergeometry in (2.3.1) is an "on-shell" supergeometry. We must find a supergeometry consistent with the Norström theory as the analog of (2.2.14).

We now wish to construct the linearized torsion and curvature supertensors with the property that when all fermions are set to zero, the theory smoothly maps to the linearization of the non-supersymmetrical theory described in section 2.1.

For this purpose, we introduce eleven-dimensional supergravity covariant derivatives linear in the infinitesimal conformal compensator Ψ given by

$$
\nabla_{\alpha} = \mathrm{D}_{\alpha} + \frac{1}{2}\Psi\mathrm{D}_{\alpha} + l_0(\mathrm{D}_{\beta}\Psi)(\gamma^{\underline{de}})_{\alpha}{}^{\beta}\mathcal{M}_{\underline{de}} \quad , \tag{2.3.2}
$$

$$
\begin{aligned}
\nabla_{\underline{a}} = {} & \partial_{\underline{a}} + \Psi\partial_{\underline{a}} + il_1(\gamma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_{\alpha}\Psi)\mathrm{D}_{\beta} + l_2(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \\
& + il_3(\gamma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta}(\mathrm{D}_{\alpha}\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{de}} \quad ,
\end{aligned}
\tag{2.3.3}
$$

where the "bare" superderivative operators D_{α} satisfy

$$
\{\mathrm{D}_{\alpha}, \mathrm{D}_{\beta}\} = i(\gamma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}} \quad , \tag{2.3.4}
$$

and the torsion tensors and Riemann curvature tensors can be obtained via

$$\begin{aligned}[][\nabla_\alpha, \nabla_\beta\} &= T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\beta}{}^{\gamma}\nabla_\gamma + \frac{1}{2}R_{\alpha\beta\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\nabla_\alpha, \nabla_{\underline{b}}\} &= T_{\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\alpha\underline{b}}{}^{\gamma}\nabla_\gamma + \frac{1}{2}R_{\alpha\underline{b}\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\nabla_{\underline{a}}, \nabla_{\underline{b}}\} &= T_{\underline{a}\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} + T_{\underline{a}\underline{b}}{}^{\gamma}\nabla_\gamma + \frac{1}{2}R_{\underline{a}\underline{b}\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad . \end{aligned} \tag{2.3.5}$$

The commutation relations of the operators with the 11D Lorentz generators satisfy

$$[\mathcal{M}_{\underline{a}\underline{b}}, \mathrm{D}_\alpha\} \;=\; \frac{1}{2}(\gamma_{\underline{a}\underline{b}})_\alpha{}^\beta \mathrm{D}_\beta \quad , \tag{2.3.6}$$

$$\big[\mathcal{M}_{\underline{a}\underline{b}}, [\mathcal{M}_{\underline{c}\underline{d}}, \mathrm{D}_\alpha\}\big\} + \big[\mathcal{M}_{\underline{c}\underline{d}}, [\mathrm{D}_\alpha, \mathcal{M}_{\underline{a}\underline{b}}\}\big\} + \big[\mathrm{D}_\alpha, [\mathcal{M}_{\underline{a}\underline{b}}, \mathcal{M}_{\underline{c}\underline{d}}\}\big\} \;=\; 0 \quad , \tag{2.3.7}$$

in addition to the relations seen in (2.1.1) and (2.1.2).

By imposing the constraints

$$T_{\underline{a}\underline{b}}{}^{\underline{c}} \;=\; 0 \quad , \quad T_{\alpha\beta}{}^{\underline{c}} \;=\; i(\gamma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.3.8}$$

we obtain the following parameterization results

$$l_0 \;=\; \frac{1}{10} \quad , \quad l_1 \;=\; \frac{1}{4} \quad , \quad l_2 \;=\; -1 \quad , \quad l_3 \;=\; 0 \quad . \tag{2.3.9}$$

In turn, these lead to a set of results that express the torsion and curvature tensors solely in terms of Ψ and its derivatives. We give these in the following.

For the components of the torsion, we find the results seen in (2.3.10) - (2.3.15).

$$T_{\alpha\beta}{}^{\underline{c}} \;=\; i(\gamma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.3.10}$$

$$T_{\alpha\beta}{}^{\gamma} \;=\; \frac{3}{40}(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad , \tag{2.3.11}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} \;=\; \frac{3}{4}\delta_{\underline{b}}{}^{\underline{c}}(\mathrm{D}_\alpha\Psi) + \frac{9}{20}(\gamma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi) \quad , \tag{2.3.12}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\gamma} \;=\; & i\frac{1}{128}\Big[-(\gamma_{\underline{b}})_\alpha{}^\gamma C^{\delta\epsilon} + \frac{1}{2}(\gamma^{[2]})_\alpha{}^\gamma(\gamma_{\underline{b}[2]})^{\delta\epsilon} - \frac{1}{3!}(\gamma_{\underline{b}[3]})_\alpha{}^\gamma(\gamma^{[3]})^{\delta\epsilon} + \frac{1}{3!}(\gamma^{[3]})_\alpha{}^\gamma(\gamma_{\underline{b}[3]})^{\delta\epsilon} \\ & - \frac{1}{4!}(\gamma_{\underline{b}[4]})_\alpha{}^\gamma(\gamma^{[4]})^{\delta\epsilon}\Big](\mathrm{D}_\delta\mathrm{D}_\epsilon\Psi) + \frac{1}{8}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\Psi) + \frac{3}{8}(\gamma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\Psi) \quad , \end{aligned} \tag{2.3.13}$$

$$T_{\underline{a}\underline{b}}{}^{\underline{c}} \;=\; 0 \quad , \tag{2.3.14}$$

$$T_{\underline{ab}}{}^{\gamma} \; = \; - \, i\frac{1}{4}(\gamma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\mathrm{D}_{\delta}\Psi) \quad . \tag{2.3.15}$$

For the components of the curvature, we find the results seen in (2.3.16) - (2.3.18).

$$\begin{aligned} R_{\alpha\beta}{}^{\underline{de}} \; &= \; \frac{1}{80}\Big[(\gamma^{\underline{de}})_{\alpha\beta}C^{\gamma\delta} + (\gamma_{[1]})_{\alpha\beta}(\gamma^{[1]\underline{de}})^{\gamma\delta} - \frac{1}{2}(\gamma_{[2]})_{\alpha\beta}(\gamma^{[2]\underline{de}})^{\gamma\delta} \\ &\qquad - \frac{1}{3!}(\gamma^{\underline{de}[3]})_{\alpha\beta}(\gamma_{[3]})^{\gamma\delta} \\ &\qquad + \frac{1}{5!4!}\epsilon^{\underline{de}[5][4]}(\gamma_{[5]})_{\alpha\beta}(\gamma_{[4]})^{\gamma\delta}\Big](\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi) \quad , \end{aligned} \tag{2.3.16}$$

$$R_{\alpha\underline{b}}{}^{\underline{de}} \; = \; - \, (\partial^{[\underline{d}}\mathrm{D}_{\alpha}\Psi)\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\gamma^{\underline{de}})_{\alpha}{}^{\delta}(\partial_{\underline{b}}\mathrm{D}_{\delta}\Psi) \quad , \tag{2.3.17}$$

$$R_{\underline{ab}}{}^{\underline{de}} \; = \; - \, (\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{2.3.18}$$

In reaching (2.3.10) - (2.3.18), we used the Fierz identities listed in Appendix A.3.

It is the last equation that ensures that we have reached our goal. Namely, the choice of constraints in (2.3.9) has led to a linearized super Riemann curvature tensor expressed solely in terms of an infinitesimal superfield Ψ that has the exact form of the first term in the non-supersymmetrical Riemann curvature tensor given in (2.1.8). Recall that the supersymmetrical theory here is linearized, so to make a proper comparison to the bosonic theory, that should also be linearized. When this is done, there is a matching of the terms.

We should note the work in [56] also constructs a fully non-linear 11D supergeometry in terms of a finite scalar compensator. However, its linearization is different from the one obtained here. In the next three sections, we will obtain new and never before presented results of this nature for the 10D, $\mathcal{N}$ = 2A, 10D, $\mathcal{N}$ = 2B, and 10D, $\mathcal{N}$ = 1 supergeometries that possess the purely bosonic linearized results as in the linearization of (2.1.8). The Fierz identities used for simplifying the torsions and curvatures in ten dimensions are listed in Appendix A.2.

2.4 Linearized Nordström Supergravity in 10D, $\mathcal{N}$ = 1 Supergeometry

We begin this discussion by pointing out the on-shell description of 10D, $\mathcal{N}$ = 1 superspace supergravity. A set of torsion and curvature supertensors can be written in the form

$$
\begin{aligned}
T_{\alpha\beta}{}^{\underline{c}} &= i\,(\sigma^{\underline{c}})_{\alpha\beta} \quad , \quad T_{\alpha\underline{b}}{}^{\underline{c}} \;=\; 0 \quad , \\
T_{\alpha\beta}{}^{\gamma} &= -\,\tfrac{1}{2}\sqrt{\tfrac{1}{2}}\left[\delta_{(\alpha}{}^{\gamma}\,\delta_{\beta)}{}^{\epsilon} \;+\; (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\,\epsilon}\right]\chi_{\epsilon} \quad , \\
T_{\alpha\underline{b}}{}^{\gamma} &= -\tfrac{1}{24}(\sigma_{\underline{b}}\,\sigma^{\underline{cde}})_{\alpha}{}^{\gamma}\left[e^{\Phi}\,H_{\underline{cde}} - i\,\tfrac{1}{8}\,(\,\chi\,\sigma_{\underline{cde}}\,\chi)\right] \\
&\quad -\,\tfrac{1}{48}(\sigma^{\underline{cde}}\,\sigma_{\underline{b}})_{\alpha}{}^{\gamma}\left[e^{\Phi}\,H_{\underline{cde}} - i\,\tfrac{1}{16}\,(\,\chi\,\sigma_{\underline{cde}}\,\chi)\right] \quad , \qquad (2.4.1) \\
R_{\alpha\beta\underline{de}} &= -i\,\tfrac{1}{4}\,(\sigma^{\underline{c}})_{\alpha\beta}\left[3e^{-\Phi}\,H_{\underline{cde}} - i\,\tfrac{5}{16}\,(\,\chi\,\sigma_{\underline{cde}}\,\chi)\right] \\
&\quad -\,i\,\tfrac{1}{24}\,(\sigma^{\underline{abc}}{}_{\underline{de}})_{\alpha\beta}\left[e^{-\Phi}\,H_{\underline{abc}} - i\,\tfrac{3}{16}\,(\,\chi\,\sigma_{\underline{abc}}\,\chi)\right] \quad , \\
R_{\alpha\underline{cde}} &= -i\,\tfrac{1}{2}\left[(\sigma_{\underline{c}})_{\alpha\gamma}\,T_{\underline{de}}{}^{\gamma} - (\sigma_{\underline{d}})_{\alpha\gamma}\,T_{\underline{ec}}{}^{\gamma} - (\sigma_{\underline{e}})_{\alpha\gamma}\,T_{\underline{cd}}{}^{\gamma}\right] \quad .
\end{aligned}
$$

as was noted in the work of [102,103]. In these expressions $H_{\underline{abc}}$ refers to the supercovariantized field strength of a two-form $B_{\underline{ab}}$. The results in (2.4.1) are the 10D, $\mathcal{N}=1$ analogs of the results in (2.2.12) for the 4D, $\mathcal{N}=1$ superspace geometry. That is the component fields embedded in this supergeometry must obey a set of mass-shell conditions. To release these conditions, one must find the 10D, $\mathcal{N}=1$ analogs of the equations in (2.2.9) and (2.2.10). However, as our goal once more is to find a supergeometry consistent with the Norström theory, we seek the analogs of (2.2.14).

The covariant derivatives linear in the conformal compensator Ψ are given by

$$
\nabla_{\alpha} \;=\; \mathrm{D}_{\alpha} + l_0\Psi\mathrm{D}_{\alpha} + l_1(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{2.4.2}
$$

$$
\begin{aligned}
\nabla_{\underline{a}} \;=\;& \partial_{\underline{a}} + l_2\Psi\partial_{\underline{a}} + il_3(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_{\alpha}\Psi)\mathrm{D}_{\beta} + l_4(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\
&+ il_5(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta}(\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi)\mathcal{M}_{\underline{de}} \quad ,
\end{aligned} \tag{2.4.3}
$$

and similar to the case of the eleven dimensional theory, here we have

$$
\{\mathrm{D}_{\alpha}, \mathrm{D}_{\beta}\} \;=\; i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}} \quad . \tag{2.4.4}
$$

The commutation relations of Poincare generators in 10D

$$
[\mathcal{M}_{\underline{ab}}, \mathrm{D}_{\alpha}\} \;=\; \frac{1}{2}(\sigma_{\underline{ab}})_{\alpha}{}^{\beta}\mathrm{D}_{\beta} \quad , \tag{2.4.5}
$$

is similar to the eleven dimensional case. Also the equation in (2.3.7) is valid in all ten dimensional theories. There will be some slight modifications for the dotted and barred spinor indices in type IIA and IIB supergravity, respectively.

By adoption of the constraints

$$T_{\underline{a}\underline{b}}{}^{\underline{c}} = 0 \quad , \quad T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.4.6}$$

we obtain the following parameterization results

$$l_0 = \frac{1}{2} \quad , \quad l_1 = \frac{1}{10} \quad , \quad l_2 = 1 \quad , \quad l_3 = -\frac{2}{5} \quad , \quad l_4 = -1 \quad , \quad l_5 = 0 \quad . \tag{2.4.7}$$

As the consequence of this choice of parameters, we find the torsion supertensors as below.

$$\begin{aligned}
T_{\alpha\beta}{}^{\underline{c}} &= i(\sigma^{\underline{c}})_{\alpha\beta} \quad , & (2.4.8)\\
T_{\alpha\beta}{}^{\gamma} &= 0 \quad , & (2.4.9)\\
T_{\alpha\underline{b}}{}^{\underline{c}} &= \frac{3}{5}\left[\delta_{\underline{b}}{}^{\underline{c}}\delta_{\alpha}{}^{\delta} + (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\delta}\right](\mathrm{D}_{\delta}\Psi) \quad , & (2.4.10)\\
T_{\alpha\underline{b}}{}^{\gamma} &= i\frac{1}{80}\left[-(\sigma^{[2]})_{\alpha}{}^{\gamma}(\sigma_{\underline{b}[2]})^{\beta\delta} + \frac{1}{3}(\sigma_{\underline{b}[3]})_{\alpha}{}^{\gamma}(\sigma^{[3]})^{\beta\delta}\right](\mathrm{D}_{\beta}\mathrm{D}_{\delta}\Psi) & \\
&\quad - \frac{3}{10}\delta_{\alpha}{}^{\gamma}(\partial_{\underline{b}}\Psi) + \frac{3}{10}(\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\gamma}(\partial_{\underline{c}}\Psi) \quad , & (2.4.11)\\
T_{\underline{a}\underline{b}}{}^{\underline{c}} &= 0 \quad , & (2.4.12)\\
T_{\underline{a}\underline{b}}{}^{\gamma} &= i\frac{2}{5}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\mathrm{D}_{\delta}\Psi) \quad . & (2.4.13)
\end{aligned}$$

For the components of the curvatures, we find the results seen below.

$$\begin{aligned}
R_{\alpha\beta}{}^{\underline{d}\underline{e}} &= -i\frac{6}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) & \\
&\quad - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{d}\underline{e}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{d}\underline{e}})^{\gamma\delta}\right](\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi) \quad , & (2.4.14)\\
R_{\alpha\underline{b}}{}^{\underline{d}\underline{e}} &= -(\mathrm{D}_{\alpha}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\sigma^{\underline{d}\underline{e}})_{\alpha}{}^{\gamma}(\partial_{\underline{b}}\mathrm{D}_{\gamma}\Psi) \quad , & (2.4.15)\\
R_{\underline{a}\underline{b}}{}^{\underline{d}\underline{e}} &= -(\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . & (2.4.16)
\end{aligned}$$

It has long been suggested [104] that a superfield with the structure of $G_{\underline{abc}}$ should appear in the off-shell structure of 10D, $\mathcal{N}=1$ supergeometry and that it was related by a superdifferential operator to an underlying unconstrained prepotential $V_{\underline{abc}}$ analogous to $H^{\underline{m}}$ that appears in 4D, $\mathcal{N}=1$ supergravity. However, there are reasons to believe [102, 103] that $V_{\underline{abc}}$ must be related to an even more fundamental spinorial prepotential $\Psi_{\underline{ab}}{}^{\alpha}$. In the equations of (2.4.11) and (2.4.14) the superfield $G_{\underline{abc}} \equiv (\sigma_{\underline{abc}})^{\gamma\delta}(\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi)$ has precisely the structure suggested in the work by Howe, Nicolai, and Van Proeyen.

2.5 Linearized Nordström Supergravity in 10D, $\mathcal{N}=2$A Supergeometry

We repeat the discussions as seen in the previous two sections with a beginning of the on-shell description of 10D, $\mathcal{N}=2$A superspace supergravity. A set of torsion and curvature supertensors can be written in the forms

$$
\begin{aligned}
T_{\alpha\dot{\beta}}{}^{\underline{c}} \;&=\; T_{\dot{\alpha}\dot{\beta}}{}^{\gamma} \;=\; T_{\alpha\dot{\beta}}{}^{\gamma} \;=\; T_{\dot{\alpha}\beta}{}^{\dot{\gamma}} \;=\; T_{\alpha\beta}{}^{\dot{\gamma}} \;=\; 0 \quad , \\
T_{\alpha\underline{b}}{}^{\underline{c}} \;&=\; T_{\dot{\alpha}\underline{b}}{}^{\underline{c}} \;=\; T_{\underline{ab}}{}^{\underline{c}} \;=\; 0 \quad , \\
T_{\alpha\beta}{}^{\underline{c}} \;&=\; i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \quad T_{\dot{\alpha}\dot{\beta}}{}^{\underline{c}} \;=\; i(\sigma^{\underline{c}})_{\dot{\alpha}\dot{\beta}} \quad , \\
T_{\alpha\beta}{}^{\gamma} \;&=\; \Big[\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} \;+\; (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\Big]\chi_{\delta} \quad , \\
T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} \;&=\; \Big[\delta_{(\dot{\alpha}}{}^{\dot{\gamma}}\delta_{\dot{\beta})}{}^{\dot{\delta}} \;+\; (\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{a}})^{\dot{\gamma}\dot{\delta}}\Big]\chi_{\dot{\delta}} \quad , \\
T_{\alpha\underline{b}}{}^{\gamma} \;&=\; -\frac{1}{8}(\sigma^{\underline{de}})_{\alpha}{}^{\gamma}H_{\underline{bde}} \quad , \quad C_{\alpha\dot{\alpha}}C^{\gamma\dot{\gamma}}T_{\dot{\gamma}\underline{b}}{}^{\dot{\alpha}} \;=\; -\frac{1}{8}(\sigma^{\underline{de}})_{\alpha}{}^{\gamma}G_{\underline{bde}} \quad , \\
C_{\gamma\dot{\gamma}}T_{\alpha\underline{b}}{}^{\dot{\gamma}} \;&=\; \frac{1}{16}(\sigma_{\underline{b}})_{\alpha\delta}\Big[(\sigma^{[2]})_{\gamma}{}^{\delta}K_{[2]} \;-\; \tfrac{1}{12}(\sigma^{[4]})_{\gamma}{}^{\delta}D_{[4]}\Big] \quad , \\
C^{\alpha\dot{\alpha}}T_{\dot{\alpha}\underline{b}}{}^{\gamma} \;&=\; -\frac{1}{16}(\sigma_{\underline{b}})^{\alpha\delta}\Big[(\sigma^{[2]})_{\delta}{}^{\gamma}K_{[2]} \;-\; \tfrac{1}{12}(\sigma^{[4]})_{\delta}{}^{\gamma}D_{[4]}\Big] \quad ,
\end{aligned}
\tag{2.5.1}
$$

as was noted in the work of [105]. In these expressions $H_{\underline{abc}}$ refers to the supercovariantized field strengths of a two-form $B_{\underline{ab}}$ gauge field, and

$$
\begin{aligned}
K_{\underline{ab}} &= \; e^{-\Phi}F_{\underline{ab}} \;-\; \chi_{\alpha}(\sigma_{\underline{ab}})_{\beta}{}^{\alpha}\chi^{\beta} \quad , \\
D_{[4]} &= \; 2e^{-\Phi}\tilde{F}_{[4]} \;+\; \chi_{\alpha}(\sigma_{[4]})_{\beta}{}^{\alpha}\chi^{\beta} \quad .
\end{aligned}
\tag{2.5.2}
$$

with $F_{\underline{ab}}$ and $\tilde{F}_{[4]}$ denoting supercovariantized field strengths for a gauge 1-form and a gauge 3-form respectively. The results in (2.5.1) are the 10D, $\mathcal{N}$ = 2A analogs of the results in (2.2.12) for the 4D, $\mathcal{N}$ = 1 superspace geometry. Those are the component fields embedded in this supergeometry must obey a set of mass-shell conditions. To release these conditions, one must find the 10D, $\mathcal{N}$ = 2A analogs of the equations in (2.2.9) and (2.2.10). Again the goal must be to find a supergeometry consistent with the Norström theory, we seek the analogs of (2.2.14). In analogy with the 3-form gauge field sector of 11D, $\mathcal{N}$ = 1 supergravity, the gauge fields components are:

$$
\begin{aligned}
&F_{\alpha\dot{\beta}} = C_{\alpha\dot{\beta}}e^{\Phi} \quad , \qquad F_{\alpha\beta} = F_{\dot{\alpha}\dot{\beta}} = 0 \quad , \\
&F_{\underline{c}\alpha} = ie^{\Phi}(\sigma_{\underline{c}})_{\alpha\beta}\chi^{\beta} \quad , \qquad F_{\underline{c}\dot{\alpha}} = iC_{\alpha\dot{\alpha}}e^{\Phi}(\sigma_{\underline{c}})^{\alpha\beta}\chi_{\beta} \quad , \\
&G_{\alpha\beta\gamma} = G_{\underline{a}\beta\dot{\gamma}} = G_{\underline{ab}\gamma} = G_{\underline{ab}\dot{\gamma}} = 0 \quad , \\
&G_{\underline{c}\alpha\beta} = i(\sigma_{\underline{c}})_{\alpha\beta} \quad , \qquad G_{\underline{c}\dot{\alpha}\dot{\beta}} = -i(\sigma_{\underline{c}})_{\dot{\alpha}\dot{\beta}} \quad , \\
&\tilde{F}_{\alpha\beta\gamma\underline{d}} = \tilde{F}_{\alpha\beta\underline{cd}} = \tilde{F}_{\dot{\alpha}\dot{\beta}\underline{cd}} = 0 \quad , \\
&\tilde{F}_{\alpha\dot{\beta}\underline{cd}} = e^{\Phi}(\sigma_{\underline{cd}})_{\alpha}{}^{\beta}C_{\beta\dot{\beta}} \quad , \\
&\tilde{F}_{\alpha\underline{bcd}} = -ie^{\Phi}(\sigma_{\underline{bcd}})_{\alpha\beta}\chi^{\beta} \quad , \qquad \tilde{F}_{\dot{\alpha}\underline{bcd}} = iC_{\alpha\dot{\alpha}}e^{\Phi}(\sigma_{\underline{bcd}})^{\alpha\beta}\chi_{\beta} \quad .
\end{aligned}
\tag{2.5.3}
$$

Φ denotes a dilaton superfield, and χ_{α} is its partner dilatino. All of the equations in (2.5.1) - (2.5.3) describe the on-shell 10D, $\mathcal{N}$ = 2A theory, i.e. these are the analogs of (2.2.12).

The covariant derivatives linear in the conformal compensator Ψ are given by

$$
\nabla_{\alpha} = \mathrm{D}_{\alpha} + \frac{1}{2}\Psi\mathrm{D}_{\alpha} + l_0(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{ab}} \tag{2.5.4}
$$

$$
\nabla_{\dot{\alpha}} = \mathrm{D}_{\dot{\alpha}} + \frac{1}{2}\Psi\mathrm{D}_{\dot{\alpha}} + l_0(\sigma^{\underline{ab}})_{\dot{\alpha}}{}^{\dot{\beta}}(\mathrm{D}_{\dot{\beta}}\Psi)\mathcal{M}_{\underline{ab}} \tag{2.5.5}
$$

$$
\begin{aligned}
\nabla_{\underline{a}} = &\, \partial_{\underline{a}} + l_1\Psi\partial_{\underline{a}} + il_2(\sigma_{\underline{a}})^{\delta\gamma}(\mathrm{D}_{\delta}\Psi)\mathrm{D}_{\gamma} + il_3(\sigma_{\underline{a}})^{\dot{\delta}\dot{\gamma}}(\mathrm{D}_{\dot{\delta}}\Psi)\mathrm{D}_{\dot{\gamma}} + l_4(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\
&+ il_5(\sigma_{\underline{a}}{}^{\underline{cd}})^{\gamma\delta}(\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi)\mathcal{M}_{\underline{cd}} + il_6(\sigma_{\underline{a}}{}^{\underline{cd}})^{\dot{\gamma}\dot{\delta}}(\mathrm{D}_{\dot{\gamma}}\mathrm{D}_{\dot{\delta}}\Psi)\mathcal{M}_{\underline{cd}}
\end{aligned}
\tag{2.5.6}
$$

where the Type IIA supersymmetry algebra

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} \;=\; i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}} \;\;,\;\; \{\mathrm{D}_{\dot\alpha}, \mathrm{D}_{\dot\beta}\} \;=\; i(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}\partial_{\underline{a}} \;\;,\;\; \{\mathrm{D}_\alpha, \mathrm{D}_{\dot\beta}\} \;=\; 0 \tag{2.5.7}$$

is satisfied by the bare derivative operators.

By adopting the constraints

$$T_{\underline{ab}}{}^{\underline{c}} = 0 \quad , \quad T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \quad T_{\dot\alpha\dot\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\dot\alpha\dot\beta} \quad , \tag{2.5.8}$$

we obtain the following paramaterization values

$$l_0 = \frac{1}{10} \;\;,\;\; l_1 = 1 \;\;,\;\; l_2 = l_3 = -\frac{1}{5} \;\;,\;\; l_4 = -1 \;\;,\;\; l_5 = l_6 = 0 \;\; . \tag{2.5.9}$$

As the consequence of this choice of parameters, we find the torsion supertensors given below.

$$T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.5.10}$$

$$T_{\alpha\beta}{}^{\gamma} = \frac{1}{5}(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad , \tag{2.5.11}$$

$$T_{\alpha\beta}{}^{\dot\gamma} = -\frac{1}{5}(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\dot\gamma\dot\delta}(\mathrm{D}_{\dot\delta}\Psi) \quad , \tag{2.5.12}$$

$$T_{\dot\alpha\dot\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\dot\alpha\dot\beta} \quad , \tag{2.5.13}$$

$$T_{\dot\alpha\dot\beta}{}^{\gamma} = -\frac{1}{5}(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}(\sigma_{\underline{a}})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad , \tag{2.5.14}$$

$$T_{\dot\alpha\dot\beta}{}^{\dot\gamma} = \frac{1}{5}(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}(\sigma_{\underline{a}})^{\dot\gamma\dot\delta}(\mathrm{D}_{\dot\delta}\Psi) \quad , \tag{2.5.15}$$

$$T_{\alpha\dot\beta}{}^{\underline{c}} = 0 \quad , \tag{2.5.16}$$

$$T_{\alpha\dot\beta}{}^{\gamma} = \frac{1}{2}\left[\delta_\alpha{}^\gamma(\mathrm{D}_{\dot\beta}\Psi) + \frac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\gamma(\sigma_{\underline{ab}})_{\dot\beta}{}^{\dot\delta}(\mathrm{D}_{\dot\delta}\Psi)\right] \quad , \tag{2.5.17}$$

$$T_{\alpha\dot\beta}{}^{\dot\gamma} = \frac{1}{2}\left[\delta_{\dot\beta}{}^{\dot\gamma}(\mathrm{D}_\alpha\Psi) + \frac{1}{10}(\sigma^{\underline{ab}})_{\dot\beta}{}^{\dot\gamma}(\sigma_{\underline{ab}})_\alpha{}^\delta(\mathrm{D}_\delta\Psi)\right] \quad , \tag{2.5.18}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = \frac{4}{5}\delta_{\underline{b}}{}^{\underline{c}}(\mathrm{D}_\alpha\Psi) + \frac{2}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\delta(\mathrm{D}_\delta\Psi) \quad , \tag{2.5.19}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\gamma} = {} & i\frac{1}{80}\left[-\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} + \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right](\mathrm{D}_\beta\mathrm{D}_\delta\Psi) \\ & - \frac{2}{5}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\Psi) + \frac{2}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\Psi) \quad , \end{aligned} \tag{2.5.20}$$

$$
\begin{aligned}
T_{\alpha \underline{b}}{}^{\dot{\gamma}} &= -i\frac{1}{80}\Big[(\sigma_{\underline{b}})_{\alpha}{}^{\dot{\gamma}}C^{\beta\dot{\delta}} - (\sigma^{\underline{c}})_{\alpha}{}^{\dot{\gamma}}(\sigma_{\underline{bc}})^{\beta\dot{\delta}} - \frac{1}{3!}(\sigma^{[3]})_{\alpha}{}^{\dot{\gamma}}(\sigma_{\underline{b}[3]})^{\beta\dot{\delta}} + \frac{1}{2}(\sigma_{\underline{b}[2]})_{\alpha}{}^{\dot{\gamma}}(\sigma^{[2]})^{\beta\dot{\delta}} \\
&\quad + \frac{1}{4!}(\sigma_{\underline{b}[4]})_{\alpha}{}^{\dot{\gamma}}(\sigma^{[4]})^{\beta\dot{\delta}}\Big](\mathrm{D}_{\beta}\mathrm{D}_{\dot{\delta}}\Psi) \quad , && (2.5.21)\\
T_{\dot{\alpha}\underline{b}}{}^{\underline{c}} &= \frac{4}{5}\delta_{\underline{b}}{}^{\underline{c}}(\mathrm{D}_{\dot{\alpha}}\Psi) + \frac{2}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_{\dot{\alpha}}{}^{\dot{\delta}}(\mathrm{D}_{\dot{\delta}}\Psi) \quad , && (2.5.22)\\
T_{\dot{\alpha}\underline{b}}{}^{\gamma} &= -i\frac{1}{80}\Big[(\sigma_{\underline{b}})_{\dot{\alpha}}{}^{\gamma}C^{\delta\dot{\beta}} + (\sigma^{\underline{c}})_{\dot{\alpha}}{}^{\gamma}(\sigma_{\underline{bc}})^{\delta\dot{\beta}} - \frac{1}{3!}(\sigma^{[3]})_{\dot{\alpha}}{}^{\gamma}(\sigma_{\underline{b}[3]})^{\delta\dot{\beta}} - \frac{1}{2}(\sigma_{\underline{b}[2]})_{\dot{\alpha}}{}^{\gamma}(\sigma^{[2]})^{\delta\dot{\beta}} \\
&\quad + \frac{1}{4!}(\sigma_{\underline{b}[4]})_{\dot{\alpha}}{}^{\gamma}(\sigma^{[4]})^{\delta\dot{\beta}}\Big](\mathrm{D}_{\delta}\mathrm{D}_{\dot{\beta}}\Psi) \quad , && (2.5.23)\\
T_{\dot{\alpha}\underline{b}}{}^{\dot{\gamma}} &= i\frac{1}{80}\left[-\frac{1}{2}(\sigma^{[2]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma_{\underline{b}[2]})^{\dot{\beta}\dot{\delta}} + \frac{1}{3!}(\sigma_{\underline{b}[3]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma^{[3]})^{\dot{\beta}\dot{\delta}}\right](\mathrm{D}_{\dot{\beta}}\mathrm{D}_{\dot{\delta}}\Psi) \\
&\quad - \frac{2}{5}\delta_{\dot{\alpha}}{}^{\dot{\gamma}}(\partial_{\underline{b}}\Psi) + \frac{2}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_{\dot{\alpha}}{}^{\dot{\gamma}}(\partial_{\underline{c}}\Psi) \quad , && (2.5.24)\\
T_{\underline{ab}}{}^{\underline{c}} &= 0 \quad , && (2.5.25)\\
T_{\underline{ab}}{}^{\gamma} &= i\frac{1}{5}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\mathrm{D}_{\delta}\Psi) \quad , && (2.5.26)\\
T_{\underline{ab}}{}^{\dot{\gamma}} &= i\frac{1}{5}(\sigma_{[\underline{a}})^{\dot{\gamma}\dot{\delta}}(\partial_{\underline{b}]}\mathrm{D}_{\dot{\delta}}\Psi) \quad . && (2.5.27)
\end{aligned}
$$

For the components of the curvatures, we find the results seen below.

$$
R_{\alpha\beta}{}^{\underline{de}} = -i\frac{6}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta}\right](\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi) \quad , \tag{2.5.28}
$$

$$
R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} = -i\frac{6}{5}(\sigma^{[\underline{d}})_{\dot{\alpha}\dot{\beta}}(\partial^{\underline{e}]}\Psi) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\dot{\alpha}\dot{\beta}}(\sigma_{[3]})^{\dot{\gamma}\dot{\delta}} + (\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{a}}{}^{\underline{de}})^{\dot{\gamma}\dot{\delta}}\right](\mathrm{D}_{\dot{\gamma}}\mathrm{D}_{\dot{\delta}}\Psi) \quad , \tag{2.5.29}
$$

$$
\begin{aligned}
R_{\alpha\dot{\beta}}{}^{\underline{de}} &= \frac{1}{40}\Big[-C_{\alpha\dot{\beta}}(\sigma^{\underline{de}})^{\gamma\dot{\delta}} + (\sigma^{\underline{de}})_{\alpha\dot{\beta}}C^{\gamma\dot{\delta}} - \frac{1}{2}(\sigma_{[2]})_{\alpha\dot{\beta}}(\sigma^{\underline{de}[2]})^{\gamma\dot{\delta}} \\
&\quad + \frac{1}{2}(\sigma^{\underline{de}[2]})_{\alpha\dot{\beta}}(\sigma_{[2]})^{\gamma\dot{\delta}} + \frac{1}{4!4!}\epsilon^{\underline{de}[4][\bar{4}]}(\sigma_{[4]})_{\alpha\dot{\beta}}(\sigma_{[\bar{4}]})^{\gamma\dot{\delta}}\Big](\mathrm{D}_{\gamma}\mathrm{D}_{\dot{\delta}}\Psi) \quad ,
\end{aligned} \tag{2.5.30}
$$

$$
\begin{aligned}
R_{\alpha\underline{b}}{}^{\underline{de}} &= -(\mathrm{D}_{\alpha}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\gamma}(\partial_{\underline{b}}\mathrm{D}_{\gamma}\Psi) \quad , && (2.5.31)\\
R_{\dot{\alpha}\underline{b}}{}^{\underline{de}} &= -(\mathrm{D}_{\dot{\alpha}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\sigma^{\underline{de}})_{\dot{\alpha}}{}^{\dot{\gamma}}(\partial_{\underline{b}}\mathrm{D}_{\dot{\gamma}}\Psi) \quad , && (2.5.32)\\
R_{\underline{ab}}{}^{\underline{de}} &= -(\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . && (2.5.33)
\end{aligned}
$$

2.6 Linearized Nordström Supergravity in 10D, $\mathcal{N} = 2$B Supergeometry

Now for a final time we replicate the discussions as seen in the previous three sections with a beginning of the on-shell description of 10D, $\mathcal{N} = 2$B superspace supergravity here.

A set of torsion and curvature supertensors can be written in the form

$$
\begin{aligned}
T_{\alpha\bar{\beta}}{}^{\underline{c}} &= i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \quad T_{\alpha\beta}{}^{\underline{c}} = T_{\bar{\alpha}\bar{\beta}}{}^{\underline{c}} = 0 \quad , \quad T_{\underline{a}\underline{b}}{}^{\underline{c}} = 0 \quad , \\
T_{\alpha\beta}{}^{\gamma} &= T_{\bar{\alpha}\bar{\beta}}{}^{\gamma} = T_{\alpha\bar{\beta}}{}^{\bar{\gamma}} = \Big[\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} + (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\Big]\Lambda_{\delta} \quad , \\
T_{\alpha\beta}{}^{\bar{\gamma}} &= T_{\bar{\alpha}\beta}{}^{\gamma} = T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}} = \Big[\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} + (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\Big]\overline{\Lambda}_{\delta} \quad ,
\end{aligned}
\tag{2.6.1}
$$

$$
\begin{aligned}
T_{\alpha\underline{b}}{}^{\bar{\gamma}} &= \frac{1}{24}(\sigma_{\underline{b}})_{\alpha\delta}(\sigma^{[3]})^{\delta\gamma}\Big[e^{-2\Phi}(1+\overline{W})\overline{G}_{[3]} - e^{-2\Phi}(1-\overline{W})G_{[3]} - i(\sigma_{[3]})^{\epsilon\lambda}(\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda})\Big] \\
&\quad + \frac{1}{96}(\sigma^{[3]})_{\alpha\delta}(\sigma_{\underline{b}})^{\delta\gamma}(G_{[3]} + \overline{G}_{[3]}) \quad , \\
T_{\bar{\alpha}\underline{b}}{}^{\gamma} &= \frac{1}{24}(\sigma_{\underline{b}})_{\alpha\delta}(\sigma^{[3]})^{\delta\gamma}\Big[e^{-2\Phi}(1+W)G_{[3]} - e^{-2\Phi}(1-W)\overline{G}_{[3]} + i(\sigma_{[3]})^{\epsilon\lambda}(\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda})\Big] \\
&\quad + \frac{1}{96}(\sigma^{[3]})_{\alpha\delta}(\sigma_{\underline{b}})^{\delta\gamma}(G_{[3]} + \overline{G}_{[3]}) \quad , \\
T_{\alpha\underline{b}}{}^{\gamma} &= -T_{\bar{\alpha}\underline{b}}{}^{\bar{\gamma}} = \frac{1}{4}(\sigma_{\underline{b}})_{\alpha\delta}(\sigma^{\underline{d}})^{\delta\gamma}\Big[e^{-2\Phi}\nabla_{\underline{d}}(W-\overline{W}) + i\frac{7}{4}(\sigma_{\underline{d}})^{\epsilon\lambda}\Lambda_{\epsilon}\overline{\Lambda}_{\lambda}\Big] \\
&\qquad + i\frac{1}{48}(\sigma_{[4]})_{\alpha}{}^{\gamma}\Big[\,\frac{1}{8}(\sigma_{\underline{b}}{}^{[4]})^{\epsilon\lambda}\Lambda_{\epsilon}\overline{\Lambda}_{\lambda} - \frac{5}{3}e^{-2\Phi}\tilde{F}_{\underline{b}}{}^{[4]}\,\Big] \quad ,
\end{aligned}
\tag{2.6.2}
$$

$$
\begin{aligned}
R_{\alpha\beta\underline{cd}} &= i\frac{1}{12}(\sigma_{\underline{cd}}{}^{[3]})_{\alpha\beta}\Big\{e^{-2\Phi}(1+\overline{W})\overline{G}_{[3]} - e^{-2\Phi}(1-\overline{W})G_{[3]} - i(\sigma_{[3]})^{\epsilon\lambda}\big[\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda}\big] \\
&\qquad - \frac{1}{4}(G_{[3]} + \overline{G}_{[3]})\Big\} \\
&\quad - i\frac{1}{2}(\sigma^{\underline{e}})_{\alpha\beta}\Big\{e^{-2\Phi}(1+\overline{W})\overline{G}_{\underline{cde}} - e^{-2\Phi}(1-\overline{W})G_{\underline{cde}} - i(\sigma_{\underline{cde}})^{\epsilon\lambda}\big[\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda}\big] \\
&\qquad + \frac{1}{4}(G_{\underline{cde}} + \overline{G}_{\underline{cde}})\Big\} \quad ,
\end{aligned}
\tag{2.6.3}
$$

$$
\begin{aligned}
R_{\bar{\alpha}\bar{\beta}\underline{cd}} &= i\frac{1}{12}(\sigma_{\underline{cd}}{}^{[3]})_{\alpha\beta}\Big\{e^{-2\Phi}(1+W)G_{[3]} - e^{-2\Phi}(1-W)\overline{G}_{[3]} + i(\sigma_{[3]})^{\epsilon\lambda}\big[\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda}\big] \\
&\qquad - \frac{1}{4}(G_{[3]} + \overline{G}_{[3]})\Big\} \\
&\quad - i\frac{1}{2}(\sigma^{\underline{e}})_{\alpha\beta}\Big\{e^{-2\Phi}(1+W)G_{\underline{cde}} - e^{-2\Phi}(1-W)\overline{G}_{\underline{cde}} + i(\sigma_{\underline{cde}})^{\epsilon\lambda}\big[\Lambda_{\epsilon}\Lambda_{\lambda} - \overline{\Lambda}_{\epsilon}\overline{\Lambda}_{\lambda}\big] \\
&\qquad + \frac{1}{4}(G_{\underline{cde}} + \overline{G}_{\underline{cde}})\Big\} \quad .
\end{aligned}
\tag{2.6.4}
$$

as was noted in the portion of the work in [105] devoted to type IIB supergravity. We will end our discussion here. As the astute reader can note the expressions are of increasing complication. But the central message of the expressions in (2.6.1) - (2.6.4) is that the on-shell description of the 10D, $\mathcal{N} = 2$B theory exists in perfect analogy with the on-shell description of 4D, $\mathcal{N} = 1$ superspace given by the equations in (2.2.12).

Now for the covariant derivative operators linear in the conformal compensator Ψ and necessary for a Nordström theory may be given by

$$\nabla_\alpha = \mathrm{D}_\alpha + \frac{1}{2}\Psi\mathrm{D}_\alpha + l_0(\sigma^{\underline{ab}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{ab}} \qquad (2.6.5)$$

$$\nabla_{\bar\alpha} = \overline{\mathrm{D}}_\alpha + \frac{1}{2}\overline{\Psi}\overline{\mathrm{D}}_\alpha + \bar{l}_0(\sigma^{\underline{ab}})_\alpha{}^\beta(\overline{\mathrm{D}}_\beta\overline{\Psi})\mathcal{M}_{\underline{ab}} \qquad (2.6.6)$$

$$\begin{aligned}\nabla_{\underline{a}} = {} & \partial_{\underline{a}} + l_1\Psi\partial_{\underline{a}} + l_2\overline{\Psi}\partial_{\underline{a}} + il_3(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\overline{\Psi})\overline{\mathrm{D}}_\beta + il_4(\sigma_{\underline{a}})^{\alpha\beta}(\overline{\mathrm{D}}_\alpha\Psi)\mathrm{D}_\beta \\ & + il_5(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\overline{\mathrm{D}}_\beta + il_6(\sigma_{\underline{a}})^{\alpha\beta}(\overline{\mathrm{D}}_\alpha\overline{\Psi})\mathrm{D}_\beta \\ & + il_7(\sigma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta}(\mathrm{D}_\alpha\overline{\mathrm{D}}_\beta\overline{\Psi})\mathcal{M}_{\underline{de}} + il_8(\sigma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta}(\overline{\mathrm{D}}_\alpha\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{de}} \\ & + l_9(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} + l_{10}(\partial_{\underline{c}}\overline{\Psi})\mathcal{M}_{\underline{a}}{}^{\underline{c}} \end{aligned} \qquad (2.6.7)$$

with the Type IIB supersymmetry algebra

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = 0 \quad , \quad \{\overline{\mathrm{D}}_\alpha, \overline{\mathrm{D}}_\beta\} = 0 \quad , \quad \{\mathrm{D}_\alpha, \overline{\mathrm{D}}_\beta\} = i(\sigma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}} \quad . \qquad (2.6.8)$$

By adopting the constraints

$$T_{\underline{ab}}{}^{\underline{c}} = 0 \quad , \quad T_{\alpha\bar\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \qquad (2.6.9)$$

we have the following parametrization results

$$l_1 = l_2 = \frac{1}{2} \ , \ l_3 = l_4 = -\frac{1}{32} \ , \ l_5 = l_6 = -\frac{27}{160} \ ,$$

$$l_7 = l_8 = 0 \quad , \ l_9 = l_{10} = -\frac{1}{2} \ . \qquad (2.6.10)$$

As the consequence of this choice of parameters, we find the torsion supertensors given below.

$$T_{\alpha\beta}{}^{\underline{c}} = 0 \quad , \qquad (2.6.11)$$

$$T_{\alpha\beta}{}^\gamma = \frac{2}{5}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad , \qquad (2.6.12)$$

$$T_{\alpha\beta}{}^{\bar\gamma} = 0 \quad , \qquad (2.6.13)$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\underline{c}} = 0 \quad , \tag{2.6.14}$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\gamma} = 0 \quad , \tag{2.6.15}$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}} = \frac{2}{5}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad , \tag{2.6.16}$$

$$T_{\alpha\bar{\beta}}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.6.17}$$

$$\begin{aligned} T_{\alpha\bar{\beta}}{}^{\gamma} = & -\frac{1}{320}\Big[(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + \frac{1}{24}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\Big](\overline{\mathrm{D}}_\delta\overline{\Psi}) \\ & + \frac{1}{192}\Big[-(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + \frac{1}{40}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\Big](\overline{\mathrm{D}}_\delta\Psi) \quad , \end{aligned} \tag{2.6.18}$$

$$\begin{aligned} T_{\alpha\bar{\beta}}{}^{\bar{\gamma}} = & -\frac{1}{320}\left[(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + \frac{1}{24}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\right](\mathrm{D}_\delta\Psi) \\ & + \frac{1}{192}\left[-(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + \frac{1}{40}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\right](\mathrm{D}_\delta\overline{\Psi}) \quad , \end{aligned} \tag{2.6.19}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = \left[\frac{53}{160}\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + \frac{59}{160}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\right](\mathrm{D}_\gamma\Psi) + \left[\frac{15}{32}\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + \frac{1}{32}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\right](\mathrm{D}_\gamma\overline{\Psi}) \quad , \tag{2.6.20}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\gamma} = & -\frac{31}{64}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\Psi) + \frac{27}{320}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\overline{\Psi}) + \frac{15}{64}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\Psi) + \frac{53}{320}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\overline{\Psi}) \\ & - i\frac{1}{512}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\Big](\mathrm{D}_\beta\overline{\mathrm{D}}_\delta\Psi) \\ & - i\frac{27}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\Big](\mathrm{D}_\beta\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad , \end{aligned} \tag{2.6.21}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\bar{\gamma}} = & -i\frac{1}{512}\left[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right](\mathrm{D}_\beta\mathrm{D}_\delta\overline{\Psi}) \\ & - i\frac{27}{2560}\left[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right](\mathrm{D}_\beta\mathrm{D}_\delta\Psi) \quad , \end{aligned} \tag{2.6.22}$$

$$T_{\bar{\alpha}\underline{b}}{}^{\underline{c}} = \left[\frac{15}{32}\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + \frac{1}{32}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\right](\overline{\mathrm{D}}_\gamma\Psi) + \left[\frac{53}{160}\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + \frac{59}{160}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\right](\overline{\mathrm{D}}_\gamma\overline{\Psi}) \quad , \tag{2.6.23}$$

$$\begin{aligned} T_{\bar{\alpha}\underline{b}}{}^{\gamma} = & -i\frac{1}{512}\left[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right](\overline{\mathrm{D}}_\beta\overline{\mathrm{D}}_\delta\Psi) \\ & - i\frac{27}{2560}\left[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right](\overline{\mathrm{D}}_\beta\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad , \end{aligned} \tag{2.6.24}$$

$$\begin{aligned} T_{\bar{\alpha}\underline{b}}{}^{\bar{\gamma}} = & -\frac{31}{64}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\overline{\Psi}) + \frac{27}{320}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\Psi) + \frac{15}{64}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\overline{\Psi}) + \frac{53}{320}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\Psi) \\ & - i\frac{1}{512}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\Big](\overline{\mathrm{D}}_\beta\mathrm{D}_\delta\overline{\Psi}) \\ & - i\frac{27}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\Big](\overline{\mathrm{D}}_\beta\mathrm{D}_\delta\Psi) \quad , \end{aligned} \tag{2.6.25}$$

$$T_{\underline{a}\underline{b}}{}^{\underline{c}} = 0 \quad , \tag{2.6.26}$$

$$T_{\underline{a}\underline{b}}{}^{\gamma} = i\frac{1}{32}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\overline{\mathrm{D}}_\delta\Psi) + i\frac{27}{160}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad , \tag{2.6.27}$$

$$T_{\underline{a}\underline{b}}{}^{\bar{\gamma}} = i\frac{1}{32}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\mathrm{D}_\delta\overline{\Psi}) + i\frac{27}{160}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\mathrm{D}_\delta\Psi) \quad . \tag{2.6.28}$$

For the components of the curvatures, we find the results seen below.

$$R_{\alpha\beta}{}^{\underline{de}} = \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} - (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta}\right](\mathrm{D}_\gamma \mathrm{D}_\delta \Psi) \quad , \tag{2.6.29}$$

$$R_{\bar{\alpha}\bar{\beta}}{}^{\underline{de}} = \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} - (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta}\right](\overline{\mathrm{D}}_\gamma \overline{\mathrm{D}}_\delta \overline{\Psi}) \quad , \tag{2.6.30}$$

$$\begin{aligned} R_{\alpha\bar{\beta}}{}^{\underline{de}} = & -i\frac{3}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}(\Psi+\overline{\Psi})) - i\frac{1}{10}(\sigma^{\underline{def}})_{\alpha\beta}(\partial_{\underline{f}}(\Psi+\overline{\Psi})) \\ & - \frac{1}{80}\left[(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta} - \frac{1}{2}(\sigma^{[2][\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}]}{}_{[2]})^{\gamma\delta} - \frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\right](\overline{\mathrm{D}}_\gamma \mathrm{D}_\delta \Psi) \\ & - \frac{1}{80}\left[(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta} - \frac{1}{2}(\sigma^{[2][\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}]}{}_{[2]})^{\gamma\delta} - \frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\right](\mathrm{D}_\gamma \overline{\mathrm{D}}_\delta \overline{\Psi}) \quad , \end{aligned} \tag{2.6.31}$$

$$R_{\alpha\underline{b}}{}^{\underline{de}} = -\frac{1}{2}(\mathrm{D}_\alpha \partial^{[\underline{d}}(\Psi+\overline{\Psi}))\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\mathrm{D}_\gamma\Psi) \quad , \tag{2.6.32}$$

$$R_{\bar{\alpha}\underline{b}}{}^{\underline{de}} = -\frac{1}{2}(\overline{\mathrm{D}}_\alpha \partial^{[\underline{d}}(\Psi+\overline{\Psi}))\delta_{\underline{b}}{}^{\underline{e}]} + \frac{1}{5}(\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\overline{\mathrm{D}}_\gamma\overline{\Psi}) \quad , \tag{2.6.33}$$

$$R_{\underline{ab}}{}^{\underline{de}} = -\frac{1}{2}(\partial_{[\underline{a}}\partial^{[\underline{d}}(\Psi+\overline{\Psi}))\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{2.6.34}$$

2.7 Higher Dimensional Component Considerations

In the following four subsections, we will appropriately adapt these results to the cases of eleven and ten-dimensional formulations appropriate for Nordström supergravity in those contexts. There are four steps:

(a.) define the Nordström SG linearized superspace supercovariant derivatives in terms of a scalar prepotential leading to component fields,

(b.) express the geometrical tensors of each respective superspace in terms of the component fields presented in the previous part,

(c.) express the "composition rules" of the parameters of general coordinates, local Lorentz, and local SUSY transformations, and

(d.) write the component level SUSY transformation laws that we undertake in each of the four cases of 11D, $\mathcal{N}=1$, 10D, $\mathcal{N}=1$, 10D, $\mathcal{N}=2$A, and 10D, $\mathcal{N}=2$B theories.

2.7.1 Adaptation To 11D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 1

In the case of the 11D N(ordström)-SG covariant derivatives we define

$$\nabla_\alpha \ = \mathrm{D}_\alpha + \frac{1}{2}\Psi \mathrm{D}_\alpha + \frac{1}{10}(\gamma^{\underline{de}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{de}} \quad , \tag{2.7.1}$$

$$\nabla_{\underline{a}} \ = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} + i\frac{1}{4}(\gamma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\mathrm{D}_\beta - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{2.7.2}$$

and "split" the spatial 11D N-SG covariant derivative into two parts

$$\nabla_{\underline{a}}| \ = \ \mathbf{D}_{\underline{a}} + \psi_{\underline{a}}{}^\gamma\nabla_\gamma| \quad . \tag{2.7.3}$$

On taking the $\theta \to 0$ limit the latter terms allows an identification with the gravitino and the leading term in this limit yields a component-level linearized gravitationally covariant derivative operator given by

$$\mathbf{D}_{\underline{a}} \ = e_{\underline{a}} + \phi_{\underline{a}}{}^\iota\mathcal{M}_\iota \ = \ \partial_{\underline{a}} + \Psi\partial_{\underline{a}} + \phi_{\underline{a}}{}^\iota\mathcal{M}_\iota \quad . \tag{2.7.4}$$

By comparison of the LHS to the RHS of (2.7.4), we see that a linearized frame field $e_{\underline{a}}{}^{\underline{m}} = (1 + \Psi)\delta_{\underline{a}}{}^{\underline{m}}$ emerges to describe a scalar graviton. Finally, comparison of the coefficient of the Lorentz generator $\mathcal{M}_\iota$ as it appears in the latter two forms of (2.7.4) informs us the spin connection is given by

$$\phi_{\underline{c}}{}^{\underline{de}} \ = \ -\frac{1}{2}\delta_{\underline{c}}{}^{[\underline{d}}(\partial^{\underline{e}]}\Psi) \quad . \tag{2.7.5}$$

Comparing the result in (2.7.2) with the one in (2.7.3) a component gravitino is identified via

$$\psi_{\underline{a}}{}^\gamma \ = \ i\frac{1}{4}(\gamma_{\underline{a}})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad . \tag{2.7.6}$$

However, as this expression contains an explicit γ-matrix we see that it really defines the non-conformal *spin*-$\frac{1}{2}$ part of the gravitino to be

$$\psi_\beta \ \equiv \ (\gamma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^\gamma \quad . \tag{2.7.7}$$

This is to be expected. As a Nordström type theory only contains a scalar graviton, it follows only the "γ-trace" of the gravitino can occur. So then we have

$$\mathrm{D}_\beta \Psi \;=\; i\frac{4}{11}(\gamma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^\gamma \;\equiv\; i\frac{4}{11}\psi_\beta \quad , \tag{2.7.8}$$

in the $\theta \to 0$ limit.

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:

$$K \;=\; C^{\gamma\delta}(\mathrm{D}_\gamma \mathrm{D}_\delta \Psi) \quad , \quad K_{[3]} \;=\; (\gamma_{[3]})^{\gamma\delta}(\mathrm{D}_\gamma \mathrm{D}_\delta \Psi) \quad , \quad K_{[4]} \;=\; (\gamma_{[4]})^{\gamma\delta}(\mathrm{D}_\gamma \mathrm{D}_\delta \Psi) \quad , \tag{2.7.9}$$

or in other words,

$$\frac{1}{2}\mathrm{D}_{[\gamma}\mathrm{D}_{\delta]}\Psi \;=\; \frac{1}{32}\Big\{C_{\gamma\delta}K - \frac{1}{3!}(\gamma^{[3]})_{\gamma\delta}K_{[3]} + \frac{1}{4!}(\gamma^{[4]})_{\gamma\delta}K_{[4]}\Big\} \quad . \tag{2.7.10}$$

We emphasize that the component fields (the K's) are defined by the $\theta \to 0$ limit of these equations. The results in (2.7.9) and (2.7.10) follow as results from a Fierz identity

$$\delta_{[\gamma}{}^\alpha \delta_{\delta]}{}^\beta \;=\; \frac{1}{16}\Big\{C_{\gamma\delta}C^{\alpha\beta} - \frac{1}{3!}(\gamma^{[3]})_{\gamma\delta}(\gamma_{[3]})^{\alpha\beta} + \frac{1}{4!}(\gamma^{[4]})_{\gamma\delta}(\gamma_{[4]})^{\alpha\beta}\Big\} \quad , \tag{2.7.11}$$

valid for 11D spinors.

2.7.2 Adaptation To 11D, $\mathcal{N} = 1$ Component/Superspace Results: Step 2

Torsions:

$$T_{\alpha\beta}{}^{\underline{c}} = i(\gamma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.7.12}$$

$$T_{\alpha\beta}{}^{\gamma} = i\frac{3}{110}(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\gamma\delta}\psi_\delta \quad , \tag{2.7.13}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = i\frac{3}{11}\Big[\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\beta + \frac{3}{5}(\gamma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\beta\Big]\psi_\beta \quad , \tag{2.7.14}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\gamma} = i\frac{1}{128}\Big[&- (\gamma_{\underline{b}})_\alpha{}^\gamma K + \frac{1}{2}(\gamma^{[2]})_\alpha{}^\gamma K_{\underline{b}[2]} - \frac{1}{3!}(\gamma_{\underline{b}[3]})_\alpha{}^\gamma K^{[3]} + \frac{1}{3!}(\gamma^{[3]})_\alpha{}^\gamma K_{\underline{b}[3]} \\ &- \frac{1}{4!}(\gamma_{\underline{b}[4]})_\alpha{}^\gamma K^{[4]}\Big] + \frac{1}{8}\Big[\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + 3(\gamma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big](\partial_{\underline{c}}\Psi) \quad , \end{aligned} \tag{2.7.15}$$

$$T_{\underline{a}\underline{b}}{}^{\underline{c}} = 0 \quad , \tag{2.7.16}$$

$$T_{\underline{a}\underline{b}}{}^{\gamma} = \frac{1}{11}(\gamma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\psi_\delta) \quad . \tag{2.7.17}$$

Curvatures:

$$\begin{aligned} R_{\alpha\beta}{}^{\underline{de}} =& \frac{1}{80}\Big[(\gamma^{\underline{de}})_{\alpha\beta}K + (\gamma_{[1]})_{\alpha\beta}K^{[1]\underline{de}} - \frac{1}{3!}(\gamma^{\underline{de}[3]})_{\alpha\beta}K_{[3]} - \frac{1}{2}(\gamma_{[2]})_{\alpha\beta}K^{[2]\underline{de}} \\ &+ \frac{1}{5!4!}\epsilon^{\underline{de}[5][4]}(\gamma_{[5]})_{\alpha\beta}K_{[4]}\Big] \quad , \end{aligned} \tag{2.7.18}$$

$$R_{\alpha\underline{b}}{}^{\underline{de}} = i\frac{4}{11}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_\alpha) + \frac{1}{5}(\gamma^{\underline{de}})_\alpha{}^\delta(\partial_{\underline{b}}\psi_\delta)\Big] \quad , \tag{2.7.19}$$

$$R_{\underline{a}\underline{b}}{}^{\underline{de}} = -(\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{2.7.20}$$

2.7.3 Adaptation To 11D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 3

Parameter Composition Rules:

$$\xi^{\underline{m}} = -i\epsilon_1{}^\alpha\epsilon_2{}^\beta(\gamma^{\underline{c}})_{\alpha\beta}\delta_{\underline{c}}{}^{\underline{m}}(1+\Psi) \quad , \tag{2.7.21}$$

$$\begin{aligned} \lambda^{\underline{de}} =& -\frac{1}{80}\epsilon_1{}^\alpha\epsilon_2{}^\beta\Big[(\gamma^{\underline{de}})_{\alpha\beta}K + (\gamma_{[1]})_{\alpha\beta}K^{[1]\underline{de}} - \frac{1}{3!}(\gamma^{\underline{de}[3]})_{\alpha\beta}K_{[3]} - \frac{1}{2}(\gamma_{[2]})_{\alpha\beta}K^{[2]\underline{de}} \\ &+ \frac{1}{5!4!}\epsilon^{\underline{de}[5][4]}(\gamma_{[5]})_{\alpha\beta}K_{[4]}\Big] + i\frac{1}{2}\epsilon_1{}^\alpha\epsilon_2{}^\beta(\gamma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) \quad , \end{aligned} \tag{2.7.22}$$

$$\epsilon^\delta = i\frac{1}{11}\epsilon_1^\alpha\epsilon_2^\beta\left[(\gamma^{[1]})_{\alpha\beta}(\gamma_{[1]})^{\delta\epsilon} - \frac{3}{10}(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\delta\epsilon}\right]\psi_\epsilon \quad . \tag{2.7.23}$$

2.7.4 Adaptation To 11D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 4

SUSY transformation laws:

$$\delta_Q e_{\underline{a}}{}^{\underline{m}} = -i\frac{4}{11}\epsilon^\beta\left[\delta_{\underline{a}}{}^{\underline{d}}\delta_\beta{}^\gamma + \frac{1}{5}(\gamma_{\underline{a}}{}^{\underline{d}})_\beta{}^\gamma\right]\delta_{\underline{d}}{}^{\underline{m}}\psi_\gamma \quad , \tag{2.7.24}$$

$$\begin{aligned} \delta_Q\psi_{\underline{a}}{}^\delta =& (1+\Psi)\partial_{\underline{a}}\epsilon^\delta - \epsilon^\delta(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\ &- i\frac{1}{128}\epsilon^\beta\Big[-(\gamma_{\underline{a}})_\beta{}^\delta K + \frac{1}{2}(\gamma^{[2]})_\beta{}^\delta K_{\underline{a}[2]} - \frac{1}{3!}(\gamma_{\underline{a}[3]})_\beta{}^\delta K^{[3]} + \frac{1}{3!}(\gamma^{[3]})_\beta{}^\delta K_{\underline{a}[3]} \\ &- \frac{1}{4!}(\gamma_{\underline{a}[4]})_\beta{}^\delta K^{[4]}\Big] - \frac{1}{8}\epsilon^\beta\Big[\delta_{\underline{a}}{}^{\underline{c}}\delta_\beta{}^\delta + 3(\gamma_{\underline{a}}{}^{\underline{c}})_\beta{}^\delta\Big](\partial_{\underline{c}}\Psi) \quad , \end{aligned} \tag{2.7.25}$$

$$\delta_Q\phi_{\underline{a}}{}^{\underline{de}} = -i\frac{4}{11}\epsilon^\beta\left[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_\beta) + \frac{1}{5}(\gamma^{\underline{de}})_\beta{}^\delta(\partial_{\underline{a}}\psi_\delta)\right] \quad . \tag{2.7.26}$$

In the remaining subsections of this section, the steps described for the case of the 11D, $\mathcal{N}=1$ theory above will be repeated, essentially line by line, in each of the cases for 10D, $\mathcal{N}=1$, 10D, $\mathcal{N}=2$A, and 10D, $\mathcal{N}=2$B superspaces. This will imply a certain repetitive nature to the respective presentation. There will only be slight variations in explicit details. We are able to minimize this very slightly by noting the result in (2.7.4) applies universally in all three cases. So we will not explicitly rewrite it nor its resultant implications several more times.

2.7.5 Adaptation To 10D, $\mathcal{N}=1$ Component/Superspace Results: Step 1

In the case of 10D $\mathcal{N}=1$ N-SG covariant derivatives we define

$$\nabla_\alpha = \mathrm{D}_\alpha + \frac{1}{2}\Psi\mathrm{D}_\alpha + \frac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{2.7.27}$$

$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\frac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\mathrm{D}_\beta - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{2.7.28}$$

and "split" the spatial 10D $\mathcal{N}=1$ N-SG covariant derivative into two parts

$$\nabla_{\underline{a}}| = \mathbf{D}_{\underline{a}} + \psi_{\underline{a}}{}^\gamma\nabla_\gamma| \quad . \tag{2.7.29}$$

Comparing the result (2.7.28) in with the one in (2.7.29) a component gravitino is identified via

$$\psi_{\underline{a}}{}^\gamma = -i\frac{2}{5}(\sigma_{\underline{a}})^{\gamma\delta}(\mathrm{D}_\delta\Psi) \quad . \tag{2.7.30}$$

However, as this expression contains an explicit σ-matrix we see that it defines the non-conformal *spin*-$\frac{1}{2}$ part of the gravitino to be

$$\psi_\beta \equiv (\sigma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^\gamma \quad . \tag{2.7.31}$$

and it follows only the "σ-trace" of the gravitino can occur. So then we have

$$\mathrm{D}_\beta\Psi = i\frac{1}{4}(\sigma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^\gamma \equiv i\frac{1}{4}\psi_\beta \quad , \tag{2.7.32}$$

in the $\theta \to 0$ limit.

The complete specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. We take advantage of the 10D Fierz identity

$$\delta_{[\gamma}{}^{\alpha}\delta_{\delta]}{}^{\beta} \;=\; \frac{1}{48}\,(\sigma^{[3]})_{\gamma\delta}(\gamma_{[3]})^{\alpha\beta} \quad , \tag{2.7.33}$$

valid for 10D spinors, so we may define a bosonic field:

$$G_{[3]} \;=\; (\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi) \quad , \tag{2.7.34}$$

or in other words,

$$\frac{1}{2}\mathrm{D}_{[\gamma}\mathrm{D}_{\delta]}\Psi \;=\; \frac{1}{16\times 3!}(\sigma^{[3]})_{\gamma\delta}G_{[3]} \quad . \tag{2.7.35}$$

We emphasize that the component field (the G) is defined by the $\theta \to 0$ limit of these equations.

2.7.6 Adaptation To 10D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 2

Torsions:

$$T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.7.36}$$

$$T_{\alpha\beta}{}^{\gamma} = 0 \quad , \tag{2.7.37}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = i\frac{3}{20}\left[\delta_{\underline{b}}{}^{\underline{c}}\delta_{\alpha}{}^{\delta} + (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\delta}\right]\psi_{\delta} \quad , \tag{2.7.38}$$

$$T_{\alpha\underline{b}}{}^{\gamma} = i\frac{1}{80}\Big[-(\sigma^{[2]})_{\alpha}{}^{\gamma}G_{\underline{b}[2]} + \frac{1}{3}(\sigma_{\underline{b}[3]})_{\alpha}{}^{\gamma}G^{[3]}\Big] - \frac{3}{10}\Big[\delta_{\underline{b}}{}^{\underline{c}}\delta_{\alpha}{}^{\gamma} - (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\gamma}\Big](\partial_{\underline{c}}\Psi) \quad , \tag{2.7.39}$$

$$T_{\underline{ab}}{}^{\underline{c}} = 0 \quad , \tag{2.7.40}$$

$$T_{\underline{ab}}{}^{\gamma} \;=\; -\frac{1}{10}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\psi_{\delta}) \quad . \tag{2.7.41}$$

Curvatures:

$$R_{\alpha\beta}{}^{\underline{de}} \;=\; -i\frac{6}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) - \frac{1}{40}\Big[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}G_{[3]} + (\sigma_{[1]})_{\alpha\beta}G^{[1]\underline{de}}\Big] \quad , \tag{2.7.42}$$

$$R_{\alpha\underline{b}}{}^{\underline{de}} = i\frac{1}{4}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\alpha}) + \frac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\gamma}(\partial_{\underline{b}}\psi_{\gamma})\Big] \quad , \tag{2.7.43}$$

$$R_{\underline{ab}}{}^{\underline{de}} \;=\; -(\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{2.7.44}$$

2.7.7 Adaptation To 10D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 3

Parameter Composition Rules:

$$\xi^{\underline{m}} = -i\epsilon_1{}^{\alpha}\epsilon_2{}^{\beta}(\sigma^{\underline{c}})_{\alpha\beta}\delta_{\underline{c}}{}^{\underline{m}}(1+\Psi) \quad , \tag{2.7.45}$$

$$\lambda^{\underline{de}} = \frac{1}{40}\epsilon_1{}^{\alpha}\epsilon_2{}^{\beta}\Big[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}G_{[3]} + (\sigma_{[1]})_{\alpha\beta}G^{[1]\underline{de}}\Big] + i\frac{17}{10}\epsilon_1{}^{\alpha}\epsilon_2{}^{\beta}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) \quad , \tag{2.7.46}$$

$$\epsilon^{\delta} = -i\frac{1}{10}\epsilon_1{}^{\alpha}\epsilon_2{}^{\beta}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\epsilon}\psi_{\epsilon} \quad . \tag{2.7.47}$$

2.7.8 Adaptation To 10D, $\mathcal{N}$ = 1 Component/Superspace Results: Step 4

SUSY transformation laws:

$$\delta_Q e_{\underline{a}}{}^{\underline{m}} = -i\frac{1}{4}\epsilon^{\beta}\left[\delta_{\underline{a}}{}^{\underline{d}}\delta_{\beta}{}^{\gamma} + \frac{1}{5}(\sigma_{\underline{a}}{}^{\underline{d}})_{\beta}{}^{\gamma}\right]\delta_{\underline{d}}{}^{\underline{m}}\psi_{\gamma} \quad , \tag{2.7.48}$$

$$\begin{aligned}\delta_Q\psi_{\underline{a}}{}^{\delta} =& (1+\Psi)\partial_{\underline{a}}\epsilon^{\delta} - \epsilon^{\delta}(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}}\\ &- i\frac{1}{80}\epsilon^{\beta}\Big[-(\sigma^{[2]})_{\beta}{}^{\delta}G_{\underline{a}[2]} + \frac{1}{3}(\sigma_{\underline{a}[3]})_{\beta}{}^{\delta}G^{[3]}\Big] + \frac{3}{10}\epsilon^{\beta}\Big[\delta_{\underline{a}}{}^{\underline{c}}\delta_{\beta}{}^{\delta} - (\sigma_{\underline{a}}{}^{\underline{c}})_{\beta}{}^{\delta}\Big](\partial_{\underline{c}}\Psi) \quad ,\end{aligned} \tag{2.7.49}$$

$$\delta_Q\phi_{\underline{a}}{}^{\underline{de}} = -i\frac{1}{4}\epsilon^{\beta}\left[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\beta}) + \frac{1}{5}(\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\psi_{\gamma})\right] \quad . \tag{2.7.50}$$

2.7.9 Adaptation To 10D, $\mathcal{N}$ = 2A Component/Superspace Results: Step 1

In the case of 10D $\mathcal{N} = 2A$ N-SG covariant derivatives we define

$$\nabla_{\alpha} = \mathrm{D}_{\alpha} + \frac{1}{2}\Psi\mathrm{D}_{\alpha} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{2.7.51}$$

$$\nabla_{\dot{\alpha}} = \mathrm{D}_{\dot{\alpha}} + \frac{1}{2}\Psi\mathrm{D}_{\dot{\alpha}} + \frac{1}{10}(\sigma^{\underline{ab}})_{\dot{\alpha}}{}^{\dot{\beta}}(\mathrm{D}_{\dot{\beta}}\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{2.7.52}$$

$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\frac{1}{5}(\sigma_{\underline{a}})^{\delta\gamma}(\mathrm{D}_{\delta}\Psi)\mathrm{D}_{\gamma} - i\frac{1}{5}(\sigma_{\underline{a}})^{\dot{\delta}\dot{\gamma}}(\mathrm{D}_{\dot{\delta}}\Psi)\mathrm{D}_{\dot{\gamma}} - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{2.7.53}$$

and "split" the spatial 10D $\mathcal{N} = 2A$ N-SG covariant derivative into three parts

$$\nabla_{\underline{a}}| = \mathbf{D}_{\underline{a}} + \psi_{\underline{a}}{}^{\gamma}\nabla_{\gamma}| + \psi_{\underline{a}}{}^{\dot{\gamma}}\nabla_{\dot{\gamma}}| \quad . \tag{2.7.54}$$

On taking the $\theta \to 0$ limit the latter terms allow an identification with the component gravitinos are identified via

$$\psi_{\underline{a}}{}^{\gamma} \; = \; - \, i\frac{1}{5}(\sigma_{\underline{a}})^{\gamma\delta}(\mathrm{D}_{\delta}\Psi) \quad , \quad \psi_{\underline{a}}{}^{\dot{\gamma}} \; = \; -i\frac{1}{5}(\sigma_{\underline{a}})^{\dot{\gamma}\dot{\delta}}(\mathrm{D}_{\dot{\delta}}\Psi) \quad . \tag{2.7.55}$$

However, as this expression contains an explicit σ-matrix we see that it really defines the non-conformal *spin*-$\frac{1}{2}$ part of the gravitino to be

$$\psi_{\beta} \; \equiv (\sigma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^{\gamma} \quad , \quad \psi_{\dot{\beta}} \; \equiv \; (\sigma^{\underline{a}})_{\dot{\beta}\dot{\gamma}}\psi_{\underline{a}}{}^{\dot{\gamma}} \quad . \tag{2.7.56}$$

It follows only the "σ-trace" of the gravitino can occur. So then we have

$$\mathrm{D}_{\beta}\Psi \; = i\frac{1}{2}(\sigma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^{\gamma} \; \equiv \; i\frac{1}{2}\psi_{\beta} \quad , \quad \mathrm{D}_{\dot{\beta}}\Psi \; = \; i\frac{1}{2}(\sigma^{\underline{a}})_{\dot{\beta}\dot{\gamma}}\psi_{\underline{a}}{}^{\dot{\gamma}} \; \equiv \; i\frac{1}{2}\psi_{\dot{\beta}} \quad , \tag{2.7.57}$$

in the $\theta \to 0$ limit.

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:

$$G_{[3]} \; = (\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_{\gamma}\mathrm{D}_{\delta}\Psi) \quad , \quad H_{[3]} \; = \; (\sigma_{[3]})^{\dot{\gamma}\dot{\delta}}(\mathrm{D}_{\dot{\gamma}}\mathrm{D}_{\dot{\delta}}\Psi) \quad , \tag{2.7.58}$$

$$N \; = C^{\gamma\dot{\delta}}(\mathrm{D}_{\gamma}\mathrm{D}_{\dot{\delta}}\Psi) \quad , \quad N_{[2]} \; = \; (\sigma_{[2]})^{\gamma\dot{\delta}}(\mathrm{D}_{\gamma}\mathrm{D}_{\dot{\delta}}\Psi) \quad , \quad N_{[4]} \; = \; (\sigma_{[4]})^{\gamma\dot{\delta}}(\mathrm{D}_{\gamma}\mathrm{D}_{\dot{\delta}}\Psi) \quad , \tag{2.7.59}$$

or in other words,

$$\frac{1}{2}\mathrm{D}_{[\gamma}\mathrm{D}_{\delta]}\Psi \; = \; \frac{1}{16 \times 3!}(\sigma^{[3]})_{\gamma\delta}G_{[3]} \quad , \quad \frac{1}{2}\mathrm{D}_{[\dot{\gamma}}\mathrm{D}_{\dot{\delta}]}\Psi \; = \; \frac{1}{16 \times 3!}(\sigma^{[3]})_{\dot{\gamma}\dot{\delta}}H_{[3]} \quad , \tag{2.7.60}$$

and

$$\mathrm{D}_{\gamma}\mathrm{D}_{\dot{\delta}}\Psi \; = \; \frac{1}{16}\Big\{C_{\gamma\dot{\delta}}N + \frac{1}{2!}(\sigma^{[2]})_{\gamma\dot{\delta}}N_{[2]} + \frac{1}{4!}(\sigma^{[4]})_{\gamma\dot{\delta}}N_{[4]}\Big\} \quad . \tag{2.7.61}$$

We emphasize that the component fields (the G's, H's, and N's) are defined by the $\theta \to 0$ limit of these equations.

2.7.10 Adaptation To 10D, $\mathcal{N} = 2A$ Component/Superspace Results: Step 2

Torsions:

$$T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.7.62}$$

$$T_{\alpha\beta}{}^{\gamma} = i\frac{1}{10}(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\gamma\delta}\psi_{\delta} \quad , \tag{2.7.63}$$

$$T_{\alpha\beta}{}^{\dot{\gamma}} = -i\frac{1}{10}(\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}})^{\dot{\gamma}\dot{\delta}}\psi_{\dot{\delta}} \quad , \tag{2.7.64}$$

$$T_{\dot{\alpha}\dot{\beta}}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\dot{\alpha}\dot{\beta}} \quad , \tag{2.7.65}$$

$$T_{\dot{\alpha}\dot{\beta}}{}^{\gamma} = -i\frac{1}{10}(\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{a}})^{\gamma\delta}\psi_{\delta} \quad , \tag{2.7.66}$$

$$T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}} = i\frac{1}{10}(\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{a}})^{\dot{\gamma}\dot{\delta}}\psi_{\dot{\delta}} \quad , \tag{2.7.67}$$

$$T_{\alpha\dot{\beta}}{}^{\underline{c}} = 0 \quad , \tag{2.7.68}$$

$$T_{\alpha\dot{\beta}}{}^{\gamma} = i\frac{1}{4}\left[\delta_{\alpha}{}^{\gamma}\delta_{\dot{\beta}}{}^{\dot{\delta}} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\gamma}(\sigma_{\underline{ab}})_{\dot{\beta}}{}^{\dot{\delta}}\right]\psi_{\dot{\delta}} \quad , \tag{2.7.69}$$

$$T_{\alpha\dot{\beta}}{}^{\dot{\gamma}} = i\frac{1}{4}\left[\delta_{\dot{\beta}}{}^{\dot{\gamma}}\delta_{\alpha}{}^{\delta} + \frac{1}{10}(\sigma^{\underline{ab}})_{\dot{\beta}}{}^{\dot{\gamma}}(\sigma_{\underline{ab}})_{\alpha}{}^{\delta}\right]\psi_{\delta} \quad , \tag{2.7.70}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = i\frac{1}{5}\Big[2\delta_{\underline{b}}{}^{\underline{c}}\delta_{\alpha}{}^{\delta} + (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\delta}\Big]\psi_{\delta} \quad , \tag{2.7.71}$$

$$T_{\alpha\underline{b}}{}^{\gamma} = i\frac{1}{80}\Big[-\frac{1}{2}(\sigma^{[2]})_{\alpha}{}^{\gamma}G_{\underline{b}[2]} + \frac{1}{3!}(\sigma_{\underline{b}[3]})_{\alpha}{}^{\gamma}G^{[3]}\Big] - \frac{2}{5}\Big[\delta_{\underline{b}}{}^{\underline{c}}\delta_{\alpha}{}^{\gamma} - (\sigma_{\underline{b}}{}^{\underline{c}})_{\alpha}{}^{\gamma}\Big](\partial_{\underline{c}}\Psi) \quad , \tag{2.7.72}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\dot{\gamma}} = {} & -i\frac{1}{80}\Big[(\sigma_{\underline{b}})_{\alpha}{}^{\dot{\gamma}}N - (\sigma^{[1]})_{\alpha}{}^{\dot{\gamma}}N_{\underline{b}[1]} + \frac{1}{2}(\sigma_{\underline{b}[2]})_{\alpha}{}^{\dot{\gamma}}N^{[2]} - \frac{1}{3!}(\sigma^{[3]})_{\alpha}{}^{\dot{\gamma}}N_{\underline{b}[3]} \\ & + \frac{1}{4!}(\sigma_{\underline{b}[4]})_{\alpha}{}^{\dot{\gamma}}N^{[4]}\Big] \quad , \end{aligned} \tag{2.7.73}$$

$$T_{\dot{\alpha}\underline{b}}{}^{\underline{c}} = i\frac{1}{5}\Big[2\delta_{\underline{b}}{}^{\underline{c}}\delta_{\dot{\alpha}}{}^{\dot{\delta}} + (\sigma_{\underline{b}}{}^{\underline{c}})_{\dot{\alpha}}{}^{\dot{\delta}}\Big]\psi_{\dot{\delta}} \quad , \tag{2.7.74}$$

$$\begin{aligned} T_{\dot{\alpha}\underline{b}}{}^{\gamma} = {} & -i\frac{1}{80}\Big[(\sigma_{\underline{b}})_{\dot{\alpha}}{}^{\gamma}N + (\sigma^{[1]})_{\dot{\alpha}}{}^{\gamma}N_{\underline{b}[1]} - \frac{1}{2}(\sigma_{\underline{b}[2]})_{\dot{\alpha}}{}^{\gamma}N^{[2]} - \frac{1}{3!}(\sigma^{[3]})_{\dot{\alpha}}{}^{\gamma}N_{\underline{b}[3]} \\ & + \frac{1}{4!}(\sigma_{\underline{b}[4]})_{\dot{\alpha}}{}^{\gamma}N^{[4]}\Big] \quad , \end{aligned} \tag{2.7.75}$$

$$T_{\dot{\alpha}\underline{b}}{}^{\dot{\gamma}} = i\frac{1}{80}\Big[-\frac{1}{2}(\sigma^{[2]})_{\dot{\alpha}}{}^{\dot{\gamma}}H_{\underline{b}[2]} + \frac{1}{3!}(\sigma_{\underline{b}[3]})_{\dot{\alpha}}{}^{\dot{\gamma}}H^{[3]}\Big] - \frac{2}{5}\Big[\delta_{\underline{b}}{}^{\underline{c}}\delta_{\dot{\alpha}}{}^{\dot{\gamma}} - (\sigma_{\underline{b}}{}^{\underline{c}})_{\dot{\alpha}}{}^{\dot{\gamma}}\Big](\partial_{\underline{c}}\Psi) \quad , \tag{2.7.76}$$

$$T_{\underline{ab}}{}^{\underline{c}} = 0 \quad , \tag{2.7.77}$$

$$T_{\underline{ab}}{}^{\gamma} = -\frac{1}{10}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\psi_{\delta}) \quad , \tag{2.7.78}$$

$$T_{\underline{ab}}{}^{\dot\gamma} \;=\; -\,\frac{1}{10}(\sigma_{[\underline{a}})^{\dot\gamma\dot\delta}(\partial_{\underline{b}]}\psi_{\dot\delta}) \quad . \tag{2.7.79}$$

Curvatures:

$$\begin{aligned}
R_{\alpha\beta}{}^{\underline{de}} \;&=\; -\,i\frac{6}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}G_{[3]} + (\sigma_{[1]})_{\alpha\beta}G^{[1]\underline{de}}\right] \quad , && (2.7.80)\\
R_{\dot\alpha\dot\beta}{}^{\underline{de}} \;&=\; -\,i\frac{6}{5}(\sigma^{[\underline{d}})_{\dot\alpha\dot\beta}(\partial^{\underline{e}]}\Psi) - \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\dot\alpha\dot\beta}H_{[3]} + (\sigma_{[1]})_{\dot\alpha\dot\beta}H^{[1]\underline{de}}\right] \quad , && (2.7.81)\\
R_{\alpha\dot\beta}{}^{\underline{de}} \;&=\; \frac{1}{40}\Big[(\sigma^{\underline{de}})_{\alpha\dot\beta}N - C_{\alpha\dot\beta}N^{\underline{de}} + \frac{1}{2}(\sigma^{\underline{de}[2]})_{\alpha\dot\beta}N_{[2]}\\
&\qquad - \frac{1}{2}(\sigma_{[2]})_{\alpha\dot\beta}N^{\underline{de}[2]} + \frac{1}{4!4!}\epsilon^{\underline{de}[4][\bar 4]}(\sigma_{[4]})_{\alpha\dot\beta}N_{[\bar 4]}\Big] \quad , && (2.7.82)\\
R_{\alpha\underline{b}}{}^{\underline{de}} \;&=\; i\frac{1}{2}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_\alpha) + \frac{1}{5}(\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\psi_\gamma)\Big] \quad , && (2.7.83)\\
R_{\dot\alpha\underline{b}}{}^{\underline{de}} \;&=\; i\frac{1}{2}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\dot\alpha}) + \frac{1}{5}(\sigma^{\underline{de}})_{\dot\alpha}{}^{\dot\gamma}(\partial_{\underline{b}}\psi_{\dot\gamma})\Big] \quad , && (2.7.84)\\
R_{\underline{ab}}{}^{\underline{de}} \;&=\; -\,(\partial_{[\underline{a}}\partial^{[\underline{d}}\Psi)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . && (2.7.85)
\end{aligned}$$

2.7.11 Adaptation To 10D, $\mathcal{N}$ = 2A Component/Superspace Results: Step 3

Parameter Composition Rules:

$$\xi^{\underline{m}} \;=\; -\,i\big[\;\epsilon_1{}^\alpha\epsilon_2{}^\beta(\sigma^{\underline{c}})_{\alpha\beta} + \epsilon_1{}^{\dot\alpha}\epsilon_2{}^{\dot\beta}(\sigma^{\underline{c}})_{\dot\alpha\dot\beta}\;\big]\;\delta_{\underline{c}}{}^{\underline{m}}(1+\Psi) \quad , \tag{2.7.86}$$

$$\begin{aligned}
\lambda^{\underline{de}} \;=\; &-\,\frac{1}{40}(\epsilon_1{}^\alpha\epsilon_2{}^{\dot\beta} + \epsilon_1{}^{\dot\beta}\epsilon_2{}^\alpha)\Big[(\sigma^{\underline{de}})_{\alpha\dot\beta}N - C_{\alpha\dot\beta}N^{\underline{de}} + \frac{1}{2}(\sigma^{\underline{de}[2]})_{\alpha\dot\beta}N_{[2]}\\
&\qquad - \frac{1}{2}(\sigma_{[2]})_{\alpha\dot\beta}N^{\underline{de}[2]} + \frac{1}{4!4!}\epsilon^{\underline{de}[4][\bar 4]}(\sigma_{[4]})_{\alpha\dot\beta}N_{[\bar 4]}\Big]\\
&+ \epsilon_1{}^\alpha\epsilon_2{}^\beta\left[i\frac{17}{10}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) + \frac{1}{40}\Big[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}G_{[3]} + (\sigma_{[1]})_{\alpha\beta}G^{[1]\underline{de}}\Big]\right]\\
&+ \epsilon_1{}^{\dot\alpha}\epsilon_2{}^{\dot\beta}\left[i\frac{17}{10}(\sigma^{[\underline{d}})_{\dot\alpha\dot\beta}(\partial^{\underline{e}]}\Psi) + \frac{1}{40}\Big[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\dot\alpha\dot\beta}H_{[3]} + (\sigma_{[1]})_{\dot\alpha\dot\beta}H^{[1]\underline{de}}\Big]\right] \quad ,
\end{aligned} \tag{2.7.87}$$

$$\begin{aligned}
\epsilon^\delta \;=\; &-\,i\frac{1}{4}(\epsilon_1{}^\alpha\epsilon_2{}^{\dot\beta} + \epsilon_1{}^{\dot\beta}\epsilon_2{}^\alpha)\left[\delta_\alpha{}^\delta\delta_{\dot\beta}{}^{\dot\epsilon} + \frac{1}{10}(\sigma^{[2]})_\alpha{}^\delta(\sigma_{[2]})_{\dot\beta}{}^{\dot\epsilon}\right]\psi_{\dot\epsilon}\\
&- i\frac{1}{5}\epsilon_1{}^\alpha\epsilon_2{}^\beta(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\epsilon}\psi_\epsilon \quad .
\end{aligned} \tag{2.7.88}$$

2.7.12 Adaptation To 10D, $\mathcal{N}$ = 2A Component/Superspace Results: Step 4

SUSY transformation laws:

$$\delta_Q e_{\underline{a}}{}^{\underline{m}} = -i\frac{1}{2}\epsilon^{\beta}\left[\delta_{\underline{a}}{}^{\underline{d}}\delta_{\beta}{}^{\gamma} + \frac{1}{5}(\sigma_{\underline{a}}{}^{\underline{d}})_{\beta}{}^{\gamma}\right]\delta_{\underline{d}}{}^{\underline{m}}\psi_{\gamma} - i\frac{1}{2}\epsilon^{\dot{\beta}}\left[\delta_{\underline{a}}{}^{\underline{d}}\delta_{\dot{\beta}}{}^{\dot{\gamma}} + \frac{1}{5}(\sigma_{\underline{a}}{}^{\underline{d}})_{\dot{\beta}}{}^{\dot{\gamma}}\right]\delta_{\underline{d}}{}^{\underline{m}}\psi_{\dot{\gamma}} \quad , \tag{2.7.89}$$

$$\begin{aligned}\delta_Q \psi_{\underline{a}}{}^{\delta} =& (1+\Psi)\partial_{\underline{a}}\epsilon^{\delta} - \epsilon^{\delta}(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\ & - i\frac{1}{80}\epsilon^{\beta}\Big[-\frac{1}{2}(\sigma^{[2]})_{\beta}{}^{\delta}G_{\underline{a}[2]} + \frac{1}{3!}(\sigma_{\underline{a}[3]})_{\beta}{}^{\delta}G^{[3]}\Big] + \frac{2}{5}\epsilon^{\beta}\Big[\delta_{\underline{a}}{}^{\underline{c}}\delta_{\beta}{}^{\delta} - (\sigma_{\underline{a}}{}^{\underline{c}})_{\beta}{}^{\delta}\Big](\partial_{\underline{c}}\Psi) \\ & + i\frac{1}{80}\epsilon^{\dot{\beta}}\Big[(\sigma_{\underline{a}})_{\dot{\beta}}{}^{\delta}N + (\sigma^{[1]})_{\dot{\beta}}{}^{\delta}N_{\underline{a}[1]} - \frac{1}{2}(\sigma_{\underline{a}[2]})_{\dot{\beta}}{}^{\delta}N^{[2]} - \frac{1}{3!}(\sigma^{[3]})_{\dot{\beta}}{}^{\delta}N_{\underline{a}[3]} \\ & \qquad + \frac{1}{4!}(\sigma_{\underline{a}[4]})_{\dot{\beta}}{}^{\delta}N^{[4]}\Big] \quad ,\end{aligned} \tag{2.7.90}$$

$$\delta_Q \phi_{\underline{a}}{}^{\underline{de}} = -i\frac{1}{2}\epsilon^{\beta}\Big[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\beta}) + \frac{1}{5}(\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\psi_{\gamma})\Big] - i\frac{1}{2}\epsilon^{\dot{\beta}}\Big[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\dot{\beta}}) + \frac{1}{5}(\sigma^{\underline{de}})_{\dot{\beta}}{}^{\dot{\gamma}}(\partial_{\underline{a}}\psi_{\dot{\gamma}})\Big] \quad . \tag{2.7.91}$$

2.7.13 Adaptation To 10D, $\mathcal{N}$ = 2B Component/Superspace Results: Step 1

In the case of 10D $\mathcal{N} = 2B$ N-SG covariant derivatives we define

$$\nabla_{\alpha} = \mathrm{D}_{\alpha} + \frac{1}{2}\Psi\mathrm{D}_{\alpha} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\mathrm{D}_{\beta}\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{2.7.92}$$

$$\overline{\nabla}_{\alpha} = \overline{\mathrm{D}}_{\alpha} + \frac{1}{2}\overline{\Psi}\overline{\mathrm{D}}_{\alpha} + \frac{1}{10}(\sigma^{\underline{ab}})_{\alpha}{}^{\beta}(\overline{\mathrm{D}}_{\beta}\overline{\Psi})\mathcal{M}_{\underline{ab}} \quad , \tag{2.7.93}$$

$$\begin{aligned}\nabla_{\underline{a}} =& \partial_{\underline{a}} + \frac{1}{2}\Psi\partial_{\underline{a}} + \frac{1}{2}\overline{\Psi}\partial_{\underline{a}} - i\frac{1}{32}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_{\alpha}\overline{\Psi})\overline{\mathrm{D}}_{\beta} - i\frac{1}{32}(\sigma_{\underline{a}})^{\alpha\beta}(\overline{\mathrm{D}}_{\alpha}\Psi)\mathrm{D}_{\beta} \\ & - i\frac{27}{160}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_{\alpha}\Psi)\overline{\mathrm{D}}_{\beta} - i\frac{27}{160}(\sigma_{\underline{a}})^{\alpha\beta}(\overline{\mathrm{D}}_{\alpha}\overline{\Psi})\mathrm{D}_{\beta} \\ & - \frac{1}{2}(\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} - \frac{1}{2}(\partial_{\underline{c}}\overline{\Psi})\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad ,\end{aligned} \tag{2.7.94}$$

and "split" the spatial 10D $\mathcal{N} = 2B$ N-SG covariant derivative into three parts

$$\nabla_{\underline{a}}| = \mathbf{D}_{\underline{a}} + \psi_{\underline{a}}{}^{\gamma}\nabla_{\gamma}| + \overline{\psi}_{\underline{a}}{}^{\gamma}\overline{\nabla}_{\gamma}| \quad . \tag{2.7.95}$$

On taking the $\theta \to 0$ limit the latter terms allows an identification with the gravitino and the leading term in this limit yields a component-level linearized gravitationally covariant derivative operator given by

$$\mathbf{D}_{\underline{a}} \;=\; e_{\underline{a}} + \phi_{\underline{a}}{}^{\iota}\mathcal{M}_{\iota} \;=\; \partial_{\underline{a}} + \frac{1}{2}(\Psi + \overline{\Psi})\partial_{\underline{a}} + \phi_{\underline{a}}{}^{\iota}\mathcal{M}_{\iota} \quad . \tag{2.7.96}$$

Comparison of the LHS to the RHS of (2.7.96), we see that a linearized frame field $e_{\underline{a}}{}^{\underline{m}}$ $= (\,1 + \frac{1}{2}(\Psi + \overline{\Psi})\,)\delta_{\underline{a}}{}^{\underline{m}}$ emerges to describe a scalar graviton. Finally, comparison of the coefficient of the Lorentz generator $\mathcal{M}_{\iota}$ as it appears in the latter two forms of (2.7.96) informs us the spin connection is given by

$$\phi_{\underline{c}}{}^{\underline{de}} \;=\; -\frac{1}{4}\delta_{\underline{c}}{}^{[\underline{d}}\big(\partial^{\underline{e}]}(\Psi + \overline{\Psi})\big) \quad . \tag{2.7.97}$$

Comparing the result (2.7.94) in with the one in (2.7.95) the component gravitinos are identified via

$$\psi_{\underline{a}}{}^{\gamma} \;=\; -\,i\frac{1}{160}(\sigma_{\underline{a}})^{\gamma\delta}\big(\overline{\mathrm{D}}_{\delta}(5\Psi + 27\overline{\Psi})\big) \quad , \tag{2.7.98}$$

$$\overline{\psi}_{\underline{a}}{}^{\gamma} \;=\; -\,i\frac{1}{160}(\sigma_{\underline{a}})^{\gamma\delta}\big(\mathrm{D}_{\delta}(5\overline{\Psi} + 27\Psi)\big) \quad , \tag{2.7.99}$$

which are equivalent to

$$\overline{\mathrm{D}}_{\alpha}(5\Psi + 27\overline{\Psi}) \;=i16(\sigma^{\underline{a}})_{\alpha\gamma}\psi_{\underline{a}}{}^{\gamma} \quad , \quad \mathrm{D}_{\alpha}(5\overline{\Psi} + 27\Psi) \;=\; i16(\sigma^{\underline{a}})_{\alpha\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma} \quad . \tag{2.7.100}$$

However, as this expression contains an explicit σ-matrix we see that it really defines the non-conformal *spin*-$\frac{1}{2}$ part of the gravitino to be

$$\psi_{\beta} \;\equiv (\sigma^{\underline{a}})_{\beta\gamma}\psi_{\underline{a}}{}^{\gamma} \quad , \quad \overline{\psi}_{\beta} \;\equiv\; -(\sigma^{\underline{a}})_{\beta\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma} \quad . \tag{2.7.101}$$

Since the results in (2.7.100) are under-constrained, we are allowed to introduce a fermionic auxiliary field λ_{α} and its complex conjugate $\overline{\lambda}_{\alpha}$. So then we have

$$\overline{\mathrm{D}}_{\alpha}\Psi \;= i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\psi_{\underline{a}}{}^{\gamma} - 27\overline{\lambda}_{\alpha} \;\equiv\; i\frac{1}{2}\psi_{\alpha} - 27\overline{\lambda}_{\alpha} \quad , \tag{2.7.102}$$

$$\overline{\mathrm{D}}_{\alpha}\overline{\Psi} \;= i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\psi_{\underline{a}}{}^{\gamma} + 5\overline{\lambda}_{\alpha} \;\equiv\; i\frac{1}{2}\psi_{\alpha} + 5\overline{\lambda}_{\alpha} \quad , \tag{2.7.103}$$

$$\mathrm{D}_{\alpha}\overline{\Psi} \;= i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma} - 27\lambda_{\alpha} \;\equiv\; -i\frac{1}{2}\overline{\psi}_{\alpha} - 27\lambda_{\alpha} \quad , \tag{2.7.104}$$

$$\mathrm{D}_\alpha \Psi \;=i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma}+5\lambda_\alpha \;\equiv\; -i\frac{1}{2}\overline{\psi}_\alpha+5\lambda_\alpha \quad , \tag{2.7.105}$$

in the $\theta \to 0$ limit. Also observe that

$$\overline{\mathrm{D}}_\alpha(\overline{\Psi}-\Psi) \;=32\overline{\lambda}_\alpha \quad , \quad \mathrm{D}_\alpha(\Psi-\overline{\Psi}) \;=\; 32\lambda_\alpha \quad . \tag{2.7.106}$$

In order to complete the specification of the geometrical superfields also requires explicit definitions of the bosonic terms to second order in D-derivatives. So we define bosonic fields:

$$U_{[3]} \;=(\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_\gamma\mathrm{D}_\delta\Psi) \quad , \qquad \overline{U}_{[3]} \;=\; -\,(\sigma_{[3]})^{\gamma\delta}(\overline{\mathrm{D}}_\gamma\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad , \tag{2.7.107}$$

$$X_{[3]} \;=(\sigma_{[3]})^{\gamma\delta}(\overline{\mathrm{D}}_\gamma\overline{\mathrm{D}}_\delta\Psi) \quad , \qquad \overline{X}_{[3]} \;=\; -\,(\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_\gamma\mathrm{D}_\delta\overline{\Psi}) \quad , \tag{2.7.108}$$

$$\begin{aligned} Y_{[3]} \;&=(\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_\gamma\overline{\mathrm{D}}_\delta\Psi) & \overline{Y}_{[3]} \;&=\; -\,(\sigma_{[3]})^{\gamma\delta}(\overline{\mathrm{D}}_\gamma\mathrm{D}_\delta\overline{\Psi}) \\ &=(\sigma_{[3]})^{\gamma\delta}(\overline{\mathrm{D}}_\gamma\mathrm{D}_\delta\Psi) \quad , & &=\; -\,(\sigma_{[3]})^{\gamma\delta}(\mathrm{D}_\gamma\overline{\mathrm{D}}_\delta\overline{\Psi}) \quad . \end{aligned} \tag{2.7.109}$$

We emphasize that the component fields (the U's, X's, and Y's) are defined by the $\theta \to 0$ limit of these equations.

2.7.14 Adaptation To 10D, $\mathcal{N}$ = 2B Component/Superspace Results: Step 2

Torsions:

$$T_{\alpha\beta}{}^{\underline{c}} \;=0 \quad , \tag{2.7.110}$$

$$T_{\alpha\beta}{}^{\gamma} \;=\; -\,i\frac{1}{5}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}\overline{\psi}_\delta+2(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}\lambda_\delta \quad , \tag{2.7.111}$$

$$T_{\alpha\beta}{}^{\bar{\gamma}} \;=0 \quad , \tag{2.7.112}$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\underline{c}} \;=0 \quad , \tag{2.7.113}$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\gamma} \;=0 \quad , \tag{2.7.114}$$

$$T_{\bar{\alpha}\bar{\beta}}{}^{\bar{\gamma}} \;=i\frac{1}{5}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}\psi_\delta+2(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\gamma\delta}\overline{\lambda}_\delta \quad , \tag{2.7.115}$$

$$T_{\alpha\bar{\beta}}{}^{\underline{c}} \;=i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \tag{2.7.116}$$

$$T_{\alpha\bar{\beta}}{}^{\gamma} \;=\; -\,i\frac{1}{240}(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\psi_\delta+\frac{1}{8}\Big[(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}-\frac{1}{30}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\Big]\overline{\lambda}_\delta \quad , \tag{2.7.117}$$

$$T_{\alpha\bar{\beta}}{}^{\bar{\gamma}} = i\frac{1}{240}(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\overline{\psi}_\delta + \frac{1}{8}\Big[(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} - \frac{1}{30}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\gamma\delta}\Big]\lambda_\delta \quad , \tag{2.7.118}$$

$$T_{\alpha\underline{b}}{}^{\underline{c}} = -i\frac{1}{5}\Big[2\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big]\overline{\psi}_\gamma + \Big[-11\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big]\lambda_\gamma \quad , \tag{2.7.119}$$

$$\begin{aligned} T_{\alpha\underline{b}}{}^{\gamma} = & \frac{1}{64}\Big[-31\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + 15(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big](\partial_{\underline{c}}\Psi) + \frac{1}{320}\Big[27\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + 53(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big](\partial_{\underline{c}}\overline{\Psi}) \\ & - i\frac{1}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma\Big(5Y_{\underline{b}[2]} - 27\overline{Y}_{\underline{b}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma\Big(5Y^{[3]} - 27\overline{Y}^{[3]}\Big)\Big] \quad , \end{aligned} \tag{2.7.120}$$

$$T_{\alpha\underline{b}}{}^{\bar{\gamma}} = -i\frac{1}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma\Big(-5\overline{X}_{\underline{b}[2]} + 27U_{\underline{b}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma\Big(-5\overline{X}^{[3]} + 27U^{[3]}\Big)\Big] \quad , \tag{2.7.121}$$

$$T_{\bar{\alpha}\underline{b}}{}^{\underline{c}} = i\frac{1}{5}\Big[2\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big]\psi_\gamma + \Big[-11\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big]\overline{\lambda}_\gamma \quad , \tag{2.7.122}$$

$$T_{\bar{\alpha}\underline{b}}{}^{\gamma} = -i\frac{1}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma\Big(5X_{\underline{b}[2]} - 27\overline{U}_{\underline{b}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma\Big(5X^{[3]} - 27\overline{U}^{[3]}\Big)\Big] \quad , \tag{2.7.123}$$

$$\begin{aligned} T_{\bar{\alpha}\underline{b}}{}^{\bar{\gamma}} = & \frac{1}{64}\Big[-31\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + 15(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big](\partial_{\underline{c}}\overline{\Psi}) + \frac{1}{320}\Big[27\delta_{\underline{b}}{}^{\underline{c}}\delta_\alpha{}^\gamma + 53(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma\Big](\partial_{\underline{c}}\Psi) \\ & - i\frac{1}{2560}\Big[\frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma\Big(-5\overline{Y}_{\underline{b}[2]} + 27Y_{\underline{b}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma\Big(-5\overline{Y}^{[3]} + 27Y^{[3]}\Big)\Big] \quad , \end{aligned} \tag{2.7.124}$$

$$T_{\underline{ab}}{}^{\underline{c}} = 0 \quad , \tag{2.7.125}$$

$$T_{\underline{ab}}{}^{\gamma} = -\frac{1}{10}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\psi_\delta) \quad , \tag{2.7.126}$$

$$T_{\underline{ab}}{}^{\bar{\gamma}} = \frac{1}{10}(\sigma_{[\underline{a}})^{\gamma\delta}(\partial_{\underline{b}]}\overline{\psi}_\delta) \quad . \tag{2.7.127}$$

Curvatures:

$$R_{\alpha\beta}{}^{\underline{de}} = \frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}U_{[3]} - (\sigma_{[1]})_{\alpha\beta}U^{[1]\underline{de}}\right] \quad , \tag{2.7.128}$$

$$R_{\bar{\alpha}\bar{\beta}}{}^{\underline{de}} = -\frac{1}{40}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}\overline{U}_{[3]} - (\sigma_{[1]})_{\alpha\beta}\overline{U}^{[1]\underline{de}}\right] \quad , \tag{2.7.129}$$

$$\begin{aligned} R_{\alpha\bar{\beta}}{}^{\underline{de}} = & -i\frac{3}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}(\Psi + \overline{\Psi})) - i\frac{1}{10}(\sigma^{\underline{def}})_{\alpha\beta}(\partial_{\underline{f}}(\Psi + \overline{\Psi})) \\ & - \frac{1}{80}\Big[(\sigma_{[1]})_{\alpha\beta}\Big(Y^{[1]\underline{de}} - \overline{Y}^{[1]\underline{de}}\Big) - \frac{1}{2}(\sigma^{[2][\underline{d}})_{\alpha\beta}\Big(Y^{\underline{e}]}{}_{[2]} - \overline{Y}^{\underline{e}]}{}_{[2]}\Big) \qquad (2.7.130) \\ & - \frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}\Big(Y_{[3]} - \overline{Y}_{[3]}\Big)\Big] \quad , \qquad (2.7.131) \end{aligned}$$

$$R_{\alpha\underline{b}}{}^{\underline{de}} = -i\frac{1}{2}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\overline{\psi}_\alpha) + \frac{1}{5}(\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\overline{\psi}_\gamma)\Big] - 11\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\lambda_\alpha) + (\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\lambda_\gamma) \quad , \tag{2.7.132}$$

$$R_{\bar{\alpha}\underline{b}}{}^{\underline{de}} = i\frac{1}{2}\Big[\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_\alpha) + \frac{1}{5}(\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\psi_\gamma)\Big] - 11\delta_{\underline{b}}{}^{[\underline{d}}(\partial^{\underline{e}]}\overline{\lambda}_\alpha) + (\sigma^{\underline{de}})_\alpha{}^\gamma(\partial_{\underline{b}}\overline{\lambda}_\gamma) \quad , \tag{2.7.133}$$

$$R_{\underline{ab}}{}^{\underline{de}} = -\frac{1}{2}\big(\partial_{[\underline{a}}\partial^{[\underline{d}}(\Psi+\overline{\Psi})\big)\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{2.7.134}$$

2.7.15 Adaptation To 10D, $\mathcal{N}$ = 2B Component/Superspace Results: Step 3

Parameter Composition Rules:

$$\xi^{\underline{m}} = -i(\epsilon_1{}^\alpha\bar{\epsilon}_2{}^\beta + \bar{\epsilon}_1{}^\beta\epsilon_2{}^\alpha)(\sigma^{\underline{c}})_{\alpha\beta}\delta_{\underline{c}}{}^{\underline{m}}\Big(1+\frac{1}{2}(\Psi+\overline{\Psi})\Big) \quad , \tag{2.7.135}$$

$$\begin{aligned}\lambda^{\underline{de}} = &-(\epsilon_1{}^\alpha\bar{\epsilon}_2{}^\beta + \bar{\epsilon}_1{}^\beta\epsilon_2{}^\alpha)\Big[-i\frac{17}{20}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}(\Psi+\overline{\Psi})) - i\frac{1}{10}(\sigma^{\underline{def}})_{\alpha\beta}(\partial_{\underline{f}}(\Psi+\overline{\Psi}))\\ &\qquad -\frac{1}{80}\Big[(\sigma_{[1]})_{\alpha\beta}\Big(Y^{[1]\underline{de}} - \overline{Y}^{[1]\underline{de}}\Big) - \frac{1}{2}(\sigma^{[2][\underline{d}})_{\alpha\beta}\Big(Y^{\underline{e}]}{}_{[2]} - \overline{Y}^{\underline{e}]}{}_{[2]}\Big)\\ &\qquad -\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}\Big(Y_{[3]} - \overline{Y}_{[3]}\Big)\Big]\Big]\\ &-\frac{1}{40}\epsilon_1{}^\alpha\epsilon_2{}^\beta\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}U_{[3]} - (\sigma_{[1]})_{\alpha\beta}U^{[1]\underline{de}}\right]\\ &+\frac{1}{40}\bar{\epsilon}_1{}^\alpha\bar{\epsilon}_2{}^\beta\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}\overline{U}_{[3]} - (\sigma_{[1]})_{\alpha\beta}\overline{U}^{[1]\underline{de}}\right] \quad ,\end{aligned} \tag{2.7.136}$$

$$\begin{aligned}\epsilon^\delta = &-(\epsilon_1{}^\alpha\bar{\epsilon}_2{}^\beta + \bar{\epsilon}_1{}^\beta\epsilon_2{}^\alpha)\Big[i\frac{1}{10}\Big[(\sigma^{[1]})_{\alpha\beta}(\sigma_{[1]})^{\delta\epsilon} - \frac{1}{24}(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\delta\epsilon}\Big]\psi_\epsilon\\ &\qquad + \frac{1}{8}\Big[(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\delta\epsilon} - \frac{1}{30}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\delta\epsilon}\Big]\overline{\lambda}_\epsilon\Big]\\ &-\epsilon_1{}^\alpha\epsilon_2{}^\beta\left[-i\frac{1}{5}(\sigma^{[1]})_{\alpha\beta}(\sigma_{[1]})^{\delta\epsilon}\overline{\psi}_\epsilon + 2(\sigma^{[1]})_{\alpha\beta}(\sigma_{[1]})^{\delta\epsilon}\lambda_\epsilon\right] \quad .\end{aligned} \tag{2.7.137}$$

2.7.16 Adaptation To 10D, $\mathcal{N}$ = 2B Component/Superspace Results: Step 4

$$\begin{aligned}\delta_Q e_{\underline{a}}{}^{\underline{m}} = &-\epsilon^\beta\left[-i\frac{1}{2}\Big[\delta_{\underline{a}}{}^{\underline{d}}\delta_\beta{}^\gamma + \frac{1}{5}(\sigma_{\underline{a}}{}^{\underline{d}})_\beta{}^\gamma\Big]\overline{\psi}_\gamma + \Big[-11\delta_{\underline{a}}{}^{\underline{d}}\delta_\beta{}^\gamma + (\sigma_{\underline{a}}{}^{\underline{d}})_\beta{}^\gamma\Big]\lambda_\gamma\right]\delta_{\underline{d}}{}^{\underline{m}}\\ &-\bar{\epsilon}^\beta\left[i\frac{1}{2}\Big[\delta_{\underline{a}}{}^{\underline{d}}\delta_\beta{}^\gamma + \frac{1}{5}(\sigma_{\underline{a}}{}^{\underline{d}})_\beta{}^\gamma\Big]\psi_\gamma + \Big[-11\delta_{\underline{a}}{}^{\underline{d}}\delta_\beta{}^\gamma + (\sigma_{\underline{a}}{}^{\underline{d}})_\beta{}^\gamma\Big]\overline{\lambda}_\gamma\right]\delta_{\underline{d}}{}^{\underline{m}} \quad ,\end{aligned} \tag{2.7.138}$$

$$
\begin{aligned}
\delta_Q \psi_{\underline{a}}{}^{\delta} \; &= \Big(1+\frac{1}{2}(\Psi+\overline{\Psi})\Big)\partial_{\underline{a}}\epsilon^{\delta} - \frac{1}{2}\epsilon^{\delta}(\partial_{\underline{c}}(\Psi+\overline{\Psi}))\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\
&- \frac{1}{64}\epsilon^{\beta}\Big[-31\delta_{\underline{a}}{}^{\underline{c}}\delta_{\beta}{}^{\delta} + 15(\sigma_{\underline{a}}{}^{\underline{c}})_{\beta}{}^{\delta}\Big](\partial_{\underline{c}}\Psi) - \frac{1}{320}\epsilon^{\beta}\Big[27\delta_{\underline{a}}{}^{\underline{c}}\delta_{\beta}{}^{\delta} + 53(\sigma_{\underline{a}}{}^{\underline{c}})_{\beta}{}^{\delta}\Big](\partial_{\underline{c}}\overline{\Psi}) \\
&+ i\frac{1}{2560}\epsilon^{\beta}\Big[\frac{1}{2}(\sigma^{[2]})_{\beta}{}^{\delta}\Big(5Y_{\underline{a}[2]} - 27\overline{Y}_{\underline{a}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{a}[3]})_{\beta}{}^{\delta}\Big(5Y^{[3]} - 27\overline{Y}^{[3]}\Big)\Big] \\
&+ i\frac{1}{2560}\overline{\epsilon}^{\beta}\Big[\frac{1}{2}(\sigma^{[2]})_{\beta}{}^{\delta}\Big(5X_{\underline{a}[2]} - 27\overline{U}_{\underline{a}[2]}\Big) - \frac{1}{3!}(\sigma_{\underline{a}[3]})_{\beta}{}^{\delta}\Big(5X^{[3]} - 27\overline{U}^{[3]}\Big)\Big] \quad ,
\end{aligned}
\tag{2.7.139}
$$

$$
\begin{aligned}
\delta_Q \phi_{\underline{a}}{}^{\underline{de}} \; &= i\frac{1}{2}\epsilon^{\beta}\Big[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\overline{\psi}_{\beta}) + \frac{1}{5}(\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\overline{\psi}_{\gamma})\Big] - \epsilon^{\beta}\Big[-11\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\lambda_{\beta}) + (\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\lambda_{\gamma})\Big] \\
&- i\frac{1}{2}\overline{\epsilon}^{\beta}\Big[\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\psi_{\beta}) + \frac{1}{5}(\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\psi_{\gamma})\Big] - \overline{\epsilon}^{\beta}\Big[-11\delta_{\underline{a}}{}^{[\underline{d}}(\partial^{\underline{e}]}\overline{\lambda}_{\beta}) + (\sigma^{\underline{de}})_{\beta}{}^{\gamma}(\partial_{\underline{a}}\overline{\lambda}_{\gamma})\Big] \quad .
\end{aligned}
\tag{2.7.140}
$$

2.8 10D, $\mathcal{N} = 2$B Chiral Compensator Considerations

In the limits where all supergravity fields are set to zero, four sets of superalgebras emerge. These take the forms:

(a.) 11D, $\mathcal{N} = 1$,

$$
\{ \mathrm{D}_{\alpha} \, , \, \mathrm{D}_{\beta} \} \; = \; i\,(\gamma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad , \quad \big[\mathrm{D}_{\alpha} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \quad \big[\partial_{\underline{a}} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \tag{2.8.1}
$$

(b.) 10D, $\mathcal{N} = 1$,

$$
\{ \mathrm{D}_{\alpha} \, , \, \mathrm{D}_{\beta} \} \; = \; i\,(\sigma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad , \quad \big[\mathrm{D}_{\alpha} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \quad \big[\partial_{\underline{a}} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \tag{2.8.2}
$$

(c.) 10D, $\mathcal{N} = 2$A,

$$
\begin{aligned}
&\{ \mathrm{D}_{\alpha} \, , \, \mathrm{D}_{\beta} \} \; = \; i\,(\sigma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad , \quad \Big\{ \mathrm{D}_{\dot{\alpha}} \, , \, \mathrm{D}_{\dot{\beta}} \Big\} \; = \; i\,(\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}}\,\partial_{\underline{a}} \quad , \quad \Big\{ \mathrm{D}_{\alpha} \, , \, \mathrm{D}_{\dot{\beta}} \Big\} \; = \; 0 \quad , \\
&\big[\mathrm{D}_{\alpha} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \qquad \big[\mathrm{D}_{\dot{\alpha}} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \qquad \big[\partial_{\underline{a}} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad ,
\end{aligned}
\tag{2.8.3}
$$

(d.) 10D, $\mathcal{N} = 2$B,

$$
\begin{aligned}
&\{ \mathrm{D}_{\alpha} \, , \, \mathrm{D}_{\beta} \} \; = \; 0 \quad , \quad \{ \overline{\mathrm{D}}_{\alpha} \, , \, \overline{\mathrm{D}}_{\beta} \} \; = \; 0 \quad , \quad \{ \mathrm{D}_{\alpha} \, , \, \overline{\mathrm{D}}_{\beta} \} \; = \; i\,(\sigma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad , \\
&\big[\mathrm{D}_{\alpha} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \qquad \big[\overline{\mathrm{D}}_{\alpha} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad , \qquad \big[\partial_{\underline{a}} \, , \, \partial_{\underline{b}} \big] \; = \; 0 \quad ,
\end{aligned}
\tag{2.8.4}
$$

We next introduce a complex superfield denoted by Ω_d into each of these d-dimensional superspaces and seek to probe the implications of imposing a first-order differential equation on this superfield that utilizes any of the spinorial derivatives above.

For either the 11D, $\mathcal{N}=1$ or 10D, $\mathcal{N}=1$ superspaces we have

$$\mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \mathrm{D}_\alpha \mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \{ \, \mathrm{D}_\alpha ~,~ \mathrm{D}_\beta \} \, \Omega_d ~=~ 0 ~~\to~~ \partial_{\underline{c}} \, \Omega_d ~=~ 0 ~~, \tag{2.8.5}$$

and by analogy for the 10D, $\mathcal{N}=2$A superspace we find

$$\begin{gathered} \mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \mathrm{D}_\alpha \mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \{ \, \mathrm{D}_\alpha ~,~ \mathrm{D}_\beta \} \, \Omega_d ~=~ 0 ~~\to~~ \partial_{\underline{c}} \, \Omega_d ~=~ 0 ~~, \\ \mathrm{D}_{\dot\beta} \, \Omega_d ~=~ 0 ~~\to~~ \mathrm{D}_{\dot\alpha} \mathrm{D}_{\dot\beta} \, \Omega_d ~=~ 0 ~~\to~~ \{ \, \mathrm{D}_{\dot\alpha} ~,~ \mathrm{D}_{\dot\beta} \} \, \Omega_d ~=~ 0 ~~\to~~ \partial_{\underline{c}} \, \Omega_d ~=~ 0 ~~, \end{gathered} \tag{2.8.6}$$

Thus, from (2.8.5) and (2.8.6) we find the superfield Ω_d in each of these d-dimensional superspaces must be a constant. However, upon repeating these considerations for the 10D, $\mathcal{N}=2$B superspace we find

$$\begin{aligned} \mathrm{D}_\beta \, \Omega_d ~&=~ 0 ~~\to~~ \mathrm{D}_\alpha \mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \{ \, \mathrm{D}_\alpha ~,~ \mathrm{D}_\beta \} \, \Omega_d ~=~ 0 ~~\to~~ 0 ~=~ 0 ~~, \\ \overline{\mathrm{D}}_\beta \, \Omega_d ~&=~ 0 ~~\to~~ \overline{\mathrm{D}}_\alpha \overline{\mathrm{D}}_\beta \, \Omega_d ~=~ 0 ~~\to~~ \{ \, \overline{\mathrm{D}}_\alpha ~,~ \overline{\mathrm{D}}_\beta \} \, \Omega_d ~=~ 0 ~~\to~~ 0 ~=~ 0 ~~, \end{aligned} \tag{2.8.7}$$

which shows that the superfield Ω_d, in this case, can be a non-trivial representation of the translation operator.

The differential equation

$$\overline{\mathrm{D}}_\beta \, \Omega_d ~=~ 0 ~~, \tag{2.8.8}$$

in the context of four dimensions implies that Ω_d is a “chiral superfield.” On the other hand the differential equation

$$\mathrm{D}_\beta \, \Omega_d ~=~ 0 ~~, \tag{2.8.9}$$

in the context of four dimensions implies that Ω_d is a “anti-chiral superfield.” While it is not possible to simultaneously impose both conditions because a chiral superfield is the complex conjugate of an anti-chiral one, either one or the other can be imposed. This also means that neither the chiral nor the anti-chiral condition can be applied to a real superfield.

Let us return to the results shown in (2.7.105) by focusing only on the equations that contain λ_α

$$\begin{aligned} \mathrm{D}_\alpha \Psi &= i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma} + 5\lambda_\alpha \equiv -i\frac{1}{2}\overline{\psi}_\alpha + 5\lambda_\alpha \quad , \\ \mathrm{D}_\alpha \overline{\Psi} &= i\frac{1}{2}(\sigma^{\underline{a}})_{\alpha\gamma}\overline{\psi}_{\underline{a}}{}^{\gamma} - 27\lambda_\alpha \equiv -i\frac{1}{2}\overline{\psi}_\alpha - 27\lambda_\alpha \quad , \end{aligned} \tag{2.8.10}$$

since the remaining equations can be obtained by complex conjugation. In all the other cases we have explored, there is no spinor field such as λ_α. Taking the difference of the two equations that appear in (2.8.10), we may obtain

$$i\,\tfrac{1}{32}\,\mathrm{D}_\alpha\big(\,\Psi \;-\; \overline{\Psi}\,\big) \;=\; i\,\lambda_\alpha \quad . \tag{2.8.11}$$

However, the quantity $i\,(\Psi \;-\; \overline{\Psi})$ is a real superfield. The requirement that $\lambda_\alpha = 0$ is equivalent to the imposition of an anti-chirality condition on a real superfield and this condition possesses no non-trivial solution.

The inability to introduce such a chiral superfield distinguishes the type 2B theory from the other higher dimensional constructions we have considered. At first order in the θ-expansion of Ψ both the spin-1/2 portion of the gravitino $\overline{\psi}_{\underline{a}}{}^{\gamma}$ *and* a separate spin-1/2 auxiliary spinor λ_α must exist.

2.9 Conclusion

This chapter gives a proposal for descriptions of Nordström supergravity in eleven and ten dimensions, as well as the component level descriptions that follow from the superfield equations we have presented. Our work is based on the assumption that in each of the cases of 11D, $\mathcal{N} = 1$, 10D, $\mathcal{N} = 1$, 10D, $\mathcal{N} = 2$A and 10D, $\mathcal{N} = 2$B, a single scalar superfield is required to provide such a description as this was the case for both ordinary gravitation as well as 4D, $\mathcal{N} = 1$ supergravity. We remark that our work is but a foundation as in future extensions of this work we plan to continue this exploration.

Having obtained the results for the theories in ten and eleven-dimensional superspaces, we can compare those results with the ones seen in section 2.2. Looking back at (2.2.8), with a bit of effort, one can show that the condition $H^{\underline{a}} = 0$ causes only modification in the form of the equations. Namely the terms $W_{\alpha\beta\gamma}$ will vanish under this restriction. It is thus pointedly seen that all the basic superfields (i. e. R and $G_{\underline{a}}$) in the algebra of the superspace supergravity covariant derivatives are bosonic. This is to be compared to the results shown in (2.2.14) where a fermionic superfield T_α appears. In all of the higher dimensional theories, such superfields appear ubiquitously.

In the future, we will also address the very important quest of whether there exists a superspace action for the Nordström supergravity theories in higher dimensions. It is clear that in order for this to be the case, it is necessary that the scalar superfield should satisfy some superdifferential constraints. The expectation is suggested by the structure of the 4D, $\mathcal{N} = 1$ theory. We remind the reader that the irreducible theories require that the superfield X is subject to some differential constraints. So it is natural to expect this to extend into the higher dimensional theories.

Our approach also raises an interesting question about Superstring Theories, M-Theory, and F-Theory. Do these theories also possess consistent truncation limits that include Nordström supergravity theories in their low energy limits? If the answer is affirmative, such limits might provide laboratories in which to investigate these more complicated mathematical structures.

Chapter 3

The Salam-Strathdee Riddle

This chapter is based on [57, 58].

This chapter aims to set up the "Salam-Strathdee Riddle" explicitly. In section 3.1, we discuss the numbers of independent bosonic and fermionic components in a scalar superfield. Section 3.2 is a transitional one where we review the $\theta-$expansion of a real scalar superfield in 4D, $\mathcal{N} = 1$ theory and introduce the higher dimensional adinkra technology.

In section 3.3, we will follow the route of the traditional θ-expansion as applied to the 11D, $\mathcal{N} = 1$ scalar superfield. A discussion involving a recursion formula used to move in a level-by-level manner up the θ-expansion of the superfield is presented and the role of Duffin-Kemmer-Petiau fields is noted. The recursive procedure is applied up to quartic order to show the calculational complications that occur in starting with a superfield and then abstracting the component field content from such a systematic starting point.

3.1 Superfield Diophantine Considerations

Via the use of simple toroidal compactification, one can count the numbers of independent bosonic and fermionic component fields that occur in a scalar superfield. As a fermionic coordinate cannot be squared, this means in the Grassmann coordinate expansion of a superfield, any one specific fermionic coordinate can only occur to the zeroth power or the first power. As component fields occur as coefficients of monomials in superspace Grassmann coordinates, counting the latter is the same as the former... as long as the superfield is not subject to any spinorial "supercovariant derivative" constraints.

Each higher dimensional superspace with D bosonic dimensions, for purposes of counting, is equivalent to some value of d, where d is the number of independent equivalent real one-dimensional fermionic coordinates on which the superfield depends. The total number of independent monomials is given by 2^d. Next, to count the total number of bosonic components n_B in the scalar superfield, we simply divide by a factor of two. This same argument applies to the total number of fermionic components n_F in the scalar superfield. Thus we have $n_B = n_F = 2^{d-1}$. A few cases are shown in the table below.

D	d	2^d	n_B	n_F
4	4	16	8	8
5	8	256	128	128
10	16	65,536	32,768	32,768
11	32	4,294,967,296	2,147,483,648	2,147,483,648

Table 3.1: Number of independent components in *unconstrained* scalar superfields in D dimensional spacetime

While superfields easily provide a methodology for finding collections of component fields that are representations of spacetime supersymmetry, one thing that superfields do not yield so easily is a theory of the constraints that provides irreducible representations. For any constrained superfield, it must be the case that the number of component fields does *not depend solely* on the parameter d. This is most certainly true for the maximally constrained and therefore minimal irreducible representations. This is illustrated by comparing the final two columns of the first three rows in Tables 3.1 and 3.2 that $n_B = n_F \neq n_{B(min)} = n_{F(min)}$.

4D Minimal Off-Shell Supermultiplet	d	2^{d-1}	$n_{B(min)}$	$n_{F(min)}$
$\mathcal{N} = 1$ Chiral	4	8	4	4
$\mathcal{N} = 2$ Vector	8	128	8	8
$\mathcal{N} = 4$ SG	16	32,768	128	128

Table 3.2: Number of independent components in *maximally constrained* superfields

3.2 4D Scalar Superfield Decomposition

The off-shell four-dimensional vector supermultiplet is well understood now to be a part of the unconstrained superfield as given below. In 4D, $\mathcal{N} = 1$ real superspace, the spinor index (the Greek index) on θ^α runs from 1 to 4, and we have 8 bosons and 8 fermions, as counted in Table 3.1. The $\theta-$expansion of a scalar superfield is

$$\begin{aligned}\mathcal{V}(x^{\underline{a}},\theta^\alpha) \;=&\; f(x^{\underline{a}}) \;+\; \theta^\alpha\,\psi_\alpha(x^{\underline{a}}) \;+\; \theta^\alpha\theta^\beta C_{\alpha\beta}\,g(x^{\underline{a}}) \\ &+\; \theta^\alpha\theta^\beta i(\gamma^5)_{\alpha\beta}\,h(x^{\underline{a}}) \;+\; \theta^\alpha\theta^\beta i(\gamma^5\gamma^{\underline{b}})_{\alpha\beta}\,v_{\underline{b}}(x^{\underline{a}}) \\ &+\; \theta^\alpha\theta^\beta\theta^\gamma C_{\alpha\beta}C_{\gamma\delta}\,\chi^\delta(x^{\underline{a}}) \;+\; \theta^\alpha\theta^\beta\theta^\gamma\theta^\delta C_{\alpha\beta}C_{\gamma\delta}\,N(x^{\underline{a}})\end{aligned} \tag{3.2.1}$$

where $f(x^{\underline{a}})$, $g(x^{\underline{a}})$, $h(x^{\underline{a}})$, $v_{\underline{b}}(x^{\underline{a}})$, and $N(x^{\underline{a}})$ are bosonic component fields with 8 d.o.f.[1] in total; $\psi_\alpha(x^{\underline{a}})$ and $\chi^\delta(x^{\underline{a}})$ are fermionic component fields with 8 d.o.f. in total. We will discuss this $\theta-$expansion result from different perspectives.

3.2.1 Group Theory Perspective

From the perspective of group theory, we can translate the scalar superfield decomposition problem to the irreducible decomposition problem of representations in $\mathfrak{so}(4)$. First, we can write the general expression of the $\theta-$expansion of a superfield

$$\begin{aligned}\mathcal{V}(x^{\underline{a}},\theta^\alpha) \;=&\; v^{(0)}(x^{\underline{a}}) \;+\; \theta^\alpha\,v^{(1)}_\alpha(x^{\underline{a}}) \;+\; \theta^\alpha\theta^\beta\,v^{(2)}_{\alpha\beta}(x^{\underline{a}}) \\ &+\; \theta^\alpha\theta^\beta\theta^\gamma\,v^{(3)}_{\alpha\beta\gamma}(x^{\underline{a}}) \;+\; \theta^\alpha\theta^\beta\theta^\gamma\theta^\delta\,v^{(4)}_{\alpha\beta\gamma\delta}(x^{\underline{a}}) \quad .\end{aligned} \tag{3.2.2}$$

Due to the antisymmetric property of the Grassmann coordinates θ^α, the quantities θ^α, $\theta^\alpha\theta^\beta$, $\theta^\alpha\theta^\beta\theta^\gamma$, and $\theta^\alpha\theta^\beta\theta^\gamma\theta^\delta$ have 4, 6, 4, and 1 degrees of freedom, respectively. They can be interpreted as representations of $\mathfrak{so}(4)$ with 4, 6, 4, and 1 dimensions. We use level-n to denote the $\theta-$monomial with n θs. The problem is reduced to do the irreducible decompositions of these representations and the results can be listed as

$$\mathcal{V} \;=\; \begin{cases} \text{Level}-0 & \{1\} \quad , \\ \text{Level}-1 & \{4\} \quad , \\ \text{Level}-2 & \{1\}\oplus\{4\}\oplus\{1\} \quad , \\ \text{Level}-3 & \{4\} \quad , \\ \text{Level}-4 & \{1\} \quad . \end{cases} \tag{3.2.3}$$

[1] We use the notation d.o.f to indicate "degrees of freedom."

Note that level-4 and 3 are conjugate to level-0 and 1, respectively, while level-2 is self-conjugate. The conjugates of $\{1\}$ and $\{4\}$ are still $\{1\}$ and $\{4\}$. Level-2 has two $\{1\}$'s[2]. Although in the group theory context, $\mathfrak{so}(4)$ only has one $\{1\}$, here two $\{1\}$'s represent two different $\theta-$monomials as you will see shortly.

In order to distinguish between bosonic irreps and fermionic irreps, we color their dimensions: blue if bosonic and red if fermionic. In the rest of the paper, we will use these conventions.

Recall that in 4D, $\mathcal{N}=1$ real superspace, we can create the covariant gamma matrices which are 4×4 real matrices. The basis of the space of matrices over these spinors is summarized in Table 3.3. It yields analytical expressions of the $\theta-$monomials below:

Basis	$C_{\alpha\beta}$	$(\gamma^{\underline{a}})_{\alpha\beta}$	$(\gamma^{[2]})_{\alpha\beta}$	$i(\gamma^5)_{\alpha\beta}$	$i(\gamma^5\gamma^{\underline{a}})_{\alpha\beta}$
Sym/Antisym	A	S	S	A	A
Count	1	4	6	1	4

Table 3.3: Summary of 4D Basis: all of these are 4×4 real matrices

- Level-0: no θ
- Level-1: $\{4\} = \theta^\alpha$
- Level-2: $\{1\} = \theta^\alpha\theta^\beta C_{\alpha\beta}$, $\{4\} = \theta^\alpha\theta^\beta i(\gamma^5\gamma^{\underline{a}})_{\alpha\beta}$, $\{1\} = \theta^\alpha\theta^\beta i(\gamma^5)_{\alpha\beta}$
- Level-3: $\{4\} = \theta^\alpha\theta^\beta\theta^\gamma C_{\alpha\beta}C_{\gamma\delta}$
- Level-4: $\{1\} = \theta^\alpha\theta^\beta\theta^\gamma\theta^\delta C_{\alpha\beta}C_{\gamma\delta}$

Thus, Equation (3.2.2) can be rewritten as

$$\begin{aligned}\mathcal{V}(x^{\underline{a}},\theta^\alpha) &= v^{(0)}(x^{\underline{a}}) + \theta^\alpha\, v_\alpha^{(1)}(x^{\underline{a}}) + \theta^\alpha\theta^\beta \Big[C_{\alpha\beta}\, v_1^{(2)}(x^{\underline{a}}) + i(\gamma^5)_{\alpha\beta}\, v_2^{(2)}(x^{\underline{a}}) \\ &\quad + i(\gamma^5\gamma^{\underline{b}})_{\alpha\beta}\, v_{\underline{b}}^{(2)}(x^{\underline{a}}) \Big] + \theta^\alpha\theta^\beta\theta^\gamma C_{\alpha\beta}C_{\gamma\delta}\, v^{(3)\delta}(x^{\underline{a}}) \\ &\quad + \theta^\alpha\theta^\beta\theta^\gamma\theta^\delta C_{\alpha\beta}C_{\gamma\delta}\, v^{(4)}(x^{\underline{a}}) \quad .\end{aligned} \tag{3.2.4}$$

[2] If one is exercising extra care, the $\{1\}$'s at level-2 are not identical. In fact, one of these corresponds to a scalar field while the other corresponds to a pseudoscalar. However, for our purposes, we can neglect this difference. This distinction is shown in Table 3.3 where the "count" is the same for both, but their projections to γ-matrices are distinct.

Note that applying the following replacements

$$
\begin{aligned}
&v^{(0)}(x^{\underline{a}}) \to f(x^{\underline{a}}) \quad , \quad v^{(1)}_{\alpha}(x^{\underline{a}}) \to \psi_{\alpha}(x^{\underline{a}}) \quad , \quad v^{(2)}_{1}(x^{\underline{a}}) \to g(x^{\underline{a}}) \quad , \quad v^{(2)}_{2}(x^{\underline{a}}) \to h(x^{\underline{a}}) \quad , \\
&v^{(2)}_{\underline{b}}(x^{\underline{a}}) \to v_{\underline{b}}(x^{\underline{a}}) \quad , \quad v^{(3)\delta}(x^{\underline{a}}) \to \chi^{\delta}(x^{\underline{a}}) \quad , \quad v^{(4)}(x^{\underline{a}}) \to N(x^{\underline{a}}) \quad ,
\end{aligned}
\tag{3.2.5}
$$

will reproduce the result in Equation (3.2.1).

3.2.2 Graph Theory Perspective: Adinkra

From the perspective of graph theory, particularly adinkra diagrams, we can define a four-dimensional adinkra based on Equation (3.2.3). First of all, the adinkra diagram carries information about component fields rather than $\theta-$monomials, so we need to translate Equation (3.2.3) to field variable language. Consider a variable with one upstairs spinor index χ^{α} and assign irrep $\{4\}$ to this field. What is the irrep corresponding to the variable with one downstairs spinor index χ_{β}? Since

$$
\chi_{\beta} ~=~ \chi^{\alpha} C_{\alpha\beta} \quad , \tag{3.2.6}
$$

where $C_{\alpha\beta}$ is the spinor metric, the irrep of χ_{β} is still $\{4\}$. Generally speaking, in 4D, $\mathcal{N}=1$ real notation, the irreps corresponding to component fields are the same as their $\theta-$monomials.

Then we can use open nodes to denote bosonic component fields and put the dimensions of their corresponding irreps in the centers of the open nodes. For fermionic component fields, we use closed nodes with a similar convention for dimensionality. The level number represents the height assignment and it increases with height. Black edges connect nodes in adjacent levels and represent supersymmetric transformation operations on the component fields. We can also interpret the adinkra using the idea in [106] (i.e. adinkra nodes are the $\theta \to 0$ limit of superfields), as illustrated in the following table.

Since the physical d.o.f. of θ^{α} is $-1/2$, the height assignment describes the physical d.o.f. of component fields as well. The corresponding adinkra diagram is Figure 3.1. The graph shows the "f" bosonic component field at the lowest level, the "ψ" fermionic

Level	Adinkra nodes	Component fields	Irrep(s) in $\mathfrak{so}(4)$
0	$\mathcal{V}\vert$	$f(x^{\underline{a}})$	$\{1\}$
1	$\mathrm{D}_\alpha\mathcal{V}\vert$	$\psi_\alpha(x^{\underline{a}})$	$\{4\}$
2	$\mathrm{D}_{[\alpha}\mathrm{D}_{\beta]}\mathcal{V}\vert$	$g(x^{\underline{a}})$, $h(x^{\underline{a}})$, $v_{\underline{b}}(x^{\underline{a}})$	$\{1\}$, $\{1\}$, $\{4\}$
3	$\mathrm{D}_{[\alpha}\mathrm{D}_{\beta}\mathrm{D}_{\gamma]}\mathcal{V}\vert$	$\chi^\delta(x^{\underline{a}})$	$\{4\}$
4	$\mathrm{D}_{[\alpha}\mathrm{D}_{\beta}\mathrm{D}_{\gamma}\mathrm{D}_{\delta]}\mathcal{V}\vert$	$N(x^{\underline{a}})$	$\{1\}$

Table 3.4: Explicit Relations between Adinkra Nodes, Component Fields, and Irreps

field at level one, the "g," "h," and "$v_{\underline{b}}$" bosonic component fields at level two, the "χ" fermionic field at level three, and finally the "N" bosonic component field at level four.

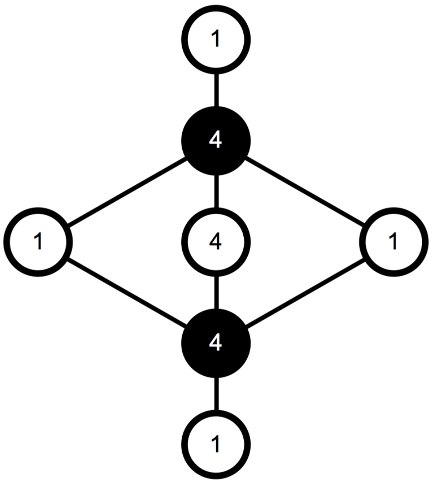


Figure 3.1: Adinkra Diagram for 4D, $\mathcal{N} = 1$

Since the 4D, $\mathcal{N} = 1$ theory can be truncated to a 1D, $\mathcal{N} = 4$ theory, we can discuss the relation between the 4D, $\mathcal{N} = 1$ adinkra and the 1D, $\mathcal{N} = 4$ adinkra as shown in Figure 3.2. The 1D, $\mathcal{N} = 4$ maximal supermultiplet and the corresponding adinkras are well-studied. The graph shown to the left graph in Figure 3.2 illustrates the 1D $\mathcal{N} = 4$ system.

By comparison, we see that starting from the 1D, $\mathcal{N} = 4$ adinkra, if we aggregate a set of bosons or fermions into a single node, then a set of corresponding links will be merged (we use black links to replace them), and thus emerges the 4D, $\mathcal{N} = 1$ adinkra. Of course, to reach this goal we must decide to enforce some rules. Four and only four black nodes are merged at the first and third levels. At the second level, the only merging choices are either one or four nodes as permitted in an "aggregated" node. From this procedure, although we have a large number of different 1D, $\mathcal{N} = 4$ adinkras, they all collapse into the same 4D, $\mathcal{N} = 1$ version. It is useful to also recall that the aggregation of nodes leads to "dimensional enhancement" [107–109] that allows the adinkra nodes to carry representations of the four-dimensional Lorentz group.

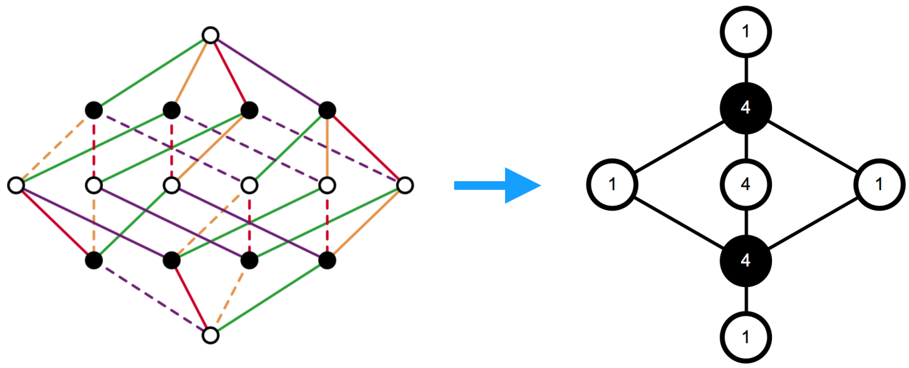


Figure 3.2: From 1D, $\mathcal{N} = 4$ Adinkra to 4D, $\mathcal{N} = 1$ Adinkra

Before we leave this section, there is another relevant comment. Figure 3.1 provides a representation of the data needed to construct the component field description of the 4D, $\mathcal{N} = 1$ unconstrained superfield $\mathcal{V}$. However, this superfield is well known to provide a *reducible* representation of 4D, $\mathcal{N} = 1$ supersymmetry. The *irreducible representations* contained in $\mathcal{V}$ are the two adinkra diagrams on the bottom of Figure 3.3.

It is useful to know one of the nodes at the lowest level of the bottom right graph in Figure 3.3 (i.e. chiral supermultiplet) must be regarded as the spacetime integral of a spin-0 field. The origin of this field was from the initial starting point as being one of the aggregated parts of the "4" at the middle level of the $\mathcal{V}$ adinkra. The bottom left graph in Figure 3.3 has nodes associated with the component fields of the vector supermultiplet where the gauge field is restricted to the Coulomb gauge. Refer to Appendix B for details.

In 4D, there is a comprehensive understanding of how to start with a reducible representation such as the real scalar superfield $\mathcal{V}$ and "break" it apart into its irreducible components. However, the extension of such a procedure is totally unknown for the cases of higher dimensional superfields.

For a general representation of spacetime supersymmetry, there is currently no understanding of how to carry out the process which has been accomplished in the realm

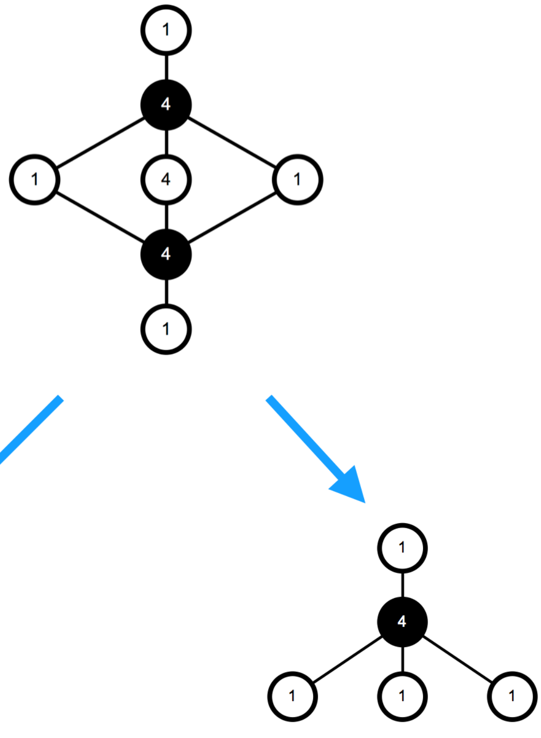


Figure 3.3: From 4D, $\mathcal{N} = 1$ Reducible Adinkra to Irreducible Adinkras (Vector and Chiral Supermultiplets)

of the 4D, $\mathcal{N} = 1$ supersymmetry. Resolving this class of problems in the realm of supersymmetric representation theory is a primary motivation for the adinkra approach to the study of superfields.

If we borrow the language of genetics, adinkras play the role of genomes for superfields. The problem of finding the irreducible off-shell representations of all superfields is equivalent to a genome-editing problem. We are still without the mathematical analog of a Cas9[3] capable of beginning with a *reducible* graph, that is a viable off-shell representation of spacetime supersymmetry, and ending with off-spring *irreducible* graphs that also provides viable off-shell representations of spacetime supersymmetry. An example of the content of the preceding sentence is shown in Figure 3.3.

[3] Cas9 (CRISPR associated protein 9) is a protein which plays a vital role in the immunological defense of certain bacteria against DNA viruses, and its main function is to cut DNA and therefore alter a cell's genome. See for the Wikipedia - Cas9, https://en.wikipedia.org/wiki/Cas9.

3.3 Traditional Path to 11D Superfield Component Decompositions

In order to construct 11D superfield component decompositions, can we directly follow the route of the traditional θ-expansion? To see why this problem is unsolved since the 1970s, we will explicitly calculate irreducible θ-monomials in this section via the traditional path.

If we start from constructing the irreducible $\theta-$monomials to understand the eleven-dimensional scalar superfield decomposition, two uniqueness problems will show up: (1) $\theta-$monomials have multiple expressions from the cubic level; (2) gamma matrix multiplications have multiple expressions. To illustrate the first problem, the quadratic level and cubic level will be discussed in detail in the following sections. For the second one, all gamma matrix multiplication results are listed in Appendix A.3.5. Moreover, constructing irreducible $\theta-$monomials requires a number of Fierz identities as shown shortly.

Each higher dimensional superspace with D bosonic dimensions, for purposes of counting, is equivalent to some value of d, which is the number of real components of θ. This is shown in Table 3.1. So for the case of the 11D, $\mathcal{N} = 1$ theory, the real *unconstrained* scalar superfield Ψ contains 2,147,483,648 bosonic and 2,147,483,648 fermionic degrees of freedom that are representations of supersymmetry. While superfields easily provide a methodology for finding collections of components in principle, actually obtaining those component fields is not as easy as it might first appear. This is especially true in the eleven-dimensional case.

In the rest of this section we are going to discuss the complications of applying the most straightforward θ-expansions in the eleven-dimensional superspace. The discussion is meant to provide an explicit demonstration of the difficulties one encounters in such a program.

A naive expansion of a real scalar superfield $\mathcal{V}$ can be expressed as

$$
\begin{aligned}
\mathcal{V}(\theta,\, x) \;=\; & \varphi^{(0)}(x) \;+\; \theta^{\alpha}\,\varphi^{(1)}_{\alpha}(x) \;+\; \Theta^{(1)}\,\varphi^{(2)}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\varphi^{(2)}_{\underline{abc}}(x) \;+\; \Theta^{(3)\,\underline{abcd}}\,\varphi^{(2)}_{\underline{abcd}}(x) \\
& +\; \Theta^{(1)}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha\,\underline{abc}}(x) \;+\; \Theta^{(3)\,\underline{abcd}}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha\,\underline{abcd}}(x) \\
& +\; \Theta^{(1)}\,\Theta^{(1)}\,\varphi^{(4)}(x) \;+\; \Theta^{(1)}\,\Theta^{(2)\,\underline{abc}}\,\varphi^{(4)}_{\underline{abc}}(x) \;+\; \Theta^{(1)}\,\Theta^{(3)\,\underline{abcd}}\,\varphi^{(4)}_{\underline{abcd}}(x) \\
& +\; \Theta^{(2)\,\underline{abc}}\,\Theta^{(2)\,\underline{def}}\,\varphi^{(4)}_{\underline{abc}\,\underline{def}}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\Theta^{(3)\,\underline{defg}}\,\varphi^{(4)}_{\underline{abc}\,\underline{defg}}(x) \\
& +\; \Theta^{(3)\,\underline{abcd}}\,\Theta^{(3)\,\underline{efgh}}\,\varphi^{(4)}_{\underline{abcd}\,\underline{efgh}}(x) \;+\; \dots
\end{aligned}
\tag{3.3.1}
$$

up to the fourth-order of the Grassmann coordinates. The number n in the superscripts of component fields $\varphi^{(n)}_{[\text{indices}]}(x)$ indicates the θ-order. In writing this expression we have introduced "auxiliary nilpotent coordinates" defined in (3.3.8) below.

There exists a recursion formula that can be applied at any non-trivial order n in the θ-expansion to derive the form of the terms at order $(n+1)$ in the θ-expansion. We can begin this by looking at the term linear in θ,

$$
\mathcal{V}(linear) \;=\; \theta^{\alpha}\varphi^{(1)}_{\alpha}(x) \quad , \tag{3.3.2}
$$

and next observe the quadratic terms may be generated by a simple replacement in this expression.

$$
\varphi^{(1)}_{\alpha}(x) \;\rightarrow\; \Big[C_{\alpha\beta}\,\varphi^{(2)}(x) \;+\; (\gamma^{\underline{abc}})_{\alpha\beta}\,\varphi^{(2)}_{\underline{abc}}(x) \;+\; (\gamma^{\underline{abcd}})_{\alpha\beta}\,\varphi^{(2)}_{\underline{abcd}}(x) \Big]\,\theta^{\beta} \;\equiv\; (\mathcal{B})_{\alpha\beta}\theta^{\beta} \quad . \tag{3.3.3}
$$

The quantity $(\mathcal{B})_{\alpha\beta}$ is a Duffin-Kemmer-Petiau [110–112] field. Therefore under the action of this replacement, we find

$$
\varphi^{(1)}_{\alpha}(x) \;\rightarrow\; (\mathcal{B})_{\alpha\beta}\theta^{\beta} \,:\, \mathcal{V}(linear) \;\rightarrow\; \mathcal{V}(quadratic) \quad , \tag{3.3.4}
$$

with $\mathcal{V}(quadratic)$ given by

$$
\mathcal{V}(quadratic) \;=\; \Theta^{(1)}\,\varphi^{(2)}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\varphi^{(2)}_{\underline{abc}}(x) \;+\; \Theta^{(3)\,\underline{abcd}}\,\varphi^{(2)}_{\underline{abcd}}(x) \quad . \tag{3.3.5}
$$

To continue, we take the component fields $\varphi^{(2)}(x)$, $\varphi^{(2)}_{\underline{abc}}(x)$, $\varphi^{(2)}_{\underline{abcd}}(x)$ and make the simultaneous replacements

$$\varphi^{(2)}(x) \;\to\; \theta^{\alpha}\varphi^{(3)}_{\alpha}(x) \;\;,\;\; \varphi^{(2)}_{\underline{abc}}(x) \;\to\; \theta^{\alpha}\varphi^{(3)}_{\alpha\,\underline{abc}}(x) \;\;,\;\; \varphi^{(2)}_{\underline{abcd}}(x) \;\to\; \theta^{\alpha}\varphi^{(3)}_{\alpha\,\underline{abcd}}(x) \quad (3.3.6)$$

which yield the cubic order terms

$$\mathcal{V}(cubic) \;=\; \Theta^{(1)}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha\,\underline{abc}}(x) \;+\; \Theta^{(3)\,\underline{abcd}}\,\theta^{\alpha}\,\varphi^{(3)}_{\alpha\,\underline{abcd}}(x) \quad (3.3.7)$$

that appear on the second line of (3.3.1).

The general rule is that if one starts with the component fields at Level-n, *where n is even*, of the scalar superfield, then to obtain the component fields at Level-$(n+1)$ one simply replaces the starting component fields by a θ-coordinate whose index is contracted against a new fermionic field in a manner that is consistent with Lorentz symmetry.

Also, a general rule is that if one starts with the component fields at Level-n, *where n is odd*, of the scalar superfield, then to obtain the component fields at Level-$(n+1)$ one simply replaces the starting component fields by a new DKP field times a θ-coordinate whose index is contracted against one index on new DKP fields in a manner that is consistent with Lorentz symmetry.

Although one can carry out this procedure to define the component fields to all orders in the θ-expansion, it is highly inefficient and redundant. This redundancy occurs due to the equivalence of many terms obtained as well as the vanishing of many terms both by the use of Fierz identities. There is also the issue of irreducibility that must be enforced. We next turn to the issue of irreducibility.

3.3.1 Quadratic Level

We denote the 32-component Majorana Grassmann coordinate living in 11 dimensional spacetime by θ^{α}. Since $C_{\alpha\beta}$, $(\gamma^{[3]})_{\alpha\beta}$ and $(\gamma^{[4]})_{\alpha\beta}$ are the antisymmetric elements in the covering Clifford algebra over 11D, we can define all possible quadratic θ-monomials as follows.

$$\begin{aligned} \{1\} \qquad \Theta^{(1)} &= C_{\alpha\beta}\,\theta^\alpha\theta^\beta \quad , \\ \{165\} \qquad \Theta^{(2)\,\underline{abc}} &= (\gamma^{\underline{abc}})_{\alpha\beta}\,\theta^\alpha\theta^\beta \quad , \\ \{330\} \qquad \Theta^{(3)\,\underline{abcd}} &= (\gamma^{\underline{abcd}})_{\alpha\beta}\,\theta^\alpha\theta^\beta \quad . \end{aligned} \tag{3.3.8}$$

Now if we look at the quadratic θ-terms, we see the total number of bosonic component fields at this level can be found by simply counting the number of independent quadratic θ-monomials

$$\{32\} \wedge \{32\} \;=\; \frac{\{32\} \;\times\; \{31\}}{2} \;=\; \{496\} \;=\; \{1\} \,\oplus\, \{165\} \,\oplus\, \{330\} \quad , \tag{3.3.9}$$

so all is well as this equation gives the complete decomposition of the product of two Grassmann coordinates into irreducible representations of the 11D Lorentz group.

3.3.2 Cubic Level

We can construct cubic θ-monomials from all the possible quadratic θ-monomials as listed in Equation (3.3.8). Since $C_{\alpha\beta}$, $(\gamma^{[3]})_{\alpha\beta}$ and $(\gamma^{[4]})_{\alpha\beta}$ are the antisymmetric Clifford algebra elements in 11D, we can write all the possible cubic monomials starting with no free Lorentz vector index and going up to four free Lorentz vector indices. All of the possible *irreducible* cubic θ-monomials can be written as

$$\begin{array}{ll} \{5,280\} & \big[\,\Theta^{(3)\,\underline{abcd}}\,\theta_\alpha\,\big]_{IR} \quad , \\ \{3,520\} & \big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{d}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \quad , \quad \big[\,\Theta^{(2)\,\underline{abc}}\,\theta_\alpha\,\big]_{IR} \quad , \\ \{1,408\} & \big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{cd}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \quad , \quad \big[\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{c}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \quad , \\ \{320\} & \big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{bcd}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \quad , \quad \big[\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{bc}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \quad , \\ \{32\} & \Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{abcd}})_{\alpha\beta}\,\theta^\beta \quad , \quad \Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{abc}})_{\alpha\beta}\,\theta^\beta \quad , \quad \Theta^{(1)}\,\theta_\alpha \quad . \end{array} \tag{3.3.10}$$

where the notation $[\quad]_{IR}$ simply means that a single γ-trace of the expression is by definition equal to zero. We will discuss each representation (except $\{5,280\}$) one by one in the following subsections. We will explain each dimension from the corresponding irreducibility condition, and prove that different versions in each representation are actually equivalent. We will show that the cubic monomials of {320} vanish. We argue that {5, 280} cubic monomials also vanish in a similar way. Another strong reason for {5, 280} to vanish is $5,280 > 4,960$ (see equation below). We can then decompose all the cubic θ-monomials by

$$\{32\} \wedge \{32\} \wedge \{32\} = \frac{\{32\} \times \{31\} \times \{30\}}{3 \times 2} = \{4,960\} = \{32\} \oplus \{1,408\} \oplus \{3,520\} \tag{3.3.11}$$

where the left hand side simply counts the number of independent cubic θ-monomials one can write, and the rightmost part contains {32}, {1, 408}, and {3, 520} which are irreducible representations of the 11D Lorentz group.

{32} Cubic Monomials We have three versions of expressions of cubic θ-monomials with no free vector index and one free spinor index as indicated in Equation (3.3.10), which are

$$V1 = \Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{abcd}})_{\alpha\beta}\,\theta^{\beta} = -\frac{1}{3!}\,(\mathcal{A}_3)_{[\delta\epsilon\beta]\alpha}\,\theta^{\delta}\theta^{\epsilon}\theta^{\beta} \quad , \tag{3.3.12}$$

$$V2 = \Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{abc}})_{\alpha\beta}\,\theta^{\beta} = -\frac{1}{3!}\,(\mathcal{A}_2)_{[\delta\epsilon\beta]\alpha}\,\theta^{\delta}\theta^{\epsilon}\theta^{\beta} \quad , \tag{3.3.13}$$

$$V3 = \Theta^{(1)}\,\theta_{\alpha} = \frac{1}{3!}\,(\mathcal{A}_1)_{[\delta\epsilon\beta]\alpha}\,\theta^{\delta}\theta^{\epsilon}\theta^{\beta} \quad . \tag{3.3.14}$$

The degrees of freedom of these monomials are thus 32, and so they are in the spinorial representation {32}. Here we define these three objects

$$\begin{aligned} (\mathcal{A}_1)_{[\delta\epsilon\beta]\alpha} &= C_{[\delta\epsilon}\,C_{\beta]\alpha} \quad , \\ (\mathcal{A}_2)_{[\delta\epsilon\beta]\alpha} &= (\gamma^{\underline{abc}})_{[\delta\epsilon}\,(\gamma_{\underline{abc}})_{\beta]\alpha} \quad , \\ (\mathcal{A}_3)_{[\delta\epsilon\beta]\alpha} &= (\gamma^{\underline{abcd}})_{[\delta\epsilon}\,(\gamma_{\underline{abcd}})_{\beta]\alpha} \quad . \end{aligned} \tag{3.3.15}$$

To find whether $V1$, $V2$, and $V3$ are related, we examine the objects $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$. Since $C_{\alpha\beta}$, $(\gamma^{[3]})_{\alpha\beta}$ and $(\gamma^{[4]})_{\alpha\beta}$ form the complete basis for the antisymmetric elements in 11D Clifford algebra with two spinor indices, we can expand the $\mathcal{A}$ objects into this basis, i.e. find the Fierz identities. The Fierz identities in Appendix A.3.1 tell us that the objects $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are related by a system of linear equations (we suppress spinor indices here for simplicity, as all of them have the same structure),

$$\begin{aligned} 31\mathcal{A}_1 &= \frac{1}{3!}\mathcal{A}_2 - \frac{1}{4!}\mathcal{A}_3 \quad , \\ 37\mathcal{A}_2 &= 990\mathcal{A}_1 - \frac{11}{4}\mathcal{A}_3 \quad , \\ 13\mathcal{A}_3 &= -\,3960\mathcal{A}_1 - 44\mathcal{A}_2 \quad . \end{aligned} \tag{3.3.16}$$

This makes sense as $\mathcal{A}_1$, $\mathcal{A}_2$ and $\mathcal{A}_3$ are the only three objects of this spinor index structure with no free vector indices and at least one Clifford element being antisymmetric. By solving these linear equations, we get

$$\begin{cases} \mathcal{A}_2 &= 66\mathcal{A}_1 \quad , \\ \mathcal{A}_3 &= -\,528\mathcal{A}_1 \quad , \end{cases} \tag{3.3.17}$$

which means all three of the expressions of the cubic θ-monomials are equivalent up to a multiplicative constant

$$V1 \;=\; -8\,V2 \;=\; 528\,V3 \quad . \tag{3.3.18}$$

We can therefore take any of the cubic θ-monomial constructed from linear combinations of $V1$, $V2$, and $V3$ as the fermionic irreducible representation {32} of so(11).

{320} Cubic Monomials For one free vector index, we have two expressions of cubic θ-monomials as suggested in Equation (3.3.10). We can write all the terms from the index structures as

$$\big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{bcd}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \;=\; \tilde{k}_0\,\Big\{\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{bcd}})_{\alpha\beta}\,\theta^\beta \;+\; k_1\,\Theta^{(3)\,\underline{bcde}}\,(\gamma^{\underline{a}}{}_{\underline{bcde}})_{\alpha\beta}\,\theta^\beta\Big\} \quad , \tag{3.3.19}$$

$$\big[\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{bc}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \;=\; \tilde{l}_0\,\Big\{\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{bc}})_{\alpha\beta}\,\theta^\beta \;+\; l_1\,\Theta^{(2)\,\underline{bcd}}\,(\gamma^{\underline{a}}{}_{\underline{bcd}})_{\alpha\beta}\,\theta^\beta\Big\} \quad . \tag{3.3.20}$$

The irreducibility conditions of setting the single γ-traces to zero are

$$(\gamma_{\underline{a}})_\alpha{}^\gamma\,\big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{bcd}})_{\gamma\beta}\,\theta^\beta\,\big]_{IR} \;=0 \quad , \tag{3.3.21}$$

$$(\gamma_{\underline{a}})_\alpha{}^\gamma \left[\Theta^{(2)\,\underline{abc}} (\gamma_{\underline{bc}})_{\gamma\beta}\, \theta^\beta \right]_{IR} = 0 \quad . \tag{3.3.22}$$

Thus, these cubic monomials with one free vector index have $32 \times 11 - 32 = 320$ degrees of freedom and are in the {320} representation. From the irreducibility conditions, we can fix the relative coefficients to $k_1 = -\frac{1}{7}$ and $l_1 = -\frac{1}{8}$. Without loss of generality, we omit the overall coefficients $\tilde{k}_0$ and $\tilde{l}_0$. Therefore, we can write

$$V1 = -\frac{1}{3!}(\mathcal{D}_1 + \tfrac{1}{7}\mathcal{D}_2)^{\underline{a}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.23}$$

$$V2 = \frac{1}{3!}(\mathcal{C}_1 + \tfrac{1}{8}\mathcal{C}_2)^{\underline{a}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.24}$$

where we define five objects

$$\begin{aligned}
(\mathcal{B})^{\underline{a}}_{[\delta\epsilon\beta]\alpha} &= C_{[\delta\epsilon}(\gamma^{\underline{a}})_{\beta]\alpha} \quad , \\
(\mathcal{C}_1)^{\underline{a}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{\underline{a}[2]})_{[\delta\epsilon}(\gamma_{[2]})_{\beta]\alpha} \quad , \\
(\mathcal{C}_2)^{\underline{a}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[3]})_{[\delta\epsilon}(\gamma^{\underline{a}}{}_{[3]})_{\beta]\alpha} \quad , \\
(\mathcal{D}_1)^{\underline{a}}_{[\delta\epsilon\beta]]\alpha} &= (\gamma^{\underline{a}[3]})_{[\delta\epsilon}(\gamma_{[3]})_{\beta]\alpha} \quad , \\
(\mathcal{D}_2)^{\underline{a}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[4]})_{[\delta\epsilon}(\gamma^{\underline{a}}{}_{[4]})_{\beta]\alpha} \quad .
\end{aligned} \tag{3.3.25}$$

When we consider all the objects with one free vector index constructed by two Clifford algebra basis elements with one of them being antisymmetric, we find the additional object $\mathcal{B}$ as defined above. From our past experience, we know that it has to occur in our Fierz expansions. The relevant Fierz identities are listed in Appendix A.3.2. They can be rewritten into a system of linear equations of objects $\mathcal{B}$, $\mathcal{C}_1$, $\mathcal{C}_2$, $\mathcal{D}_1$, and $\mathcal{D}_2$ (vector and spinor indices suppressed) as

$$\begin{aligned}
33\mathcal{B} &= -\frac{1}{3!}\mathcal{C}_2 + \frac{1}{2}\mathcal{C}_1 - \frac{1}{4!}\mathcal{D}_2 + \frac{1}{3!}\mathcal{D}_1 \quad , \\
90\mathcal{C}_1 &= 180\mathcal{B} - 2\mathcal{C}_2 - \frac{1}{2}\mathcal{D}_2 + 2\mathcal{D}_1 \quad , \\
5\mathcal{C}_2 &= -90\mathcal{B} - 3\mathcal{C}_1 + \frac{1}{4}\mathcal{D}_2 - \mathcal{D}_1 \quad , \\
5\mathcal{D}_1 &= 90\mathcal{B} - \mathcal{C}_2 + 3\mathcal{C}_1 - \frac{1}{4}\mathcal{D}_2 \quad , \\
17\mathcal{D}_2 &= -2520\mathcal{B} + 28\mathcal{C}_2 - 84\mathcal{C}_1 - 28\mathcal{D}_1 \quad .
\end{aligned} \tag{3.3.26}$$

The solutions are

$$\begin{cases} \mathcal{D}_1 \;=\; -\dfrac{1}{7}\mathcal{D}_2 \;=\; 48\mathcal{B} \quad , \\ \mathcal{C}_1 \;=\; -\dfrac{1}{8}\mathcal{C}_2 \;=\; 6\mathcal{B} \quad . \end{cases} \tag{3.3.27}$$

Therefore, from Equations (3.3.23) and (3.3.24), it is very clear that

$$V1 \;=\; V2 \;=\; 0 \quad . \tag{3.3.28}$$

This suggests that there exists no cubic θ-monomial in the {320} irreducible representation of so(11).

Another way of seeing that {320} cubic θ-monomials do not exist is the above constructed monomials fail to satisfy the irreducibility condition. From Equation (3.3.27), we see that $V1$ and $V2$ are clearly proportional to $\mathcal{B}$, for example. The irreducibility conditions in (3.3.21) and (3.3.22) thus read

$$(\gamma_{\underline{a}})_\alpha{}^\gamma\,(\mathcal{B})^{\underline{a}}_{[\delta\epsilon\beta]\gamma} \;=\; (\gamma_{\underline{a}})_\alpha{}^\gamma\,C_{[\delta\epsilon}(\gamma^{\underline{a}})_{\beta]\gamma} \;=\; -11C_{[\delta\epsilon}C_{\beta]\alpha} \;=\; -\,11(\mathcal{A}_1)_{[\delta\epsilon\beta]\alpha} \;\neq\; 0 \quad , \tag{3.3.29}$$

as $\mathcal{A}_1 \neq 0$ numerically (otherwise, the {32} cubic monomials would also vanish). Doing this with $\mathcal{C}$'s or $\mathcal{D}$'s will give us other linear combinations of $\mathcal{A}$'s, which would not vanish also as all $\mathcal{A}$'s are proportional to $\mathcal{A}_1$.

{1, 408} Cubic Monomials There are two versions of expressions of cubic θ-monomials with two antisymmetric vector indices as listed in Equation (3.3.10). They can be expanded in the following basis

$$\begin{aligned}\big[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{cd}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \;=\; \tilde{g}_0\Big\{&\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{cd}})_{\alpha\beta}\,\theta^\beta \;-\; \tfrac{1}{7}\,\Theta^{(3)\,[\underline{a}|\underline{cde}}\,(\gamma^{|\underline{b}]}{}_{\underline{cde}})_{\alpha\beta}\,\theta^\beta \\ &-\; \tfrac{4}{7}\,\tfrac{1}{5!4!}\,\epsilon^{[5]\underline{abcdef}}\,\Theta^{(3)}_{\underline{cdef}}\,(\gamma_{[5]})_{\alpha\beta}\,\theta^\beta\Big\} \quad ,\end{aligned} \tag{3.3.30}$$

$$\begin{aligned}\big[\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{c}})_{\alpha\beta}\,\theta^\beta\,\big]_{IR} \;=\; \tilde{h}_0\Big\{&\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{c}})_{\alpha\beta}\,\theta^\beta \;-\; \tfrac{1}{8}\,\Theta^{(2)\,[\underline{a}|\underline{cd}}\,(\gamma^{|\underline{b}]}{}_{\underline{cd}})_{\alpha\beta}\,\theta^\beta \\ &-\; \tfrac{1}{56}\Theta^{(2)\,\underline{cde}}\,(\gamma^{\underline{ab}}{}_{\underline{cde}})_{\alpha\beta}\,\theta^\beta\Big\} \quad ,\end{aligned} \tag{3.3.31}$$

where the relative coefficients are fixed by the irreducibility conditions

$$(\gamma_{\underline{b}})_\alpha{}^\gamma\,[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{cd}})_{\gamma\beta}\,\theta^\beta\,]_{IR}\;=0\quad, \tag{3.3.32}$$

$$(\gamma_{\underline{b}})_\alpha{}^\gamma\,[\,\Theta^{(2)\,\underline{abc}}\,(\gamma_{\underline{c}})_{\gamma\beta}\,\theta^\beta\,]_{IR}\;=0\quad. \tag{3.3.33}$$

From the conditions we know that these cubic θ-monomials have $32\times\frac{11\times10}{2}-32\times11=$ $1,408$ degrees of freedom, and thus they live in the $\{1,408\}$ representation. By omitting the overall coefficients $\tilde{g}_0$ and $\tilde{h}_0$, we can write the two versions as

$$V1\;=\frac{1}{3!}(\mathcal{G}_1\;-\;\tfrac{1}{7}\,\mathcal{G}_2\;-\;\tfrac{4}{7}\,\mathcal{G}_3)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta\quad, \tag{3.3.34}$$

$$V2\;=\frac{1}{3!}(\mathcal{F}_1\;+\;\tfrac{1}{8}\,\mathcal{F}_2\;-\;\tfrac{1}{56}\mathcal{F}_3)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta\quad, \tag{3.3.35}$$

where we define seven objects

$$\begin{aligned}
(\mathcal{E})^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=C_{[\delta\epsilon}(\gamma^{\underline{ab}})_{\beta]\alpha}\quad,\\
(\mathcal{F}_1)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=(\gamma^{\underline{ab}[1]})_{[\delta\epsilon}(\gamma_{[1]})_{\beta]\alpha}\quad,\\
(\mathcal{F}_2)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=(\gamma^{[2][\underline{a}})_{[\delta\epsilon}(\gamma^{\underline{b}]}{}_{[2]})_{\beta]\alpha}\quad,\\
(\mathcal{F}_3)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=(\gamma^{[3]})_{[\delta\epsilon}(\gamma^{\underline{ab}}{}_{[3]})_{\beta]\alpha}\quad,\\
(\mathcal{G}_1)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=(\gamma^{\underline{ab}[2]})_{[\delta\epsilon}(\gamma_{[2]})_{\beta]\alpha}\quad,\\
(\mathcal{G}_2)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=(\gamma^{[3][\underline{a}})_{[\delta\epsilon}(\gamma^{\underline{b}]}{}_{[3]})_{\beta]\alpha}\quad,\\
(\mathcal{G}_3)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\;&=\frac{1}{4!5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\delta\epsilon}(\gamma^{[5]})_{\beta]\alpha}\quad.
\end{aligned} \tag{3.3.36}$$

An additional object $\mathcal{E}$ is defined to span the entire basis, which plays a similar role to $\mathcal{B}$ in the $\{320\}$ representation. The Fierz expansions of all these objects as listed in

Appendix A.3.3 give us the system of linear equations

$$
\begin{aligned}
5\mathcal{G}_1 &= 9\mathcal{E} + \mathcal{F}_2 + 5\mathcal{F}_1 - \mathcal{G}_3 \quad , \\
2\mathcal{G}_2 &= 63\mathcal{E} + \mathcal{F}_3 - 21\mathcal{F}_1 - 3\mathcal{G}_3 \\
50\mathcal{F}_1 &= -18\mathcal{E} - \mathcal{F}_3 + 5\mathcal{F}_2 - \mathcal{G}_2 + 5\mathcal{G}_1 - 2\mathcal{G}_3 \quad , \\
2\mathcal{F}_2 &= 9\mathcal{E} + 5\mathcal{F}_1 + \mathcal{G}_1 - \mathcal{G}_3 \quad , \\
5\mathcal{F}_3 &= 63\mathcal{E} - 21\mathcal{F}_1 + \mathcal{G}_2 - 3\mathcal{G}_3 \quad , \\
33\mathcal{E} &= \frac{1}{3!}\mathcal{F}_3 + \frac{1}{2}\mathcal{F}_2 - \mathcal{F}_1 - \mathcal{G}_3 + \frac{1}{3!}\mathcal{G}_2 + \frac{1}{2}\mathcal{G}_1 \quad .
\end{aligned} \tag{3.3.37}
$$

By solving these linear equations, we obtain

$$
\begin{cases}
\mathcal{F}_2 = 12\mathcal{E} + 4\mathcal{F}_1 \quad , \\
\mathcal{F}_3 = 30\mathcal{E} - 6\mathcal{F}_1 \quad , \\
\mathcal{G}_1 = 6\mathcal{E} + 2\mathcal{F}_1 \quad , \\
\mathcal{G}_2 = 60\mathcal{E} - 12\mathcal{F}_1 \quad , \\
\mathcal{G}_3 = -9\mathcal{E} - \mathcal{F}_1 \quad ,
\end{cases} \tag{3.3.38}
$$

which means

$$V1 = \frac{1}{7}(3\mathcal{E} + 5\mathcal{F}_1)^{ab}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.39}$$

$$V2 = \frac{3}{56}(3\mathcal{E} + 5\mathcal{F}_1)^{ab}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad . \tag{3.3.40}$$

It is of note that here we also have the freedom to choose any other two objects as our basis. For example, we can choose $\mathcal{F}_1$ and $\mathcal{F}_2$ instead. Then we will find

$$V1 = \frac{1}{28}(16\mathcal{F}_1 + \mathcal{F}_2)^{ab}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.41}$$

$$V2 = \frac{3}{224}(16\mathcal{F}_1 + \mathcal{F}_2)^{ab}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.42}$$

$V1$ and $V2$ are always proportional to each other,

$$V1 = \frac{8}{3}V2 \quad , \tag{3.3.43}$$

thus they are equivalent and there's only one $\{1, 408\}$ irreducible representation sitting in the cubic sector.

Let's check the irreducibility condition. If we choose $\mathcal{E}$ and $\mathcal{F}_1$ as the basis, the conditions in (3.3.32) and (3.3.33) translate to

$$(\gamma_{\underline{b}})_\alpha{}^\gamma \big(3\mathcal{E} + 5\mathcal{F}_1\big)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta = 0 \quad . \tag{3.3.44}$$

After some quick calculations, we can simplify this condition as

$$-30C_{[\delta\epsilon}(\gamma^{\underline{a}})_{\beta]\alpha} + 5(\gamma^{\underline{a}[2]})_{[\delta\epsilon}(\gamma_{[2]})_{\beta]\alpha} = \big(-30\mathcal{B} + 5\mathcal{C}_1\big)^{\underline{a}}_{[\delta\epsilon\beta]\alpha} = 0 \quad , \tag{3.3.45}$$

which is exactly satisfied by Equation (3.3.27). If we choose another basis, like $\mathcal{F}_1$ and $\mathcal{F}_2$, and simplify the irreducible condition, we will get the relation between objects $\mathcal{B}$ and $\mathcal{D}_2$ in Equation (3.3.27).

With the experiences of checking the irreducibility conditions in $\{320\}$ and $\{1, 408\}$, we observe a nested structure. In Equations (3.3.15), (3.3.25) and (3.3.36), we considered all the objects constructed by two Clifford elements with one of them being antisymmetric on spinor indices, and with zero, one and two free vector indices for $\{32\}$, $\{320\}$ and $\{1, 408\}$ respectively. Applying an irreducibility condition involves doing a single γ-trace, which contracts one vector index out. Therefore, the irreducibility condition of $\{320\}$ in Equation (3.3.29) can be written in terms of objects in $\{32\}$, and different versions of irreducibility conditions of $\{1, 408\}$ (such as Equation (3.3.45)) can be written in terms of objects in $\{320\}$, which in turn give us the relations between the $\{320\}$ objects in Equation (3.3.27).

We can then comment further on why $\{320\}$ cubic monomials must vanish. We observe that there is one non-vanishing independent object in $\{32\}$, therefore the irreducibility condition in $\{320\}$ implies that there are more than one independent objects in $\{320\}$. Meanwhile, there are two independent objects in $\{1, 408\}$, and the irreducibility condition in $\{1, 408\}$ implies that there is only one independent object in $\{320\}$. Thus, we reach a contradiction. Therefore, sandwiching from $\{32\}$ and $\{1, 408\}$ would force the {320} objects to vanish.

Following this line of logical arguments, the next representation with three free vector indices, {3, 520}, would have to have three independent objects if not vanishing, and its irreducibility conditions should reduce to the relations between objects in {1, 408} in Equation (3.3.38), as we will see.

{3, 520} Cubic Monomials The two versions of cubic θ-monomials with three totally antisymmetric vector indices in Equation (3.3.10) can be expressed as

$$\begin{aligned}\left[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{d}})_{\alpha\beta}\,\theta^{\beta}\,\right]_{IR} \;&= \tilde{m}_0 \left\{\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{d}})_{\alpha\beta}\,\theta^{\beta} \;-\; \tfrac{1}{14}\,\Theta^{(3)\,[\underline{ab}|\underline{de}}\,(\gamma^{|\underline{c}]}{}_{\underline{de}})_{\alpha\beta}\,\theta^{\beta}\right.\\ &\qquad -\; \tfrac{1}{84}\,\Theta^{(3)\,[\underline{a}|\underline{def}}\,(\gamma^{|\underline{bc}]}{}_{\underline{def}})_{\alpha\beta}\,\theta^{\beta}\\ &\qquad \left. +\; \tfrac{4}{35}\,\tfrac{1}{4!4!}\,\epsilon^{[4]\underline{abcdefg}}\,\Theta^{(3)}{}_{\underline{defg}}\,(\gamma_{[4]})_{\alpha\beta}\,\theta^{\beta}\right\} \quad ,\end{aligned} \tag{3.3.46}$$

$$\begin{aligned}\left[\,\Theta^{(2)\,\underline{abc}}\,\theta_{\alpha}\,\right]_{IR} \;&= \tilde{n}_0 \left\{\Theta^{(2)\,\underline{abc}}\,\theta_{\alpha} \;+\; \tfrac{1}{16}\,\Theta^{(2)\,[\underline{ab}|\underline{d}}\,(\gamma^{|\underline{c}]}{}_{\underline{d}})_{\alpha\beta}\,\theta^{\beta}\right.\\ &\qquad +\; \tfrac{1}{112}\,\Theta^{(2)\,[\underline{a}|\underline{de}}\,(\gamma^{|\underline{bc}]}{}_{\underline{de}})_{\alpha\beta}\,\theta^{\beta}\\ &\qquad \left. -\; \tfrac{1}{56}\,\tfrac{1}{5!3!}\,\epsilon^{[5]\underline{abcdef}}\,\Theta^{(2)}{}_{\underline{def}}\,(\gamma_{[5]})_{\alpha\beta}\,\theta^{\beta}\right\} \quad ,\end{aligned} \tag{3.3.47}$$

where the relative coefficients are fixed by the irreducibility conditions

$$(\gamma_{\underline{c}})_{\alpha}{}^{\gamma}\left[\,\Theta^{(3)\,\underline{abcd}}\,(\gamma_{\underline{d}})_{\gamma\beta}\,\theta^{\beta}\,\right]_{IR} \;= 0 \quad , \tag{3.3.48}$$

$$(\gamma_{\underline{c}})_{\alpha}{}^{\gamma}\left[\,\Theta^{(2)\,\underline{abc}}\,\theta_{\gamma}\,\right]_{IR} \;= 0 \quad . \tag{3.3.49}$$

Let us do the counting. By subtracting the degrees of freedom from the irreducibility conditions, these cubic θ-monomials have $32 \times \frac{11\times10\times9}{3\times2} - 32 \times \frac{11\times10}{2} = 3,520$ degrees of freedom. Hence they sit in the {3, 520} representation. Again, we can omit the overall coefficients $\tilde{m}_0$ and $\tilde{n}_0$ and write

$$V1 \;= \frac{1}{3!}\,(\mathcal{H}_1 \;+\; \tfrac{1}{14}\,\mathcal{H}_2 \;+\; \tfrac{1}{84}\,\mathcal{H}_3 \;-\; \tfrac{4}{35}\,\mathcal{H}_4)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha}\;\theta^{\delta}\theta^{\epsilon}\theta^{\beta} \quad , \tag{3.3.50}$$

$$V2 \;= \frac{1}{3!}\,(\mathcal{I}_0 \;+\; \tfrac{1}{16}\,\mathcal{I}_1 \;-\; \tfrac{1}{112}\,\mathcal{I}_2 \;-\; \tfrac{1}{56}\,\mathcal{I}_3)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha}\;\theta^{\delta}\theta^{\epsilon}\theta^{\beta} \quad , \tag{3.3.51}$$

where we define nine objects

$$
\begin{aligned}
(\mathcal{J})^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= C_{[\delta\epsilon}(\gamma^{\underline{abc}})_{\beta]\alpha} \quad , \\
(\mathcal{H}_1)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{\underline{abc}[1]})_{[\delta\epsilon}(\gamma_{[1]})_{\beta]\alpha} \quad , \\
(\mathcal{H}_2)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[2][\underline{ab}}})_{[\delta\epsilon}(\gamma^{\underline{c}]}{}_{[2]})_{\beta]\alpha} \quad , \\
(\mathcal{H}_3)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[3][\underline{a}}})_{[\delta\epsilon}(\gamma^{\underline{bc}]}{}_{[3]})_{\beta]\alpha} \quad , \\
(\mathcal{H}_4)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= \frac{1}{4!4!}\,\epsilon^{\underline{abc}[4][\bar{4}]}\,(\gamma_{[4]})_{[\delta\epsilon}(\gamma_{[\bar{4}]})_{\beta]\alpha} \\
(\mathcal{I}_0)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{\underline{abc}})_{[\delta\epsilon}C_{\beta]\alpha} \quad , \\
(\mathcal{I}_1)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[1][\underline{ab}}})_{[\delta\epsilon}(\gamma^{\underline{c}]}{}_{[1]})_{\beta]\alpha} \quad , \\
(\mathcal{I}_2)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= (\gamma^{[2][\underline{a}}})_{[\delta\epsilon}(\gamma^{\underline{bc}]}{}_{[2]})_{\beta]\alpha} \quad , \\
(\mathcal{I}_3)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha} &= \frac{1}{3!5!}\,\epsilon^{\underline{abc}[3][5]}\,(\gamma_{[3]})_{[\delta\epsilon}(\gamma_{[5]})_{\beta]\alpha} \quad ,
\end{aligned}
\tag{3.3.52}
$$

with an additional $\mathcal{J}$ playing the role of $\mathcal{B}$ and $\mathcal{E}$ in representations $\{320\}$ and $\{1,408\}$. Again, relevant Fierz identities are listed in Appendix A.3.4. We can rewrite them as

$$
\begin{aligned}
33\mathcal{I}_0 &= -\mathcal{J} + \mathcal{I}_3 + \frac{1}{4}\mathcal{I}_2 - \frac{1}{2}\mathcal{I}_1 - \mathcal{H}_4 + \frac{1}{12}\mathcal{H}_3 + \frac{1}{4}\mathcal{H}_2 + \mathcal{H}_1 \quad , \\
33\mathcal{J} &= -\mathcal{I}_3 + \frac{1}{4}\mathcal{I}_2 + \frac{1}{2}\mathcal{I}_1 - \mathcal{I}_0 - \mathcal{H}_4 - \frac{1}{12}\mathcal{H}_3 + \frac{1}{4}\mathcal{H}_2 - \mathcal{H}_1 \quad , \\
26\mathcal{H}_1 &= -8\mathcal{J} - 2\mathcal{I}_3 - \mathcal{I}_2 + 3\mathcal{I}_1 + 8\mathcal{I}_0 + \frac{1}{6}\mathcal{H}_3 + \mathcal{H}_2 \quad , \\
38\mathcal{I}_1 &= 48\mathcal{J} - 12\mathcal{I}_3 - 2\mathcal{I}_2 - 48\mathcal{I}_0 - \frac{1}{3}\mathcal{H}_3 + 2\mathcal{H}_1 \quad , \\
28\mathcal{H}_2 &= 336\mathcal{J} - 24\mathcal{I}_3 + 4\mathcal{I}_2 + 28\mathcal{I}_1 + 336\mathcal{I}_0 - 48\mathcal{H}_4 + \frac{2}{3}\mathcal{H}_3 + 168\mathcal{H}_1 \quad , \\
28\mathcal{I}_2 &= 336\mathcal{J} + 24\mathcal{I}_3 - 28\mathcal{I}_1 + 336\mathcal{I}_0 - 48\mathcal{H}_4 - \frac{2}{3}\mathcal{H}_3 + 4\mathcal{H}_2 - 168\mathcal{H}_1 \quad .
\end{aligned}
\tag{3.3.53}
$$

Solving these linear equations gives us

$$
\begin{cases}
\mathcal{I}_1 = 4\mathcal{J} - 4\mathcal{I}_0 + 2\mathcal{H}_1 \quad , \\
\mathcal{I}_2 = 20\mathcal{J} + 28\mathcal{I}_0 - 8\mathcal{H}_1 \quad , \\
\mathcal{I}_3 = -4\mathcal{J} + 4\mathcal{I}_0 - \mathcal{H}_1 \quad , \\
\mathcal{H}_2 = 28\mathcal{J} + 20\mathcal{I}_0 + 8\mathcal{H}_1 \quad , \\
\mathcal{H}_3 = -120\mathcal{J} + 120\mathcal{I}_0 + 12\mathcal{H}_1 \quad , \\
\mathcal{H}_4 = -5\mathcal{J} - 5\mathcal{I}_0 \quad .
\end{cases}
\tag{3.3.54}
$$

Here we choose $\mathcal{J}$, $\mathcal{I}_0$ and $\mathcal{H}_1$ as our basis since they are the three simplest objects. Then

$$V1 \;=\; \frac{4}{7}(2\mathcal{J}+6\mathcal{I}_0+3\mathcal{H}_1)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.55}$$

$$V2 \;=\; \frac{1}{14}(2\mathcal{J}+6\mathcal{I}_0+3\mathcal{H}_1)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha}\theta^\delta\theta^\epsilon\theta^\beta \quad , \tag{3.3.56}$$

which means

$$V1 \;=\; 8\,V2 \quad , \tag{3.3.57}$$

and that there is precisely one independent $\{3, 520\}$ representation sitting in the space of cubic monomials.

Now rewrite the irreducibility conditions in (3.3.48) and (3.3.49) to

$$(\gamma_{\underline{c}})_\alpha{}^\gamma\,\big(2\mathcal{J}+6\mathcal{I}_0+3\mathcal{H}_1\big)^{\underline{abc}}_{[\delta\epsilon\beta]\alpha}\,\theta^\delta\theta^\epsilon\theta^\beta = 0 \quad . \tag{3.3.58}$$

This condition can be simplified as

$$-6\,C_{[\delta\epsilon}(\gamma^{\underline{ab}})_{\beta]\alpha}+(\gamma^{\underline{ab}[2]})_{[\delta\epsilon}(\gamma_{[2]})_{\beta]\alpha}-2\,(\gamma^{\underline{ab}[1]})_{[\delta\epsilon}(\gamma_{[1]})_{\beta]\alpha} = \big(-6\mathcal{E}+\mathcal{G}_1-2\mathcal{F}_1\big)^{\underline{ab}}_{[\delta\epsilon\beta]\alpha} = 0 \quad , \tag{3.3.59}$$

which exactly satisfies Equation (3.3.38), as predicted in the last subsection.

The importance of the observations so far in this section is that an expansion to third order of a real scalar superfield $\mathcal{V}$ must be written in terms of complete sets of irreducible monomials. So to this order we have

$$\begin{aligned}\mathcal{V}(\theta,\,x) \;=&\; \varphi^{(0)}(x) \;+\; \theta^\alpha\,\varphi^{(1)}_\alpha(x) \;+\; \Theta^{(1)}\,\varphi^{(2)}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\varphi^{(2)}_{\underline{abc}}(x) \;+\; \Theta^{(3)\,\underline{abcd}}\,\varphi^{(2)}_{\underline{abcd}}(x)\\ &+\; \Theta^{(1)}\,\theta^\alpha\,\varphi^{(3)}_\alpha(x) \;+\; \big[\Theta^{(2)\,\underline{abc}}\,\theta^\alpha\big]_{IR}\,\varphi^{(3)}_{\alpha\,\underline{abc}}(x) \;+\; \big[\Theta^{(3)\,\underline{abcd}}\,\theta^\alpha\big]_{IR}\,\varphi^{(3)}_{\alpha\,\underline{abcd}}(x)\\ &+\; \dots \quad .\end{aligned} \tag{3.3.60}$$

3.3.3 Quartic Level

At fourth-order, a new problem appears in the expression in (3.3.1). Let us first pick out the relevant terms,

$$\begin{aligned}\mathcal{V}(quartic) &= \Theta^{(1)}\,\Theta^{(1)}\,\varphi^{(4)}(x) \;+\; \Theta^{(1)}\,\Theta^{(2)\,\underline{abc}}\,\varphi^{(4)}_{\underline{abc}}(x) \;+\; \Theta^{(1)}\,\Theta^{(3)\,\underline{abcd}}\,\varphi^{(4)}_{\underline{abcd}}(x) \\ &\quad+\; \Theta^{(2)\,\underline{abc}}\,\Theta^{(2)\,\underline{def}}\,\varphi^{(4)}_{\underline{abc}\,\underline{def}}(x) \;+\; \Theta^{(2)\,\underline{abc}}\,\Theta^{(3)\,\underline{defg}}\,\varphi^{(4)}_{\underline{abc}\,\underline{defg}}(x) \\ &\quad+\; \Theta^{(3)\,\underline{abcd}}\,\Theta^{(3)\,\underline{efgh}}\,\varphi^{(4)}_{\underline{abcd}\,\underline{efgh}}(x) \;+\; \dots \quad .\end{aligned} \tag{3.3.61}$$

The problem is seen by the following argument. From the antisymmetry of the θ-coordinates, we know the number of degrees of freedom at this order is given by

$$\{32\} \wedge \{32\} \wedge \{32\} \wedge \{32\} \;=\; \frac{\{32\} \;\times\; \{31\} \;\times\; \{30\} \;\times\; \{29\}}{4 \;\times\; 3 \;\times\; 2} \;=\; \{35,960\} \quad . \tag{3.3.62}$$

Next, one can count the degrees of freedom of the bosonic component fields at the fourth-order superfield expansion in (3.3.61). We have

$$\begin{array}{ll} \{1\} & \Theta^{(1)}\,\Theta^{(1)} \quad , \\ \{165\} & \Theta^{(1)}\,\Theta^{(2)\,\underline{abc}} \quad , \\ \{330\} & \Theta^{(1)}\,\Theta^{(3)\,\underline{abcd}} \quad , \\ \{13,695\} & \Theta^{(2)\,\underline{abc}}\,\Theta^{(2)\,\underline{def}} \quad , \\ \{54,450\} & \Theta^{(2)\,\underline{abc}}\,\Theta^{(3)\,\underline{defg}} \quad , \\ \{54,615\} & \Theta^{(3)\,\underline{abcd}}\,\Theta^{(3)\,\underline{efgh}} \quad . \end{array} \tag{3.3.63}$$

The last three numbers come from

$$\begin{aligned}\{13,695\} &= [\,\{165\} \;\otimes\; \{165\}\,]_S \;=\; \frac{\{165\} \;\times\; \{166\}}{2} \quad , \\ \{54,450\} &= \{165\} \;\otimes\; \{330\} \quad , \\ \{54,615\} &= [\,\{330\} \;\otimes\; \{330\}\,]_S \;=\; \frac{\{330\} \;\times\; \{331\}}{2} \quad ,\end{aligned} \tag{3.3.64}$$

where $[\;\;]_S$ means the symmetric part of the product, as $\Theta^{(2)}$ and $\Theta^{(3)}$ carry two spinor indices and they commute with themselves. Note that these three numbers are not the dimensions of any $\mathfrak{so}(11)$ irrep. They are reducible representations that carry more than the necessary irreducible components contained in the superfield. This is clear as symmetry properties are not fully utilized to constrain the degrees of freedom. For example in the second line, the spinor indices on $\Theta^{(2)}$ and $\Theta^{(3)}$ could also be swapped, but the tensor product among them clearly include the maximal degrees of freedom. Also, in the first and third lines, one only considered exchanging the positions of the entire $\Theta^{(2)}$'s or $\Theta^{(3)}$'s, but not the possibilities of exchanging individual Lorentz indices. Moreover,

$$\{1\} \oplus \{165\} \oplus \{330\} \oplus \{13,695\} \oplus \{54,450\} \oplus \{54,615\} = \{123,256\} \quad . \tag{3.3.65}$$

Clearly, the dimension of $\{123,256\}$ far exceeds that of $\{35,960\}$.

Therefore, by following the path in the cubic sector, one immediate thought is to write down all possible index structures, i.e. first constructing the irreducible quartic monomials just as was done for the cubic order. That would essentially be breaking down the reducible representations $\{13,695\}$, $\{54,450\}$, and $\{54,615\}$ into sums of $\mathfrak{so}(11)$ irreducible representations. In other words, one would allow the component fields $\varphi^{(4)}_{\underline{abc}\,\underline{def}}(x)$, $\varphi^{(4)}_{\underline{abc}\,\underline{defg}}(x)$, and $\varphi^{(4)}_{\underline{abcd}\,\underline{efgh}}(x)$ to admit linear constraints that identify various irreducible component fields with each other. Due to the overwhelming excess of irreducible components, one could hope to ameliorate this situation by finding that some of the bosonic fields are actually zero. As with the cubic sector, there exist a sufficient number of Fierz identities such that when these fourth-order terms are expanded over an irreducible basis, one finds relations between the seemingly independent ones in such a way that some terms vanish and the counting of the remaining field components adds to 35,960. That would be an excessive amount of tedious calculations.

In later chapters, one would see how these terrible decompositions could be turned into neat group-theory problems that could be solved by efficient algorithms.

The main message of this section of our work is that explicit θ-expansion of the eleven dimensional scalar superfield is considerably more complicated than in lower dimensions. One must contend with four separate problems:

(a.) there are multiple equivalent ways to express the required θ-monomials,

(b.) some apparently reasonable monomial combinations actually vanish,

(c.) the requirement of the irreducibility of the θ-monomial expansion requires carefully constructed combinations, and

(d.) the over abundance of bosonic fields encountered from the most obvious θ-expansion.

The resolution of the first two of these problems relies of the derivation of Fierz identities. With regard to the third problem, the only methodology known to use is brute force establishment of their existences. The final problem requires a careful choice of constraints.

At all higher orders, up to the sixteenth, in the θ-expansion this problem of deriving explicit irreducible θ-monomials occurs. Above this order, the form of the required higher-order irreducible θ-monomials can be deduced from the lower order irreducible θ-monomials. For terms in odd orders, an actual derivation of the irreducible θ-monomials involves derivations of Fierz identities as we explicitly demonstrated at cubic order. For terms in even orders, an actual derivation of the irreducible θ-monomials involves derivations along the lines we implicitly discussed at quartic order. To our knowledge, these impediments have not been recognized previously in the literature. A most disappointing realization would be that all of these need to be sorted out before any discussion of dynamics using off-shell 11D, $\mathcal{N} = 1$ superfields.

All of these point to the fact that superfields and their accompanying θ-expansions become increasing unwieldy as a "technological platform" for the study of supermultiplets in higher dimensions. This is due to the fact that in order to calculate efficiently, polynomials $\left[\theta^1 \cdots \theta^p\right]_{IR}$ at Level-p are required to be explicitly constructed.

For all the reasons enunciated in an illustrative manner throughout this section, we are motivated to search for an approach that is free of these burdensome complications.

Chapter 4

The Ectoplasmic Conjecture

This chapter is based on [113].

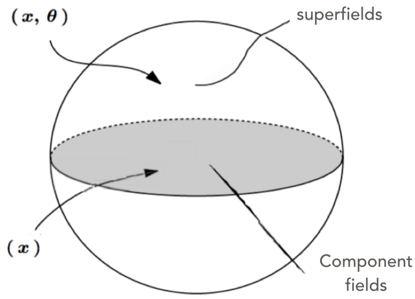
This chapter aims to provide a big picture of how the so-called ectoplasmic conjecture opens a new doorway to address the forty-year-old problem discussed in chapter 3. We will start from the superspace version of the ectoplasmic conjecture which was first introduced in [114]. Then its branching rule realizations and implications for superfields give a clear methodology on how to find out the complete component field content for a superfield in practice.

4.1 The Ectoplasmic Conjecture

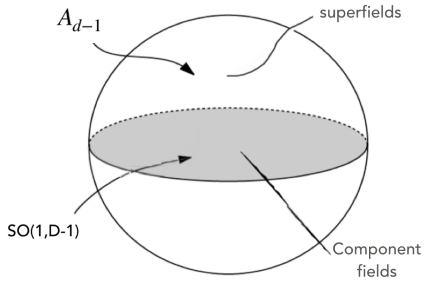
In a past work [114], there was presented a graphical visualization of the superspace that appears below in Fig. 4.1 (a). One can think of the volume of the sphere as representing the entirety of the superspace and the equatorial plane represents the bosonic subspace. Superfields "live" in the whole of the Salam-Strathdee superspace, and in order to extract the component fields, one needs to carry out the θ-expansion as Taylor series and then set $\theta \to 0$ (a "projection" process) in the coefficients of the expansion.

Recently [40, 57–59, 62] we noted a more mathematically concrete manner for utilization of this construct and asserted the branching rules for $\mathfrak{su}(d) \supset \mathfrak{so}(\mathrm{D})$ can be applied to realize the projection process.

In the second of these diagrams, two parameters appear. The first, d, denotes the minimal number of independent real θ-coordinates that must be introduced so the spinorial



(A) A representation of superspace [114]



(B) A realization of "projection" [40, 57–59, 62]

Figure 4.1: The Ectoplasmic Conjecture

superspace coordinates can be identified with a spin bundle of the double cover representation of the Lorentz rotations of a D-dimensional Minkowski space. The number of spatial dimensions, D, of the Minkowski space manifold is the second parameter. The parameter d indicates which of the $\mathcal{A}_{d-1}$-series Lie algebras is relevant to the construction of SUSY representations. There exists a function $d(\mathrm{D})$ that relates the d-parameter to the D-parameter.

In order to gain insight into this function, it is simplest to study it in the case where $\mathrm{D} = 1$, i.e. no spatial directions. In this case, the parameter d can be conveniently related to $\mathcal{N}$ where the former parameter denotes the number of independent bosons (or fermions) required for a faithful linear representation of SUSY in the $\mathrm{D} = 1$ theory. The quantity $\mathcal{N}$ is the number of "supercharges' in our description of this function. In the language of physics, $\mathcal{N}$ is the "degrees of extension' of the supersymmetry generators. Thus, the parameter d becomes a function of $\mathcal{N}$.

When the results for d are written more explicitly for values $1 \leq \mathcal{N} \leq 16$, they can be expressed as shown in the table that follows. From this table, we see there are some

Table 4.1: Number of 1D Supercharges vs. Number of Bosons (or Fermions)

$\mathcal{N}$	1	2	3	4	5	6	7	8	9	10	11	12	13	14	15	16
d	1	2	4	4	8	8	8	8	16	32	64	64	128	128	128	128

values of $\mathcal{N}$ such that as one goes from $\mathcal{N}$ to $\mathcal{N}+1$, the size of the parameter d "jumps" by a factor of two. This occurs for:

$$\mathcal{N} \ = \ 1, \ 2, \ 4, \ 8, \ 9, \ 10, \ 12, \ 16, \ 17, \ 18, \ 20, \ 25, \ldots \tag{4.1.1}$$

and we can borrow language from nuclear physics[1] and call these "magical" values of $\mathcal{N}$.

To efficiently describe this behavior, it is convenient to introduce the function $\widehat{d}(n)$ defined as

$$\begin{aligned} \widehat{d}(n) \ &= \ \left[\delta_{n,\,1} \ + \sum_{p=0}^{2} 2^{p+1} \left(\sum_{q=1}^{2^p} \delta_{n,\,q+2^p} \right) \right] \\ &= \ [\, \delta_{n,\,1} \ + \ 2 \ (\, \delta_{n,\,2}\,) \ + \ 4 \ (\, \delta_{n,\,3} \ + \ \delta_{n,\,4}\,) \ + \ 8 \ (\, \delta_{n,\,5} \ + \ \delta_{n,\,6} \ + \ \delta_{n,\,7} \ + \ \delta_{n,\,8}\,)\,] \quad . \end{aligned} \tag{4.1.2}$$

in terms of the Kronecker delta symbol. This function was first noted in tabular form in the work of [76]. Since we define the domain of this function to be the counting integers from one to eight we can display it as a table also and so it is very clear that for a

Table 4.2: n vs. $\widehat{d}(n)$

n	1	2	3	4	5	6	7	8
$\widehat{d}(n)$	1	2	4	4	8	8	8	8

restricted domain of $\mathcal{N}$, the quantities d and $\widehat{d}$ are equal. Next the parameter $\mathcal{N}$ can be projected onto a space of two integers denoted by m and n via the equation $\mathcal{N} = 8\,m + n$ where $m = 0, 1, 2, 3, \ldots$ and $1 \le n \le 8$ and using this one can write

$$d(\mathcal{N}) \ = \ 2^{4m}\, \widehat{d}(n) \quad , \tag{4.1.3}$$

which is valid for all values of $\mathcal{N}$.

[1] See the webpage at `https://en.wikipedia.org/wiki/Magic_number_(physics)` online.

4.2 Branching Rule Realizations

In the work [40,57,58,62], we proposed branching rules for $\mathfrak{su}(d) \supset \mathfrak{so}(\mathrm{D})$ to decipher the complete set of component fields residing in the scalar superfield in arbitrary D dimensions and in this context d is the dimension of the minimal irreducible spinor representation in $\mathfrak{so}(\mathrm{D})$.

In the work [59], in order to translate Lorentz irreducible representations into field variables, we introduced irreducible Bosonic Young Tableau (BYT) and irreducible Spinorial Young Tableau (SYT). The irreducibility conditions of BYT's are effectuated by the branching rules for $\mathfrak{su}(\mathrm{D}) \supset \mathfrak{so}(\mathrm{D})$ in spacetime dimension D. The bosonic tying rules are graphical rules for the same branching rules $\mathfrak{su}(\mathrm{D}) \supset \mathfrak{so}(\mathrm{D})$. Below is the table of branching rules used in various dimensions. It is also useful to note that for all values of

D	Branching Rules $\to$ component fields	Branching Rules $\to$ BYT
11	$\mathcal{A}_{31} \supset \mathcal{B}_5$	$\mathcal{A}_{10} \supset \mathcal{B}_5$
10	$\mathcal{A}_{15} \supset \mathcal{D}_5$	$\mathcal{A}_9 \supset \mathcal{D}_5$
9	$\mathcal{A}_{15} \supset \mathcal{B}_4$	$\mathcal{A}_8 \supset \mathcal{B}_4$
8	$\mathcal{A}_{15} \supset \mathcal{D}_4$	$\mathcal{A}_7 \supset \mathcal{D}_4$
7	$\mathcal{A}_{15} \supset \mathcal{B}_3$	$\mathcal{A}_6 \supset \mathcal{B}_3$
6	$\mathcal{A}_7 \supset \mathcal{D}_3 = \mathcal{A}_3$	$\mathcal{A}_5 \supset \mathcal{D}_3$
5	$\mathcal{A}_7 \supset \mathcal{B}_2 \cong \mathcal{C}_2$	$\mathcal{A}_4 \supset \mathcal{B}_2$
4	$\mathcal{A}_3 \supset \mathcal{D}_2 \cong \mathcal{A}_1 \times \mathcal{A}_1$	$\mathcal{A}_3 \supset \mathcal{D}_2$
3	$\mathcal{A}_1 \cong \mathcal{B}_1$	$\mathcal{A}_2 \supset \mathcal{B}_1$
2	$\mathfrak{su}(2) \supset \mathfrak{so}(2) \cong \mathfrak{u}_1$	$\mathfrak{su}(2) \supset \mathfrak{u}_1$

Table 4.3: A list of branching rules used to obtain scalar superfield decompositions as well as define BYT's and determine their irreducibility conditions in various spacetime dimension D's

$d > 4$, the injections $\mathcal{A}_{d-1} \supset \mathcal{A}_3$ describe 4D, $\mathcal{N}$-extended SUSY where $\mathcal{N} = d/4$.

When $\mathrm{D} = 3$, the Lorentz group is $SO(1,2)$ and the corresponding fundamental representation acts on a real (Majorana) two-component spinor $\psi^\alpha = (\psi^+, \psi^-)$. The $SU(2)$ group, which corresponds to the A_1 algebra, is the double copy of $SO(3)$. When $\mathrm{D} = 2$, like in 10 dimensions, we have Majorana-Weyl spinors which have d.o.f. $= 1$. So we actually consider the Dirac spinors in this case.

The first two columns of Table 4.3 can be summarized more simply by relating the number of bosonic dimensions to the number of fermionic dimensions in a simpler table.

and this shows the same sort of jumping behavior seen for the 1D discussion, starting

Table 4.4: Number of Bosonic Dimensions vs. Number of Fermionic Dimensions

D	3	4	5	6	7	8	9	10	11
d	2	4	8	8	16	16	16	16	32

above the value of D = 3

$$\mathrm{D} \ = \ 3, \ 4, \ 6, \ 10, \ldots \tag{4.2.1}$$

as the "magical dimensions" similar to the D = 1 case. Here, the parameter D can be projected onto a space of two integers denoted by r and s via the equation $\mathrm{D} = 8\,r\, +\, s\, +\, 2$ where r and s take on the same values as m and n previously. Thus, using this one can write

$$d(\mathrm{D}) \ = \ 2^{4r+1}\, \widehat{d}(s) \quad , \tag{4.2.2}$$

which we conjecture is valid for all values of D.

The formula in Eq.(4.2.2) applies to the case of a realization of one supersymmetry in a space-time dimension of D. It is also possible to realize $\mathcal{N}$ such supersymmetry generators in a space-time of dimension D. However, we will not explore this topic here.

The equations Eq.(4.1.2) and Eq.(4.2.2) provide an example of the "SUSY holography" conjecture discussed in the work [74]. The mathematical function, $\widehat{d}$, that applies to the dimensions of the fermionic space for a 1D, $\mathcal{N}$-extended SUSY theory also applies to the fermionic space associated with the D-dimensional theory and minimal SUSY. We will talk more about the "SUSY holography" conjecture in chapter 9 later.

Finally, the function $\widehat{d}(n)$ may be related to the Radon-Hurwitz number and we leave this question open for the mathematically minded reader.

A feature that is apparent from examining the middle column of the Table 4.3 is that consistently above dimension three, there is an (expected) alternation between the $\mathcal{B}$ and $\mathcal{D}$ Lie algebras. Based on this observation and further component level evidence (to be discussed later), we are prompted to make a conjecture about the structure of superspace associated with a minimal realization of SUSY for arbitrary values of D.

The irreducible conditions of SYTs are given by the tensor product rules of its corresponding BYTs and the fundamental spinorial irreducible representation of $\mathfrak{so}(\mathrm{D})$, which is $\{d\}$. The combinatorics correspond to requiring all possible gamma-traces to be zero.

4.3 Implications for Superfields

The points indicated in the last section have implications for superfields, to which we now turn.

Ectoplasmic Conjecture: Representation Theory Version

Minkowski superspaces, with spacetime metric signatures of $(1, \mathrm{D}-1)$*, are fibrations that occur by injections of* $\mathcal{A}_{d-1} \supset \mathcal{B}_{(\mathrm{D}-1)/2}$ *(for* D *odd) or* $\supset \mathcal{D}_{\mathrm{D}/2}$ *(for* D *even) representations in Cartan Classification's of compact Lie algebras and where* d *and* D *are determined by the function* $d(\mathrm{D})$ *that possesses Bott Periodicity.*

It is useful at this point to provide the evidence that led us to this conjecture.

A Minkowski superspace is has traditionally been described by a bosonic coordinate $x^{\underline{a}}$ (where the index $\underline{a}$ takes on values $\underline{a} = 0, 1, \ldots, \mathrm{D}-1$) and a fermionic coordinate θ^α (where the index α takes on values $a = 0, 1, \ldots, d$). A scalar superfield $\mathcal{V}(x^{\underline{a}}, \theta^\alpha)$ has traditionally been described as a Taylor Series expansion in the 'θ-coordinates.' One is thus led to consider quantities of the forms

$$\mathcal{V}(x^{\underline{a}}, \theta^\alpha) ~=~ v^{(0)}(x) ~+~ \sum_{n=1}^{d} v^{(n)}{}_{\alpha_1 \cdots \alpha_n}(x)\, \theta^{\alpha_1} \cdots \theta^{\alpha_n} \quad . \tag{4.3.1}$$

For the sake of simplicity, we always (at least initially) assume that the 'θ-coordinates' are real anticommuting Grassmann numbers that form the minimal irreducible representation of the spin-group associated with the bosonic space with coordinates $x^{\underline{a}}$. Next one can introduce a 'level parameter,' denoted by ℓ, whose role is to account for how the functions $v^{(0)}(x)$, and $v^{(n)}{}_{\alpha_1 \cdots \alpha_n}(x)$ are associated with the various powers of θ^α. The function $v^{(0)}(x)$ is not associated with any powers of θ^α, so we associated $(\ell)^0$ with it.

The functions ${v^{(1)}}_{\alpha_1}(x)$ are associated with the first powers of the θ^α-coordinates, so we associated $(\ell)^1$ with them, and so forth for the remaining functions.

At level ℓ in the expansion of Eq.(4.3.1), due to the anticommutativity of the d independent Grassmann coordinates, there occur $d!/\ell!(d-\ell)!$ degrees of freedom. Let us consider only the even values of ℓ as a beginning point since in this case the functions ${v^{(n)}}_{\alpha_1\cdots\alpha_n}(x)$ are bosons. These bosonic fields are defined over the D-dimension Minkowski space and must therefore occur as some bosonic Lorentz representations $\{\mathcal{R}\}$ of the manifold. Such representations possess a dimensionality (i.e. degrees of freedom) that can be denoted by $d_{\{\mathcal{R}\}}$. Therefore, one can conclude an equation expressed as

$$\frac{d!}{\ell!(d-\ell)!} = \sum_{\mathcal{R}} b_{\{\mathcal{R}\}}\, d_{\{\mathcal{R}\}} \quad , \tag{4.3.2}$$

(where $b_{\{\mathcal{R}\}}$ denotes a set of integers) must be satisfied. By similar reasoning applied to odd values of ℓ, one concludes an equation expressed as

$$\frac{d!}{\ell!(d-\ell)!} = \sum_{\mathcal{R}} b_{\{\mathcal{R}\}}\, d_{\{\mathcal{R}\}} \quad , \tag{4.3.3}$$

(where $b_{\{\mathcal{R}\}}$ again denotes a set of integers) must be satisfied for fermionic representations $\{\mathcal{R}\}$.

If we now turn to representation theory, the factors of $d!/\ell!(d-\ell)!$ are recognizable as the dimensions of forms in the $\mathcal{A}_{d-1}$ series of compact Lie Algebras. We are thus able to write

$$\frac{[dim_{Fund}(\mathcal{A}_{d-1})]!}{\ell!\,[dim_{Fund}(\mathcal{A}_{d-1})-\ell]!} = \sum_{\mathcal{R}} b_{\{\mathcal{R}\}}\, d_{\{\mathcal{R}\}} \quad , \tag{4.3.4}$$

for even values of ℓ

$$\frac{[dim_{Fund}(\mathcal{A}_{d-1})]!}{\ell!\,[dim_{Fund}(\mathcal{A}_{d-1})-\ell]!} = \sum_{\mathcal{R}} b_{\{\mathcal{R}\}}\, d_{\{\mathcal{R}\}} \quad , \tag{4.3.5}$$

and for odd values of ℓ.

A surmise from this can be drawn, it is the ℓ-forms of $\mathcal{A}_{d-1}$ Lie Algebra that 'control' the component field representations that occur in the conventional θ-expansions of superfields.

However, one still requires a mathematical principle to determine the $b_{\{\mathcal{R}\}}$ and $b_{\{\mathcal{R}\}}$ coefficients on the right hand side of these equations. We concluded in the works of [57–59, 62] that it is the branching rules of the injections of $\mathcal{A}_{d-1} \supset \mathcal{B}_{(\mathrm{D}-1)/2}$ (for D odd) or $\supset \mathcal{D}_{\mathrm{D}/2}$ (for D even) representations applied to the ℓ-forms that 'control' the values of the $b_{\{\mathcal{R}\}}$ and $b_{\{\mathcal{R}\}}$ coefficients.

Stated in the simplest possible way, branching rules for the ℓ-forms for $\mathcal{A}_{d-1} \supset \mathcal{B}_{(\mathrm{D}-1)/2}$ (for D odd) or $\supset \mathcal{D}_{\mathrm{D}/2}$ (for D even) determine the component spectrum of superfields for the D-dimensional superspace with simple supersymmetry.

The appearance of these ℓ-forms provides additional insights for the construction of efficient algorithms that lend to the transparent exposure and manipulation of the component field content of superfields. It is well known that Young Tableaux provide a useful visualization of the representations of Lie Algebras. So the $\mathcal{A}_{d-1}$ ℓ-forms may be represented by single column tableaux. On the other hand, there also exist Young Tableau representations of the bosonic and fermionic field representations of $\mathcal{B}_{(\mathrm{D}-1)/2}$ (for D odd) or $\supset \mathcal{D}_{\mathrm{D}/2}$ (for D even) representations. This implies the existence of a graphical visualization basis for describing the component field content of superfields in arbitrary dimensions based on these Young Tableaux representations. In particular, the multiplication rules for Young Tableaux provide a basis for understanding how the multiplications of superfields are determined.

Chapter 5

Superfield Component Decompositions in 10D Superspaces

This chapter is based on [57].

In this chapter, the first complete and explicit SO(1,9) Lorentz descriptions of all component fields contained in $\mathcal{N} = 1$, $\mathcal{N} = 2$A, and $\mathcal{N} = 2$B unconstrained scalar 10D superfields are presented. These are made possible by the discovery of the relation of the superfield component expansion as a consequence of the branching rules of irreducible representations in one ordinary Lie algebra into one of its Lie subalgebras. Adinkra graphs for ten-dimensional superspaces are defined for the first time, whose nodes depict spin bundle representations of SO(1,9). A consequential deliverable of this advance is it provides the first explicit, in terms of component fields, examples of off-shell 10D Nordström SG theories that are reducible but with finite numbers of fields in the manner of Breitenlohner. Also an analog of Breitenlohner's approach is implemented to scan for superfields that contain graviton(s) and gravitino(s), which are the candidates for the superconformal prepotential superfields of 10D off-shell supergravity theories and Yang-Mills theories.

We organize this current chapter in the manner described below. In section 5.1, we present the decomposition result of the scalar superfield in 10D, $\mathcal{N} = 1$ theory. Two different approaches that lead to the same result are presented. One we call the "handicraft approach,[1]" where we introduce Bosonic and Fermionic Young Tableaux. The other is the application of branching rules, the restrictions of representations from a Lie algebra $\mathfrak{g}$ to one of its subalgebra $\mathfrak{h}$. The complete ten-dimensional $\mathcal{N}=1$ adinkra is drawn.

In section 5.2, we start from the well-studied off-shell description of 4D, $\mathcal{N}=1$ supergravity established by Breitenlohner and show how to apply his idea to carry out constructions of candidates for the prepotential and Yang-Mills supermultiplets in 10D, $\mathcal{N}=1$ theory. In section 5.3, we present the methodology and results of constructing the 10D, $\mathcal{N}=2$A scalar superfield from the 10D, $\mathcal{N}=1$ scalar superfield, as well as the discussion of the search of prepotential supermultiplet in Type IIA superspace. The ten-dimensional IIA adinkra diagram is shown at low orders. However, its complete structure is given in the form of a list of the component field representations it contains. These same steps are repeated in section 5.4 that gives the methodology and explicit decomposition results as well as the discussion of constructing prepotential supermultiplet in Type IIB superspace. The ten-dimensional IIB adinkra diagram is also shown at low orders.

5.1 10D, $\mathcal{N}=1$ Scalar Superfield Decomposition

In the case of the 10D, $\mathcal{N}=1$ theory, the number of independent Grassmann coordinates is $2^{10/2-1}=16$ due to the Majorana-Weyl condition. Then the superspace has coordinates $(x^{\underline{a}},\theta^{\alpha})$, where $\underline{a}=0,1,\dots,9$ and $\alpha=1,\dots,16$. Hence, the θ-expansion of the ten-dimensional scalar superfield begins at Level-0 and continues to Level-16, where Level-n corresponds to the order $\mathcal{O}(\theta^n)$. The unconstrained real scalar superfield $\mathcal{V}$ contains 2^{16-1} bosonic and 2^{16-1} fermionic components, as counted in Table 3.1. Let's express the 10D, $\mathcal{N}=1$ scalar superfield as

$$\mathcal{V}(x^{\underline{a}},\,\theta^{\alpha}) \;=\; \varphi^{(0)}(x^{\underline{a}}) \;+\; \theta^{\alpha}\,\varphi^{(1)}_{\alpha}(x^{\underline{a}}) \;+\; \theta^{\alpha}\theta^{\beta}\,\varphi^{(2)}_{\alpha\beta}(x^{\underline{a}}) \;+\; \dots \quad . \tag{5.1.1}$$

[1] The term "handicraft approach" comes for P. van Nieuwenhuizen who coined it to refer to a mathematical "Rube Goldberg" type of approach that nevertheless yields a correct result in the context of supersymmetry calculations.

We can decompose θ-monomials $\theta^{[\alpha_1} \cdots \theta^{\alpha_n]}$ into a direct sum of irreducible representations of Lorentz group SO(1,9). With the antisymmetric property of Grassmann coordinates, we have

$$\mathcal{V} = \begin{cases} \text{Level} - 0 & \{1\} \quad , \\ \text{Level} - 1 & \{16\} \quad , \\ \text{Level} - 2 & \{16\} \wedge \{16\} \quad , \\ \text{Level} - 3 & \{16\} \wedge \{16\} \wedge \{16\} \quad , \\ \quad \vdots & \vdots \\ \text{Level} - n & \underbrace{\{16\} \wedge \ \dots \ \wedge \{16\}}_{n \text{ times}} \quad , \\ \quad \vdots & \vdots \\ \text{Level} - 16 & \{1\} \quad . \end{cases} \tag{5.1.2}$$

All even levels are bosonic representations, while all odd levels are fermionic representations. Note that in a 16-dimensional Grassmann space, the Hodge-dual of a p-form is a $(16 - p)$-form. Therefore, level-$(16 - n)$ is the dual of level-n for $n = 0, \dots, 8$, and they have the same dimensions. By simple use of the values of the function "16 choose n," these dimensions are found to be the ones that follow,

$$\mathcal{V} = \begin{cases} \text{Level} - 0 & 1 \quad , \\ \text{Level} - 1 & 16 \quad , \\ \text{Level} - 2 & 120 \quad , \\ \text{Level} - 3 & 560 \quad , \\ \quad \vdots & \vdots \\ \text{Level} - n & \frac{16!}{n!(16-n)!} \quad , \\ \quad \vdots & \vdots \\ \text{Level} - 16 & 1 \quad . \end{cases} \tag{5.1.3}$$

The $\theta-$monomials contained within the 10D scalar superfield $\mathcal{V}$ can be specified by the irreducible representations of $\mathfrak{so}(1,9)$ as follows. In Appendix C, the irreps with small dimensions are listed.

$$\mathcal{V} = \begin{cases} \text{Level} - 0 & \{\mathsf{1}\} \quad , \\ \text{Level} - 1 & \{\mathsf{16}\} \quad , \\ \text{Level} - 2 & \{\mathsf{120}\} \quad , \\ \text{Level} - 3 & \{\overline{\mathsf{560}}\} \quad , \\ \text{Level} - 4 & \{\mathsf{770}\} \oplus \{\mathsf{1050}\} \quad , \\ \text{Level} - 5 & \{\mathsf{672}\} \oplus \{\mathsf{3696}\} \quad , \\ \text{Level} - 6 & \{\mathsf{3696'}\} \oplus \{\mathsf{4312}\} \quad , \\ \text{Level} - 7 & \{\overline{\mathsf{2640}}\} \oplus \{\overline{\mathsf{8800}}\} \quad , \\ \text{Level} - 8 & \{\mathsf{660}\} \oplus \{\mathsf{4125}\} \oplus \{\mathsf{8085}\} \quad , \\ \text{Level} - 9 & \{\mathsf{2640}\} \oplus \{\mathsf{8800}\} \quad , \\ \text{Level} - 10 & \{\overline{\mathsf{3696'}}\} \oplus \{\mathsf{4312}\} \quad , \\ \text{Level} - 11 & \{\overline{\mathsf{672}}\} \oplus \{\overline{\mathsf{3696}}\} \quad , \\ \text{Level} - 12 & \{\mathsf{770}\} \oplus \{\overline{\mathsf{1050}}\} \quad , \\ \text{Level} - 13 & \{\mathsf{560}\} \quad , \\ \text{Level} - 14 & \{\mathsf{120}\} \quad , \\ \text{Level} - 15 & \{\overline{\mathsf{16}}\} \quad , \\ \text{Level} - 16 & \{\mathsf{1}\} \quad . \end{cases} \tag{5.1.4}$$

Note that Level-16 to Level-9 are the conjugates of Level-0 to Level-7 respectively, and Level-8 is self-conjugate. That's the meaning of the duality mentioned above. It is also clear that the dimensions given by $\frac{16!}{n!(16-n)!}$ "align" with the order of the irreducible representations for the first four levels (and the corresponding dual levels) of the 10D, $\mathcal{N}=1$ scalar superfield $\mathcal{V}$. When the dimension of the wedge product does not align with the dimension of an irreducible representation, it must be the case that the sum of a judiciously chosen subset of irreducible representation is equal to $\frac{16!}{n!(16-n)!}$ at the n-th level.

One may wonder why "bar" representations occur in 10D, $\mathcal{N} = 1$ theory with the absence of dotted spinor indices. Let us turn to the $\{16\}$ and $\{\overline{16}\}$ representations. If we define χ^{α} as the $\{16\}$ representation, then it follows that $\chi^{\dot{\alpha}}$ is the $\{\overline{16}\}$ representation. They both have upper spinor indices. However, it is important to keep in mind that the spinor metric for 10D takes the form $C_{\alpha\dot{\beta}}$ and this means that it is possible to define another spinor that has a subscript index via the equation

$$\chi_{\alpha} ~=~ \chi^{\dot{\beta}} C_{\alpha\dot{\beta}} ~~~, \tag{5.1.5}$$

where obviously χ_{α} has a subscript undotted spinor index. So if we define χ^{α} with a superscript as the $\{16\}$ representation, then it follows that χ_{α} with a subscript is the $\{\overline{16}\}$ representation!

This provides a way to understand the meaning of the "bar" representations shown in the studies of the ten-dimensional superfield. This also tells us the irreps corresponding to the component fields are nothing but the conjugate of the irreps corresponding to the $\theta-$monomials.

We can also rewrite the scalar superfield decomposition result in Equation (5.1.4) in Dynkin labels as follows.

$$\mathcal{V} = \begin{cases} \text{Level} - 0 & (00000) \quad , \\ \text{Level} - 1 & (00001) \quad , \\ \text{Level} - 2 & (00100) \quad , \\ \text{Level} - 3 & (01010) \quad , \\ \text{Level} - 4 & (02000) \oplus (10020) \quad , \\ \text{Level} - 5 & (00030) \oplus (11010) \quad , \\ \text{Level} - 6 & (01020) \oplus (20100) \quad , \\ \text{Level} - 7 & (30001) \oplus (10110) \quad , \\ \text{Level} - 8 & (40000) \oplus (00200) \oplus (20011) \quad , \\ \text{Level} - 9 & (30010) \oplus (10101) \quad , \\ \text{Level} - 10 & (01002) \oplus (20100) \quad , \\ \text{Level} - 11 & (00003) \oplus (11001) \quad , \\ \text{Level} - 12 & (02000) \oplus (10002) \quad , \\ \text{Level} - 13 & (01001) \quad , \\ \text{Level} - 14 & (00100) \quad , \\ \text{Level} - 15 & (00010) \quad , \\ \text{Level} - 16 & (00000) \quad . \end{cases} \tag{5.1.6}$$

In the following subsections, we will present two different approaches to obtain the result in Equation (5.1.4). We will also draw the 10D, $\mathcal{N} = 1$ adinkra diagram as we did in the 4D, $\mathcal{N} = 1$ case.

In the discussion above, it was noted that the problem of finding a judicious choice of irreducible representations of the Lorentz group appropriate for level-n of the superfield is at the crux of identifying what component fields actually appear at the level-n. In the following discussions, we will show two ways to carry out this process.

5.1.1 Handicraft Approach: Fermionic Young Tableaux

In this section, we will utilize the fermionic Young Tableau (FYT) and its application to obtaining the irreducible Lorentz decompositions of the component fields appearing in the θ-expansion of the ten-dimensional scalar superfield[2]. Since in superspace, there are not only spacetime coordinates but also Grassmann coordinates, we introduce the fermionic Young Tableau as an extension of the normal (bosonic) Young Tableau. In order to distinguish the bosonic Young Tableaux from the fermionic Young Tableaux, we use different colored boxes: Young Tableaux with blue boxes are bosonic and the ones with red boxes are fermionic. Namely, when calculating the dimension of a representation associated with a Young Tableau, we put "10" into the first box if it is bosonic and "16" if it is fermionic in 10D.

One can start with the quadratic level. In $\mathfrak{so}(10)$, we can use Young Tableaux to denote *reducible* representations. The rules of tensor product of two Young Tableaux are still valid. Thus, we have

$$\square \otimes \square = \square\!\square \oplus \begin{array}{c}\square\\[-4pt]\square\end{array} \quad , \tag{5.1.7}$$

where entries in $\square\!\square$ are completely symmetric in a corresponding set of spinor indices and entries in $\begin{array}{c}\square\\[-4pt]\square\end{array}$ are completely antisymmetric in these same spinor indices. Therefore, the dimensions of these two representations are 136 and 120 respectively. Moreover, $\square\!\square$ and $\begin{array}{c}\square\\[-4pt]\square\end{array}$ are the only Young Tableaux that contain two boxes. By using the Mathematica package LieART [72], one obtains[3] the following results about tensor product decomposition in $\mathfrak{so}(10)$:

$$\{\mathsf{16}\} \otimes \{\mathsf{16}\} = \{\mathsf{10}\} \oplus \{\mathsf{120}\} \oplus \{\overline{\mathsf{126}}\} \quad . \tag{5.1.8}$$

[2] As we noted in our introduction, the use of Young Tableaux for supersymmetric representation theory was pioneered in the literature [115–117] some years ago.

[3] Of course simply ways can be used to find these results also.

Then we know the decompositions of $\ydiagram{2}$ and $\ydiagram{1,1}$

$$
\begin{aligned}
\ydiagram{2} &= \{10\} \oplus \{\overline{126}\} \quad , \qquad (5.1.9)\\
\ydiagram{1,1} &= \{120\} \quad , \qquad (5.1.10)
\end{aligned}
$$

entirely from dimensionality. This exercise also teaches the lesson that

$$
\{16\} \wedge \{16\} = \{120\} \quad . \qquad (5.1.11)
$$

Note that the sigma matrices with five vector indices satisfy the self-dual / anti-self-dual identities

$$
\begin{aligned}
(\sigma_{[5]})_{\alpha\beta} &= \frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})_{\alpha\beta} \quad , & (\sigma_{[5]})^{\dot{\alpha}\dot{\beta}} &= \frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})^{\dot{\alpha}\dot{\beta}} \quad , \\
(\sigma_{[5]})^{\alpha\beta} &= -\frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})^{\alpha\beta} \quad , & (\sigma_{[5]})_{\dot{\alpha}\dot{\beta}} &= -\frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})_{\dot{\alpha}\dot{\beta}} \quad .
\end{aligned}
\qquad (5.1.12)
$$

Thus, the degrees of freedom are halved. We denote the 5-forms satisfying the self-dual condition as $\{126\}$, and the 5-forms satisfying the anti-self-dual condition as $\{\overline{126}\}$. Then we have $\ydiagram{1,1,1,1,1} = \frac{10\times9\times8\times7\times6}{5!} = \{252\} = \{126\} \oplus \{\overline{126}\}$. This discussion reveals that the reducible $\{252\}$ may be graphically reduced according to

$$
\ydiagram{1,1,1,1,1} = \ydiagram{1,1,1,1,1}_{\,+} \oplus \ydiagram{1,1,1,1,1}_{\,-} \quad , \qquad (5.1.13)
$$

as an image. The level-2 result that we need for the scalar superfield is $\{16\} \wedge \{16\}$, which is the totally antisymmetric piece $\ydiagram{1,1}$.

Next, we go to the cubic level. The tensor product of three $\{16\}$ is

$$
\begin{aligned}
\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} &= \left[\ydiagram{1} \otimes \ydiagram{1}\right] \otimes \ydiagram{1} \\
&= \ydiagram{3} \oplus 2\,\ydiagram{2,1} \oplus \ydiagram{1,1,1} \quad .
\end{aligned}
\qquad (5.1.14)
$$

Therefore we obtain

$$\ydiagram{3} \oplus \ydiagram{2,1} = \ydiagram{2} \otimes \ydiagram{1} = \{\overline{16}\} \oplus 2\{\overline{144}\} \oplus \{\overline{672}\} \oplus \{\overline{1200}\} \quad , \tag{5.1.15}$$

$$\ydiagram{2,1} \oplus \ydiagram{1,1,1} = \ydiagram{1,1} \otimes \ydiagram{1} = \{\overline{16}\} \oplus \{\overline{144}\} \oplus \{\overline{560}\} \oplus \{\overline{1200}\} \quad . \tag{5.1.16}$$

To solve for $\ydiagram{1,1,1}$, first note that

$$\ydiagram{1,1,1} \supseteq \ydiagram{2,1} \oplus \ydiagram{1,1,1} - \ydiagram{3} \oplus \ydiagram{2,1} = \{\overline{560}\} \quad , \tag{5.1.17}$$

where "−" means complement including duplicates if we treat each direct sum of irreps as a set of irreps. Now note that the dimension of $\ydiagram{1,1,1}$ is exactly 560. Therefore, we can solve for all the decompositions in cubic level,

$$\ydiagram{3} = \{\overline{144}\} \oplus \{\overline{672}\} \quad , \tag{5.1.18}$$

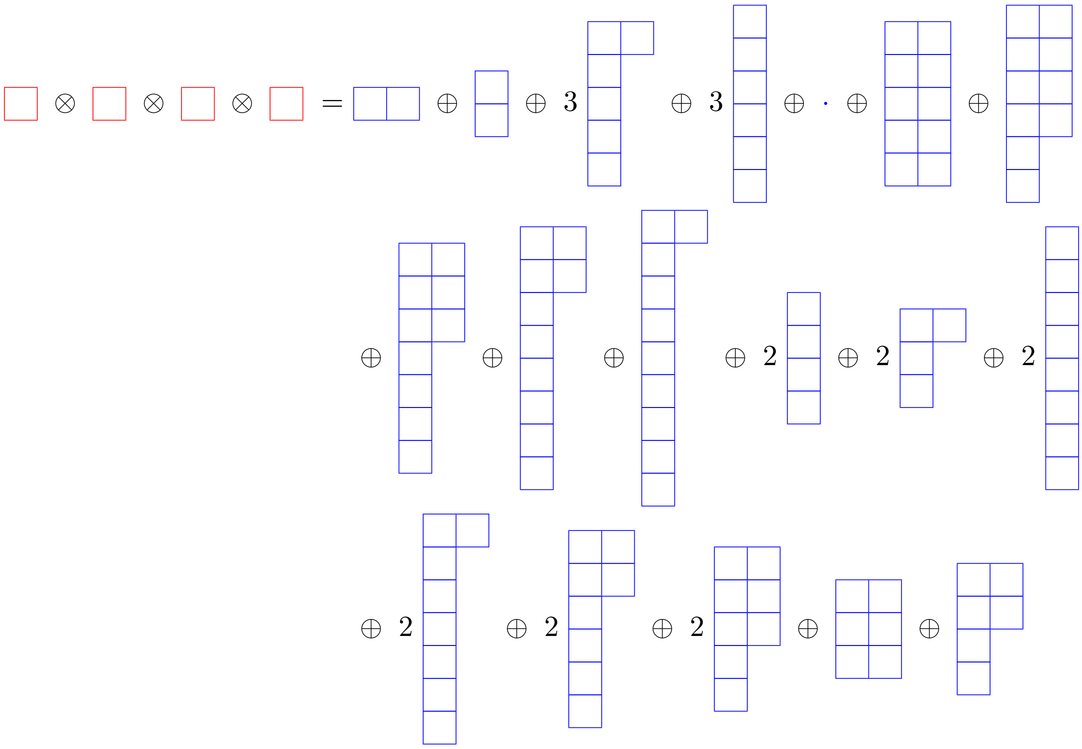
$$\ydiagram{2,1} = \{\overline{16}\} \oplus \{\overline{144}\} \oplus \{\overline{1200}\} \quad , \tag{5.1.19}$$

$$\ydiagram{1,1,1} = \{\overline{560}\} \quad . \tag{5.1.20}$$

When we look at the quartic level, we find

$$\begin{aligned} \ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} &= \left[\ydiagram{1} \otimes \ydiagram{1} \otimes \ydiagram{1} \right] \otimes \ydiagram{1} \\ &= \left[\ydiagram{1} \otimes \ydiagram{1} \right] \otimes \left[\ydiagram{1} \otimes \ydiagram{1} \right] \\ &= \ydiagram{4} \oplus 3\ \ydiagram{3,1} \oplus 2\ \ydiagram{2,2} \oplus 3\ \ydiagram{2,1,1} \oplus \ydiagram{1,1,1,1} \quad . \end{aligned} \tag{5.1.21}$$

By the hook dimension formula, the tensors with four spinorial indices above have dimensions 3,876, 9,180, 5,440, 7,140, and 1,820 respectively. It is more useful to express these in terms of tensors with vector indices. By using the second line of Equation (5.1.21), from Equations (5.1.9) and (5.1.10), we find



(5.1.22)

However, the r.h.s. of Equation (5.1.21) and (5.1.22) are not irreducible representations! By LieART, we obtain the irreducible decomposition

$$\begin{aligned}\square \otimes \square \otimes \square \otimes \square = {} & (2)\{\mathsf{1}\} \oplus (6)\{\mathsf{45}\} \oplus (3)\{\mathsf{54}\} \oplus (8)\{\mathsf{210}\} \oplus \{\mathsf{770}\} \\ & \oplus (6)\{\mathsf{945}\} \oplus \{\mathsf{1,050}\} \oplus (6)\{\overline{\mathsf{1,050}}\} \oplus \{\overline{\mathsf{2,772}}\} \\ & \oplus (2)\{\mathsf{4,125}\} \oplus (3)\{\mathsf{5,940}\} \oplus (3)\{\overline{\mathsf{6,930}}\}\end{aligned} \tag{5.1.23}$$

In subsequent work, we will develop graphical techniques to get from Equation (5.1.21) and (5.1.22) to Equation (5.1.23).

Similar to Equations (5.1.15) and (5.1.16), we have 4 independent equations with 5 different Fermionic Young Tableaux with 4 boxes. Note that in level-n, the number of Young Tableaux with n-boxes is the number of integer partitions of n, $p(n)$. In level-5, 6, 7, and 8, there are 7, 11, 15, and 22 different types of Young Tableaux respectively.

The number of independent equations in level-n is $p(n)-1$. This method becomes increasingly tedious. We won't go through the details of the subsequent levels. Although the systems of equations always seem underconstrained, in all levels the restrictions from dimensionality are able to nail down all the solutions in the 10D, $\mathcal{N}=1$ case, and give us back the scalar superfield decomposition in Equation (5.1.4). Thus we call this a handicraft approach.

5.1.2 *Branching Rules for* $\mathfrak{su}(16) \supset \mathfrak{so}(10)$

At level-n of the 10D, $\mathcal{N}=1$ scalar superfield, we want to decompose the totally antisymmetric product $\{16\}^{\wedge n}$ to $\mathfrak{so}(10)$ irreps. We are essentially looking at all the one-column Young Tableaux with 16 filled at the first box,

$$\cdot \quad , \quad \begin{array}{|c|}\hline 16 \\ \hline \end{array} \quad , \quad \begin{array}{|c|}\hline 16 \\ \hline 15 \\ \hline \end{array} \quad , \quad \cdots \quad , \quad \begin{array}{|c|}\hline 16 \\ \hline \vdots \\ \hline 1 \\ \hline \end{array} \quad . \tag{5.1.24}$$

They have dimensions $\begin{pmatrix} 16 \\ n \end{pmatrix}$, $n=0,1,\ldots,16$. One natural way is to interpret these Young Tableaux as $\mathfrak{su}(16)$ irreps (while they do not correspond to irreps when interpreted in the $\mathfrak{so}(10)$ context), and consider how they branch into $\mathfrak{so}(10)$ irreps. We summarize the relevant branching rules in the following table. They give us the decomposition in Equation (5.1.4).

To our knowledge, the expansion of component fields of a superfield arising from the branching rules of one ordinary Lie algebra over one of its Lie subalgebras has never been noted in any of the prior literature concerning superfields.

This is one of the major discoveries we are reporting in this research paper. It provides a clean, precise, and new way to define the component fields in superfields. It is very satisfying also from the point of view of our use of the "handicraft" techniques discussed in the last section. The two methods yield the same conclusions at all levels. However, the method in this section is both labor-saving and in a sense more mathematically rigorous.

Level	Irrep in $\mathfrak{su}(16)$	Irrep(s) in $\mathfrak{so}(10)$
0	$\{1\}$	$\{\mathsf{1}\}$
1	$\{16\}$	$\{\mathsf{16}\}$
2	$\{120\}$	$\{\mathsf{120}\}$
3	$\{560\}$	$\{\overline{\mathsf{560}}\}$
4	$\{1,820\}$	$\{\mathsf{770}\} \oplus \{\mathsf{1,050}\}$
5	$\{4,368\}$	$\{\mathsf{672}\} \oplus \{\mathsf{3,696}\}$
6	$\{8,008\}$	$\{\mathsf{3,696'}\} \oplus \{\mathsf{4,312}\}$
7	$\{11,440\}$	$\{\overline{\mathsf{2,640}}\} \oplus \{\overline{\mathsf{8,800}}\}$
8	$\{12,870\}$	$\{\mathsf{660}\} \oplus \{\mathsf{4,125}\} \oplus \{\mathsf{8,085}\}$

Table 5.1: Summary of $\mathfrak{su}(16)$ to $\mathfrak{so}(10)$ branching rules for level-0 to 8 in 10D, $\mathcal{N} = 1$ scalar superfield

5.1.3 10D, $\mathcal{N} = 1$ Adinkra Diagram

In a manner similar to how we drew the 4D, $\mathcal{N} = 1$ adinkra, we are now in position to explicitly demonstrate the 10D, $\mathcal{N} = 1$ adinkra by the same process. The adinkra with irrep dimensionality shown in each node is drawn in Figure 5.1, while the one with the corresponding Dynkin labels appears in Figure 5.2.

Readers may immediately notice that not all nodes in adjacent levels are connected. As we know, the black edges represent the supersymmetry transformation, a more careful calculation should be done to determine whether two nodes are connected (namely related to each other via susy transformations). A detailed discussion on this issue can be found in [58] and here we will just present the final results without going through all calculations.

Considering the case of 4D, $\mathcal{N} = 1$ supersymmetry, the graphs shown in Figure 5.1 and Figure 5.2 for the 10D, $\mathcal{N} = 1$ scalar superfield should be considered as the equivalence to the graph shown in Figure 3.1 for the 4D, $\mathcal{N} = 1$ scalar superfield. The obvious difference in "heights" was to be expected. However, a surprising feature is the maximum width of the 10D, $\mathcal{N} = 1$ scalar superfield adinkra is exactly the same as that of the 4D,

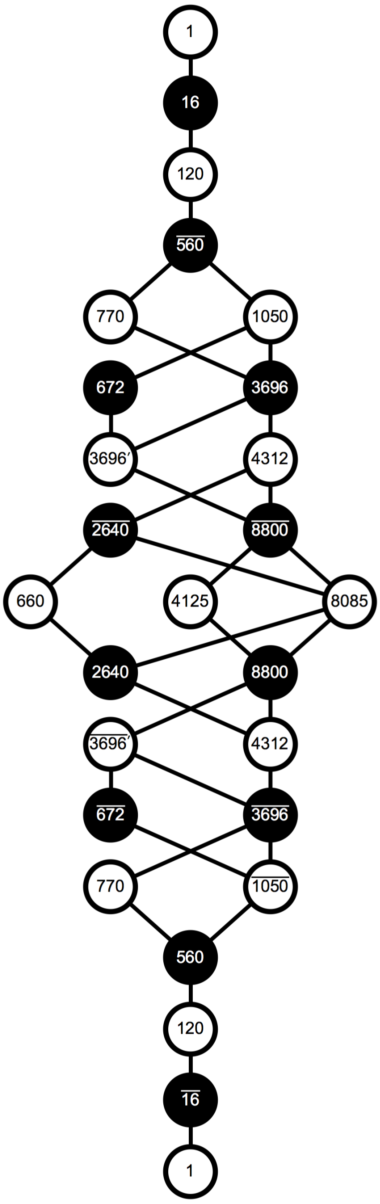


Figure 5.1: Adinkra with Dimensionality Indicated In Nodes

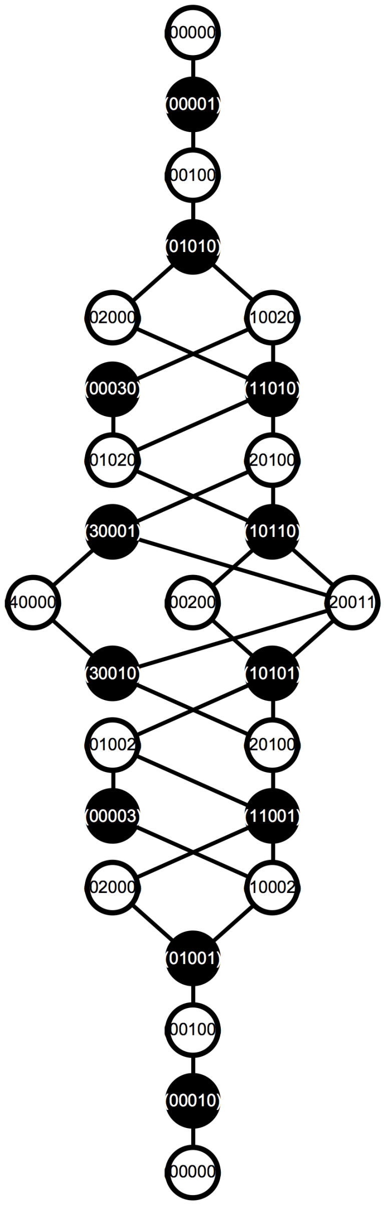


Figure 5.2: Adinkra with Dynkin Labels Indicated In Nodes

$\mathcal{N} = 1$ scalar superfield, namely only three bosonic nodes. Another surprising feature of the 10D, $\mathcal{N} = 1$ scalar superfield is its economy. It only contains 15 independent bosonic field representations and 12 independent fermionic field representations.

A great and obvious challenge here is to discover the analog of the rules that govern the splitting that was described as to how 4D, $\mathcal{N} = 1$ chiral (irreducible) and 4D, $\mathcal{N} = 1$ vector (irreducible) adinkras can be obtained from a 4D, $\mathcal{N} = 1$ real scalar (reducible) adinkra.

5.1.4 The 10D, $\mathcal{N} = 1$ Adinkra & Nordström SG Components

In our previous work of [36] (see materials in chapter 2), we found that the construction of 10D, $\mathcal{N} = 1$ supergravitation at the level of superfields and in terms of linearized supergeometry is very direct and without obstructions. It involved the introduction of a 10D, $\mathcal{N} = 1$ scalar superfield Ψ that appears in linearized frame operators in the forms

$$\mathrm{E}_\alpha = \mathrm{D}_\alpha + \tfrac{1}{2}\Psi \mathrm{D}_\alpha \quad , \tag{5.1.25}$$

$$\mathrm{E}_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\tfrac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\mathrm{D}_\beta \quad . \tag{5.1.26}$$

Taking the $\theta \to 0$ limit in the second of these equations implies that

$$\mathrm{E}_{\underline{a}}\Big| = \Big[\, 1 \;+\; \Psi \,\Big]\Big|\delta_{\underline{a}}{}^{\underline{m}}\partial_{\underline{m}} \;+\; \Big[-i\tfrac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\Big]\Big|\,\mathrm{D}_\beta \quad , \tag{5.1.27}$$

where $\Big|$ is a notation used to indicate the taking of the limit. The left hand side of this equation can be defined as

$$\mathrm{E}_{\underline{a}}\Big| = \mathrm{e}_{\underline{a}}{}^{\underline{m}}\partial_{\underline{m}} \;+\; \tilde{\psi}_{\underline{a}}{}^{\beta}\,\mathrm{D}_\beta \quad , \tag{5.1.28}$$

with $\mathrm{e}_{\underline{a}}{}^{\underline{m}}$ describing the 10D graviton (expressed as a frame field) and $\tilde{\psi}_{\underline{a}}{}^{\beta}$ describing the 10D gravitino. Combining the last two equations yields

$$\mathrm{e}_{\underline{a}}{}^{\underline{m}} \;=\; \Big[\, 1 \;+\; \Psi \,\Big]\Big|\delta_{\underline{a}}{}^{\underline{m}} \quad , \quad \tilde{\psi}_{\underline{a}}{}^{\beta} \;=\; \Big[-i\tfrac{2}{5}(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\Big]\Big| \quad . \tag{5.1.29}$$

In Nordström theory, only the non-conformal spin-0 part of graviton described by a scalar component field and the non-conformal spin-$\frac{1}{2}$ part of the gravitino

$$\psi_\beta \equiv (\sigma^{\underline{a}})_{\beta\gamma}\tilde{\psi}_{\underline{a}}{}^\gamma \quad , \tag{5.1.30}$$

which is described by a $\{\overline{16}\}$ component field show up. Both of these are appropriate for a Nordström-type theory.

Two of the results in the work of [36] take the respective forms

$$T_{\alpha\underline{b}}{}^\gamma = -\frac{3}{10}\delta_\alpha{}^\gamma(\partial_{\underline{b}}\Psi) + \frac{3}{10}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\gamma(\partial_{\underline{c}}\Psi) + i\frac{1}{160}\left[-(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} + \frac{1}{3}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\right]G_{\beta\delta} \quad , \tag{5.1.31}$$

and

$$R_{\alpha\beta}{}^{\underline{de}} = -i\frac{6}{5}(\sigma^{[\underline{d}})_{\alpha\beta}(\partial^{\underline{e}]}\Psi) - \frac{1}{80}\left[\frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} + (\sigma^{\underline{a}})_{\alpha\beta}(\sigma_{\underline{a}}{}^{\underline{de}})^{\gamma\delta}\right]G_{\gamma\delta} \quad . \tag{5.1.32}$$

where $G_{\alpha\beta} = ([\mathrm{D}_\alpha, \mathrm{D}_\beta]\Psi)$. In the $\theta \to 0$ limit, the first two terms of each of these equations involve the spacetime derivative of the scalar graviton in the theory. The final two terms in each of these equations involve the quantity $G_{\alpha\beta} = ([\mathrm{D}_\alpha, \mathrm{D}_\beta]\Psi)$ corresponding to the Level-2 $\{120\}$ component field node shown in either of the adinkras shown in Figure 5.1 or Figure 5.2.

Application of a spinorial derivative to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-3 $\{560\}$ fermionic component field node. Application of two spinorial derivatives to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_{\gamma_2}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-4 $\{770\}$ and $\{\overline{1050}\}$ bosonic component field nodes. Application of three spinorial derivatives to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_{\gamma_2}\mathrm{D}_{\gamma_3}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-5 $\{\overline{672}\}$ and $\{\overline{3696}\}$ fermionic component field nodes. Application of four spinorial derivatives to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_{\gamma_2}\mathrm{D}_{\gamma_3}\mathrm{D}_{\gamma_4}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-6 $\{\overline{3696}\}$ and $\{4312\}$ bosonic component field nodes. Application of five spinorial derivatives to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_{\gamma_2}\mathrm{D}_{\gamma_3}\mathrm{D}_{\gamma_4}\mathrm{D}_{\gamma_5}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-7 $\{\overline{2640}\}$ and $\{\overline{8800}\}$ fermionic component field nodes. Application of six spinorial derivatives to $G_{\alpha\beta}$ (i.e. $\mathrm{D}_{[\gamma_1}\mathrm{D}_{\gamma_2}\mathrm{D}_{\gamma_3}\mathrm{D}_{\gamma_4}\mathrm{D}_{\gamma_5}\mathrm{D}_{\gamma_6}\mathrm{D}_\alpha\mathrm{D}_{\beta]}\Psi$) yields the Level-8 $\{\overline{660}\}$, $\{\overline{4125}\}$, and $\{\overline{8085}\}$ bosonic component field nodes.

From the form of the adinkras, we know that the application of more than six fermionic derivatives to $G_{\alpha\beta}$ (or eight to Ψ) only leads to the dual representations occurring. So for example seven fermionic derivatives applied to $G_{\alpha\beta}$ (or nine to Ψ) must lead to the dual representations of those that occurred with the application of five fermionic derivatives (or seven to Ψ). Thus, all the component fields of 10D, $\mathcal{N}=1$ Nordström SG are obtained from (5.1.29) as well as applying all possible spinorial derivatives to the superspace covariant supergravity field strength $G_{\alpha\beta}$ followed by taking the $\theta \to 0$ limit.

Thus, the knowledge of the 10D, $\mathcal{N}=1$ scalar superfield adinkras provides an avenue to the complete component field description of the corresponding reducible Nordström supergravity theory derived from superspace geometry [36].

5.2 10D Breitenlohner Approach

In this section, we will apply the Breitenlohner approach to construct candidates for the 10D Type I superconformal multiplet. The principle of this approach is to attach bosonic and fermionic indices on the 10D scalar superfield and search for the traceless graviton and the traceless gravitino.

Recall that the first off-shell description of 4D, $\mathcal{N}=1$ supergravity was actually carried out by Breitenlohner [37], who took an approach equivalent to starting with the component fields of the Wess-Zumino gauge 4D, $\mathcal{N}=1$ vector supermultiplet $(v_{\underline{a}}, \lambda_\beta, \mathrm{d})$ together with their familiar SUSY transformation laws

$$
\begin{aligned}
\mathrm{D}_\alpha v_{\underline{a}} &= (\gamma_{\underline{a}})_\alpha{}^\beta \lambda_\beta \quad , \\
\mathrm{D}_\alpha \lambda_\beta &= -i\tfrac{1}{4}([\gamma^{\underline{a}}, \gamma^{\underline{b}}])_{\alpha\beta}(\partial_{\underline{a}} v_{\underline{b}} - \partial_{\underline{b}} v_{\underline{a}}) + (\gamma^5)_{\alpha\beta}\,\mathrm{d} \quad , \\
\mathrm{D}_\alpha \mathrm{d} &= i(\gamma^5\gamma^{\underline{a}})_\alpha{}^\beta \partial_{\underline{a}}\lambda_\beta \quad ,
\end{aligned}
\tag{5.2.1}
$$

followed by choosing as the gauge group the spacetime translations, SUSY generators, and the spin angular momentum generators as well as allowing additional internal symmetries. For the spacetime translation, this requires a series of replacements of the fields according to

$$
v_{\underline{a}} \to h_{\underline{a}\,\underline{b}} \quad , \quad \lambda_\beta \to \psi_{\underline{b}\beta} \quad , \quad \mathrm{d} \to A_{\underline{b}} \quad ,
\tag{5.2.2}
$$

(in the notation in [37], $A_{\underline{b}}$ is $B^5{}_\mu$). Because the vector supermultiplet is off-shell (up to WZ gauge transformations), the resulting supergravity theory is off-shell and includes a redundant set of auxiliary component fields, i. e. this is not an irreducible description of supergravity. But as seen from (5.2.2), the supergravity fields are all present, and together with the remaining component fields a complete superspace geometry can be constructed.

The key of this process is that if you look at Equation (3.2.3), there is a $\{4\}$ irrep (which is a vector gauge field) in level-2. When you consider the expansion of the superfield with one vector index $H^{\underline{a}}$ which is also called the prepotential, you will find

$$\{4\} \otimes \{4\} = \{1\} \oplus \{3\} \oplus \{\overline{3}\} \oplus \{9\} \quad , \tag{5.2.3}$$

where $\{9\}$ is the traceless graviton, as the degree of freedom is $9 = \frac{4\times 5}{2} - 1$.

In 10D, $\mathcal{N} = 1$ theory, we want to search for the traceless graviton $h_{\underline{ab}}$ and the traceless gravitino $\psi_{\underline{a}}{}^{\beta}$. Let us first figure out which irreducible representations do they correspond to. The graviton has two symmetric vector indices, which corresponds to $\frac{10\times 11}{2} = 55$, and can be decomposed into the traceless part and the trace,

$$\begin{aligned} \tilde{h}_{\underline{ab}} &= h_{\underline{ab}} + \eta_{\underline{ab}}\, h \quad , \\ \{55\} &= \{54\} \oplus \{1\} \quad . \end{aligned} \tag{5.2.4}$$

where both $\{54\}$ and $\{1\}$ are SO(1,9) irreps, while $\{55\}$ is not. In the following, we will denote the traceless graviton irrep to be $\{\mathbf{54}\}$, where the green color is added just to highlight this irrep. To determine the irreducible representations corresponding to the gravitino $\tilde{\psi}_{\underline{a}}{}^{\beta}$, first note that it has one vector index which corresponds to $\{10\}$ and one *upper* spinor index which corresponds to $\{16\}$. We can then decompose the gravitino into its traceless part and its σ-trace part by the tensor product of these two irreps,

$$\begin{aligned} \tilde{\psi}_{\underline{a}}{}^{\beta} &= \psi_{\underline{a}}{}^{\beta} + \tfrac{1}{10}(\sigma_{\underline{a}})^{\beta\gamma}\psi_{\gamma} \quad , \\ \{10\} \otimes \{16\} &= \{\overline{144}\} \oplus \{\overline{16}\} \quad , \end{aligned} \tag{5.2.5}$$

where the non-conformal spin-$\frac{1}{2}$ σ-trace part ψ_{γ} is defined in Equation (5.1.30). The traceless part ${\psi_{\underline{a}}}^{\beta}$ of the gravitino is conformal, and this is what we should search for in the superconformal prepotential multiplet.

Now, if we hope to follow the exact same method in Equation (5.2.3), we would immediately confront the problem of the lacking of $\{10\}$ in the expansion in Equation (5.1.4). Therefore, we cannot introduce the vector superfield as the prepotential and construct the off-shell supergravity in the 10D, $\mathcal{N} = 1$ case right away. Instead, we attach other combinations of bosonic and fermionic indices onto the scalar superfields and search for traceless gravitons $\{54\}$ and gravitinos $\{\overline{144}\}$ directly.

Given the Lorentz component content of a scalar superfield, how do we know the components of the superfields with various bosonic (or fermionic) indices? Let's write down the scalar superfield $\theta-$expansion again,

$$\mathcal{V} \;=\; \varphi^{(0)} \;+\; \theta^{\alpha}\,\varphi^{(1)}_{\alpha} \;+\; \theta^{\alpha}\theta^{\beta}\,\varphi^{(2)}_{\alpha\beta} \;+\; \dots \quad . \tag{5.2.6}$$

Attaching indices onto the scalar superfield would mean

$$\mathcal{V}_{[\text{indices}]} \;=\; \varphi^{(0)}_{[\text{indices}]} \;+\; \theta^{\alpha}\,\varphi^{(1)}_{\alpha\,[\text{indices}]} \;+\; \theta^{\alpha}\theta^{\beta}\,\varphi^{(2)}_{\alpha\beta\,[\text{indices}]} \;+\; \dots \quad . \tag{5.2.7}$$

where [indices] could be any combination of bosonic and/or fermionic indices, and it would correspond to a (sum of) bosonic or fermionic irreducible representation(s) in SO(1,9). If we let the level-n $\theta-$monomial decompositions of 10D, $\mathcal{N} = 1$ scalar superfield be ℓ_n for $n = 0, 1, \dots, 16$, the level-n component field content would be $\overline{\ell_n}$. For simplicity, consider only the [indices] that corresponds to one and only one irreducible representation $\{\text{irrep}\}$. Then the level-n component field content of $\mathcal{V}_{[\text{indices}]}$ would be $\overline{\ell_n} \otimes \{\text{irrep}\}$. We denote the *component* content of the entire superfield by the expression $\mathcal{V} \otimes \{\text{irrep}\}$, which we will use throughout the following sections.

In the following two sections, we will study the expansions of some bosonic and fermionic superfields respectively to see whether we can identify possible off-shell supergravity supermultiplet(s).

5.2.1 Bosonic Superfields

We will start by attaching bosonic indices on the scalar superfield, which is equivalent to tensoring the corresponding bosonic irreps to the component decomposition of the scalar superfield. Here we explicitly study these bosonic irreps: $\{10\}$, $\{45\}$, $\{54\}$, $\{120\}$, $\{126\}$, $\{\overline{126}\}$, $\{210\}$, $\{210'\}$, $\{320\}$, $\{660\}$ and $\{770\}$. The summary of their expansions is listed in Table 5.2. The third column shows which tensor product contributes to $\{\mathbf{54}\}$, and the decomposition results show that each one only provides one $\{\mathbf{54}\}$.

Dynkin Label	Irrep	Is there $\{\mathbf{54}\}$?
(10000)	$\{10\}$	no
(01000)	$\{45\}$	no
(20000)	$\{54\}$	level-0: $\{1\} \otimes \{54\}$ level-4: $\{770\} \otimes \{54\}$ level-8: $\{660\} \otimes \{54\}$
(00100)	$\{120\}$	level-2: $\{120\} \otimes \{120\}$ level-6: $\{4312\} \otimes \{120\}$
(00020)	$\{126\}$	no
(00002)	$\{\overline{126}\}$	no
(00011)	$\{210\}$	level-4: $\{\overline{1050}\} \otimes \{210\}$ level-8: $\{8085\} \otimes \{210\}$
(30000)	$\{210'\}$	no
(11000)	$\{320\}$	level-2: $\{120\} \otimes \{320\}$ level-6: $\{4312\} \otimes \{320\}$
(40000)	$\{660\}$	level-8: $\{660\} \otimes \{660\}$
(02000)	$\{770\}$	level-4: $\{770\} \otimes \{770\}$ level-8: $\{4125\} \otimes \{770\}$

Table 5.2: Summary of tensoring bosonic irreps onto the scalar superfield

Of course, the result for $\mathcal{V} \otimes \{54\}$ is not surprising and can be removed as a candidate for a conformal 10D, $\mathcal{N} = 1$ supergravity prepotential. All of the other entries in the table do provide candidate prepotentials. It is noticeable that several of these candidates ($\mathcal{V} \otimes \{210\}$, $\mathcal{V} \otimes \{660\}$, and $\mathcal{V} \otimes \{770\}$) allow conformal graviton embeddings at the eighth level. If Wess-Zumino gauges exist to eliminate lower order bosons, these imply "short" supergravity multiplets.

Now, one can look into one of those bosonic superfields with traceless gravitons $\{\mathbf{54}\}$ in detail as an example. The irreps shown here are *component* fields. The reader can find other bosonic superfields with traceless gravitons in Appendix of [57].

$$
\mathcal{V}\otimes\{120\} = \left\{
\begin{array}{ll}
\text{Level} - 0 & \{120\} \quad , \\
\text{Level} - 1 & \{16\} \oplus \{144\} \oplus \{560\} \oplus \{1200\} \quad , \\
\text{Level} - 2 & \{1\} \oplus \{45\} \oplus \{54\} \oplus (2)\{210\} \oplus \{770\} \oplus \{945\} \oplus \{1050\} \\
 & \oplus\{\overline{1050}\} \oplus \{4125\} \oplus \{5940\} \quad , \\
\text{Level} - 3 & \{\overline{16}\} \oplus (2)\{\overline{144}\} \oplus (2)\{\overline{560}\} \oplus \{\overline{672}\} \oplus \{\overline{720}\} \oplus (2)\{\overline{1200}\} \oplus \{\overline{1440}\} \\
 & \oplus(2)\{\overline{3696}\} \oplus \{\overline{8064}\} \oplus \{\overline{8800}\} \oplus \{\overline{11088}\} \oplus \{\overline{25200}\} \quad , \\
\text{Level} - 4 & (2)\{120\} \oplus \{\overline{126}\} \oplus (2)\{320\} \oplus (3)\{1728\} \oplus (2)\{2970\} \\
 & \oplus(3)\{3696'\} \oplus \{\overline{3696}'\} \oplus (2)\{4312\} \oplus \{4410\} \oplus \{\overline{4950}\} \\
 & \oplus\{\overline{6930}'\} \oplus \{10560\} \oplus \{34398\} \oplus (2)\{36750\} \oplus \{\overline{48114}\} \quad , \\
\text{Level} - 5 & \{144\} \oplus (3)\{560\} \oplus (2)\{720\} \oplus \{1200\} \oplus (2)\{1440\} \oplus \{2640\} \\
 & \oplus(3)\{3696\} \oplus (2)\{5280\} \oplus (2)\{8064\} \oplus (4)\{8800\} \oplus \{11088\} \\
 & \oplus(2)\{15120\} \oplus \{25200\} \oplus \{29568\} \oplus (2)\{34992\} \oplus \{38016\} \\
 & \oplus\{43680\} \oplus \{49280\} \oplus \{144144\} \quad , \\
\text{Level} - 6 & \{54\} \oplus \{210\} \oplus \{660\} \oplus (2)\{770\} \oplus (2)\{945\} \oplus \{1050\} \oplus (3)\{\overline{1050}\} \\
 & \oplus(2)\{1386\} \oplus \{\overline{2772}\} \oplus (2)\{4125\} \oplus (3)\{5940\} \oplus (2)\{\overline{6930}\} \\
 & \oplus(4)\{8085\} \oplus \{8910\} \oplus \{14784\} \oplus \{16380\} \oplus \{17325\} \oplus \{\overline{17325}\} \\
 & \oplus(3)\{17920\} \oplus \{23040\} \oplus (3)\{\overline{23040}\} \oplus \{\overline{50688}\} \oplus (2)\{72765\} \\
 & \oplus\{73710\} \oplus \{112320\} \oplus \{\overline{128700}\} \oplus \{143000\} \quad , \\
\text{Level} - 7 & \{\overline{144}\} \oplus \{\overline{560}\} \oplus \{\overline{672}\} \oplus (2)\{\overline{720}\} \oplus (2)\{\overline{1200}\} \oplus \{\overline{1440}\} \oplus (3)\{\overline{2640}\} \\
 & \oplus(4)\{\overline{3696}\} \oplus \{\overline{7920}\} \oplus \{\overline{8064}\} \oplus (4)\{\overline{8800}\} \oplus (3)\{\overline{11088}\} \\
 & \oplus(3)\{\overline{15120}\} \oplus \{\overline{17280}\} \oplus \{\overline{23760}\} \oplus (3)\{\overline{25200}\} \oplus \{\overline{30800}\} \\
 & \oplus\{\overline{34992}\} \oplus (3)\{\overline{38016}\} \oplus \{\overline{43680}\} \oplus \{\overline{48048}\} \oplus (2)\{\overline{49280}\} \\
 & \oplus\{\overline{55440}\} \oplus \{\overline{124800}\} \oplus \{\overline{144144}\} \oplus \{\overline{196560}\} \oplus \{\overline{205920}\} \quad , \\
\text{Level} - 8 & \{120\} \oplus \{210'\} \oplus \{320\} \oplus (3)\{1728\} \oplus (2)\{2970\} \oplus (2)\{3696'\} \\
 & \oplus(2)\{\overline{3696}'\} \oplus (5)\{4312\} \oplus \{4410\} \oplus (2)\{4608\} \oplus (2)\{4950\} \\
 & \oplus(2)\{\overline{4950}\} \oplus (3)\{10560\} \oplus (2)\{27720\} \oplus (3)\{28160\} \oplus \{34398\} \\
 & \oplus(4)\{36750\} \oplus \{42120\} \oplus (2)\{48114\} \oplus (2)\{\overline{48114}\} \oplus \{64680\} \\
 & \oplus\{68640\} \oplus \{70070'\} \oplus \{90090\} \oplus \{\overline{90090}\} \oplus \{192192\} \oplus \{299520\} \quad , \\
\vdots & \vdots
\end{array}
\right.
\tag{5.2.8}
$$

where Level-9 to Level-16 are conjugate of Level-7 to Level-0 respectively.

The superfield $\mathcal{V} \otimes \{120\}$ can be interpreted as $\mathcal{V}_{\underline{abc}}$, a three form (three totally antisymmetric Lorentz indices). It provides four possibilities for the embedding of traceless gravitons (at level-2, level-6, level-10, and level-14). Note that if one finds a $\{54\}$ in level-n, one finds a $\{\overline{144}\}$ at level-$(n+1)$. This holds for all bosonic superfields listed in Table 5.2. We can see this relation between graviton and gravitino clearly by considering the SUSY transformation law of the graviton in 10D $\mathcal{N} = 1$ theory. We know the undotted D-operator acting on the graviton represented by $h_{\underline{ab}}$ gives a term proportional to the undotted gravitino in the on-shell case (in the off-shell case, there are several auxiliary fields showing up in the r.h.s. besides the undotted gravitino)

$$\mathrm{D}_{\alpha} h_{\underline{ab}} \;\propto\; (\sigma_{(\underline{a}})_{\alpha\beta}\, \psi_{\underline{b})}{}^{\beta} \quad . \tag{5.2.9}$$

Note that the undotted spinor index on the gravitino is a superscript. According to our convention, $\psi_{\underline{b}}{}^{\beta}$ corresponds to the irrep $\{\overline{144}\}$. Recall that acting the D-operator once will add one to the level. Thus, Equation (5.2.9) is exactly the mathematical expression of the statement that if you find a $\{54\}$ in level-n, you will find a $\{\overline{144}\}$ in level-$(n+1)$.

Based on Equation (5.2.8), we can see four possible embeddings for the graviton, ten possible embeddings for the gravitino, along with a number of auxiliary fields. The superfield $\mathcal{V}_{\underline{abc}}$ is also the simplest nontrivial bosonic superfield that contains traceless gravitons and gravitinos and can be used as the starting point to construct supergravity. In fact, as long ago as 1982, Howe, Nicolai, and Van Proeyen [104] had suggested this particular superfield might be an appropriate point from which to construct a prepotential for 10D, $\mathcal{N} = 1$ supergravity.

5.2.2 Fermionic Superfields

Now we investigate fermionic superfields by attaching spinor indices onto the scalar superfield. Here are the spinorial irreps we will explicitly explore: $\{16\}$, $\{\overline{16}\}$, $\{144\}$, $\{\overline{144}\}$, $\{560\}$, $\{\overline{560}\}$, $\{672\}$, $\{\overline{672}\}$, $\{720\}$, and $\{\overline{720}\}$. Before we present our results, some words of motivation will likely serve the cause of explaining the purpose of doing so.

Ever since the work of [118], it has been known that the prepotential superfield description of 4D, $\mathcal{N}$ = 2 supergravity is described by a fundamental dynamical entity that is fermionic. As a 4D, $\mathcal{N}$ = 2 supersymmetrical theory can be descended from a 6D minimally supersymmetrical one, this clearly points out the possibility that higher-dimensional theories can possess fermionic supergravity prepotentials.

There is a second reason why the study of fermionic superfields is suggested. Some time ago [102], efforts were undertaken to investigate the structure of 1-form 10D, $\mathcal{N}$ = 1 gauge theories as theories involving superspace superconnections over fiber bundles. This study reached a conclusion that *all* off-shell theories of this type *must* include a bosonic component field $\{126\}$. This observation was later confirmed by a number of subsequent studies [119–122]. This leads to a powerful restriction if the scanning process is applied to the 1-form 10D, $\mathcal{N}$ = 1 gauge theories. At the same level in the expansion in Grassmann coordinates, both the $\{10\}$ and the $\{126\}$ SO(1,9) representations must be present. In addition, there must also occur the $\{16\}$ at one level higher in the expansion to accommodate an accompanying gaugino. In a subsequent section, we will discuss the relevance of the features uncovered to the case of the 1-form 10D, $\mathcal{N}$ = 1 gauge theory. Furthermore, in order to reach the maximal level of simplification in our presentation, we will only consider the abelian case for the 1-form 10D, $\mathcal{N}$ = 1 gauge theory.

We will begin by only considering the feature revealed by our tensoring that are relevant to the case of 10D, $\mathcal{N}$ = 1 supergravity theory. The summary of the expansions for the fermionic cases is listed in Table 5.3. The third column shows which tensor product contributes $\{\mathbf{54}\}$ and the decomposition results show that each one only contributes one $\{\mathbf{54}\}$.

Now, one can look at one of those fermionic superfields with traceless gravitons $\{\mathbf{54}\}$ in some detail as an example. The cases of other fermionic superfields with embeddings for traceless gravitons are given in Appendix of [57]. Generally for $\mathcal{V} \otimes \{\mathsf{irrep}\}$, here we only list level-0 to level-8 results, and its level-9 to level-16 are conjugate of level-7 to level-0 of $\mathcal{V} \otimes \overline{\{\mathsf{irrep}\}}$.

Dynkin Label	Irrep	Is there $\{\mathbf{54}\}$?
(00001)	$\{16\}$	no
(00010)	$\{\overline{16}\}$	no
(10010)	$\{144\}$	level-1: $\{\overline{16}\} \otimes \{144\}$ level-5: $\{\overline{3696}\} \otimes \{144\}$
(10001)	$\{\overline{144}\}$	level-3: $\{560\} \otimes \{\overline{144}\}$ level-7: $\{2640\} \otimes \{\overline{144}\}$
(01001)	$\{560\}$	level-5: $\{\overline{3696}\} \otimes \{560\}$
(01010)	$\{\overline{560}\}$	level-3: $\{560\} \otimes \{\overline{560}\}$ level-7: $\{8800\} \otimes \{\overline{560}\}$
(00030)	$\{672\}$	no
(00003)	$\{\overline{672}\}$	no
(20001)	$\{720\}$	level-1: $\{\overline{16}\} \otimes \{720\}$ level-5: $\{\overline{3696}\} \otimes \{720\}$
(20010)	$\{\overline{720}\}$	level-3: $\{560\} \otimes \{\overline{720}\}$ level-7: $\{2640\} \otimes \{\overline{720}\}$

Table 5.3: Summary of tensoring spinorial irreps onto the scalar superfield

$\mathcal{V}\otimes\{560\}$ can be interpreted as $\mathcal{V}_{\underline{ab}}{}^{\gamma}$, a fermionic superfield with two antisymmetric bosonic indices and one spinor index satisfying

$$(\sigma^{\underline{a}})_{\gamma\delta}\mathcal{V}_{\underline{ab}}{}^{\gamma} \;=\; 0 \quad . \tag{5.2.10}$$

It provides three possible embeddings for traceless gravitons in total (in level-5, level-9, and level-13). Note that if you find a $\{54\}$ in level-n, you can find a $\{\overline{144}\}$ in level-$(n+1)$. This holds for all fermionic superfields listed in Table 5.3, which is consistent with Equation (5.2.9).

There exists a work [102] in the literature from 1987, where the proposal for the use of a fermionic 10D, $\mathcal{N}=1$ supergravity prepotential was first given. However, the proposed prepotential was of the form $\mathcal{V}_{\underline{a}}{}^{\gamma}$ which is equivalent to $\mathcal{V}\otimes\{\overline{144}\}$.

Based on the decomposition result of $\mathcal{V}\otimes\{560\}$, we see three possible embeddings for gravitons, fifteen possible embeddings for gravitinos, and auxiliary fields. The superfield $\mathcal{V}_{\underline{ab}}{}^{\gamma}$ is also the simplest nontrivial fermionic superfield that contains traceless gravitons and gravitinos and can be used to construct supergravity.

5.2.3 10D, $\mathcal{N}=1$ Yang-Mills Supermultiplet

The work previously described in this chapter on fermionic superfields also opens the doorway for using the techniques developed so far in this paper to the issue of scanning the component level description of a superspace connection $\Gamma_{\underline{A}}=(\Gamma_{\alpha},\Gamma_{\underline{a}})$ used to define a 1-form U(1) superspace covariant derivatives

$$\nabla_{\underline{A}} \;=\; \mathrm{D}_{\underline{A}} \;+\; ig\Gamma_{\underline{A}}\,\mathrm{t} \quad , \tag{5.2.11}$$

where t denotes the U(1) generator and g is a coupling constant. The commutators of two covariant derivatives take the forms [102]

$$[\nabla_{\alpha},\nabla_{\beta}\} \;=\; i(\sigma^{\underline{c}})_{\alpha\beta}\nabla_{\underline{c}} \;+\; g\frac{1}{5!}(\sigma^{[5]})_{\alpha\beta}f_{[5]}\,\mathrm{t} \quad , \tag{5.2.12}$$

$$[\nabla_{\alpha},\nabla_{\underline{b}}\} \;=\; -ig\left[\lambda_{\alpha\underline{b}} \;+\; i\frac{1}{\sqrt{2}}(\sigma_{\underline{b}})_{\alpha\gamma}W^{\gamma}\right]\mathrm{t} \quad , \tag{5.2.13}$$

$$(\sigma^{\underline{b}})^{\alpha\beta}\lambda_{\alpha\underline{b}} \;=\; 0 \quad , \tag{5.2.14}$$

$$[\nabla_{\underline{a}}, \nabla_{\underline{b}}\} \;=\; igF_{\underline{ab}}\,\mathrm{t} \quad . \tag{5.2.15}$$

The equations below are consistent with the solutions of Bianchi identities [102]

$$i(\sigma^{\underline{a}})_{\delta\beta}\nabla_{\underline{a}}W^{\beta} \;=\; -\frac{1}{2\sqrt{2}\times 7!}\Big[\, i(\sigma_{\underline{a}})^{\beta\epsilon}(\sigma^{[4]})_{\epsilon}{}^{\gamma}(\sigma^{\underline{ab}})_{\delta}{}^{\lambda}\nabla_{\lambda}\nabla_{\beta}\nabla_{\gamma}f_{\underline{b}[4]} + 8(\sigma^{[4]})_{\delta}{}^{\beta}\{\nabla^{\underline{b}}, \nabla_{\beta}\}f_{\underline{b}[4]}\,\Big] \quad , \tag{5.2.16}$$

and

$$\begin{aligned}\nabla^{\underline{b}}F_{\underline{bc}} \;=&\; i\frac{1}{8\times 5!}(\sigma^{\underline{b}})^{\alpha\gamma}(\sigma^{[4]})_{\gamma}{}^{\beta}\nabla_{\underline{b}}\nabla_{\alpha}\nabla_{\beta}f_{\underline{c}[4]} \\ &+\; \frac{1}{32\times 7!}\Big[(\sigma_{\underline{c}})^{\alpha\epsilon}(\sigma^{\underline{ab}})_{\epsilon}{}^{\beta}(\sigma_{\underline{a}})^{\gamma\lambda}(\sigma^{[4]})_{\lambda}{}^{\delta}\nabla_{\alpha}\nabla_{\beta}\nabla_{\gamma}\nabla_{\delta}f_{\underline{b}[4]} \\ &\qquad -\; i8(\sigma_{\underline{c}})^{\alpha\gamma}(\sigma^{[4]})_{\gamma}{}^{\beta}\nabla_{\alpha}\{\nabla^{\underline{b}}, \nabla_{\beta}\}f_{\underline{b}[4]}\Big] \quad .\end{aligned} \tag{5.2.17}$$

The results immediately above are the most important. They show that if the component field variable $f_{[5]}$ is set to zero on the RHS of these equations, then the spinorial photino field W^{β} and the bosonic Maxwell field $F_{\underline{bc}}$ must both satisfy their equations of motion. Stated another way, for these two-component fields to be off-shell, it requires the presence of the condition $f_{[5]} \neq 0$.

In the work of [102], the structures of the spinorial gauge connection superfield Γ_{α} and its gauge parameter K up to level-4 in θ are given by

$$\begin{aligned}\Gamma_{\alpha} \;=&\; \phi_{\alpha} \;+\; i\frac{1}{2}\theta^{\beta}\Big[\, (\sigma^{\underline{c}})_{\alpha\beta}A_{\underline{c}} \;+\; (\sigma^{[3]})_{\alpha\beta}\phi_{[3]} \;-\; (\sigma^{[5]})_{\alpha\beta}f_{[5]}\,\Big] \;+\; i\theta^{\beta}\theta^{\gamma}(\sigma^{[3]})_{\beta\gamma}\Sigma_{\alpha[3]} \\ &+\; \theta^{\beta}\theta^{\gamma}(\sigma^{[3]})_{\beta\gamma}\theta^{\delta}\Big[\, (\sigma^{\underline{c}})_{\alpha\delta}B_{\underline{c}[3]} \;+\; (\sigma^{[3']})_{\alpha\delta}B_{[3'][3]} \;+\; (\sigma^{[5]})_{\alpha\delta}B_{[5][3]}\,\Big] \\ &+\; \theta^{\beta}\theta^{\gamma}(\sigma^{[3]})_{\beta\gamma}\theta^{\delta}\theta^{\epsilon}(\sigma^{[3']})_{\delta\epsilon}\Sigma_{\alpha[3][3']} \;+\; \mathcal{O}(\theta^5) \quad ,\end{aligned} \tag{5.2.18}$$

$$\begin{aligned}K \;=&\; k \;+\; \theta^{\alpha}\pi_{\alpha} \;+\; i\theta^{\beta}\theta^{\gamma}(\sigma^{[3]})_{\beta\gamma}H_{[3]} \;+\; \theta^{\beta}\theta^{\gamma}(\sigma^{[3]})_{\beta\gamma}\theta^{\alpha}\pi_{\alpha[3]} \\ &+\; \theta^{\alpha}\theta^{\beta}(\sigma^{[3]})_{\alpha\beta}\theta^{\gamma}\theta^{\delta}(\sigma^{[3']})_{\gamma\delta}H_{[3][3']} \;+\; \mathcal{O}(\theta^5) \quad ,\end{aligned} \tag{5.2.19}$$

respectively. The irreducible component fields in Figure 5.1 and Figure 5.2 are contained in K in the following way. The component fields in K at levels 0, 1, 2, and 3 in the θ-expansion correspond to the nodes at the same height assignments in the adinkra. The component field $H_{[3][3']}$ at level-4 in the θ-expansion of K is to be expanded *solely* over the the irreducible representations shown at the fourth level of the adinkra. The *component* content of the connection superfield Γ_α is encoded by tensoring the components of the scalar superfield with a lower spinor index, i.e. $\mathcal{V} \otimes \{\overline{16}\}$.

$$
\Gamma_\alpha = \begin{cases}
\text{Level} - 0 & \{\overline{16}\} \quad , \\
\text{Level} - 1 & \{10\} \oplus \{120\} \oplus \{126\} \quad , \\
\text{Level} - 2 & \{16\} \oplus \{144\} \oplus \{560\} \oplus \{1200\} \quad , \\
\text{Level} - 3 & \{45\} \oplus \{210\} \oplus \{770\} \oplus \{945\} \oplus \{\overline{1050}\} \oplus \{5940\} \quad , \\
\text{Level} - 4 & \{\overline{144}\} \oplus \{\overline{560}\} \oplus \{\overline{672}\} \oplus \{\overline{1200}\} \oplus (2)\{\overline{3696}\} \oplus \{\overline{8064}\} \oplus \{\overline{11088}\} \quad , \\
\text{Level} - 5 & \{\overline{126}\} \oplus \{320\} \oplus \{1728\} \oplus \{2970\} \oplus (2)\{\overline{3696'}\} \oplus \{4312\} \oplus \{4410\} \\
 & \oplus\{\overline{4950}\} \oplus \{\overline{6930'}\} \oplus \{36750\} \quad , \\
\text{Level} - 6 & \{560\} \oplus \{720\} \oplus \{1440\} \oplus \{2640\} \oplus \{3696\} \oplus \{5280\} \oplus \{8064\} \\
 & \oplus(2)\{8800\} \oplus \{15120\} \oplus \{34992\} \oplus \{38016\} \quad , \\
\text{Level} - 7 & \{660\} \oplus \{945\} \oplus \{\overline{1050}\} \oplus \{1386\} \oplus \{4125\} \oplus \{5940\} \oplus \{\overline{6930}\} \\
 & \oplus(2)\{8085\} \oplus \{14784\} \oplus \{17325\} \oplus \{17920\} \oplus \{\overline{23040}\} \oplus \{72765\} \quad , \\
\text{Level} - 8 & \{\overline{720}\} \oplus \{\overline{1200}\} \oplus (2)\{\overline{2640}\} \oplus \{\overline{3696}\} \oplus \{\overline{7920}\} \oplus (2)\{\overline{8800}\} \\
 & \oplus\{\overline{11088}\} \oplus \{\overline{15120}\} \oplus \{\overline{25200}\} \oplus \{\overline{30800}\} \oplus \{\overline{38016}\} \oplus \{\overline{49280}\} \quad , \\
\text{Level} - 9 & \{210'\} \oplus \{1728\} \oplus \{2970\} \oplus \{3696'\} \oplus (2)\{4312\} \oplus \{4608\} \oplus \{4950\} \\
 & \oplus\{\overline{4950}\} \oplus \{10560\} \oplus \{27720\} \oplus \{28160\} \oplus \{36750\} \oplus \{48114\} \quad , \\
\text{Level} - 10 & \{672\} \oplus \{720\} \oplus \{1200\} \oplus \{2640\} \oplus (2)\{3696\} \oplus \{8800\} \oplus \{11088\} \\
 & \oplus\{15120\} \oplus \{17280\} \oplus \{25200\} \oplus \{38016\} \quad , \\
\text{Level} - 11 & \{770\} \oplus \{945\} \oplus (2)\{1050\} \oplus \{1386\} \oplus \{2772\} \oplus \{5940\} \oplus \{6930\} \\
 & \oplus\{8085\} \oplus \{17920\} \oplus \{23040\} \quad , \\
\text{Level} - 12 & (2)\{\overline{560}\} \oplus \{\overline{720}\} \oplus \{\overline{1440}\} \oplus \{\overline{3696}\} \oplus \{\overline{5280}\} \oplus \{\overline{8064}\} \oplus \{\overline{8800}\} \quad , \\
\text{Level} - 13 & \{120\} \oplus \{126\} \oplus \{320\} \oplus \{1728\} \oplus \{2970\} \oplus \{3696'\} \quad , \\
\text{Level} - 14 & \{16\} \oplus \{144\} \oplus \{560\} \oplus \{1200\} \quad , \\
\text{Level} - 15 & \{1\} \oplus \{45\} \oplus \{210\} \quad , \\
\text{Level} - 16 & \{\overline{16}\} \quad .
\end{cases}
\tag{5.2.20}
$$

By comparing this with Equation (5.2.18), we read that in level-0 ϕ_α corresponds to $\{\overline{16}\}$; level-1 $A_{\underline{c}}$, $\phi_{[3]}$ and $f_{[5]}$ correspond to $\{10\}$, $\{120\}$ and $\{126\}$; level-2 $\Sigma_{\alpha[3]}$ corresponds to $\{120\} \otimes \{\overline{16}\} \;=\; \overline{\ell_2} \otimes \{\overline{16}\}$ which is exactly the decomposition at level-2 of (5.2.20), where ℓ_2 is the level-2 $\theta-$monomial of the scalar superfield. One point to note is that $f_{[5]}$ is $\{126\}$ but not $\{\overline{126}\}$. This is because it is contracted with $(\sigma^{[5]})_{\alpha\beta}$ which a self-dual 5-form by Equation (5.1.12). The conditions for embedding the component fields into Γ_α requires at some level-n (here $n = 1$) there must occur $\{10\}$ and the $\{126\}$ while at the level-$(n + 1)$ there must occur a $\{16\}$, as we noted earlier.

5.3 10D, $\mathcal{N}$ = 2A Scalar Superfield Decomposition and Superconformal Multiplet

5.3.1 Methodology for 10D, $\mathcal{N} = 2A$ Scalar Superfield Construction

One can introduce two sets of 10D spinor coordinates denoted by θ^{α} and $\theta^{\dot{\alpha}}$ so that a 10D, $\mathcal{N} = (1,1)$ superfield can be expressed in the form $\mathcal{V}(x^{\underline{a}},\, \theta^{\alpha},\, \theta^{\dot{\alpha}})$. It is possible to organize the expansion so that it takes the form

$$
\mathcal{V}(x^{\underline{a}},\, \theta^{\alpha},\, \theta^{\dot{\alpha}}) \;=\; \mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\, \mathcal{V}^{(1)}_{\dot{\alpha}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \mathcal{V}^{(2)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \dots \quad , \tag{5.3.1}
$$

and the point is that each of the expansion coefficients $\mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha})$, $\mathcal{V}^{(1)}_{\dot{\alpha}}(x^{\underline{a}},\, \theta^{\alpha})$, $\mathcal{V}^{(2)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}},\, \theta^{\alpha})$, ... are 10D, $\mathcal{N} = 1$ superfields. More explicitly, one can write

$$
\begin{aligned}
\mathcal{V}(x^{\underline{a}},\, \theta^{\alpha},\, \theta^{\dot{\alpha}}) \;&= \mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\, \mathcal{V}^{(1)}_{\dot{\alpha}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \mathcal{V}^{(2)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}},\, \theta^{\alpha}) \;+\; \dots \\
&= \big[\; \varphi^{(0)}(x^{\underline{a}}) \;+\; \theta^{\alpha}\, \varphi^{(1)}_{\alpha}(x^{\underline{a}}) \;+\; \theta^{\alpha}\theta^{\beta}\, \varphi^{(2)}_{\alpha\beta}(x^{\underline{a}}) \;+\; \dots \;\big] \\
&\quad +\; \theta^{\dot{\alpha}}\, \big[\; \varphi^{(0)}_{\dot{\alpha}}(x^{\underline{a}}) \;+\; \theta^{\alpha}\, \varphi^{(1)}_{\alpha\dot{\alpha}}(x^{\underline{a}}) \;+\; \theta^{\alpha}\theta^{\beta}\, \varphi^{(2)}_{\alpha\beta\dot{\alpha}}(x^{\underline{a}}) \;+\; \dots \;\big] \\
&\quad +\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \big[\; \varphi^{(0)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}}) \;+\; \theta^{\alpha}\, \varphi^{(1)}_{\alpha\dot{\alpha}\dot{\beta}}(x^{\underline{a}}) \;+\; \theta^{\alpha}\theta^{\beta}\, \varphi^{(2)}_{\alpha\beta\dot{\alpha}\dot{\beta}}(x^{\underline{a}}) \;+\; \dots \;\big] \\
&\quad +\; \dots \\
&= \varphi^{(0)}(x^{\underline{a}}) \\
&\quad +\; \theta^{\alpha}\, \varphi^{(1)}_{\alpha}(x^{\underline{a}}) \;+\; \theta^{\dot{\alpha}}\, \varphi^{(0)}_{\dot{\alpha}}(x^{\underline{a}}) \\
&\quad +\; \theta^{\alpha}\theta^{\beta}\, \varphi^{(2)}_{\alpha\beta}(x^{\underline{a}}) \;+\; \theta^{\dot{\alpha}}\theta^{\alpha}\, \varphi^{(1)}_{\alpha\dot{\alpha}}(x^{\underline{a}}) \;+\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \varphi^{(0)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}}) \\
&\quad +\; \theta^{\alpha}\theta^{\beta}\theta^{\gamma}\, \varphi^{(3)}_{\alpha\beta\gamma}(x^{\underline{a}}) \;+\; \theta^{\dot{\alpha}}\theta^{\alpha}\theta^{\beta}\, \varphi^{(2)}_{\alpha\beta\dot{\alpha}}(x^{\underline{a}}) \;+\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\theta^{\alpha}\, \varphi^{(1)}_{\alpha\dot{\alpha}\dot{\beta}}(x^{\underline{a}}) \\
&\quad +\; \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\theta^{\dot{\gamma}}\, \varphi^{(0)}_{\dot{\alpha}\dot{\beta}\dot{\gamma}}(x^{\underline{a}}) \;+\; \dots \quad .
\end{aligned} \tag{5.3.2}
$$

Attaching n totally antisymmetric dotted θ's to an undotted $\theta-$monomial corresponds to tensoring the $\{\overline{16}\}^{\wedge n}$ representation on it, i.e. tensoring the conjugate of the nth level of the $\theta-$monomial of the 10D, $\mathcal{N} = 1$ scalar superfield on it. Therefore, if we denote level-n $\theta-$monomial of 10D, $\mathcal{N} = 1$ scalar superfield as ℓ_n, we can obtain the

10D, $\mathcal{N} = 2$A scalar superfield $\theta-$monomial decomposition by

$$\mathcal{V} = \begin{cases} \text{Level} - 0 & \ell_0 \otimes \overline{\ell_0} \quad , \\ \text{Level} - 1 & \ell_1 \otimes \overline{\ell_0} \oplus \ell_0 \otimes \overline{\ell_1} \quad , \\ \text{Level} - 2 & \ell_2 \otimes \overline{\ell_0} \oplus \ell_1 \otimes \overline{\ell_1} \oplus \ell_0 \otimes \overline{\ell_2} \quad , \\ \text{Level} - 3 & \ell_3 \otimes \overline{\ell_0} \oplus \ell_2 \otimes \overline{\ell_1} \oplus \ell_1 \otimes \overline{\ell_2} \oplus \ell_0 \otimes \overline{\ell_3} \quad , \\ \quad \vdots & \vdots \\ \text{Level} - n & \ell_n \otimes \overline{\ell_0} \oplus \ell_{n-1} \otimes \overline{\ell_1} \oplus \ldots \oplus \ell_1 \otimes \overline{\ell_{n-1}} \oplus \ell_0 \otimes \overline{\ell_n} \quad , \\ \quad \vdots & \vdots \\ \text{Level} - 16 & \ell_{16} \otimes \overline{\ell_0} \oplus \ell_{15} \otimes \overline{\ell_1} \oplus \ldots \oplus \ell_1 \otimes \overline{\ell_{15}} \oplus \ell_0 \otimes \overline{\ell_{16}} \quad , \\ \quad \vdots & \vdots \end{cases} \tag{5.3.3}$$

and level-17 to 32 are the conjugates of level-15 to 0 respectively. From Equation (5.3.3), it's clear that each level is self-conjugate, and therefore level-17 to 32 are the same as level-15 to 0. Moreover, the decompositions of component fields are the same as that of the $\theta-$monomials.

5.3.2 10D, $\mathcal{N} = 2$A Scalar Superfield Decomposition Results and Superconformal Multiplet

Based on Equations (5.1.4) and (5.3.3), one can directly obtain the scalar superfield decomposition in 10D, $\mathcal{N} = 2$A. The results from level-0 to level-16 are listed below. We use green color to highlight the irrep $\{\mathbf{54}\}$ which corresponds to the traceless graviton in 10D. Unlike in 10D, $\mathcal{N} = 1$ theory, one can find possible embeddings for gravitons and gravitinos in the $\theta-$expansion of the scalar 10D, $\mathcal{N} = 2$A superfield. We can translate irreps into component fields and see 72 graviton embeddings, 280 gravitino embeddings, and associated hosts of auxiliary fields. The scalar superfield is the simplest superfield that contains traceless gravitons and gravitinos and can be used to construct supergravity.

Note that if you find a $\{\mathbf{54}\}$ in level-n, you can find $\{144\}$ and $\{\overline{144}\}$ in level-$(n+1)$. This is also consistent with SUSY transformation laws of the graviton $h_{\underline{ab}}$ in 10D, $\mathcal{N} = 2$A theory. Acting the undotted D-operator on the graviton gives a term proportional to the undotted gravitino in the on-shell case (in the off-shell case, there are several auxiliary fields showing up in the r.h.s. besides the undotted gravitino)

$$\mathrm{D}_\alpha h_{\underline{a}\underline{b}} \;\propto\; (\sigma_{(\underline{a}})_{\alpha\beta}\, \psi_{\underline{b})}{}^{\beta} \quad . \tag{5.3.4}$$

Note that the gravitino here has an undotted superscript spinor index and hence corresponds to irrep $\{\overline{144}\}$ in our convention. Meanwhile, the dotted D-operator acting on the graviton satisfies

$$\mathrm{D}_{\dot\alpha} h_{\underline{a}\underline{b}} \;\propto\; (\sigma_{(\underline{a}})_{\dot\alpha\dot\beta}\, \psi_{\underline{b})}{}^{\dot\beta} \quad , \tag{5.3.5}$$

which gives a term proportional to the dotted gravitino in the on-shell case. The superscript dotted spinor index on the gravitino indicates that it corresponds to irrep $\{144\}$ in our convention.

- Level-0: $\{1\}$
- Level-1: $\{16\} \oplus \{\overline{16}\}$
- Level-2: $\{1\} \oplus \{45\} \oplus (2)\{120\} \oplus \{210\}$
- Level-3: $\{16\} \oplus \{\overline{16}\} \oplus (2)\{560\} \oplus (2)\{\overline{560}\} \oplus \{144\} \oplus \{\overline{144}\} \oplus \{1200\} \oplus \{\overline{1200}\}$
- Level-4: $\{1\} \oplus \{45\} \oplus \{54\} \oplus (2)\{120\} \oplus \{126\} \oplus \{\overline{126}\} \oplus (2)\{210\} \oplus (2)\{320\} \oplus (3)\{770\} \oplus \{945\} \oplus (2)\{1050\} \oplus (2)\{\overline{1050}\} \oplus (2)\{1728\} \oplus (2)\{2970\} \oplus \{3696'\} \oplus \{\overline{3696'}\} \oplus \{4125\} \oplus \{5940\}$
- Level-5: $\{16\} \oplus \{\overline{16}\} \oplus (2)\{144\} \oplus (2)\{\overline{144}\} \oplus (4)\{560\} \oplus (4)\{\overline{560}\} \oplus (2)\{672\} \oplus (2)\{\overline{672}\} \oplus (2)\{720\} \oplus (2)\{\overline{720}\} \oplus (2)\{1200\} \oplus (2)\{\overline{1200}\} \oplus (2)\{1440\} \oplus (2)\{\overline{1440}\} \oplus (4)\{3696\} \oplus (4)\{\overline{3696}\} \oplus \{5280\} \oplus \{\overline{5280}\} \oplus (2)\{8064\} \oplus (2)\{\overline{8064}\} \oplus (2)\{8800\} \oplus (2)\{\overline{8800}\} \oplus \{11088\} \oplus \{\overline{11088}\} \oplus \{25200\} \oplus \{\overline{25200}\}$
- Level-6: $\{1\} \oplus (2)\{45\} \oplus \{54\} \oplus (4)\{120\} \oplus \{126\} \oplus \{\overline{126}\} \oplus (3)\{210\} \oplus (4)\{320\} \oplus (4)\{770\} \oplus (5)\{945\} \oplus (4)\{1050\} \oplus (4)\{\overline{1050}\} \oplus (3)\{1386\} \oplus (6)\{1728\} \oplus \{2772\} \oplus \{\overline{2772}\} \oplus (4)\{2970\} \oplus (5)\{3696'\} \oplus (5)\{\overline{3696'}\} \oplus \{4125\} \oplus (6)\{4312\} \oplus (2)\{4410\} \oplus \{4950\} \oplus \{\overline{4950}\} \oplus (6)\{5940\} \oplus (2)\{6930\} \oplus (2)\{\overline{6930}\} \oplus \{6930'\} \oplus \{\overline{6930'}\} \oplus \{7644\} \oplus$

$(4)\{8085\}\oplus\{8910\}\oplus(2)\{10560\}\oplus(4)\{17920\}\oplus(2)\{23040\}\oplus(2)\{\overline{23040}\}\oplus(2)\{34398\}\oplus$
$(4)\{36750\}\oplus\{48114\}\oplus\{\overline{48114}\}\oplus\{72765\}\oplus\{73710\}$

- Level-7: $(2)\{16\}\oplus(2)\{\overline{16}\}\oplus(4)\{144\}\oplus(4)\{\overline{144}\}\oplus(7)\{560\}\oplus(7)\{\overline{560}\}\oplus(2)\{672\}\oplus$
$(2)\{\overline{672}\}\oplus(5)\{720\}\oplus(5)\{\overline{720}\}\oplus(6)\{1200\}\oplus(6)\{\overline{1200}\}\oplus(4)\{1440\}\oplus(4)\{\overline{1440}\}\oplus$
$(4)\{2640\}\oplus(4)\{\overline{2640}\}\oplus(10)\{3696\}\oplus(10)\{\overline{3696}\}\oplus(2)\{5280\}\oplus(2)\{\overline{5280}\}\oplus(5)\{8064\}\oplus$
$(5)\{\overline{8064}\}\oplus(10)\{8800\}\oplus(10)\{\overline{8800}\}\oplus(6)\{11088\}\oplus(6)\{\overline{11088}\}\oplus(5)\{15120\}\oplus$
$(5)\{\overline{15120}\}\oplus(2)\{17280\}\oplus(2)\{\overline{17280}\}\oplus(5)\{25200\}\oplus(5)\{\overline{25200}\}\oplus\{29568\}\oplus$
$\{\overline{29568}\}\oplus(4)\{34992\}\oplus(4)\{\overline{34992}\}\oplus(4)\{38016\}\oplus(4)\{\overline{38016}\}\oplus(3)\{43680\}\oplus$
$(3)\{\overline{43680}\}\oplus(2)\{49280\}\oplus(2)\{\overline{49280}\}\oplus\{55440\}\oplus\{\overline{55440}\}\oplus\{70560\}\oplus\{\overline{70560}\}\oplus$
$(2)\{144144\}\oplus(2)\{\overline{144144}\}\oplus\{205920\}\oplus\{\overline{205920}\}$

- Level-8: $(2)\{1\}\oplus(2)\{10\}\oplus(3)\{45\}\oplus(4)\{54\}\oplus(6)\{120\}\oplus(3)\{126\}\oplus(3)\{\overline{126}\}\oplus$
$(7)\{210\}\oplus(4)\{210'\}\oplus(6)\{320\}\oplus(5)\{660\}\oplus(8)\{770\}\oplus(9)\{945\}\oplus(8)\{1050\}\oplus$
$(8)\{\overline{1050}\}\oplus(6)\{1386\}\oplus(14)\{1728\}\oplus\{2772\}\oplus\{\overline{2772}\}\oplus(12)\{2970\}\oplus(8)\{3696'\}\oplus$
$(8)\{\overline{3696'}\}\oplus(10)\{4125\}\oplus(12)\{4312\}\oplus(6)\{4410\}\oplus(4)\{4608\}\oplus(7)\{4950\}\oplus(7)\{\overline{4950}\}\oplus$
$(14)\{5940\}\oplus(4)\{6930\}\oplus(4)\{\overline{6930}\}\oplus(2)\{6930'\}\oplus(2)\{\overline{6930'}\}\oplus(2)\{7644\}\oplus(16)\{8085\}\oplus$
$(5)\{8910\}\oplus(10)\{10560\}\oplus(3)\{14784\}\oplus(4)\{16380\}\oplus(3)\{17325\}\oplus(3)\{\overline{17325}\}\oplus$
$(12)\{17920\}\oplus(8)\{23040\}\oplus(8)\{\overline{23040}\}\oplus(6)\{27720\}\oplus(6)\{28160\}\oplus(4)\{34398\}\oplus$
$(16)\{36750\}\oplus(2)\{37632\}\oplus(3)\{46800\}\oplus(3)\{\overline{46800}\}\oplus(5)\{48114\}\oplus(5)\{\overline{48114}\}\oplus$
$(2)\{50688\}\oplus(2)\{\overline{50688}\}\oplus\{52920\}\oplus(4)\{64680\}\oplus(4)\{68640\}\oplus(9)\{72765\}\oplus$
$(6)\{73710\}\oplus(2)\{90090\}\oplus(2)\{\overline{90090}\}\oplus(3)\{112320\}\oplus(2)\{128700\}\oplus(2)\{\overline{128700}\}\oplus$
$(5)\{143000\}\oplus\{150150\}\oplus\{\overline{150150}\}\oplus\{174636\}\oplus\{189189\}\oplus(2)\{192192\}\oplus\{242550\}\oplus$
$\{\overline{242550}\}\oplus\{274560\}\oplus(2)\{299520\}\oplus(2)\{380160\}$

- Level-9: $(4)\{16\}\oplus(4)\{\overline{16}\}\oplus(8)\{144\}\oplus(8)\{\overline{144}\}\oplus(12)\{560\}\oplus(12)\{\overline{560}\}\oplus(3)\{672\}\oplus$
$(3)\{\overline{672}\}\oplus(11)\{720\}\oplus(11)\{\overline{720}\}\oplus(13)\{1200\}\oplus(13)\{\overline{1200}\}\oplus(9)\{1440\}\oplus(9)\{\overline{1440}\}\oplus$
$(12)\{2640\}\oplus(12)\{\overline{2640}\}\oplus(20)\{3696\}\oplus(20)\{\overline{3696}\}\oplus(4)\{5280\}\oplus(4)\{\overline{5280}\}\oplus$
$(4)\{7920\}\oplus(4)\{\overline{7920}\}\oplus(12)\{8064\}\oplus(12)\{\overline{8064}\}\oplus(24)\{8800\}\oplus(24)\{\overline{8800}\}\oplus$
$(16)\{11088\}\oplus(16)\{\overline{11088}\}\oplus(15)\{15120\}\oplus(15)\{\overline{15120}\}\oplus(3)\{17280\}\oplus(3)\{\overline{17280}\}\oplus$
$(3)\{23760\}\oplus(3)\{\overline{23760}\}\oplus(16)\{25200\}\oplus(16)\{\overline{25200}\}\oplus\{29568\}\oplus\{\overline{29568}\}\oplus$
$(6)\{30800\}\oplus(6)\{\overline{30800}\}\oplus(11)\{34992\}\oplus(11)\{\overline{34992}\}\oplus(15)\{38016\}\oplus(15)\{\overline{38016}\}\oplus$

$(2)\{39600\} \oplus (2)\{\overline{39600}\} \oplus (10)\{43680\} \oplus (10)\{\overline{43680}\} \oplus (5)\{48048\} \oplus (5)\{\overline{48048}\} \oplus (12)\{49280\} \oplus (12)\{\overline{49280}\} \oplus (5)\{55440\} \oplus (5)\{\overline{55440}\} \oplus (3)\{70560\} \oplus (3)\{\overline{70560}\} \oplus (2)\{102960\} \oplus (2)\{\overline{102960}\} \oplus (4)\{124800\} \oplus (4)\{\overline{124800}\} \oplus (10)\{144144\} \oplus (10)\{\overline{144144}\} \oplus (3)\{155232\} \oplus (3)\{\overline{155232}\} \oplus (2)\{164736\} \oplus (2)\{\overline{164736}\} \oplus (3)\{196560\} \oplus (3)\{\overline{196560}\} \oplus \{198000\} \oplus \{\overline{198000}\} \oplus (8)\{205920\} \oplus (8)\{\overline{205920}\} \oplus \{258720\} \oplus \{\overline{258720}\} \oplus \{274560'\} \oplus \{\overline{274560'}\} \oplus \{332640\} \oplus \{\overline{332640}\} \oplus (2)\{364000\} \oplus (2)\{\overline{364000}\} \oplus (2)\{388080\} \oplus (2)\{\overline{388080}\} \oplus (2)\{529200\} \oplus (2)\{\overline{529200}\} \oplus \{769824\} \oplus \{\overline{769824}\}$

- Level-10: $(2)\{1\} \oplus (2)\{10\} \oplus (6)\{45\} \oplus (4)\{54\} \oplus (12)\{120\} \oplus (4)\{126\} \oplus (4)\{\overline{126}\} \oplus (11)\{210\} \oplus (4)\{210'\} \oplus (12)\{320\} \oplus (5)\{660\} \oplus (11)\{770\} \oplus (21)\{945\} \oplus (14)\{1050\} \oplus (14)\{\overline{1050}\} \oplus (15)\{1386\} \oplus (26)\{1728\} \oplus \{2772\} \oplus \{\overline{2772}\} \oplus (20)\{2970\} \oplus (19)\{3696'\} \oplus (19)\{\overline{3696'}\} \oplus (14)\{4125\} \oplus (30)\{4312\} \oplus (12)\{4410\} \oplus (10)\{4608\} \oplus (13)\{4950\} \oplus (13)\{\overline{4950}\} \oplus (30)\{5940\} \oplus (12)\{6930\} \oplus (12)\{\overline{6930}\} \oplus (4)\{6930'\} \oplus (4)\{\overline{6930'}\} \oplus (6)\{7644\} \oplus (31)\{8085\} \oplus (9)\{8910\} \oplus (22)\{10560\} \oplus (3)\{12870\} \oplus (15)\{14784\} \oplus (8)\{16380\} \oplus (9)\{17325\} \oplus (9)\{\overline{17325}\} \oplus (28)\{17920\} \oplus (20)\{23040\} \oplus (20)\{\overline{23040}\} \oplus (14)\{27720\} \oplus (18)\{28160\} \oplus (14)\{34398\} \oplus (38)\{36750\} \oplus (4)\{37632\} \oplus (6)\{42120\} \oplus (6)\{46800\} \oplus (6)\{\overline{46800}\} \oplus (16)\{48114\} \oplus (16)\{\overline{48114}\} \oplus (2)\{48510\} \oplus \{50050\} \oplus \{\overline{50050}\} \oplus (6)\{50688\} \oplus (6)\{\overline{50688}\} \oplus \{52920\} \oplus (12)\{64680\} \oplus (14)\{68640\} \oplus (7)\{70070\} \oplus (4)\{70070'\} \oplus (28)\{72765\} \oplus (15)\{73710\} \oplus (4)\{81081\} \oplus (11)\{90090\} \oplus (11)\{\overline{90090}\} \oplus (7)\{112320\} \oplus \{124740\} \oplus (6)\{128700\} \oplus (6)\{\overline{128700}\} \oplus (20)\{143000\} \oplus (3)\{150150\} \oplus (3)\{\overline{150150}\} \oplus \{165165\} \oplus (6)\{174636\} \oplus (6)\{189189\} \oplus (12)\{192192\} \oplus (6)\{199017\} \oplus (6)\{\overline{199017}\} \oplus (4)\{207360\} \oplus (3)\{210210\} \oplus (2)\{216216\} \oplus (2)\{\overline{216216}\} \oplus (4)\{242550\} \oplus (4)\{\overline{242550}\} \oplus (2)\{270270\} \oplus (5)\{274560\} \oplus (2)\{275968\} \oplus (2)\{\overline{275968}\} \oplus (12)\{299520\} \oplus \{371250\} \oplus \{\overline{371250}\} \oplus (2)\{376320\} \oplus (8)\{380160\} \oplus (4)\{436590\} \oplus \{577500\} \oplus (2)\{590490\} \oplus \{630630\} \oplus \{\overline{630630}\} \oplus (2)\{705600\} \oplus (2)\{\overline{705600}\} \oplus \{848925\} \oplus \{\overline{848925}\} \oplus \{945945\} \oplus (4)\{1048576\} \oplus (2)\{1064448\} \oplus \{1299078\}$

- Level-11: $(4)\{16\} \oplus (4)\{\overline{16}\} \oplus (12)\{144\} \oplus (12)\{\overline{144}\} \oplus (22)\{560\} \oplus (22)\{\overline{560}\} \oplus (8)\{672\} \oplus (8)\{\overline{672}\} \oplus (19)\{720\} \oplus (19)\{\overline{720}\} \oplus (21)\{1200\} \oplus (21)\{\overline{1200}\} \oplus (16)\{1440\} \oplus (16)\{\overline{1440}\} \oplus (18)\{2640\} \oplus (18)\{\overline{2640}\} \oplus (40)\{3696\} \oplus (40)\{\overline{3696}\} \oplus (11)\{5280\} \oplus (11)\{\overline{5280}\} \oplus (8)\{7920\} \oplus (8)\{\overline{7920}\} \oplus (23)\{8064\} \oplus (23)\{\overline{8064}\} \oplus (43)\{8800\} \oplus (43)\{\overline{8800}\} \oplus (30)\{11088\} \oplus (30)\{\overline{11088}\} \oplus (33)\{15120\} \oplus (33)\{\overline{15120}\} \oplus (9)\{17280\} \oplus$

$(9)\{\overline{17280}\} \oplus (2)\{20592\} \oplus (2)\{\overline{20592}\} \oplus (8)\{23760\} \oplus (8)\{\overline{23760}\} \oplus (32)\{25200\} \oplus$
$(32)\{\overline{25200}\} \oplus (4)\{29568\} \oplus (4)\{\overline{29568}\} \oplus (12)\{30800\} \oplus (12)\{\overline{30800}\} \oplus (26)\{34992\} \oplus$
$(26)\{\overline{34992}\} \oplus (35)\{38016\} \oplus (35)\{\overline{38016}\} \oplus (3)\{39600\} \oplus (3)\{\overline{39600}\} \oplus (22)\{43680\} \oplus$
$(22)\{\overline{43680}\} \oplus (16)\{48048\} \oplus (16)\{\overline{48048}\} \oplus (25)\{49280\} \oplus (25)\{\overline{49280}\} \oplus (13)\{55440\} \oplus$
$(13)\{\overline{55440}\} \oplus (6)\{70560\} \oplus (6)\{\overline{70560}\} \oplus (3)\{80080\} \oplus (3)\{\overline{80080}\} \oplus (6)\{102960\} \oplus$
$(6)\{\overline{102960}\} \oplus (15)\{124800\} \oplus (15)\{\overline{124800}\} \oplus (4)\{129360\} \oplus (4)\{\overline{129360}\} \oplus (29)\{144144\} \oplus$
$(29)\{\overline{144144}\} \oplus (9)\{155232\} \oplus (9)\{\overline{155232}\} \oplus (9)\{164736\} \oplus (9)\{\overline{164736}\} \oplus (11)\{196560\} \oplus$
$(11)\{\overline{196560}\} \oplus (2)\{196560'\} \oplus (2)\{\overline{196560'}\} \oplus \{198000\} \oplus \{\overline{198000}\} \oplus (22)\{205920\} \oplus$
$(22)\{\overline{205920}\} \oplus \{224224\} \oplus \{\overline{224224}\} \oplus (7)\{258720\} \oplus (7)\{\overline{258720}\} \oplus (3)\{274560'\} \oplus$
$(3)\{\overline{274560'}\} \oplus (2)\{332640\} \oplus (2)\{\overline{332640}\} \oplus (2)\{343200\} \oplus (2)\{\overline{343200}\} \oplus (10)\{364000\} \oplus$
$(10)\{\overline{364000}\} \oplus (5)\{388080\} \oplus (5)\{\overline{388080}\} \oplus (2)\{388080'\} \oplus (2)\{\overline{388080'}\} \oplus (2)\{443520\} \oplus$
$(2)\{\overline{443520}\} \oplus \{458640\} \oplus \{\overline{458640}\} \oplus (3)\{465696\} \oplus (3)\{\overline{465696}\} \oplus (10)\{529200\} \oplus$
$(10)\{\overline{529200}\} \oplus \{758912\} \oplus \{\overline{758912}\} \oplus \{764400\} \oplus \{\overline{764400}\} \oplus (7)\{769824\} \oplus$
$(7)\{\overline{769824}\} \oplus \{784784\} \oplus \{\overline{784784}\} \oplus (2)\{905520\} \oplus (2)\{\overline{905520}\} \oplus (2)\{1260000\} \oplus$
$(2)\{\overline{1260000}\} \oplus (2)\{1441440\} \oplus (2)\{\overline{1441440}\} \oplus \{1544400\} \oplus \{\overline{1544400}\} \oplus \{1924560\} \oplus$
$\{\overline{1924560}\}$

- Level-12: $(2)\{1\} \oplus (2)\{10\} \oplus (6)\{45\} \oplus (9)\{54\} \oplus (12)\{120\} \oplus (9)\{126\} \oplus (9)\{\overline{126}\} \oplus$
$(16)\{210\} \oplus (8)\{210'\} \oplus (22)\{320\} \oplus (8)\{660\} \oplus (27)\{770\} \oplus (29)\{945\} \oplus (27)\{1050\} \oplus$
$(27)\{\overline{1050}\} \oplus (23)\{1386\} \oplus (42)\{1728\} \oplus (2)\{1782\} \oplus (5)\{2772\} \oplus (5)\{\overline{2772}\} \oplus$
$(38)\{2970\} \oplus (30)\{3696'\} \oplus (30)\{\overline{3696'}\} \oplus (25)\{4125\} \oplus \{4290\} \oplus (40)\{4312\} \oplus$
$(26)\{4410\} \oplus (16)\{4608\} \oplus (27)\{4950\} \oplus (27)\{\overline{4950}\} \oplus (48)\{5940\} \oplus (19)\{6930\} \oplus$
$(19)\{\overline{6930}\} \oplus (12)\{6930'\} \oplus (12)\{\overline{6930'}\} \oplus (7)\{7644\} \oplus (51)\{8085\} \oplus (18)\{8910\} \oplus$
$(34)\{10560\} \oplus (7)\{12870\} \oplus (22)\{14784\} \oplus (20)\{16380\} \oplus (19)\{17325\} \oplus (19)\{\overline{17325}\} \oplus$
$(52)\{17920\} \oplus (3)\{20790\} \oplus (3)\{\overline{20790}\} \oplus (38)\{23040\} \oplus (38)\{\overline{23040}\} \oplus (30)\{27720\} \oplus$
$(32)\{28160\} \oplus (2)\{31680\} \oplus (22)\{34398\} \oplus (70)\{36750\} \oplus (6)\{37632\} \oplus (10)\{42120\} \oplus$
$(16)\{46800\} \oplus (16)\{\overline{46800}\} \oplus (29)\{48114\} \oplus (29)\{\overline{48114}\} \oplus (8)\{48510\} \oplus (7)\{50050\} \oplus$
$(7)\{\overline{50050}\} \oplus (14)\{50688\} \oplus (14)\{\overline{50688}\} \oplus (2)\{52920\} \oplus (22)\{64680\} \oplus (34)\{68640\} \oplus$
$(13)\{70070\} \oplus (4)\{70070'\} \oplus (51)\{72765\} \oplus (32)\{73710\} \oplus (12)\{81081\} \oplus (23)\{90090\} \oplus$
$(23)\{\overline{90090}\} \oplus (2)\{90090'\} \oplus (2)\{\overline{90090'}\} \oplus \{105105\} \oplus (23)\{112320\} \oplus (3)\{123750\} \oplus$
$\{124740\} \oplus (2)\{126126\} \oplus (2)\{\overline{126126}\} \oplus (17)\{128700\} \oplus (17)\{\overline{128700}\} \oplus (43)\{143000\} \oplus$

$(4)\{144144'\} \oplus (4)\{\overline{144144'}\} \oplus (7)\{150150\} \oplus (7)\{\overline{150150}\} \oplus \{165165\} \oplus (14)\{174636\} \oplus$
$(11)\{189189\} \oplus (28)\{192192\} \oplus (14)\{199017\} \oplus (14)\{\overline{199017}\} \oplus (2)\{203840\} \oplus (12)\{207360\} \oplus$
$(7)\{210210\} \oplus (7)\{216216\} \oplus (7)\{\overline{216216}\} \oplus (2)\{219648\} \oplus (2)\{\overline{219648}\} \oplus (10)\{242550\} \oplus$
$(10)\{\overline{242550}\} \oplus (2)\{258720'\} \oplus (4)\{270270\} \oplus (13)\{274560\} \oplus (8)\{275968\} \oplus (8)\{\overline{275968}\} \oplus$
$\{294294\} \oplus \{\overline{294294}\} \oplus (30)\{299520\} \oplus (4)\{351000\} \oplus \{371250\} \oplus \{\overline{371250}\} \oplus$
$(10)\{376320\} \oplus (20)\{380160\} \oplus (14)\{436590\} \oplus (2)\{536250\} \oplus (2)\{577500\} \oplus (4)\{590490\} \oplus$
$(3)\{620928\} \oplus (4)\{630630\} \oplus (4)\{\overline{630630}\} \oplus (2)\{698880\} \oplus (2)\{\overline{698880}\} \oplus (8)\{705600\} \oplus$
$(8)\{\overline{705600}\} \oplus \{720720\} \oplus \{\overline{720720}\} \oplus (2)\{831600\} \oplus (2)\{\overline{831600}\} \oplus (3)\{848925\} \oplus$
$(3)\{\overline{848925}\} \oplus (2)\{882882\} \oplus (2)\{\overline{882882}\} \oplus (2)\{884520\} \oplus \{928125\} \oplus (6)\{945945\} \oplus$
$(16)\{1048576\} \oplus (6)\{1064448\} \oplus \{1137500\} \oplus (2)\{1281280\} \oplus (5)\{1299078\} \oplus \{1316250\} \oplus$
$\{1387386\} \oplus \{\overline{1387386}\} \oplus \{1698840\} \oplus (2)\{1774080\} \oplus \{1897280\} \oplus \{2502500\} \oplus$
$\{\overline{2502500}\} \oplus (2)\{3706560\}$

- Level-13: $(4)\{16\} \oplus (4)\{\overline{16}\} \oplus (18)\{144\} \oplus (18)\{\overline{144}\} \oplus (34)\{560\} \oplus (34)\{\overline{560}\} \oplus$
$(12)\{672\} \oplus (12)\{\overline{672}\} \oplus (28)\{720\} \oplus (28)\{\overline{720}\} \oplus (31)\{1200\} \oplus (31)\{\overline{1200}\} \oplus (24)\{1440\} \oplus$
$(24)\{\overline{1440}\} \oplus (25)\{2640\} \oplus (25)\{\overline{2640}\} \oplus (60)\{3696\} \oplus (60)\{\overline{3696}\} \oplus (19)\{5280\} \oplus$
$(19)\{\overline{5280}\} \oplus (13)\{7920\} \oplus (13)\{\overline{7920}\} \oplus (39)\{8064\} \oplus (39)\{\overline{8064}\} \oplus (64)\{8800\} \oplus$
$(64)\{\overline{8800}\} \oplus (48)\{11088\} \oplus (48)\{\overline{11088}\} \oplus (51)\{15120\} \oplus (51)\{\overline{15120}\} \oplus (16)\{17280\} \oplus$
$(16)\{\overline{17280}\} \oplus (5)\{20592\} \oplus (5)\{\overline{20592}\} \oplus (15)\{23760\} \oplus (15)\{\overline{23760}\} \oplus (52)\{25200\} \oplus$
$(52)\{\overline{25200}\} \oplus (3)\{26400\} \oplus (3)\{\overline{26400}\} \oplus (10)\{29568\} \oplus (10)\{\overline{29568}\} \oplus (20)\{30800\} \oplus$
$(20)\{\overline{30800}\} \oplus (41)\{34992\} \oplus (41)\{\overline{34992}\} \oplus (56)\{38016\} \oplus (56)\{\overline{38016}\} \oplus (6)\{39600\} \oplus$
$(6)\{\overline{39600}\} \oplus (39)\{43680\} \oplus (39)\{\overline{43680}\} \oplus (28)\{48048\} \oplus (28)\{\overline{48048}\} \oplus \{48048'\} \oplus$
$\{\overline{48048'}\} \oplus (42)\{49280\} \oplus (42)\{\overline{49280}\} \oplus (21)\{55440\} \oplus (21)\{\overline{55440}\} \oplus (10)\{70560\} \oplus$
$(10)\{\overline{70560}\} \oplus (7)\{80080\} \oplus (7)\{\overline{80080}\} \oplus (13)\{102960\} \oplus (13)\{\overline{102960}\} \oplus (28)\{124800\} \oplus$
$(28)\{\overline{124800}\} \oplus (10)\{129360\} \oplus (10)\{\overline{129360}\} \oplus (54)\{144144\} \oplus (54)\{\overline{144144}\} \oplus$
$(19)\{155232\} \oplus (19)\{\overline{155232}\} \oplus (19)\{164736\} \oplus (19)\{\overline{164736}\} \oplus (2)\{185328\} \oplus (2)\{\overline{185328}\} \oplus$
$(22)\{196560\} \oplus (22)\{\overline{196560}\} \oplus (7)\{196560'\} \oplus (7)\{\overline{196560'}\} \oplus (4)\{198000\} \oplus (4)\{\overline{198000}\} \oplus$
$(40)\{205920\} \oplus (40)\{\overline{205920}\} \oplus (3)\{224224\} \oplus (3)\{\overline{224224}\} \oplus (15)\{258720\} \oplus (15)\{\overline{258720}\} \oplus$
$(5)\{274560'\} \oplus (5)\{\overline{274560'}\} \oplus (2)\{308880\} \oplus (2)\{\overline{308880}\} \oplus (4)\{332640\} \oplus (4)\{\overline{332640}\} \oplus$
$(7)\{343200\} \oplus (7)\{\overline{343200}\} \oplus (19)\{364000\} \oplus (19)\{\overline{364000}\} \oplus (12)\{388080\} \oplus (12)\{\overline{388080}\} \oplus$
$(5)\{388080'\} \oplus (5)\{\overline{388080'}\} \oplus (5)\{443520\} \oplus (5)\{\overline{443520}\} \oplus (5)\{458640\} \oplus (5)\{\overline{458640}\} \oplus$

$(7)\{465696\} \oplus (7)\{\overline{465696}\} \oplus (2)\{498960\} \oplus (2)\{\overline{498960}\} \oplus \{524160\} \oplus \{\overline{524160}\} \oplus$
$(24)\{529200\} \oplus (24)\{\overline{529200}\} \oplus \{758912\} \oplus \{\overline{758912}\} \oplus (7)\{764400\} \oplus (7)\{\overline{764400}\} \oplus$
$\{764400'\} \oplus \{\overline{764400'}\} \oplus (16)\{769824\} \oplus (16)\{\overline{769824}\} \oplus (3)\{784784\} \oplus (3)\{\overline{784784}\} \oplus$
$\{831600'\} \oplus \{\overline{831600'}\} \oplus (6)\{905520\} \oplus (6)\{\overline{905520}\} \oplus \{917280\} \oplus \{\overline{917280}\} \oplus$
$\{1123200\} \oplus \{\overline{1123200}\} \oplus \{1153152\} \oplus \{\overline{1153152}\} \oplus (6)\{1260000\} \oplus (6)\{\overline{1260000}\} \oplus$
$(6)\{1441440\} \oplus (6)\{\overline{1441440}\} \oplus (4)\{1544400\} \oplus (4)\{\overline{1544400}\} \oplus (3)\{1924560\} \oplus$
$(3)\{\overline{1924560}\} \oplus (2)\{2274480\} \oplus (2)\{\overline{2274480}\} \oplus \{2310000\} \oplus \{\overline{2310000}\} \oplus (2)\{2402400\} \oplus$
$(2)\{\overline{2402400}\} \oplus \{4756752\} \oplus \{\overline{4756752}\}$

- Level-14: $(2)\{1\} \oplus (2)\{10\} \oplus (13)\{45\} \oplus (9)\{54\} \oplus (26)\{120\} \oplus (9)\{126\} \oplus (9)\{\overline{126}\} \oplus$
$(23)\{210\} \oplus (8)\{210'\} \oplus (28)\{320\} \oplus (8)\{660\} \oplus (27)\{770\} \oplus (46)\{945\} \oplus (32)\{1050\} \oplus$
$(32)\{\overline{1050}\} \oplus (31)\{1386\} \oplus (58)\{1728\} \oplus (2)\{1782\} \oplus (5)\{2772\} \oplus (5)\{\overline{2772}\} \oplus$
$(48)\{2970\} \oplus (43)\{3696'\} \oplus (43)\{\overline{3696'}\} \oplus (30)\{4125\} \oplus \{4290\} \oplus (60)\{4312\} \oplus$
$(32)\{4410\} \oplus (22)\{4608\} \oplus (32)\{4950\} \oplus (32)\{\overline{4950}\} \oplus (71)\{5940\} \oplus (30)\{6930\} \oplus$
$(30)\{\overline{6930}\} \oplus (15)\{6930'\} \oplus (15)\{\overline{6930'}\} \oplus (16)\{7644\} \oplus (68)\{8085\} \oplus (21)\{8910\} \oplus$
$(50)\{10560\} \oplus (11)\{12870\} \oplus (37)\{14784\} \oplus (24)\{16380\} \oplus (24)\{17325\} \oplus (24)\{\overline{17325}\} \oplus$
$(72)\{17920\} \oplus (5)\{20790\} \oplus (5)\{\overline{20790}\} \oplus (54)\{23040\} \oplus (54)\{\overline{23040}\} \oplus (40)\{27720\} \oplus$
$(46)\{28160\} \oplus (4)\{31680\} \oplus (40)\{34398\} \oplus (100)\{36750\} \oplus (12)\{37632\} \oplus (20)\{42120\} \oplus$
$(22)\{46800\} \oplus (22)\{\overline{46800}\} \oplus (46)\{48114\} \oplus (46)\{\overline{48114}\} \oplus (10)\{48510\} \oplus (9)\{50050\} \oplus$
$(9)\{\overline{50050}\} \oplus (22)\{50688\} \oplus (22)\{\overline{50688}\} \oplus (2)\{52920\} \oplus (4)\{64350\} \oplus (4)\{\overline{64350}\} \oplus$
$(30)\{64680\} \oplus (48)\{68640\} \oplus (25)\{70070\} \oplus (10)\{70070'\} \oplus \{70785\} \oplus (75)\{72765\} \oplus$
$(47)\{73710\} \oplus (20)\{81081\} \oplus (37)\{90090\} \oplus (37)\{\overline{90090}\} \oplus (3)\{90090'\} \oplus (3)\{\overline{90090'}\} \oplus$
$(2)\{102960'\} \oplus (2)\{\overline{102960'}\} \oplus (6)\{105105\} \oplus (30)\{112320\} \oplus (3)\{123750\} \oplus (4)\{124740\} \oplus$
$(3)\{126126\} \oplus (3)\{\overline{126126}\} \oplus (25)\{128700\} \oplus (25)\{\overline{128700}\} \oplus (67)\{143000\} \oplus (5)\{144144'\} \oplus$
$(5)\{\overline{144144'}\} \oplus (14)\{150150\} \oplus (14)\{\overline{150150}\} \oplus \{165165\} \oplus (26)\{174636\} \oplus (18)\{189189\} \oplus$
$(42)\{192192\} \oplus (27)\{199017\} \oplus (27)\{\overline{199017}\} \oplus (6)\{203840\} \oplus (20)\{207360\} \oplus (11)\{210210\} \oplus$
$(12)\{216216\} \oplus (12)\{\overline{216216}\} \oplus (4)\{219648\} \oplus (4)\{\overline{219648}\} \oplus (2)\{237160\} \oplus (18)\{242550\} \oplus$
$(18)\{\overline{242550}\} \oplus (2)\{258720'\} \oplus (6)\{270270\} \oplus (18)\{274560\} \oplus (14)\{275968\} \oplus (14)\{\overline{275968}\} \oplus$
$\{294294\} \oplus \{\overline{294294}\} \oplus (48)\{299520\} \oplus (8)\{351000\} \oplus (2)\{369600\} \oplus (2)\{\overline{369600}\} \oplus$
$(4)\{371250\} \oplus (4)\{\overline{371250}\} \oplus (18)\{376320\} \oplus (34)\{380160\} \oplus (4)\{420420\} \oplus (26)\{436590\} \oplus$
$\{462462\} \oplus (4)\{536250\} \oplus (5)\{577500\} \oplus (10)\{590490\} \oplus (3)\{620928\} \oplus (10)\{630630\} \oplus$

$(10)\{\overline{630630}\} \oplus \{660660\} \oplus \{\overline{660660}\} \oplus (4)\{698880\} \oplus (4)\{\overline{698880}\} \oplus (16)\{705600\} \oplus$
$(16)\{\overline{705600}\} \oplus (3)\{720720\} \oplus (3)\{\overline{720720}\} \oplus (3)\{831600\} \oplus (3)\{\overline{831600}\} \oplus (7)\{848925\} \oplus$
$(7)\{\overline{848925}\} \oplus (4)\{882882\} \oplus (4)\{\overline{882882}\} \oplus (4)\{884520\} \oplus (2)\{928125\} \oplus (15)\{945945\} \oplus$
$(28)\{1048576\} \oplus (10)\{1064448\} \oplus (5)\{1137500\} \oplus (2)\{1202850\} \oplus (2)\{\overline{1202850}\} \oplus$
$(6)\{1281280\} \oplus (9)\{1299078\} \oplus (5)\{1316250\} \oplus (2)\{1387386\} \oplus (2)\{\overline{1387386}\} \oplus$
$(2)\{1559250\} \oplus \{1683990\} \oplus \{\overline{1683990}\} \oplus (2)\{1698840\} \oplus (4)\{1774080\} \oplus \{1897280\} \oplus$
$\{1990170\} \oplus \{\overline{1990170}\} \oplus (3)\{2502500\} \oplus (3)\{\overline{2502500}\} \oplus (2)\{2520000\} \oplus \{2866500\} \oplus$
$\{3071250\} \oplus \{\overline{3071250}\} \oplus (6)\{3706560\} \oplus \{3838185\} \oplus \{4802490\}$

- Level-15: $(12)\{16\} \oplus (12)\{\overline{16}\} \oplus (25)\{144\} \oplus (25)\{\overline{144}\} \oplus (40)\{560\} \oplus (40)\{\overline{560}\} \oplus$
$(12)\{672\} \oplus (12)\{\overline{672}\} \oplus (34)\{720\} \oplus (34)\{\overline{720}\} \oplus (43)\{1200\} \oplus (43)\{\overline{1200}\} \oplus (30)\{1440\} \oplus$
$(30)\{\overline{1440}\} \oplus (32)\{2640\} \oplus (32)\{\overline{2640}\} \oplus (69)\{3696\} \oplus (69)\{\overline{3696}\} \oplus (21)\{5280\} \oplus$
$(21)\{\overline{5280}\} \oplus (18)\{7920\} \oplus (18)\{\overline{7920}\} \oplus (47)\{8064\} \oplus (47)\{\overline{8064}\} \oplus (82)\{8800\} \oplus$
$(82)\{\overline{8800}\} \oplus \{9504\} \oplus \{\overline{9504}\} \oplus (59)\{11088\} \oplus (59)\{\overline{11088}\} \oplus (62)\{15120\} \oplus (62)\{\overline{15120}\} \oplus$
$(20)\{17280\} \oplus (20)\{\overline{17280}\} \oplus (8)\{20592\} \oplus (8)\{\overline{20592}\} \oplus (19)\{23760\} \oplus (19)\{\overline{23760}\} \oplus$
$(68)\{25200\} \oplus (68)\{\overline{25200}\} \oplus (4)\{26400\} \oplus (4)\{\overline{26400}\} \oplus (12)\{29568\} \oplus (12)\{\overline{29568}\} \oplus$
$(28)\{30800\} \oplus (28)\{\overline{30800}\} \oplus (53)\{34992\} \oplus (53)\{\overline{34992}\} \oplus (72)\{38016\} \oplus (72)\{\overline{38016}\} \oplus$
$(7)\{39600\} \oplus (7)\{\overline{39600}\} \oplus (51)\{43680\} \oplus (51)\{\overline{43680}\} \oplus (35)\{48048\} \oplus (35)\{\overline{48048}\} \oplus$
$(3)\{48048'\} \oplus (3)\{\overline{48048'}\} \oplus (53)\{49280\} \oplus (53)\{\overline{49280}\} \oplus (27)\{55440\} \oplus (27)\{\overline{55440}\} \oplus$
$(2)\{68640'\} \oplus (2)\{\overline{68640'}\} \oplus (18)\{70560\} \oplus (18)\{\overline{70560}\} \oplus (9)\{80080\} \oplus (9)\{\overline{80080}\} \oplus$
$(21)\{102960\} \oplus (21)\{\overline{102960}\} \oplus \{1029'\} \oplus \{\overline{10296'}\} \oplus (39)\{124800\} \oplus (39)\{\overline{124800}\} \oplus$
$(13)\{129360\} \oplus (13)\{\overline{129360}\} \oplus (70)\{144144\} \oplus (70)\{\overline{144144}\} \oplus (26)\{155232\} \oplus$
$(26)\{\overline{155232}\} \oplus (26)\{164736\} \oplus (26)\{\overline{164736}\} \oplus (4)\{185328\} \oplus (4)\{\overline{185328}\} \oplus (30)\{196560\} \oplus$
$(30)\{\overline{196560}\} \oplus (11)\{196560'\} \oplus (11)\{\overline{196560'}\} \oplus (8)\{198000\} \oplus (8)\{\overline{198000}\} \oplus \{203280\} \oplus$
$\{\overline{203280}\} \oplus (54)\{205920\} \oplus (54)\{\overline{205920}\} \oplus (3)\{224224\} \oplus (3)\{\overline{224224}\} \oplus (21)\{258720\} \oplus$
$(21)\{\overline{258720}\} \oplus (7)\{274560'\} \oplus (7)\{\overline{274560'}\} \oplus (3)\{308880\} \oplus (3)\{\overline{308880}\} \oplus (8)\{332640\} \oplus$
$(8)\{\overline{332640}\} \oplus (13)\{343200\} \oplus (13)\{\overline{343200}\} \oplus (26)\{364000\} \oplus (26)\{\overline{364000}\} \oplus (16)\{388080\} \oplus$
$(16)\{\overline{388080}\} \oplus (9)\{388080'\} \oplus (9)\{\overline{388080'}\} \oplus (2)\{428064\} \oplus (2)\{\overline{428064}\} \oplus (7)\{443520\} \oplus$
$(7)\{\overline{443520}\} \oplus (8)\{458640\} \oplus (8)\{\overline{458640}\} \oplus (11)\{465696\} \oplus (11)\{\overline{465696}\} \oplus (4)\{498960\} \oplus$
$(4)\{\overline{498960}\} \oplus \{522720\} \oplus \{\overline{522720}\} \oplus \{524160\} \oplus \{\overline{524160}\} \oplus (34)\{529200\} \oplus$
$(34)\{\overline{529200}\} \oplus (2)\{758912\} \oplus (2)\{\overline{758912}\} \oplus (12)\{764400\} \oplus (12)\{\overline{764400}\} \oplus \{764400'\} \oplus$

$\{\overline{764400'}\} \oplus (24)\{769824\} \oplus (24)\{\overline{769824}\} \oplus (5)\{784784\} \oplus (5)\{\overline{784784}\} \oplus (3)\{831600'\} \oplus$
$(3)\{\overline{831600'}\} \oplus (8)\{905520\} \oplus (8)\{\overline{905520}\} \oplus \{917280\} \oplus \{\overline{917280}\} \oplus (2)\{990000\} \oplus$
$(2)\{\overline{990000}\} \oplus \{1121120\} \oplus \{\overline{1121120}\} \oplus \{1123200\} \oplus \{\overline{1123200}\} \oplus \{1153152\} \oplus$
$\{\overline{1153152}\} \oplus (9)\{1260000\} \oplus (9)\{\overline{1260000}\} \oplus (10)\{1441440\} \oplus (10)\{\overline{1441440}\} \oplus$
$(8)\{1544400\} \oplus (8)\{\overline{1544400}\} \oplus (5)\{1924560\} \oplus (5)\{\overline{1924560}\} \oplus (4)\{2274480\} \oplus$
$(4)\{\overline{2274480}\} \oplus (3)\{2310000\} \oplus (3)\{\overline{2310000}\} \oplus (4)\{2402400\} \oplus (4)\{\overline{2402400}\} \oplus$
$\{2469600\} \oplus \{\overline{2469600}\} \oplus \{2642640\} \oplus \{\overline{2642640}\} \oplus \{4410000\} \oplus \{\overline{4410000}\} \oplus (2)\{4756752\} \oplus$
$(2)\{\overline{4756752}\}$

- Level-16: $(11)\{1\} \oplus (10)\{10\} \oplus (13)\{45\} \oplus (16)\{54\} \oplus (26)\{120\} \oplus (16)\{126\} \oplus$
$(16)\{\overline{126}\} \oplus (30)\{210\} \oplus (14)\{210'\} \oplus (28)\{320\} \oplus (14)\{660\} \oplus (32)\{770\} \oplus (46)\{945\} \oplus$
$(38)\{1050\} \oplus (38)\{\overline{1050}\} \oplus (31)\{1386\} \oplus (64)\{1728\} \oplus (6)\{1782\} \oplus (6)\{2772\} \oplus$
$(6)\{\overline{2772}\} \oplus (58)\{2970\} \oplus (43)\{3696'\} \oplus (43)\{\overline{3696'}\} \oplus (45)\{4125\} \oplus (4)\{4290\} \oplus$
$(60)\{4312\} \oplus (36)\{4410\} \oplus (24)\{4608\} \oplus (40)\{4950\} \oplus (40)\{\overline{4950}\} \oplus (76)\{5940\} \oplus$
$(30)\{6930\} \oplus (30)\{\overline{6930}\} \oplus (16)\{6930'\} \oplus (16)\{\overline{6930'}\} \oplus (16)\{7644\} \oplus (76)\{8085\} \oplus$
$(27)\{8910\} \oplus (2)\{9438\} \oplus (60)\{10560\} \oplus (11)\{12870\} \oplus (38)\{14784\} \oplus (30)\{16380\} \oplus$
$(30)\{17325\} \oplus (30)\{\overline{17325}\} \oplus (80)\{17920\} \oplus \{19305\} \oplus (7)\{20790\} \oplus (7)\{\overline{20790}\} \oplus$
$(60)\{23040\} \oplus (60)\{\overline{23040}\} \oplus (52)\{27720\} \oplus (52)\{28160\} \oplus \{28314\} \oplus \{\overline{28314}\} \oplus$
$(4)\{31680\} \oplus (40)\{34398\} \oplus (110)\{36750\} \oplus (16)\{37632\} \oplus (20)\{42120\} \oplus (31)\{46800\} \oplus$
$(31)\{\overline{46800}\} \oplus (49)\{48114\} \oplus (49)\{\overline{48114}\} \oplus (12)\{48510\} \oplus (13)\{50050\} \oplus (13)\{\overline{50050}\} \oplus$
$(24)\{50688\} \oplus (24)\{\overline{50688}\} \oplus (7)\{52920\} \oplus (5)\{64350\} \oplus (5)\{\overline{64350}\} \oplus (36)\{64680\} \oplus$
$(56)\{68640\} \oplus (25)\{70070\} \oplus (10)\{70070'\} \oplus \{70785\} \oplus (86)\{72765\} \oplus (56)\{73710\} \oplus$
$(24)\{81081\} \oplus \{84942\} \oplus \{\overline{84942}\} \oplus (41)\{90090\} \oplus (41)\{\overline{90090}\} \oplus (6)\{90090'\} \oplus$
$(6)\{\overline{90090'}\} \oplus (2)\{102960'\} \oplus (2)\{\overline{102960'}\} \oplus (6)\{105105\} \oplus (40)\{112320\} \oplus (4)\{123750\} \oplus$
$(4)\{124740\} \oplus (5)\{126126\} \oplus (5)\{\overline{126126}\} \oplus (31)\{128700\} \oplus (31)\{\overline{128700}\} \oplus \{141570\} \oplus$
$\{\overline{141570}\} \oplus (74)\{143000\} \oplus (8)\{144144'\} \oplus (8)\{\overline{144144'}\} \oplus (15)\{150150\} \oplus (15)\{\overline{150150}\} \oplus$
$(2)\{165165\} \oplus (29)\{174636\} \oplus (19)\{189189\} \oplus (48)\{192192\} \oplus (29)\{199017\} \oplus (29)\{\overline{199017}\} \oplus$
$(8)\{203840\} \oplus (24)\{207360\} \oplus (14)\{210210\} \oplus (15)\{216216\} \oplus (15)\{\overline{216216}\} \oplus (4)\{219648\} \oplus$
$(4)\{\overline{219648}\} \oplus (2)\{237160\} \oplus (24)\{242550\} \oplus (24)\{\overline{242550}\} \oplus (6)\{258720'\} \oplus (6)\{270270\} \oplus$
$(24)\{274560\} \oplus (16)\{275968\} \oplus (16)\{\overline{275968}\} \oplus \{286650\} \oplus \{\overline{286650}\} \oplus \{294294\} \oplus$
$\{\overline{294294}\} \oplus (56)\{299520\} \oplus (14)\{351000\} \oplus (4)\{369600\} \oplus (4)\{\overline{369600}\} \oplus (4)\{371250\} \oplus$

$(4)\{\overline{371250}\}\oplus(20)\{376320\}\oplus(40)\{380160\}\oplus(5)\{420420\}\oplus(30)\{436590\}\oplus(2)\{462462\}\oplus$
$(6)\{536250\}\oplus(5)\{577500\}\oplus(10)\{590490\}\oplus(6)\{620928\}\oplus(11)\{630630\}\oplus(11)\{\overline{630630}\}\oplus$
$\{660660\}\oplus\{\overline{660660}\}\oplus\{680625\}\oplus\{\overline{680625}\}\oplus(4)\{698880\}\oplus(4)\{\overline{698880}\}\oplus(20)\{705600\}\oplus$
$(20)\{\overline{705600}\}\oplus(3)\{720720\}\oplus(3)\{\overline{720720}\}\oplus\{749112\}\oplus(6)\{831600\}\oplus(6)\{\overline{831600}\}\oplus$
$(9)\{848925\}\oplus(9)\{\overline{848925}\}\oplus(6)\{882882\}\oplus(6)\{\overline{882882}\}\oplus(6)\{884520\}\oplus(2)\{917280'\}\oplus$
$(5)\{928125\}\oplus(20)\{945945\}\oplus(32)\{1048576\}\oplus(12)\{1064448\}\oplus(6)\{1137500\}\oplus$
$(2)\{1202850\}\oplus(2)\{\overline{1202850}\}\oplus(8)\{1281280\}\oplus(12)\{1299078\}\oplus(5)\{1316250\}\oplus$
$(2)\{1387386\}\oplus(2)\{\overline{1387386}\}\oplus\{1524600\}\oplus\{\overline{1524600}\}\oplus(2)\{1559250\}\oplus\{1683990\}\oplus$
$\{\overline{1683990}\}\oplus(2)\{1698840\}\oplus(4)\{1774080\}\oplus(2)\{1897280\}\oplus\{1990170\}\oplus\{\overline{1990170}\}\oplus$
$\{2388750\}\oplus(5)\{2502500\}\oplus(5)\{\overline{2502500}\}\oplus(4)\{2520000\}\oplus\{2866500\}\oplus\{3071250\}\oplus$
$\{\overline{3071250}\}\oplus\{3320240\}\oplus\{\overline{3320240}\}\oplus(8)\{3706560\}\oplus\{3838185\}\oplus(2)\{4802490\}$

5.3.3 Adinkra Diagram for 10D, $\mathcal{N}=2A$ Superfield

In this section, we present the ten-dimensional $\mathcal{N}=2$A adinkra diagram up to the cubic level. Based on the results listed in Section 5.3.2, we can draw the complete adinkra diagram in principle: use open nodes to denote bosonic component fields and put their corresponding irreps in the center. For fermionic component fields, use closed nodes. The number of levels represents the height assignment. Black edges connect nodes in the adjacent levels, meaning SUSY transformations. Due to the limited space of the paper, we only draw the adinkra up to the cubic level.

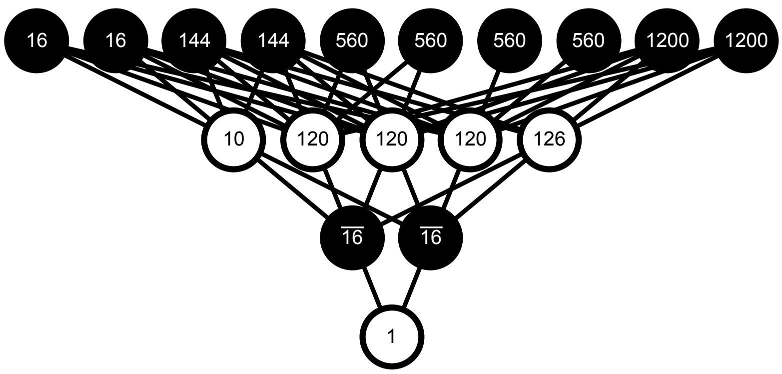


Figure 5.3: Adinkra Diagram for 10D, $\mathcal{N}=2$A At Low Orders

5.4 10D, $\mathcal{N}$ = 2B Scalar Superfield Decomposition and Superconformal Multiplet

5.4.1 Methodology for 10D, $\mathcal{N}$ = 2B Scalar Superfield Construction

The Type IIB theory means $\mathcal{N} = (2, 0)$, i.e. we have two sets of spinor coordinates denoted by θ^{α^1} and θ^{α^2}. Following the same logic as in Type IIA, one can organize the $\mathcal{N} = 2$B scalar superfield as

$$\mathcal{V}(x^{\underline{a}},\, \theta^{\alpha^1},\, \theta^{\alpha^2}) ~=~ \mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha^1}) ~+~ \theta^{\alpha^2}\, \mathcal{V}^{(1)}_{\alpha^2}(x^{\underline{a}},\, \theta^{\alpha^1}) ~+~ \theta^{\alpha^2}\theta^{\beta^2}\, \mathcal{V}^{(2)}_{\alpha^2\beta^2}(x^{\underline{a}},\, \theta^{\alpha^1}) ~+~ \dots \quad , \tag{5.4.1}$$

where the expansion coefficients $\mathcal{V}^{(0)}(x^{\underline{a}},\, \theta^{\alpha^1})$, $\mathcal{V}^{(1)}_{\alpha^2}(x^{\underline{a}},\, \theta^{\alpha^1})$, $\mathcal{V}^{(2)}_{\alpha^2\beta^2}(x^{\underline{a}},\, \theta^{\alpha^1}), \dots$ are 10D, $\mathcal{N} = 1$ superfields. Attaching another copy of n totally antisymmetric undotted θ's to an undotted $\theta-$monomial corresponds to tensoring the $\{16\}^{\wedge n}$ representation on it, i.e. the nth level of the $\theta-$monomial of the 10D, $\mathcal{N} = 1$ superfield itself, without any conjugate. Therefore, the $\mathcal{N} = 2$B scalar superfield $\theta-$monomial decomposition can be summarized as

$$\mathcal{V} ~=~ \begin{cases} \text{Level} - 0 & \ell_0 \otimes \ell_0 \quad , \\ \text{Level} - 1 & \ell_1 \otimes \ell_0 \,\oplus\, \ell_0 \otimes \ell_1 \quad , \\ \text{Level} - 2 & \ell_2 \otimes \ell_0 \,\oplus\, \ell_1 \otimes \ell_1 \,\oplus\, \ell_0 \otimes \ell_2 \quad , \\ \text{Level} - 3 & \ell_3 \otimes \ell_0 \,\oplus\, \ell_2 \otimes \ell_1 \,\oplus\, \ell_1 \otimes \ell_2 \,\oplus\, \ell_0 \otimes \ell_3 \quad , \\ \qquad \vdots & \vdots \\ \text{Level} - n & \ell_n \otimes \ell_0 \,\oplus\, \ell_{n-1} \otimes \ell_1 \,\oplus\, \dots \,\oplus\, \ell_1 \otimes \ell_{n-1} \,\oplus\, \ell_0 \otimes \ell_n \quad , \\ \qquad \vdots & \vdots \\ \text{Level} - 16 & \ell_{16} \otimes \ell_0 \,\oplus\, \ell_{15} \otimes \ell_1 \,\oplus\, \dots \,\oplus\, \ell_1 \otimes \ell_{15} \,\oplus\, \ell_0 \otimes \ell_{16} \quad , \\ \qquad \vdots & \vdots \end{cases} \tag{5.4.2}$$

and level-17 to 32 are the conjugates of level-15 to 0 respectively. The irreps corresponding to the component fields are the conjugates of the $\theta-$monomials at each level.

5.4.2 10D, $\mathcal{N} = 2B$ Scalar Superfield Decomposition Results and Superconformal Multiplet

Based on Equations (5.1.4) and (5.4.2), we can directly obtain the scalar superfield decomposition in 10D, $\mathcal{N} = 2$B. The results from level-0 to level-16 are listed below. Same as in the Type IIA case, we can find graviton embeddings and gravitino embeddings in the $\theta-$expansion of the scalar superfield. We use green color to highlight the irrep $\{54\}$ which corresponds to the traceless graviton in 10D as well. We can translate irreps into component fields and see 72 graviton embeddings, 280 gravitino embeddings, and accompanying auxiliary fields. The scalar superfield is the simplest superfield that contains traceless graviton and gravitino embeddings that can be used as a starting point to construct supergravity. The numbers of gravitons and gravitinos are the same as those in the Type IIA case.

Note that if one find a $\{54\}$ in level-n, one can find $\{144\}$ in level-$(n+1)$, which can be interpreted by SUSY transformation laws of the graviton $h_{\underline{a}\underline{b}}$ in 10D, $\mathcal{N} = 2$B theory. Since in Type IIB theory we have two copies of θ^{α} rather than θ^{α} and $\theta^{\dot{\alpha}}$, we have D-operators D_{α^i} where $i = 1, 2$. Acting the D-operators on the graviton gives a term proportional to the gravitino $\psi_{\underline{b}}{}^{\beta^i}$ in the on-shell case (in the off-shell case, there are several auxiliary fields showing up in the r.h.s. besides the undotted gravitino)

$$\mathrm{D}_{\alpha^i} h_{\underline{a}\underline{b}} \;\propto\; (\sigma_{(\underline{a}})_{\alpha^i \beta^i}\, \psi_{\underline{b})}{}^{\beta^i} \quad . \tag{5.4.3}$$

Note that the spinor indices on the gravitinos are superscript. It suggests that the gravitinos correspond to irrep $\{\overline{144}\}$ in our convention for both $i = 1$ and 2. In the context of $\theta-$monomials, the conjugate of $\{54\}$ is still $\{54\}$, and the conjugate of $\{\overline{144}\}$ is $\{144\}$.

- Level-0: $\{1\}$
- Level-1: $\{16\} \oplus \{16\}$
- Level-2: $\{10\} \oplus (3)\{120\} \oplus \{\overline{126}\}$
- Level-3: $(2)\{\overline{16}\} \oplus (2)\{\overline{144}\} \oplus (4)\{\overline{560}\} \oplus (2)\{\overline{1200}\}$

- Level-4: $\{1\} \oplus (3)\{45\} \oplus \{\mathbf{54}\} \oplus (4)\{210\} \oplus (5)\ \{770\} \oplus (3)\{945\} \oplus (5)\{1050\} \oplus \{\overline{1050}\} \oplus \{4125\} \oplus (3)\{5940\}$

- Level-5: $(2)\{16\} \oplus (6)\{144\} \oplus (6)\{560\} \oplus (6)\{672\} \oplus (2)\{720\} \oplus (6)\{1200\} \oplus (2)\{1440\} \oplus (10)\{3696\} \oplus (4)\{8064\} \oplus (2)\{8800\} \oplus (4)\ \{11088\} \oplus (2)\{25200\}$

- Level-6: $\{10\} \oplus (6)\{120\} \oplus (6)\{126\} \oplus \{\overline{126}\} \oplus \{210'\} \oplus (8)\{320\} \oplus (12)\{1728\} \oplus (9)\{2970\} \oplus (15)\{3696'\} \oplus (3)\ \{\overline{3696'}\} \oplus (10)\{4312\} \oplus (5)\ \{4410\} \oplus (5)\{4950\} \oplus \{\overline{4950}\} \oplus (5)\{6930'\} \oplus (4)\{10560\} \oplus \{27720\} \oplus (3)\{34398\} \oplus (8)\{36750\} \oplus \{46800\} \oplus (3)\ \{48114\}$

- Level-7: $(2)\{\overline{16}\} \oplus (6)\{\overline{144}\} \oplus (16)\{\overline{560}\} \oplus (12)\{\overline{720}\} \oplus (8)\{\overline{1200}\} \oplus (12)\{\overline{1440}\} \oplus (10)\{\overline{2640}\} \oplus (16)\{\overline{3696}\ \} \oplus (10)\{\overline{5280}\} \oplus (12)\{\overline{8064}\} \oplus (26)\{\overline{8800}\} \oplus (6)\ \{\overline{11088}\} \oplus (10)\{\overline{15120}\} \oplus (6)\{\overline{25200}\} \oplus (4)\{\overline{29568}\} \oplus (2)\{\overline{30800}\} \oplus (12)\{\overline{34992}\} \oplus (8)\{\overline{38016}\} \oplus (4)\ \{\overline{43680}\} \oplus (4)\{\overline{49280}\} \oplus (2)\{\overline{70560}\} \oplus (2)\{\overline{102960}\} \oplus (6)\{\overline{144144}\}$

- Level-8: $\{1\} \oplus (3)\{45\} \oplus (6)\{\mathbf{54}\} \oplus (9)\{210\} \oplus (10)\{660\} \oplus (11)\{770\} \oplus (21)\{945\} \oplus (21)\ \{1050\} \oplus (9)\{\overline{1050}\} \oplus (15)\{1386\} \oplus (5)\{2772\} \oplus (19)\{4125\} \oplus (27)\{5940\} \oplus (18)\{6930\} \oplus (3)\{\overline{6930}\} \oplus (3)\{7644\} \oplus (33)\{8085\} \oplus (6)\{8910\} \oplus (10)\{14784\} \oplus (5)\{16380\} \oplus (5)\{17325\} \oplus (8)\{\overline{17325}\} \oplus (24)\{17920\} \oplus (24)\{23040\} \oplus (8)\ \{\overline{23040}\} \oplus (8)\{50688\} \oplus \{52920\} \oplus (3)\{64350\} \oplus (3)\{70070\} \oplus (19)\{72765\} \oplus (9)\ \{73710\} \oplus \{90090'\} \oplus (6)\{112320\} \oplus (9)\ \{128700\} \oplus (8)\{143000\} \oplus (3)\{174636\} \oplus (3)\ \{199017\} \oplus (4)\{242550\}$

- Level-9: $(2)\{16\} \oplus (12)\{144\} \oplus (16)\{560\} \oplus (10)\{672\ \} \oplus (24)\{720\} \oplus (26)\{1200\} \oplus (12)\{1440\} \oplus (30)\{2640\} \oplus (42)\{3696\} \oplus (4)\{5280\} \oplus (10)\{7920\} \oplus (16)\{8064\} \oplus (48)\{8800\} \oplus (36)\ \{11088\} \oplus (30)\{15120\} \oplus (16)\{17280\} \oplus (12)\ \{23760\} \oplus (38)\{25200\} \oplus (4)\{26400\} \oplus (14)\ \{30800\} \oplus (12)\{34992\} \oplus (36)\{38016\} \oplus (18)\ \{43680\} \oplus (10)\{48048\} \oplus (20)\{49280\} \oplus (12)\ \{55440\} \oplus (2)\{68640'\} \oplus (2)\{70560\} \oplus (8)\{124800\} \oplus (16)\{144144\} \oplus (4)\{155232\} \oplus (4)\{164736\} \oplus (12)\{196560\} \oplus (6)\{196560'\} \oplus (6)\{198000\} \oplus (18)\{205920\} \oplus (6)\{258720\} \oplus (2)\{332640\} \oplus (2)\{388080'\} \oplus (6)\ \{529200\}$

- Level-10: $\{10\} \oplus (13)\{120\} \oplus (6)\{126\} \oplus (6)\ \{\overline{126}\} \oplus (15)\{210'\} \oplus (20)\{320\ \} \oplus (48)\{1728\} \oplus \{1782\} \oplus (39)\{2970\} \oplus (33)\{3696'\} \oplus (34)\{\overline{3696'}\} \oplus (63)\{4312\} \oplus$

$(21)\{4410\} \oplus (24)\{4608\} \oplus (36)\{4950\} \oplus (30)\{\overline{4950}\} \oplus (13)\{6930'\} \oplus (5)\{\overline{6930'}\} \oplus (44)\{10560\} \oplus (9)\{20790\} \oplus (39)\{27720\} \oplus (40)\{28160\} \oplus \{28314\} \oplus (21)\{34398\} \oplus (72)\{36750\} \oplus (4)\{37632\} \oplus (10)\{42120\} \oplus (18)\{46800\} \oplus (6)\{\overline{46800}\} \oplus (42)\{48114\} \oplus (27)\{\overline{48114}\} \oplus (5)\{48510\} \oplus (5)\{50050\} \oplus \{\overline{50050}\} \oplus (16)\{64680\} \oplus (29)\{68640\} \oplus (10)\{70070'\} \oplus (27)\{90090\} \oplus (15)\{\overline{90090}\} \oplus (3)\{102960'\} \oplus (6)\{144144'\} \oplus (11)\{150150\} \oplus (24)\{192192\} \oplus (9)\{216216\} \oplus \{258720'\} \oplus (24)\{299520\} \oplus \{351000\} \oplus (4)\{369600\} \oplus (3)\{371250\} \oplus (8)\{376320\} \oplus (12)\{380160\} \oplus (8)\{436590\} \oplus (3)\{590490\} \oplus (3)\{630630\} \oplus (12)\{705600\} \oplus \{831600\}$

- Level-11: $(2)\{\overline{16}\} \oplus (20)\{\overline{144}\} \oplus (30)\{\overline{560}\} \oplus (22)\{\overline{672}\} \oplus (40)\{\overline{720}\} \oplus (42)\{\overline{1200}\} \oplus (24)\{\overline{1440}\} \oplus (42)\{\overline{2640}\} \oplus (82)\{\overline{3696}\} \oplus (22)\{\overline{5280}\} \oplus (20)\{\overline{7920}\} \oplus (36)\{\overline{8064}\} \oplus (86)\{\overline{8800}\} \oplus (2)\{\overline{9504}\} \oplus (60)\{\overline{11088}\} \oplus (72)\{\overline{15120}\} \oplus (20)\{\overline{17280}\} \oplus (2)\{\overline{20592}\} \oplus (12)\{\overline{23760}\} \oplus (66)\{\overline{25200}\} \oplus (16)\{\overline{29568}\} \oplus (32)\{\overline{30800}\} \oplus (48)\{\overline{34992}\} \oplus (72)\{\overline{38016}\} \oplus (4)\{\overline{39600}\} \oplus (36)\{\overline{43680}\} \oplus (30)\{\overline{48048}\} \oplus (56)\{\overline{49280}\} \oplus (20)\{\overline{55440}\} \oplus (8)\{\overline{70560}\} \oplus (12)\{\overline{80080}\} \oplus (24)\{\overline{102960}\} \oplus (36)\{\overline{124800}\} \oplus (10)\{\overline{129360}\} \oplus (64)\{\overline{144144}\} \oplus (18)\{\overline{155232}\} \oplus (12)\{\overline{164736}\} \oplus (6)\{\overline{185328}\} \oplus (24)\{\overline{196560}\} \oplus (2)\{\overline{203280}\} \oplus (32)\{\overline{205920}\} \oplus (12)\{\overline{258720}\} \oplus (2)\{\overline{332640}\} \oplus (2)\{\overline{343200}\} \oplus (24)\{\overline{364000}\} \oplus (10)\{\overline{388080}\} \oplus (4)\{\overline{443520}\} \oplus (4)\{\overline{458640}\} \oplus (12)\{\overline{465696}\} \oplus (6)\{\overline{498960}\} \oplus (16)\{\overline{529200}\} \oplus (6)\{\overline{764400}\} \oplus (18)\{\overline{769824}\} \oplus (6)\{\overline{784784}\} \oplus (2)\{\overline{990000}\} \oplus (6)\{\overline{1260000}\} \oplus (2)\{\overline{1544400}\}$

- Level-12: $\{1\} \oplus (10)\{45\} \oplus (15)\{54\} \oplus (25)\{210\} \oplus (15)\{660\} \oplus (45)\{770\} \oplus (60)\{945\} \oplus (35)\{1050\} \oplus (66)\{\overline{1050}\} \oplus (50)\{1386\} \oplus (6)\{2772\} \oplus (16)\{\overline{2772}\} \oplus (46)\{4125\} \oplus \{4290\} \oplus (93)\{5940\} \oplus (33)\{6930\} \oplus (50)\{\overline{6930}\} \oplus (13)\{7644\} \oplus (105)\{8085\} \oplus (30)\{8910\} \oplus (15)\{12870\} \oplus (52)\{14784\} \oplus (36)\{16380\} \oplus (46)\{17325\} \oplus (30)\{\overline{17325}\} \oplus (104)\{17920\} \oplus (72)\{23040\} \oplus (80)\{\overline{23040}\} \oplus (32)\{50688\} \oplus (24)\{\overline{50688}\} \oplus \{52920\} \oplus (7)\{64350\} \oplus (33)\{70070\} \oplus (104)\{72765\} \oplus (57)\{73710\} \oplus (24)\{81081\} \oplus \{84942\} \oplus (9)\{90090'\} \oplus (3)\{105105\} \oplus (44)\{112320\} \oplus (5)\{123750\} \oplus (3)\{124740\} \oplus (5)\{126126\} \oplus \{\overline{126126}\} \oplus (38)\{128700\} \oplus (30)\{\overline{128700}\} \oplus (87)\{143000\} \oplus (27)\{174636\} \oplus (18)\{189189\} \oplus (45)\{199017\} \oplus (18)\{\overline{199017}\} \oplus (24)\{207360\} \oplus (16)\{210210\} \oplus (8)\{219648\} \oplus (21)\{242550\} \oplus (16)\{\overline{242550}\} \oplus (22)\{274560\} \oplus (24)\{275968\} \oplus (8)\{\overline{275968}\} \oplus (3)\{420420\} \oplus (3)\{577500\} \oplus (6)\{620928\} \oplus (3)\{660660\} \oplus \{680625\} \oplus$

$(8)\{698880\}\oplus(12)\{848925\}\oplus(9)\{882882\}\oplus\{928125\}\oplus(12)\{945945\}\oplus(32)\{1048576\}\oplus$ $(3)\{1137500\}\oplus(3)\{1202850\}\oplus(9)\{1299078\}\oplus(3)\{1316250\}\oplus(4)\{2502500\}$

- Level-13: $(10)\{16\}\oplus(30)\{144\}\oplus(80)\{560\}\oplus(10)\{672\}\oplus(60)\{720\}\oplus(50)\{1200\}\oplus$ $(60)\{1440\}\oplus(50)\{2640\}\oplus(110)\{3696\}\oplus(50)\{5280\}\oplus(24)\{7920\}\oplus(84)\{8064\}\oplus$ $(138)\{8800\}\oplus(84)\{11088\}\oplus(100)\{15120\}\oplus(24)\{17280\}\oplus(12)\{20592\}\oplus(36)\{23760\}\oplus$ $(96)\{25200\}\oplus(6)\{26400\}\oplus(20)\{29568\}\oplus(40)\{30800\}\oplus(90)\{34992\}\oplus(116)\{38016\}\oplus$ $(12)\{39600\}\oplus(78)\{43680\}\oplus(62)\{48048\}\oplus(2)\{48048'\}\oplus(80)\{49280\}\oplus(42)\{55440\}\oplus$ $(2)\{68640'\}\oplus(22)\{70560\}\oplus(4)\{80080\}\oplus(20)\{102960\}\oplus(48)\{124800\}\oplus(16)\{129360\}\oplus$ $(106)\{144144\}\oplus(36)\{155232\}\oplus(48)\{164736\}\oplus(48)\{196560\}\oplus(22)\{196560'\}\oplus$ $(8)\{198000\}\oplus(84)\{205920\}\oplus(10)\{224224\}\oplus(28)\{258720\}\oplus(12)\{274560'\}\oplus$ $(4)\{308880\}\oplus(6)\{332640\}\oplus(18)\{343200\}\oplus(32)\{364000\}\oplus(24)\{388080\}\oplus$ $(18)\{388080'\}\oplus(2)\{428064\}\oplus(12)\{443520\}\oplus(6)\{458640\}\oplus(12)\{465696\}\oplus$ $(2)\{522720\}\oplus(4)\{524160\}\oplus(54)\{529200\}\oplus(10)\{764400\}\oplus(4)\{764400'\}\oplus(24)\{769824\}\oplus$ $(2)\{831600'\}\oplus(16)\{905520\}\oplus(12)\{1260000\}\oplus(18)\{1441440\}\oplus(6)\{1544400\}\oplus$ $(6)\{1924560\}\oplus(6)\{2274480\}\oplus(2)\{2310000\}\oplus(6)\{2402400\}\oplus(2)\{2642640\}$

- Level-14: $(10)\{10\}\oplus(55)\{120\}\oplus(15)\{126\}\oplus(35)\{\overline{126}\}\oplus(15)\{210'\}\oplus(60)\{320\}\oplus$ $(120)\{1728\}\oplus(6)\{1782\}\oplus(105)\{2970\}\oplus(65)\{3696'\}\oplus(105)\{\overline{3696'}\}\oplus(105)\{4312\}\oplus$ $(70)\{4410\}\oplus(40)\{4608\}\oplus(60)\{4950\}\oplus(68)\{\overline{4950}\}\oplus(21)\{6930'\}\oplus(40)\{\overline{6930'}\}\oplus$ $\{9438\}\oplus(100)\{10560\}\oplus(9)\{20790\}\oplus(5)\{\overline{20790}\}\oplus(76)\{27720\}\oplus(88)\{28160\}\oplus$ $(8)\{31680\}\oplus(78)\{34398\}\oplus(203)\{36750\}\oplus(28)\{37632\}\oplus(36)\{42120\}\oplus(45)\{46800\}\oplus$ $(55)\{\overline{46800}\}\oplus(81)\{48114\}\oplus(90)\{\overline{48114}\}\oplus(21)\{48510\}\oplus(27)\{50050\}\oplus(14)\{\overline{50050}\}\oplus$ $(70)\{64680\}\oplus(99)\{68640\}\oplus(13)\{70070'\}\oplus(82)\{90090\}\oplus(63)\{\overline{90090}\}\oplus(3)\{102960'\}\oplus$ $\{141570\}\oplus(14)\{144144'\}\oplus(6)\{\overline{144144'}\}\oplus(27)\{150150\}\oplus(27)\{\overline{150150}\}\oplus(84)\{192192\}\oplus$ $(12)\{203840\}\oplus(32)\{216216\}\oplus(16)\{\overline{216216}\}\oplus(3)\{237160\}\oplus(6)\{258720'\}\oplus$ $(15)\{270270\}\oplus\{\overline{286650}\}\oplus(5)\{294294\}\oplus(96)\{299520\}\oplus(18)\{351000\}\oplus(4)\{369600\}\oplus$ $(3)\{371250\}\oplus(10)\{\overline{371250}\}\oplus(32)\{376320\}\oplus(72)\{380160\}\oplus(51)\{436590\}\oplus$ $(9)\{536250\}\oplus(18)\{590490\}\oplus(27)\{630630\}\oplus(11)\{\overline{630630}\}\oplus(36)\{705600\}\oplus$ $(24)\{\overline{705600}\}\oplus(8)\{720720\}\oplus(3)\{\overline{720720}\}\oplus(9)\{831600\}\oplus(6)\{\overline{831600}\}\oplus(9)\{884520\}\oplus$ $\{917280'\}\oplus(24)\{1064448\}\oplus(12)\{1281280\}\oplus(8)\{1387386\}\oplus\{1524600\}\oplus(3)\{1559250\}\oplus$

$(8)\ \{1774080\} \oplus (3)\{1990170\} \oplus (4)\{2520000\} \oplus (3)\ \{3071250\} \oplus \{3320240\} \oplus$
$(12)\{3706560\}$

- Level-15: $(36)\{\overline{16}\}\oplus(64)\{\overline{144}\}\oplus (90)\{\overline{560}\}\oplus(30)\{\overline{672}\}\oplus (60)\{\overline{720}\}\oplus(100)\{\overline{1200}\} \oplus$
$(60)\{\overline{1440}\}\oplus(50)\{\overline{2640}\ \}\oplus(146)\{\overline{3696}\}\oplus(30)\ \{\overline{5280}\}\oplus(32)\{\overline{7920}\}\oplus (104)\{\overline{8064}\}\oplus$
$(150)\{\overline{8800}\} \oplus (130)\{\overline{11088}\} \oplus (120)\ \{\overline{15120}\} \oplus (40)\{\overline{17280}\} \oplus (16)\{\overline{20592}\} \oplus$
$(30)\{\overline{23760}\} \oplus(140)\{\overline{25200}\}\oplus(4)\{\overline{26400}\}\oplus(16)\{\overline{29568}\}\oplus(44)\ \{\overline{30800}\}\oplus(108)\{\overline{34992}\}\oplus$
$(132)\{\overline{38016}\}\oplus(20)\{\overline{39600}\} \oplus(114)\{\overline{43680}\}\oplus(66)\{\overline{48048}\}\oplus(6)\{\overline{48048'}\}\oplus(108)\ \{\overline{49280}\}\oplus$
$(60)\{\overline{55440}\}\oplus (42)\{\overline{70560}\}\oplus(22)\{\overline{80080}\} \oplus(38)\{\overline{102960}\}\oplus(2)\{\overline{102960'}\}\oplus(80)\{\overline{124800}\}\oplus$
$(28)\{\overline{129360}\} \oplus (138)\{\overline{144144}\} \oplus (56)\{\overline{155232}\} \oplus (48)\ \{\overline{164736}\} \oplus (6)\{\overline{185328}\} \oplus$
$(48)\{\overline{196560}\} \oplus (12)\{\overline{196560'}\} \oplus (14)\{\overline{198000}\} \oplus (116)\ \{\overline{205920}\} \oplus (4)\{\overline{224224}\} \oplus$
$(42)\{\overline{258720}\} \oplus (20)\{\overline{274560'}\} \oplus (6)\{\overline{308880}\} \oplus (20)\ \{\overline{332640}\} \oplus (24)\{\overline{343200}\} \oplus$
$(58)\{\overline{364000}\} \oplus (36)\ \{\overline{388080}\} \oplus (12)\{\overline{388080'}\} \oplus (2)\{\overline{428064}\} \oplus (12)\ \{\overline{443520}\} \oplus$
$(18)\{\overline{458640}\} \oplus (18)\{\overline{465696}\} \oplus (6)\{\overline{498960}\} \oplus (64)\{\overline{529200}\} \oplus (8)\ \{\overline{758912}\} \oplus$
$(24)\{\overline{764400}\} \oplus (54)\{\overline{769824}\} \oplus (12)\ \{\overline{784784}\} \oplus (6)\{\overline{831600'}\ \} \oplus (16)\{\overline{905520}\} \oplus$
$(4)\ \{\overline{917280}\} \oplus (2)\{\overline{990000}\} \oplus (2)\{\overline{1121120}\} \oplus (4)\{\overline{1123200}\} \oplus (4)\{\overline{1153152}\} \oplus$
$(16)\ \{\overline{1260000}\}\oplus(18)\{\overline{1441440}\} \oplus(18)\{\overline{1544400}\}\oplus(12)\ \{\overline{1924560}\}\oplus(6)\{\overline{2274480}\} \oplus$
$(6)\{\overline{2310000}\} \oplus (6)\ \{\overline{2402400}\} \oplus (2)\{\overline{2469600}\} \oplus (2)\{\overline{4410000}\} \oplus (6)\ \{\overline{4756752}\}$

- Level-16: $(27)\{1\} \oplus (45)\{45\} \oplus (28)\{\mathbf{54}\} \oplus (80)\{210\} \oplus (16)\{660\} \oplus (70)\{770\} \oplus$
$(100)\{945\} \oplus (75)\{1050\} \oplus (75)\{\overline{1050}\} \oplus (57)\ \{1386\} \oplus (5)\{2772\} \oplus (5)\{\overline{2772}\} \oplus$
$(75)\{4125\} \oplus (6)\{4290\} \oplus (170)\{5940\} \oplus (60)\{6930\} \oplus (60)\{\overline{6930}\} \oplus (48)\ \{7644\} \oplus$
$(140)\{8085\}\oplus(63)\{8910\}\oplus(23)\ \{12870\}\oplus(68)\{14784\}\oplus(60)\{16380\}\oplus(51)\ \{17325\}\oplus$
$(51)\{\overline{17325}\}\oplus(160)\{17920\} \oplus\{19305\}\oplus(120)\{23040\}\oplus(120)\ \{\overline{23040}\}\oplus(48)\{50688\}\oplus$
$(48)\ \{\overline{50688}\}\oplus(15)\{52920\}\oplus(3)\{64350\} \oplus(3)\{\overline{64350}\}\oplus(43)\{70070\}\oplus(3)\ \{70785\}\oplus$
$(168)\{72765\}\oplus(126)\{73710\}\oplus(48)\ \{81081\}\oplus(6)\{90090'\}\oplus(6)\ \{\overline{90090'}\}\oplus(14)\{105105\}\oplus$
$(66)\ \{112320\}\oplus(6)\{123750\}\oplus(10)\{124740\}\oplus(9)\ \{126126\}\oplus(9)\{\overline{126126}\}\oplus(54)\ \{128700\}\oplus$
$(54)\{\overline{128700}\} \oplus (154)\ \{143000\} \oplus (8)\{165165\} \oplus (63)\{174636\} \oplus (55)\ \{189189\} \oplus$
$(57)\{199017\} \oplus (57)\{\overline{199017}\ \} \oplus (48)\{207360\} \oplus (24)\{210210\} \oplus (8)\ \{219648\} \oplus$
$(8)\{\overline{219648}\} \oplus (49)\ \{242550\} \oplus (49)\{\overline{242550}\} \oplus (54)\{274560\ \} \oplus (32)\{275968\} \oplus$
$(32)\{\overline{275968}\} \oplus(7)\{420420\}\oplus(4)\{462462\}\oplus(15)\{577500\} \oplus(6)\{620928\}\oplus(8)\{698880\}\oplus$
$(8)\ \{\overline{698880}\}\oplus\{749112\}\oplus(19)\{848925\} \oplus(19)\{\overline{848925}\}\oplus(9)\{882882\}\oplus (9)\{\overline{882882}\}\oplus$

$(9)\{928125\} \oplus (40)\ \{945945\} \oplus (64)\{1048576\} \oplus (12)\{1137500\} \oplus (3)\ \{1202850\} \oplus$
$(3)\{\overline{1202850}\} \oplus (24)\ \{1299078\} \oplus (11)\{1316250\} \oplus (3)\{1683990\} \oplus (3)\ \{\overline{1683990}\} \oplus$
$(8)\{1698840\} \oplus (6)\ \{1897280\} \oplus \{2388750\} \oplus (9)\{2502500\} \oplus (9)\ \{\overline{2502500}\} \oplus$
$(3)\{2866500\} \oplus (3)\ \{3838185\} \oplus (4)\{4802490\}$

5.4.3 Adinkra Diagram for 10D, $\mathcal{N} = 2B$ Superfield

Here we draw the ten-dimensional $\mathcal{N} = 2\text{B}$ adinkra diagram up to the cubic level.

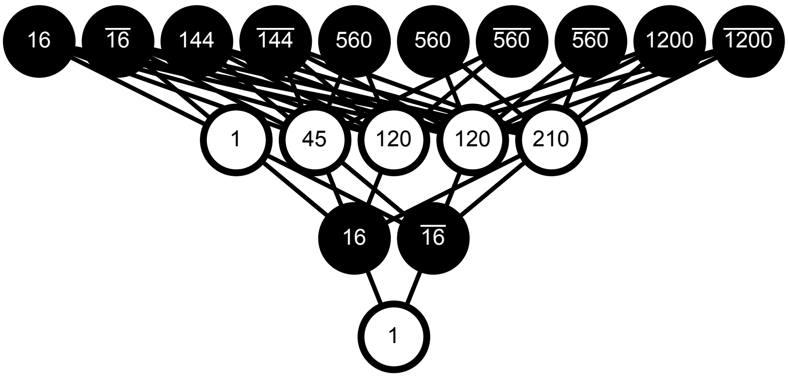


Figure 5.4: Adinkra Diagram for 10D, $\mathcal{N} = 2\text{B}$ At Low Orders

Chapter 6

Advening to Adynkrafields: Young Tableaux to Component Fields

This chapter is based on [113].

In this chapter, we will continue the discussions in Chapter 4 and illustrate how branching rules help us uncover more information for superfields as well as inspire us to construct a new formalism in which θ-monomials are replaced by Young Tableaux. We will start with the 11D scalar superfield result and define adynkra genomes and adynkrafields. Gauge group discussions are also included. 10D $\mathcal{N} = 1$ adynkra genome and the scalar adynkrafield will be presented. With all of these machineries, we will end this chapter with the complete discussion in 4D, $\mathcal{N} = 1$ superspace.

6.1 Young Tableaux & the ℓ-forms

In this section, we choose the example of the eleven-dimensional (i.e. D $=$ 11) scalar superfield at low orders to illustrate the discussion in Section 4.3. We now wish to illustrate these points by use of Young Tableaux, whose utilization in discussions of representations of compact Lie Algebras is a common and well-understood occurrence.

Starting with the result shown in Eq. (4.3.1), the expression can be written to appear as

$$\begin{aligned}\mathcal{V}(x^{\underline{a}},\,\theta^{\alpha}) \;=\; & v^{(0)}(x) \;+\; \theta^{\alpha}\, v^{(1)}_{\alpha}(x) \;+\; \theta^{\alpha}\,\theta^{\beta}\; v^{(2)}_{\alpha\beta}(x) \\ & +\; \theta^{\alpha}\,\theta^{\beta}\,\theta^{\gamma}\, v^{(3)}_{\alpha\beta\gamma}(x) \;+\; \theta^{\alpha}\,\theta^{\beta}\,\theta^{\gamma}\;\theta^{\delta} v^{(4)}_{\alpha\beta\gamma\delta}(x) \;+\; \dots\end{aligned} \tag{6.1.1}$$

and here the superscript on $x^{\underline{a}}$ takes on values $\underline{a}$ = 0, 1, . . . , 10 and the superscript on θ^α takes on values α = 1, . . . , 32. The discussion in the previous chapters informs us that the respective coefficients that precede the functions $v_\alpha^{(1)}(x)$, $v_{\alpha\beta}^{(2)}(x)$, $v_{\alpha\beta\gamma}^{(3)}(x)$, and $v_{\alpha\beta\gamma\delta}^{(4)}(x)$ should be 'mapped' onto the respective 1-forms 2-forms, 3-forms, and 4-forms in the $\mathcal{A}_{31}$ (i.e. $\mathfrak{su}(32)$) Lie algebra. Thus we have

$$v_\alpha^{(2)}(x) \to \ydiagram{1} \quad , \quad v_{\alpha\beta}^{(2)}(x) \to \ydiagram{1,1} \quad , \quad v_{\alpha\beta\gamma}^{(3)}(x) \to \ydiagram{1,1,1} \quad , \quad v_{\alpha\beta\gamma\delta}^{(4)}(x) \to \ydiagram{1,1,1,1} \tag{6.1.2}$$

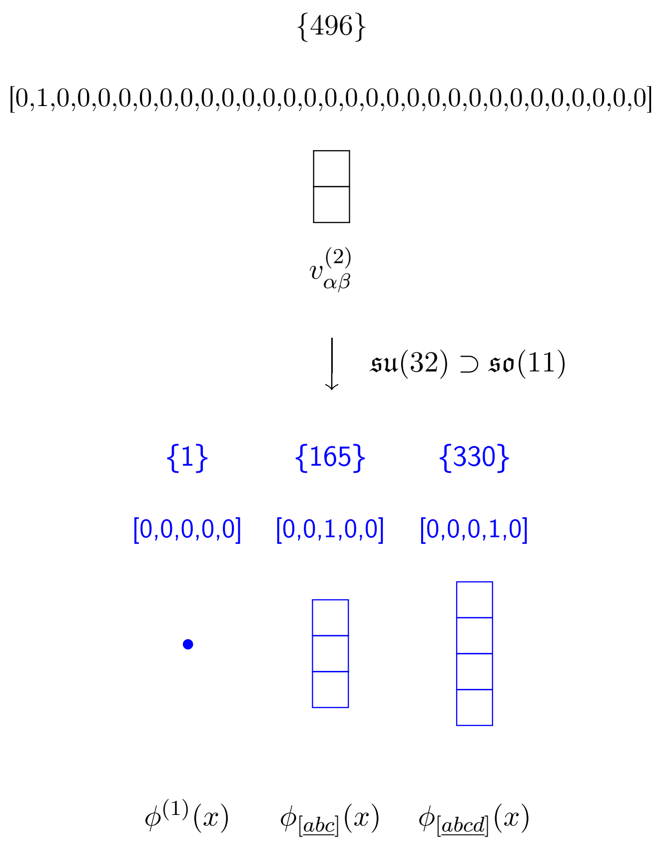
associated with the θ-dependent terms in Eq.(6.1.1). The diagrams in Eq. (6.1.2) correspond to the $\{32\}$, $\{496\}$, $\{4,960\}$, and $\{35,960\}$ representations respectively as seen by use of the Hook's Length formula applied to $\mathfrak{su}(32)$ Young Tableaux. The discussion surrounding Eq.(4.3.2) - Eq.(4.3.5) implied that the $\mathfrak{su}(32)$ Young Tableaux each need to be injected into $\mathcal{B}_5$ Young Tableaux. Analytically, as was described in the works of [57–59,62], this is done by translating the Young Tableaux into their associated Dynkin Labels and then using a projection matrix. Thus, we have demonstrated a mathematically well-grounded way to carry out the injections. However, it is simpler for purposes of presentation to describe this process graphically. For this, we need to introduce two additional types of Young Tableaux.

For the fermionic representations of $\mathcal{B}_5$ Young Tableaux, we will use tableaux that are red in color, i.e. $\ydiagram{1}$, etc.. For the bosonic representations of $\mathcal{B}_5$ Young Tableaux, we will use tableaux that are blue in color, i.e. $\ydiagram{1}$, etc.. As presaged in Eq.(4.3.2) and Eq.(4.3.3), we also adhere to a convention where $\{\mathcal{R}\}$ (with $\mathcal{R}$ a specified number) describes fermionic representations of $\mathcal{B}_5$. In a similar fashion we adhere to a convention where $\{\mathcal{R}\}$ (with $\mathcal{R}$ a specified number) describes bosonic representations of $\mathcal{B}_5$. We begin by graphically discussing the $\{32\}$ irrep below.

$$
\begin{array}{c}
\{32\} \\[1em]
[1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0] \\[1em]
\square \\[1em]
v_{\alpha}^{(1)} \\[1em]
\Big\downarrow \quad \mathfrak{su}(32) \supset \mathfrak{so}(11) \\[1em]
\{32\} \\[1em]
[0,0,0,0,1] \\[1em]
\square \\[1em]
\varphi_{\alpha}(x)
\end{array}
\tag{6.1.3}
$$

At the topmost level in the diagram associated with Eq. (6.1.3) there is shown the $\mathfrak{su}(32)$ dimensionality of the representation $\{32\}$ followed immediately below by its Dynkin Label, below that the corresponding Young Tableau, and finally the affiliated 11D component field variable. The injection is shown by the downward pointing arrow. Under that the same information (and in the same order) as was given for the $\mathfrak{su}(32)$ irrep is given for the $\mathfrak{so}(11)$ irrep.

Next the $\{496\}$ $\mathfrak{su}(32)$-irrep is treated in exactly the same way and order as shown in the image associated Eq. (6.1.4).



(6.1.4)

The obvious different feature that emerges from this injection is that there are multiple $\mathfrak{so}(11)$ irreps that appear at the endpoint. The singlet representation $\{1\}$ along with a 3-form $\{165\}$ irrep, and a 4-form $\{330\}$ irrep all occur at the $\ell = 2$ level.

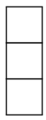
One point of convenience to note about this approach is that the 11D spacetime indices on the bosonic fields transparently appear. This is due to the use of irreducible Young Tableaux or equivalently the Dynkin Labels. As form-fields are always irreducible, there is no need to introduce further notational devices to indicate this information. As will be apparent in the next two examples, this is not always the case. Whenever the Young Tableaux associated with a field variable, arising from the injection processes, possesses multiple columns (whether blue or red) we will use a decomposition into irreps. This will be indicated by adding a subscript with the letters IR next to the Young Tableaux. To further indicate the irreducibility of field variables, we include a 'hat' over the symbol for the field index.

One other point to note is that the injection of the $\mathcal{A}_{31}$ two-form in eq.(6.1.4) induces an isomorphism to the direct sum of the three Young Tableaux that describe $\mathfrak{so}(11)$ bosonic representations. This isomorphism holds for the Dynkin Labels of course. However, it is also the origin of the 'Fierz Identities' that are ubiquitously used in the θ-expansion of superfields. In terms of the creation of algorithms, this is the source of a tremendous computational saving in deriving component-level results.

We next below consider the case of the $\mathcal{A}_{31}$ three-form in eq.(6.1.5) that describes $\mathfrak{so}(11)$ fermionic representations.

$$\{4{,}960\}$$

$$[0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$

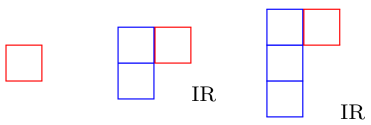


$$v^{(3)}_{\alpha\beta\gamma}$$

$$\Big\downarrow \ \mathfrak{su}(32) \supset \mathfrak{so}(11) \tag{6.1.5}$$

$$\{32\} \qquad \{1{,}408\} \qquad \{3{,}520\}$$

$$[0,0,0,0,1] \qquad [0,1,0,0,1] \qquad [0,0,1,0,1]$$



$$\widehat{\varphi}^{(1)}_{\alpha}(x) \qquad \widehat{\varphi}_{[\underline{ab}]\,\alpha}(x) \qquad \widehat{\varphi}_{[\underline{abc}]\,\alpha}(x)$$

There are several features that become apparent here.

First, when odd powers of the $\mathcal{A}_{31}$ $\square$ tableaux are used, the resulting $\mathfrak{so}(11)$ fermionic representations can be expressed using a single $\square$ tableau together with some number of $\square$ tableaux. The reason for this is that even powers of the $\mathcal{A}_{31}$ $\square$ tableaux can always be expressed totally in terms of even numbers of $\square$ tableaux via use of the "Fierz Identity isomorphisms."

A second new feature is the appearance of the IR subscript which implies that only the irreducible parts of the associated Young Tableaux are to be included. This means for the field variables, the conditions

$$(\gamma^{\underline{a}})_{\beta}{}^{\alpha}\, \widehat{\varphi}_{[\underline{ab}]\,\alpha}(x) ~=~ 0 \quad , \quad (\gamma^{\underline{a}})_{\beta}{}^{\alpha}\, \widehat{\varphi}_{[\underline{abc}]\,\alpha}(x) ~=~ 0 \quad , \tag{6.1.6}$$

are imposed. Physicists sometimes refer to this process as "pulling out gamma-traces." These conditions also imply the additional conditions

$$(\gamma^{\underline{a}}\gamma^{\underline{b}})_{\beta}{}^{\alpha}\, \widehat{\varphi}_{[\underline{ab}]\,\alpha}(x) ~=~ 0 \quad , \quad (\gamma^{\underline{a}}\gamma^{\underline{b}})_{\beta}{}^{\alpha}\, \widehat{\varphi}_{[\underline{abc}]\,\alpha}(x) ~=~ 0 \quad , \quad (\gamma^{\underline{a}}\gamma^{\underline{b}}\gamma^{\underline{c}})_{\beta}{}^{\alpha}\, \widehat{\varphi}_{[\underline{abc}]\,\alpha}(x) ~=~ 0 \quad , \tag{6.1.7}$$

must be satisfied. Due to this, there is another description available for these fields. One can introduce a fermionic field variable

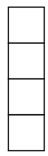
$$\Psi_{[\underline{abc}]\,\alpha}(x) ~=~ (\gamma_{[\,\underline{a}}\gamma_{\underline{b}}\gamma_{\underline{c}\,]})_{\alpha}{}^{\beta}\, \widehat{\varphi}_{\beta}(x) ~+~ (\gamma_{[\,\underline{a}|})_{\alpha}{}^{\beta}\, \widehat{\varphi}_{|\underline{b}\,\underline{c}\,]\,\beta}(x) ~+~ \widehat{\varphi}_{[\underline{abc}]\,\alpha}(x) \quad , \tag{6.1.8}$$

that does not satisfy the conditions in Eq.(6.1.6) nor in Eq.(6.1.7).

In equation (6.1.9) we present an explicit example of the level-4 component fields of an 11D unconstrained scalar superfield. The antisymmetric product of four θ's is represented by a Young tableau of a column of 4 boxes in $\mathfrak{su}(32)$, and the Lorentz irreducible representations can be decomposed via the branching rule $\mathfrak{su}(32) \supset \mathfrak{so}(11)$.

$$\{35{,}960\}$$

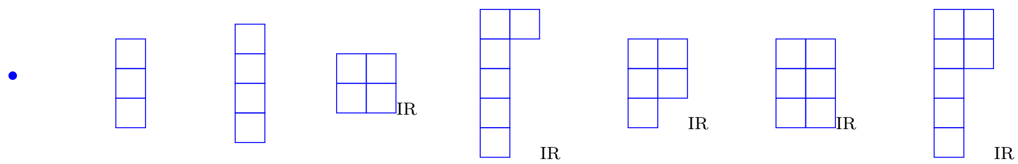
$$[0,0,0,1,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0,0]$$



$$v^{(4)}_{\alpha\beta\gamma\delta}$$

$$\Big\downarrow \;\; \mathfrak{su}(32) \supset \mathfrak{so}(11)$$

{1}	{165}	{330}	{1,144}	{4,290}	{5,005}	{7,865}	{17,160}
[0,0,0,0,0]	[0,0,1,0,0]	[0,0,0,1,0]	[0,2,0,0,0]	[1,0,0,0,2]	[0,1,1,0,0]	[0,0,2,0,0]	[0,1,0,0,2]



$$\phi^{(2)}(x) \quad \phi^{(1)}_{[\underline{abc}]}(x) \quad \phi^{(1)}_{[\underline{abcd}]}(x) \quad \widehat{\phi}_{\{\underline{ab},\underline{cd}\}}(x) \quad \widehat{\phi}_{\{\underline{a}|\underline{bcdef}\}}(x) \quad \widehat{\phi}_{\{\underline{ab}|\underline{cde}\}}(x) \quad \widehat{\phi}_{\{\underline{abc},\underline{def}\}}(x) \quad \widehat{\phi}_{\{\underline{ab}|\underline{cdefg}\}}(x) \tag{6.1.9}$$

It is seen that the {1}, {165}, and {330} Young Tableaux once more correspond to 1-form, 3-form, and 5-form field variables and as such are irreps of $\mathfrak{so}(11)$ where the number of 'boxes' in the Young Tableaux correspond to the number of '11D vector' indices on fields. Since the single-column Young Tableaux is by definition totally antisymmetric, so are the indices on the corresponding field variables.

The {1, 144} Young Tableaux is the lowest dimension irrep that does not correspond to a form. From the corresponding Young Tableaux, it can be concluded that the related field variable $\widehat{\phi}_{\{\underline{ab},\underline{cd}\}}(x)$ possesses the following symmetries under pairwise exchanges of indices:

$$\begin{array}{ccccccccc} \widehat{\phi}_{\{\underline{ab},\underline{cd}\}} &=& -\,\widehat{\phi}_{\{\underline{ba},\underline{cd}\}} &,& \widehat{\phi}_{\{\underline{ab},\underline{cd}\}} &=& -\,\widehat{\phi}_{\{\underline{ab},\underline{dc}\}} &,\\ \widehat{\phi}_{\{\underline{ab},\underline{cd}\}} &=& +\,\widehat{\phi}_{\{\underline{cb},\underline{ad}\}} &,& \widehat{\phi}_{\{\underline{ab},\underline{cd}\}} &=& +\,\widehat{\phi}_{\{\underline{ab},\underline{dc}\}} &,\\ \widehat{\phi}_{\{\underline{db},\underline{ca}\}} &=& +\,\widehat{\phi}_{\{\underline{cb},\underline{ad}\}} &,& \widehat{\phi}_{\{\underline{ab},\underline{cd}\}} &=& +\,\widehat{\phi}_{\{\underline{ac},\underline{bd}\}} &. \end{array} \tag{6.1.10}$$

The subscript IR additionally indicates that there are also a set of tracelessness conditions imposed (this is the meaning of "pulling out traces"). These are given by

$$\eta^{\underline{a}\underline{c}}\,\widehat{\phi}_{\{\underline{a}\underline{b},\underline{c}\underline{d}\}} = \eta^{\underline{a}\underline{d}}\,\widehat{\phi}_{\{\underline{a}\underline{b},\underline{c}\underline{d}\}} = \eta^{\underline{b}\underline{c}}\,\widehat{\phi}_{\{\underline{a}\underline{b},\underline{c}\underline{d}\}} = \eta^{\underline{b}\underline{d}}\,\widehat{\phi}_{\{\underline{a}\underline{b},\underline{c}\underline{d}\}} = \eta^{\underline{a}\underline{c}}\,\eta^{\underline{b}\underline{d}}\,\widehat{\phi}_{\{\underline{a}\underline{b},\underline{c}\underline{d}\}} = 0 \quad . \tag{6.1.11}$$

In all the results of Eq.(6.1.3), Eq.(6.1.4), Eq.(6.1.5), and Eq.(6.1.9), the space-time index structures of the field variables are transparently obtainable from the knowledge of the Young Tableaux (or alternately the Dynkin Labels) as shown by the extension of the reasoning that leads to the results in Eq. (6.1.10) and (6.1.11).

As the index structure of the component fields can become quite baroque and intricate, the works [57–59, 62] have supported a proposal that the 'index notation' problem can be most efficiently solved by using the Dynkin Labels as field indices. Thus, the fields in Eq.(6.1.3) through Eq.(6.1.9) can be indicated by the respective notational conventions

$$\varphi_{[0,0,0,0,1]}(x) \quad , \tag{6.1.12}$$

$$\phi_{[1,0,0,0,0]}(x) \quad , \quad \phi_{[0,0,1,0,0]}(x) \quad , \quad \phi_{[0,0,0,1,0]}(x) \quad , \tag{6.1.13}$$

$$\varphi_{[0,0,0,0,1]}(x) \quad , \quad \varphi_{[0,1,0,0,1]}(x) \quad , \quad \varphi_{[0,0,1,0,1]}(x) \quad , \tag{6.1.14}$$

$$\begin{gathered} \phi_{[1,0,0,0,0]}(x) \quad , \quad \phi_{[0,0,1,0,0]}(x) \quad , \quad \phi_{[0,0,0,1,0]}(x) \quad , \quad \phi_{[0,2,0,0,0]}(x) \quad , \\ \phi_{[1,0,0,0,2]}(x) \quad , \quad \phi_{[0,1,1,0,0]}(x) \quad , \quad \phi_{[0,0,2,0,0]}(x) \quad , \quad \phi_{[0,1,0,0,2]}(x) \quad . \end{gathered} \tag{6.1.15}$$

It is obvious that the use of Dynkin Labels has two major advantages over other choices:

(a.) there is a uniformity and simplicity in comparison to the use of another symbols for the field indices, and

(b.) these advantages are significantly present even over the standard space-time spinor and vector indices.

6.1.1 Young Tableaux Wedge Product & Adynkra Genomes

The presentation in chapter 4 that led to its major results began with Eq.(4.3.1), an expression for a Salam-Strathdee superfield. Our re-interpretation of the Taylor Series expansion as a problem in the representation theory of Lie Algebras was the key realization that led to the ability to identify the 4,294,967,296 degrees of freedom of the eleven-dimensional scalar superfield contained among its component fields with well-defined properties with respect to the Lorentz Group.

This raises the question of whether there exists a purely representation theory argument that explains the levels of the eleven-dimensional scalar superfield itself? We have concluded the answer to this question is yes and now turn to present our reasoning.

This discussion starts with a short review of the Littlewood-Richardson rule [70,123–125] that describes the multiplication of Young Tableaux. We will discuss this in the context of the results shown in Eq.(6.1.4), Eq.(6.1.5), and Eq.(6.1.9). Adhering to our convention, the fundamental $\mathcal{A}_{31}$ representation is described by the symbol $\{32\}$. The Littlewood-Richardson rule then implies

$$\begin{aligned}
\{32\} \otimes \{32\} &= \{496\} \oplus \{528\} \quad , \\
\{32\} \otimes \{32\} \otimes \{32\} &= \{5,984\} \oplus (2)\{10,912\} \oplus \{4,960\} \quad , \\
\{32\} \otimes \{32\} \otimes \{32\} \otimes \{32\} &= \{52,360\} \oplus (3)\{139,128\} \oplus (2)\{87,296\} \\
&\quad \oplus (3)\{122,760\} \oplus \{35,960\} \quad ,
\end{aligned} \tag{6.1.16}$$

The ‘smallest’ irrep at the right hand side of each line is a form, i.e. corresponds to a Young Tableaux that possesses a single column. This suggests the existence of an alternative to the Littlewood-Richardson rule that we will denote by the ‘wedge’ symbol $\wedge$ with the property

$$\begin{aligned}
\{32\} \wedge \{32\} &= \{496\} \quad , \\
\{32\} \wedge \{32\} \wedge \{32\} &= \{4,960\} \quad , \\
\{32\} \wedge \{32\} \wedge \{32\} \wedge \{32\} &= \{35,960\} \quad .
\end{aligned} \tag{6.1.17}$$

Once one becomes accepting of the operation $\wedge$, then it is natural to define an operator $\mathcal{G}$ via the equation

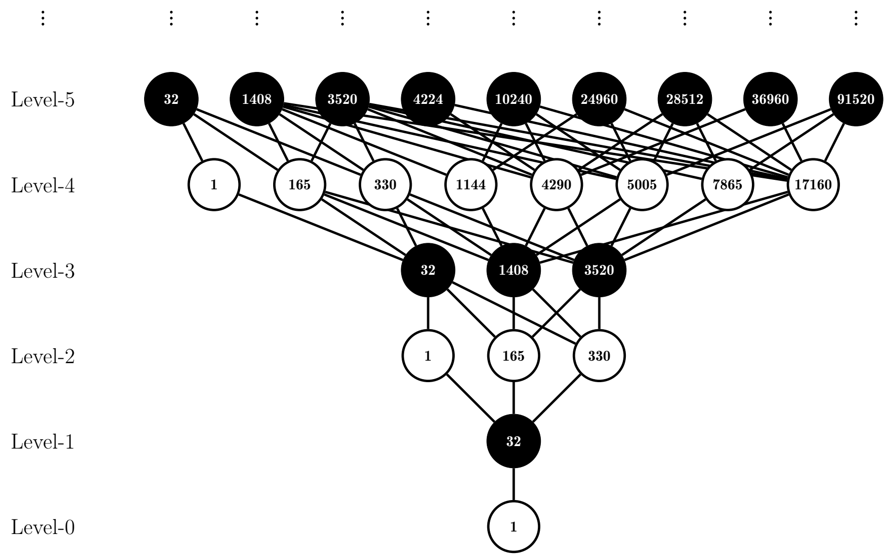
$$\mathcal{G}(\ell,\, \{32\}) \;=\; \exp_{\wedge}\left[\,\ell\,\{32\}\,\right] \supset \mathcal{B}_5 \tag{6.1.18}$$

where the definition of $exp_{\wedge}$ implies the "$\wedge$-multiplication" is to be used in the expansion of the exponential. More generally for the cases of $\mathcal{A}_{d-1}$ this definition is replaced by

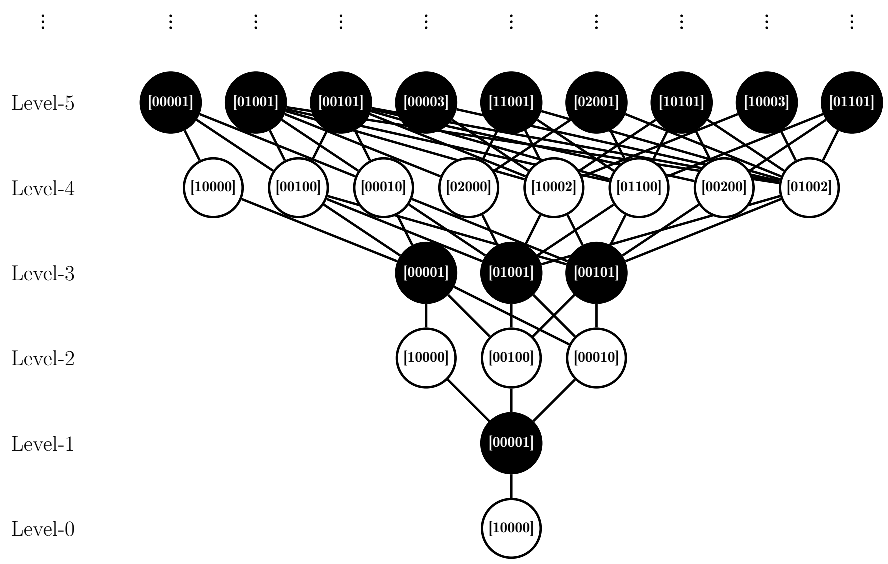
$$\mathcal{G}(\ell,\, \{_{Fund}(\mathcal{A}_{d-1})\}) \;=\; \exp_{\wedge}\left[\,\ell\,\{_{Fund}(\mathcal{A}_{d-1})\}\,\right] \;\supset \mathcal{B}\,\text{or}\,\mathcal{D} \tag{6.1.19}$$

where the choice of $\mathcal{B}$ or $\mathcal{D}$ is determined by the middle column of Table 3. Here the quantity ℓ can be interpreted as a parameter that keeps track of the order in the expansion at which the different representations occur. We have introduced the name 'adynkra genomes' for the mathematical objects in Eq.(6.1.19) that utilize Dynkin Labels to specify the fundamental $\mathcal{A}_{d-1}$ representations. When this is done, Klimyk's formula [126, 127] for multiplying Dynkin Labels is the replacement for the Littlewood-Richardson rule. This can all be shown in terms of graphs.

When the orbits of the supersymmetry generators are included in such graphs, we call these adynkras. By this means, we have created a definition of the expansion required by the Salam-Strathdee definition of superfields and one that is solely dependent on concepts that arise from representation theory. Collections of the representation can be graphically shown according to the power of ℓ at which they occur in the expansion. An adinkra using the dimensions of representations is shown in Eq.(6.1.20) and an adynkra is shown in Eq.(6.1.21)



(6.1.20)



(6.1.21)

The bottom line is that the graph shown in Eq.(6.1.21) contains precisely the same data as the expansion in Eq.(6.1.1). However, the former contains this data in a way that is superior for the construction of algorithms and codes that can be used to probe the data. Hence, transparency is created into the component field content structure of higher dimensional superfields by the introduction of adynkras, [57] - [59][1].

Finally, although the bulk of this section deals with scalar superfields, when an adynkra genome is left multiplied by any Dynkin Label, the expansion leads to the component field content for a superfield that itself is a non-trivial representation of the D-dimensional Lorentz Group.

6.2 Branching rule realizations for the off-shell & on-shell counting

In this section, we will discuss a proposal on how to consider the gauge transformations of component fields via branching rules. This is inspired by the general derivation of degrees of freedom (d.o.f.) of a s-form in D dimensions. For massless case (we only consider massless case here), off-shell d.o.f. is obtained by subtracting the gauge parameter contributions. On-shell d.o.f. is obtained by further subtracting the equation of motion contributions. One can easily check the following statement: for a massless s-form in D dimensions,

$$\begin{aligned} \text{off-shell d.o.f.} &= \begin{pmatrix} D-1 \\ s \end{pmatrix} = \frac{(D-1)!}{s!(D-1-s)!} ~, \\ \text{on-shell d.o.f.} &= \begin{pmatrix} D-2 \\ s \end{pmatrix} = \frac{(D-2)!}{s!(D-2-s)!} ~. \end{aligned} \tag{6.2.1}$$

This general fact inspires us to come up with the following proposal.

Proposal: For a component field in 11 dimensions described by the irreducible Young Tableau (equivalently the Dynkin label), its off-shell d.o.f. is equal to the dimension of the irrep in SO(10) which has the same shape of irreducible Young Tableau; its on-shell d.o.f. is equal to the dimension of the irrep in SO(9) which has the same shape of irreducible Young Tableau. The branching rules of $\mathfrak{so}(11) \supset \mathfrak{so}(10)$ convey the splitting due to the gauge redundancy.

We will illustrate this proposal with examples. Note that when the component field is s-form, the proposal collapses to (6.2.1). Moreover, we know the branching rule for

[1] Since the $\wedge$ multiplication appears in the definition of the exponentiation, we assert that no more than thirty-one such multiplications can appear and expansion thus terminates.

s-form is as below (written in terms of dimensions)

$$\left\{\begin{pmatrix} D \\ s \end{pmatrix}\right\} \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{=} \left\{\begin{pmatrix} D-1 \\ s \end{pmatrix}\right\} \oplus \left\{\begin{pmatrix} D-1 \\ s-1 \end{pmatrix}\right\} \tag{6.2.2}$$

where $\begin{pmatrix} D-1 \\ s-1 \end{pmatrix}$ is exactly the d.o.f. of the gauge parameter.

Then, let's look at the conformal graviton, conformal gravitino, and 3-form fields.

SO(11)	SO(10) (off-shell)	SO(9) (on-shell)
$h_{[2,0,0,0,0]} = \ydiagram{2}_{\rm IR} = \{65\}$	$h_{[2,0,0,0,0]} = \ydiagram{2}_{\rm IR} = \{54\}$	$h_{[2,0,0,0]} = \ydiagram{2}_{\rm IR} = \{44\}$
$\psi_{[1,0,0,0,1]} = \ydiagram{2}_{\rm IR} = \{320\}$	$\psi_{[1,0,0,0,1]} = \begin{array}{\|c\|c\|}\hline \ & 16 \\ \hline\end{array}_{\rm IR} = \{144\}$ $\psi_{[1,0,0,1,0]} = \begin{array}{\|c\|c\|}\hline \ & \overline{16} \\ \hline\end{array}_{\rm IR} = \{144\}$	$\psi_{[1,0,0,1]} = \ydiagram{2}_{\rm IR} = \{128\}$
$C_{[0,0,1,0,0]} = \ydiagram{1,1,1} = \{165\}$	$C_{[0,0,1,0,0]} = \ydiagram{1,1,1} = \{120\}$	$C_{[0,0,1,0]} = \ydiagram{1,1,1} = \{84\}$

Table 6.1: 11D Supergravity Counting

The branching rules relevant to the result presented in Table 6.1 are listed below.

$$[20000] \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{=} [20000] \oplus [10000] \oplus [00000] \tag{6.2.3}$$

$$[20000] \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{=} [2000] \oplus [1000] \oplus [0000] \tag{6.2.4}$$

$$[10001] \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{=} [10001] \oplus [10010] \oplus [00001] \oplus [00010] \tag{6.2.5}$$

$$[10001] \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{=} [1001] \oplus [0001] \tag{6.2.6}$$

$$[10010] \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{=} [1001] \oplus [0001] \tag{6.2.7}$$

$$[00100] \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{=} [00100] \oplus [01000] \tag{6.2.8}$$

$$[00100] \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{=} [0010] \oplus [0100] \tag{6.2.9}$$

where (6.2.8) is exactly (6.2.2) when $s = 3$. Look at (6.2.3), which exactly shows how the d.o.f. $h_{(\underline{a}\underline{b})}$ field splits due to the gauge redundancy. For the graviton field, we know

$$\delta\, h_{(\underline{a}\underline{b})} \;=\; \partial_{(\underline{a}}\, \xi_{\underline{b})} \qquad \xi_{\underline{b}} \;\sim\; \{11\} \tag{6.2.10}$$

Therefore, we have the following interpretation.

$$
\begin{array}{ccccc}
[00000] \oplus [20000] & \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{=} & [20000] \oplus [00000] & \oplus & [10000] \oplus [00000] \\
\{1\} \oplus \{65\} & & \{54\} \oplus \{1\} & & \{10\} \oplus \{1\} \\
h + h_{(\underline{ab})} & & \text{off-shell 11D graviton counting} & & \xi_{\underline{a}}
\end{array}
\tag{6.2.11}
$$

Let us end this section by presenting the application of this proposal to component fields occurring in the first four levels of the 11D scalar superfield. Level-0 is trivial and we will skip all $\{1\}$'s in what follows. For level-1, we have

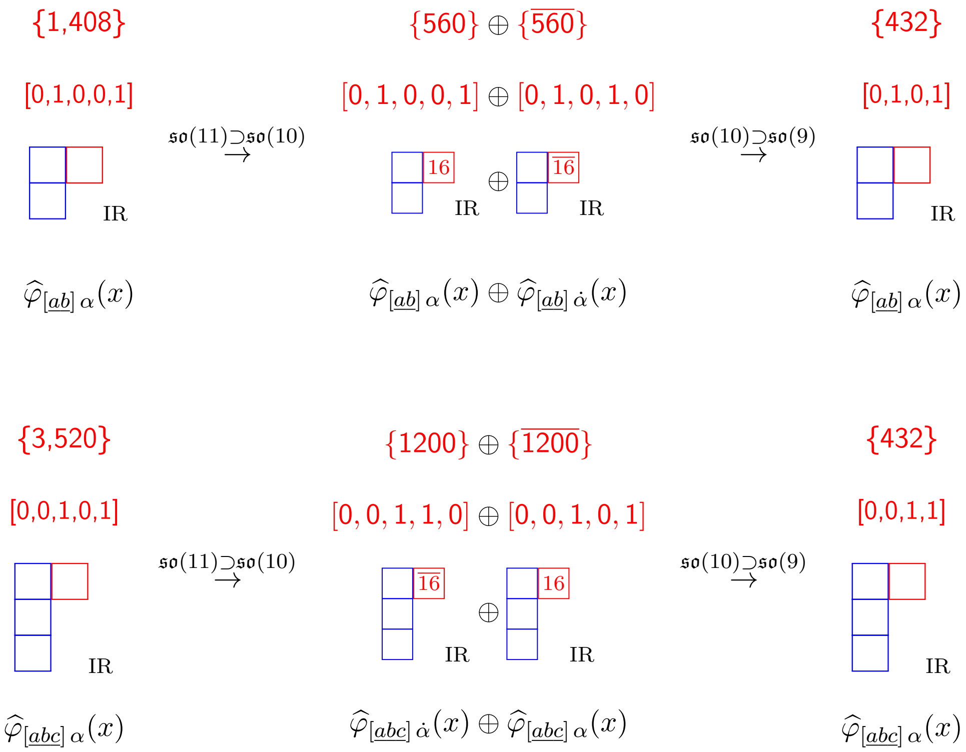
$$
\begin{array}{ccccc}
\{32\} & & \{16\} \oplus \{\overline{16}\} & & \{16\} \\
[0,0,0,0,1] & \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{\longrightarrow} & [0,0,0,0,1] \oplus [0,0,0,1,0] & \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{\longrightarrow} & [0,0,0,1] \\
\boxed{} & & \boxed{16} \oplus \boxed{\overline{16}} & & \boxed{} \\
\varphi_\alpha(x) & & \varphi_\alpha(x) \oplus \varphi_{\dot{\alpha}}(x) & & \varphi_\alpha(x)
\end{array}
\tag{6.2.12}
$$

For level-2, we have

$$
\begin{array}{ccccc}
\{165\} & & \{120\} & & \{84\} \\
[0,0,1,0,0] & \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{\longrightarrow} & [0,0,1,0,0] & \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{\longrightarrow} & [0,0,1,0] \\
\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} & & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} & & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \\
\phi_{[\underline{abc}]}(x) & & \phi_{[\underline{abc}]}(x) & & \phi_{[\underline{abc}]}(x) \\
 & & & & \\
\{330\} & & \{210\} & & \{126\} \\
[0,0,0,1,0] & \overset{\mathfrak{so}(11)\supset\mathfrak{so}(10)}{\longrightarrow} & [0,0,0,1,1] & \overset{\mathfrak{so}(10)\supset\mathfrak{so}(9)}{\longrightarrow} & [0,0,0,2] \\
\begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} & & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} & & \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \\ \hline \end{array} \\
\phi_{[\underline{abcd}]}(x) & & \phi_{[\underline{abcd}]}(x) & & \phi_{[\underline{abcd}]}(x)
\end{array}
\tag{6.2.13}
$$

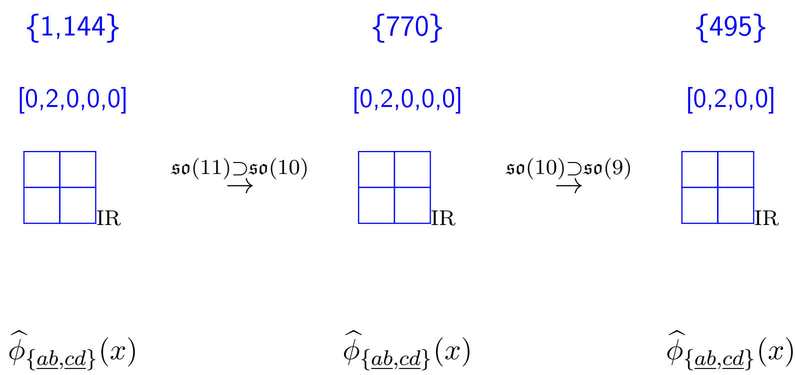
which is consistent with the counting results in (6.2.1).

For level-3, we have (since {32} has been discussed in (6.2.12), we will skip it below.)



(6.2.14)

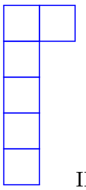
For level-4, we have (since {165} and {330} have been discussed in (6.2.13), we will skip it below.)



(6.2.15)

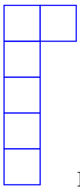
$\{4{,}290\}$ $\quad$ $\{\overline{1050}\} \oplus \{1050\}$ $\quad$ $\{924\}$

$[1,0,0,0,2]$ $\quad$ $[1,0,0,0,2] \oplus [1,0,0,2,0]$ $\quad$ $[1,0,0,2]$



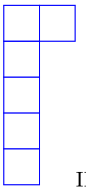
IR

$\xrightarrow{\mathfrak{so}(11)\supset\mathfrak{so}(10)}$



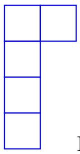
IR,+

$\oplus$



IR,−

$\xrightarrow{\mathfrak{so}(10)\supset\mathfrak{so}(9)}$

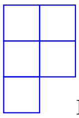


IR

$$\widehat{\phi}_{\{\underline{a}|\underline{bcdef}\}}(x) \qquad \widehat{\phi}_{\{\underline{a}|\underline{bcdef}\}^+}(x) \oplus \widehat{\phi}_{\{\underline{a}|\underline{bcdef}\}^-}(x) \qquad \widehat{\phi}_{\{\underline{a}|\underline{bcde}\}}(x) \tag{6.2.16}$$

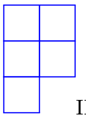
$\{5{,}005\}$ $\quad$ $\{2{,}970\}$ $\quad$ $\{1{,}650\}$

$[0,1,1,0,0]$ $\quad$ $[0,1,1,0,0]$ $\quad$ $[0,1,1,0]$



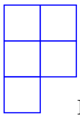
IR

$\xrightarrow{\mathfrak{so}(11)\supset\mathfrak{so}(10)}$



IR

$\xrightarrow{\mathfrak{so}(10)\supset\mathfrak{so}(9)}$

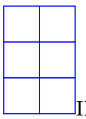


IR

$$\widehat{\phi}_{\{\underline{ab}|\underline{cde}\}}(x) \qquad \widehat{\phi}_{\{\underline{ab}|\underline{cde}\}}(x) \qquad \widehat{\phi}_{\{\underline{ab}|\underline{cde}\}}(x) \tag{6.2.17}$$

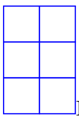
$\{7{,}865\}$ $\quad$ $\{4{,}125\}$ $\quad$ $\{1{,}980\}$

$[0,0,2,0,0]$ $\quad$ $[0,0,2,0,0]$ $\quad$ $[0,0,2,0]$



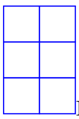
IR

$\xrightarrow{\mathfrak{so}(11)\supset\mathfrak{so}(10)}$



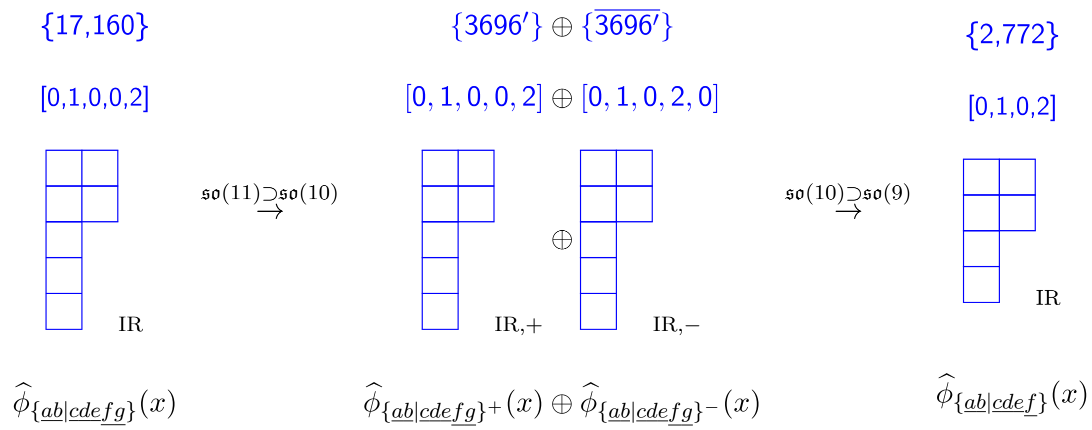
IR

$\xrightarrow{\mathfrak{so}(10)\supset\mathfrak{so}(9)}$



IR

$$\widehat{\phi}_{\{\underline{abc},\underline{def}\}}(x) \qquad \widehat{\phi}_{\{\underline{abc},\underline{def}\}}(x) \qquad \widehat{\phi}_{\{\underline{abc},\underline{def}\}}(x) \tag{6.2.18}$$



(6.2.19)

6.3 10D, $\mathcal{N} = 1$ Discussions

As stated in the previous section, the Lie Algebra realizations suggest that there exists a pure representation theory argument that explains the levels of the eleven-dimensional scalar superfield. This pushes us to think about how to construct a new formalism that is sorely based on representation theory arguments. Since the 11D scalar superfield is already a very huge system containing 4,294,967,296 degrees of freedom, we will look back to 10D, $\mathcal{N} = 1$ scalar superfield result shown in Fig. 5.2.

6.3.1 From Superfields to Adynkrafields

In [59], we constructed a one-to-one map between Dynkin Labels, irreducible bonsonic/spinorial Young Tableaux, and field variables in 10D. By applying the idea discussed in the previous section, we define a new operator, called the 10D $\mathcal{N} = 1$ adynkra genome, as

$$
\begin{aligned}
\widetilde{\mathcal{G}} = \cdot \;&\oplus\; \ell\,\boxed{16} \;\oplus\; \tfrac{1}{2}\,(\ell)^2\,\ydiagram{1,1,1}_{\rm IR} \;\oplus\; \tfrac{1}{3!}\,(\ell)^3\,\ytableaushort{{}{16},{}}_{\rm IR} \;\oplus\; \tfrac{1}{4!}\,(\ell)^4\,\ydiagram{2,2}_{\rm IR} \;\oplus\; \tfrac{1}{4!}\,(\ell)^4\,\ydiagram{2,1,1,1,1}_{{\rm IR},-} \\
&\oplus\; \tfrac{1}{5!}\,(\ell)^5\,\ytableaushort{{}{16},{},{},{},{}}_{{\rm IR},-} \;\oplus\; \tfrac{1}{5!}\,(\ell)^5\,\ytableaushort{{}{}{16},{}}_{\rm IR} \;\oplus\; \tfrac{1}{6!}\,(\ell)^6\,\ydiagram{2,2,1,1,1}_{{\rm IR},-} \;\oplus\; \tfrac{1}{6!}\,({\rm h})^6\,\ydiagram{3,1,1}_{\rm IR} \\
&\oplus\; \tfrac{1}{7!}\,(\ell)^7\,\ytableaushort{{}{}{}{16}}_{\rm IR} \;\oplus\; \tfrac{1}{7!}\,(\ell)^7\,\ytableaushort{{}{}{16},{},{}}_{\rm IR} \\
&\oplus\; \tfrac{1}{8!}\,(\ell)^8\,\ydiagram{4}_{\rm IR} \;\oplus\; \tfrac{1}{8!}\,(\ell)^8\,\ydiagram{2,2,2}_{\rm IR} \;\oplus\; \tfrac{1}{8!}\,(\ell)^8\,\ydiagram{3,1,1,1}_{\rm IR} \\
&\oplus\; \tfrac{1}{9!}\,(\ell)^9\,\ytableaushort{{}{}{}{16}}_{\rm IR} \;\oplus\; \tfrac{1}{9!}\,(\ell)^9\,\ytableaushort{{}{}{16},{},{}}_{\rm IR} \\
&\oplus\; \tfrac{1}{10!}\,(\ell)^{10}\,\ydiagram{2,2,1,1,1}_{{\rm IR},+} \;\oplus\; \tfrac{1}{10!}\,(\ell)^{10}\,\ydiagram{3,1,1}_{\rm IR} \;\oplus\; \tfrac{1}{11!}\,(\ell)^{11}\,\ytableaushort{{}{16},{},{},{},{}}_{{\rm IR},+} \;\oplus\; \tfrac{1}{11!}\,(\ell)^{11}\,\ytableaushort{{}{}{16},{}}_{\rm IR} \\
&\oplus\; \tfrac{1}{12!}\,(\ell)^{12}\,\ydiagram{2,2}_{\rm IR} \;\oplus\; \tfrac{1}{12!}\,(\ell)^{12}\,\ydiagram{2,1,1,1,1}_{{\rm IR},+} \\
&\oplus\; \tfrac{1}{13!}\,(\ell)^{13}\,\ytableaushort{{}{16},{}}_{\rm IR} \;\oplus\; \tfrac{1}{14!}\,(\ell)^{14}\,\ydiagram{1,1,1}_{\rm IR} \;\oplus\; \tfrac{1}{15!}\,(\ell)^{15}\,\boxed{16} \;\oplus\; \tfrac{1}{16!}\,(\ell)^{16}\,\cdot \qquad .
\end{aligned}
\tag{6.3.1}
$$

Attaching the 10D, $\mathcal{N}=1$ adynkra genome (6.3.1 with all component fields, we define the 10D, $\mathcal{N}=1$ adynkrafield as

$$
\begin{aligned}
\widehat{\mathcal{G}}(x) \;=\;& \Phi(x) \;+\; \ell\,\boxed{16}\,\Psi_{\alpha}(x) \;+\; \tfrac{1}{2}\,(\ell)^2\,\ydiagram{1,1,1}_{\rm IR}\;\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \;+\; \tfrac{1}{3!}\,(\ell)^3\,\ytableaushort{{}{16},{}}_{\rm IR}\;\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^{\alpha}(x) \\
&+\; \tfrac{1}{4!}\,(\ell)^4\,\ydiagram{2,2}_{\rm IR}\;\Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \;+\; \tfrac{1}{4!}\,(\ell)^4\,\ydiagram{2,1,1,1,1}_{{\rm IR},-}\;\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \\
&+\; \tfrac{1}{5!}\,(\ell)^5\,\ytableaushort{{}{16},{},{},{},{}}_{{\rm IR},-}\;\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^{\alpha}(x) \;+\; \tfrac{1}{5!}\,(\ell)^5\,\ytableaushort{{}{}{16},{}}_{\rm IR}\;\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^{\alpha}(x) \\
&+\; \tfrac{1}{6!}\,(\ell)^6\,\ydiagram{2,2,1,1,1}_{{\rm IR},-}\;\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(x) \;+\; \tfrac{1}{6!}\,(\ell)^6\,\ydiagram{3,1,1}_{\rm IR}\;\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x)
\end{aligned}
$$

$$
\begin{aligned}
&+ \tfrac{1}{7!}\,(\ell)^7 \begin{ytableau} \, & \, & \, & 16 \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(x) \;+\; \tfrac{1}{7!}\,(\ell)^7 \begin{ytableau} \, & \, & 16 \\ \, \\ \, \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\alpha}(x) \\
&+ \tfrac{1}{8!}\,(\ell)^8 \ydiagram{4}_{\,\rm IR}\, \Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x) \;+\; \tfrac{1}{8!}\,(\ell)^8 \ydiagram{2,2,2}_{\,\rm IR}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(x) \\
&+ \tfrac{1}{8!}\,(\ell)^8 \ydiagram{3,1,1,1}_{\,\rm IR}\, \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(x) \\
&+ \tfrac{1}{9!}\,(\ell)^9 \begin{ytableau} \, & \, & \, & 16 \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(x) \;+\; \tfrac{1}{9!}\,(\ell)^9 \begin{ytableau} \, & \, & 16 \\ \, \\ \, \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(x) \\
&+ \tfrac{1}{10!}\,(\ell)^{10} \ydiagram{2,2,1,1,1}_{\,{\rm IR},+}\, \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \;+\; \tfrac{1}{10!}\,(\ell)^{10} \ydiagram{3,1,1}_{\,\rm IR}\, \widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \\
&+ \tfrac{1}{11!}\,(\ell)^{11} \begin{ytableau} \, & 16 \\ \, \\ \, \\ \, \\ \, \end{ytableau}_{\,{\rm IR},+}\, \Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(x) \;+\; \tfrac{1}{11!}\,(\ell)^{11} \begin{ytableau} \, & \, & 16 \\ \, \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(x) \\
&+ \tfrac{1}{12!}\,(\ell)^{12} \ydiagram{2,2}_{\,\rm IR}\, \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(x) \;+\; \tfrac{1}{12!}\,(\ell)^{12} \ydiagram{2,1,1,1,1}_{\,{\rm IR},+}\, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(x) \\
&+ \tfrac{1}{13!}\,(\ell)^{13} \begin{ytableau} \, & 16 \\ \, \end{ytableau}_{\,\rm IR}\, \Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(x) \;+\; \tfrac{1}{14!}\,(\ell)^{14} \ydiagram{1,1,1}_{\,\rm IR}\, \widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(x) \;+\; \tfrac{1}{15!}\,(\ell)^{15} \begin{ytableau} 16 \end{ytableau}\, \Psi^{\alpha}(x) \\
&+ \tfrac{1}{16!}\,(\ell)^{16}\, \widehat{\Phi}(x) \quad .
\end{aligned}
\tag{6.3.2}
$$

6.3.2 From Adynkrafields Back To 1D Adinkras

Moreover, we can consider projecting the 10D adynkrafield back to 1D, $\mathcal{N} = 16$ superspace. The can be done by imposing the condition that all of the field variables depend solely on a time-like coordinate τ and the imposition of the condition that $(\ell)^2 = 1$. Then we have

$$
\begin{aligned}
&\widehat{\mathcal{G}}_{Adnk}(\tau) = \\
&\quad \Big\{ \Phi(\tau) + \tfrac{1}{2} \ydiagram{1,1,1}_{\,\rm IR}\, \Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau) \;+\; \tfrac{1}{4!} \ydiagram{2,2}_{\,\rm IR}\, \Phi_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(\tau) \;+\; \tfrac{1}{4!} \ydiagram{2,1,1,1,1}_{\,{\rm IR},-}\, \Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(\tau) \\
&\quad + \tfrac{1}{6!} \ydiagram{2,2,1,1,1}_{\,{\rm IR},-}\, \Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}(\tau) \;+\; \tfrac{1}{6!} \ydiagram{3,1}_{\,\rm IR}\, \Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau)
\end{aligned}
$$

$$
\begin{aligned}
&+\tfrac{1}{8!}\,\ydiagram{4}_{\rm IR}\,\Phi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(\tau)\;+\;\tfrac{1}{8!}\,\ydiagram{2,2,2}_{\rm IR}\,\Phi_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}(\tau)\;+\;\tfrac{1}{8!}\,\ydiagram{3,1,1,1}_{\rm IR}\,\Phi_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}(\tau)\\
&+\tfrac{1}{10!}\,\ydiagram{3,1,1}_{\rm IR}\,\widehat{\Phi}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau)\;+\;\tfrac{1}{10!}\,\ydiagram{2,2,1,1,1}_{\rm IR,+}\,\Phi_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(\tau)\\
&+\tfrac{1}{12!}\,\ydiagram{2,2}_{\rm IR}\,\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}(\tau)\;+\;\tfrac{1}{12!}\,\ydiagram{2,1,1,1,1}_{\rm IR,+}\,\Phi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}(\tau)\;+\;\tfrac{1}{14!}\,\ydiagram{1,1,1}_{\rm IR}\,\widehat{\Phi}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}(\tau)\\
&+\tfrac{1}{16!}\,\widehat{\Phi}(\tau)\;\Bigg\}\;+\;\ell\,\Bigg\{\;\boxed{16}\,\Psi_\alpha(\tau)\;+\;\tfrac{1}{3!}\,\ydiagram{1,1}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_1\underline{b}_1\}}{}^{\alpha}(\tau)\;+\;\tfrac{1}{5!}\,\ydiagram{1,1,1,1,1}\,\boxed{16}_{\rm IR,-}\,\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^{\alpha}(\tau)\\
&+\tfrac{1}{5!}\,\ydiagram{2,1}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}}{}^{\alpha}(\tau)\;+\;\tfrac{1}{7!}\,\ydiagram{3}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}(\tau)\;+\;\tfrac{1}{7!}\,\ydiagram{2,1,1}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\alpha}(\tau)\\
&+\tfrac{1}{9!}\,\ydiagram{3}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}(\tau)\;+\;\tfrac{1}{9!}\,\ydiagram{2,1,1}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}(\tau)\\
&+\tfrac{1}{11!}\,\ydiagram{1,1,1,1,1}\,\boxed{16}_{\rm IR,+}\,\Psi_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}(\tau)\;+\;\tfrac{1}{11!}\,\ydiagram{2}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\}\alpha}(\tau)\;+\;\tfrac{1}{13!}\,\ydiagram{1}\,\boxed{16}_{\rm IR}\,\Psi_{\{\underline{a}_1\underline{b}_1\}\alpha}(\tau)\\
&+\tfrac{1}{15!}\,\boxed{16}\,\Psi^{\alpha}(\tau)\;\Bigg\}\quad,
\end{aligned}
\tag{6.3.3}
$$

where the factorial factors can be eliminated by field re-definitions. This 1D, N = 16 valise adinkra system clearly contains 32,768 bosons and 32,768 fermions. It also contains the information associated with the Lorentz representations (via the YT's) of the original 10D, $\mathcal{N} = 1$ scalar supermultiplet for which it is the hologram. It would be great and useful if we can reverse this procedure. Namely, we start from a 1D system and recover the information in the higher-dimensional system. This is highly related to the SUSY holography program which will be discussed more in chapter 9 later.

6.4 4D, $\mathcal{N} = 1$ Superfields Discussions

Finally, this chapter ends with an application of all machinery that we have introduced so far to a toy system – 4D, $\mathcal{N} = 1$ superfields.

6.4.1 4D, $\mathcal{N}=1$ Vector Superfield

In 4D, $\mathcal{N}=1$ superspace, a superfield with a vector index, $H_{\underline{a}}$, contains components which can be defined by the equations (below we use the conventions in *Superspace* [100]),

$$
\begin{aligned}
h_{\underline{a}} &= H_{\underline{a}}| \quad , \\
h_{\alpha\beta\dot{\beta}} &= \mathrm{D}_\beta H_{\alpha\dot{\beta}}| \quad , \\
h^{(2)}{}_{\underline{a}} = \mathrm{D}^2 H_{\underline{a}}| \quad , \quad h_{\underline{ab}} &= -\tfrac{1}{2}[\mathrm{D}_\alpha, \overline{\mathrm{D}}_{\dot{\alpha}}] H_{\beta\dot{\beta}}| \quad , \\
\overline{\psi}_{\underline{a}\dot{\beta}} &= -\, i\, \mathrm{D}^2 \overline{\mathrm{D}}_{\dot{\alpha}} H_{\alpha\dot{\beta}}| \quad , \\
A_{\underline{a}} = -\tfrac{2}{3}\mathrm{D}^\beta \overline{\mathrm{D}}^2 \mathrm{D}_\beta H_{\underline{a}}| &- \tfrac{1}{6}\,\epsilon_{\underline{abcd}}\,\partial^{\underline{b}}[\mathrm{D}^\gamma, \overline{\mathrm{D}}^{\dot{\gamma}}] H^{\underline{d}}| \quad ,
\end{aligned}
\tag{6.4.1}
$$

along with the conjugate fields $\overline{h}_{\dot{\alpha}\beta\dot{\beta}}$, $\overline{h}^{(2)}{}_{\underline{a}}$ and $\psi_{\underline{a}\beta}$.

Apart from the analytic approach of writing out the θ-expansion of the superfield and acting with a number of supercovariant derivatives on it, one can deduce the structure of the component field expansion from group theory approaches as were developed recently [57–59, 62]. Especially from the work in [62], we know that a scalar superfield in 4D has components expressed in $\mathfrak{so}(4)$ irreducible representations as follows,

$$
\mathcal{V} = \begin{cases}
\text{level} - 0: & [0,0] \quad , \\
\text{level} - 1: & [1,0] \oplus [0,1] \quad , \\
\text{level} - 2: & (2)[0,0] \oplus [1,1] \quad , \\
\text{level} - 3: & [1,0] \oplus [0,1] \quad , \\
\text{level} - 4: & [0,0] \quad ,
\end{cases}
\tag{6.4.2}
$$

where the basic bosonic and spinorial Dynkin Labels correspond to index structures as

$$
\underline{a} \equiv [1,1] \quad , \quad \alpha \equiv [1,0] \quad , \quad \dot{\alpha} \equiv [0,1] \quad ,
\tag{6.4.3}
$$

This makes sense as $\underline{a} = \alpha\dot{\alpha}$, i.e.

$$
[1,1] = [1,0] \otimes [0,1] \quad .
\tag{6.4.4}
$$

This will be further elaborated below via Young Tableau notation.

The component content of $H_{\underline{a}}$ is thus $\mathcal{V} \otimes [1,1]$. Explicitly, we have

$$H_{\underline{a}} = \begin{cases} \text{level} - 0: & [1,1] \quad , \\ \text{level} - 1: & [1,0] \oplus [0,1] \oplus [1,2] \oplus [2,1] \quad , \\ \text{level} - 2: & [0,0] \oplus [2,0] \oplus [0,2] \oplus (2)[1,1] \oplus [2,2] \quad , \\ \text{level} - 3: & [1,0] \oplus [0,1] \oplus [1,2] \oplus [2,1] \quad , \\ \text{level} - 4: & [1,1] \quad . \end{cases} \tag{6.4.5}$$

To interpret these components, we will follow the techniques developed in the papers [59] and [62], but adapted to 4D in order to translate Dynkin Labels to Young Tableaux (YT), then to index notation. Next, we compare (6.4.1) and (6.4.5).

The first step is to translate Dynkin Labels to YT notation.

As discussed in [62], in 4D, $\mathcal{N} = 1$ superspace, given a bosonic irrep with Dynkin Label $[a, b]$, its corresponding irreducible bosonic Young Tableau is composed of $\min\{a, b\}$ columns of one box and $|b - a|/2$ columns of two vertical boxes. Therefore, the fundamental building blocks of an irreducible BYT (bosonic YT) are

$$\square_{\text{IR}} \equiv [1,1] \quad , \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},+} \equiv [0,2] \quad , \quad \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},-} \equiv [2,0] \quad , \tag{6.4.6}$$

where "+" indicates self-dual 2-form, and "−" indicates anti-self-dual 2-form. A generic 2-form without specific duality constraint could be written as

$$\begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR}} \equiv \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},+} \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},-} \equiv [0,2] \oplus [2,0] \quad . \tag{6.4.7}$$

For spinorial irreps, by adopting the Weyl convention, the basic SYT (spinorial YT) are

$$\boxed{2} \equiv [1,0] \quad , \quad \boxed{\bar{2}} \equiv [0,1] \quad . \tag{6.4.8}$$

For a general spinorial irrep $[a, b]$, one follows the convention in [59]. When $b > a$, one inteprets it as $[0,1]$ "+" some bosonic Dynkin Labels; when $a > b$, one inteprets it as $[1,0]$

"+" some bosonic Dynkin Labels. This ensures conjugation preserves dimensionality. An example that illustrates this well is

$$[1,2] \;=\; [1,1] \;+\; [0,1] \;=\; \ytableaushort{{}{\bar{2}}}_{\rm IR} \quad , \tag{6.4.9}$$

$$[2,1] \;=\; [1,1] \;+\; [1,0] \;=\; \ytableaushort{{}{2}}_{\rm IR} \quad . \tag{6.4.10}$$

Therefore, the Young Tableau description (which equivalently describes the index structures of corresponding component fields), of the component field content of a 4D, $\mathcal{N} = 1$ vector superfield is as seen below.

$$H_{\underline{a}} \;=\; \begin{cases} {\rm level}-0: \; \ydiagram{1}_{\rm IR} \quad , \\ {\rm level}-1: \; \ytableaushort{2} \oplus \ytableaushort{{\bar{2}}} \oplus \ytableaushort{{}{\bar{2}}}_{\rm IR} \oplus \ytableaushort{{}{2}}_{\rm IR} \quad , \\ {\rm level}-2: \; \cdot \oplus \ydiagram{1,1}_{{\rm IR},-} \oplus \ydiagram{1,1}_{{\rm IR},+} \oplus (2)\ydiagram{1}_{\rm IR} \oplus \ydiagram{2}_{\rm IR} \quad , \\ {\rm level}-3: \; \ytableaushort{2} \oplus \ytableaushort{{\bar{2}}} \oplus \ytableaushort{{}{\bar{2}}}_{\rm IR} \oplus \ytableaushort{{}{2}}_{\rm IR} \quad , \\ {\rm level}-4: \; \ydiagram{1}_{\rm IR} \quad . \end{cases} \tag{6.4.11}$$

To interpret all of these, let us utilizes the techniques in [59]. This includes Littlewood's rule for BYT [70, 123, 128–132] or its recent variant tying rule [59], and tensor product rules between bosonic YTs and SYT developed by Murnaghan, Littlewood, King and others [124, 125, 132, 133]. The relevant tying rule is

$$\ydiagram{2} \;=\; \ydiagram{2}_{\rm IR} \oplus \cdot \quad , \tag{6.4.12}$$

while the relevant spinorial tensor product rules are

$$\ydiagram{1}_{\rm IR} \otimes \ytableaushort{2} \;=\; \ytableaushort{{}{2}}_{\rm IR} \oplus \ytableaushort{{\bar{2}}} \quad , \tag{6.4.13}$$

$$\ydiagram{1}_{\rm IR} \otimes \ytableaushort{{\bar{2}}} \;=\; \ytableaushort{{}{\bar{2}}}_{\rm IR} \oplus \ytableaushort{2} \quad . \tag{6.4.14}$$

Therefore, one could rewrite (6.4.11) as[2]

$$
H_{\underline{a}} \;=\; \begin{cases} \text{level}-0: & \Box \quad , \\ \text{level}-1: & \Box \otimes \boxed{2} \;\oplus\; \Box \otimes \boxed{\bar{2}} \quad , \\ \text{level}-2: & (2)\Box \;\oplus\; \Box \otimes \Box \quad , \\ \text{level}-3: & \Box \otimes \boxed{\bar{2}} \;\oplus\; \Box \otimes \boxed{2} \quad , \\ \text{level}-4: & \Box \quad . \end{cases} \tag{6.4.15}
$$

The second step is to translate YT to index notation. The basic ones are

$$
\underline{a} \;\equiv\; \Box_{\text{IR}} \quad , \quad \alpha \;\equiv\; \boxed{2} \quad , \quad \dot{\alpha} \;\equiv\; \boxed{\bar{2}} \quad , \tag{6.4.16}
$$

hence one obtains (6.4.3). One could therefore do the final translation,

$$
H_{\underline{a}} \;=\; \begin{cases} \text{level}-0: & h_{\underline{a}} \quad , \\ \text{level}-1: & h_{\alpha\beta\dot{\beta}} \quad , \quad \bar{h}_{\dot{\alpha}\beta\dot{\beta}} \quad , \\ \text{level}-2: & h^{(2)}{}_{\underline{a}} \quad , \quad \bar{h}^{(2)}{}_{\underline{a}} \quad , \quad h_{\underline{ab}} \quad , \\ \text{level}-3: & \bar{\psi}_{\underline{a}\dot{\beta}} \quad , \quad \psi_{\underline{a}\beta} \quad , \\ \text{level}-4: & A_{\underline{a}} \quad . \end{cases} \tag{6.4.17}
$$

This conveys the exact same content of (6.4.1). Note that some of these component fields are not irreducible. If one looks at (6.4.1), one would find that the analytic definitions do not impose any irreducible condition or symmetry on the indices. Therefore the whole story is consistent.

We end this section by noting that the identification of the graviton as a linear combination of YT

$$
h_{\underline{ab}} \sim \Box\Box_{\text{IR}} \oplus \begin{smallmatrix}\Box\\ \Box\end{smallmatrix}_{\text{IR}} \oplus \quad \oplus \;\bullet\; = \Box\Box_{\text{IR}} \oplus \begin{smallmatrix}\Box\\ \Box\end{smallmatrix}_{\text{IR},+} \oplus \begin{smallmatrix}\Box\\ \Box\end{smallmatrix}_{\text{IR},-} \oplus \;\bullet\; , \tag{6.4.18}
$$

[2] Note that $\Box = \Box_{\text{IR}}$ as it is already irreducible without putting on any irreducible conditions. Thus we omit the IR subscript for a neat presentation.

similarly for the gravitino

$$\psi_{\underline{a}\,\beta} \;\sim\; \boxed{}\boxed{2}_{\rm IR} \;\oplus\; \boxed{\bar{2}} \quad , \tag{6.4.19}$$

and finally for the axial vector auxiliary field

$$A_{\underline{a}} \;\sim\; \boxed{}_{\rm IR} \quad , \tag{6.4.20}$$

those are located at three adjacent levels act as an adynkrafield "genetic marker" in four dimension for how to identify superfields that are candidates for the supergravity prepotential. With minor appropriate modifications, this statement applies to all dimensions and is the basis for the work seen in [40].

6.4.2 4D, $\mathcal{N} = 1$ Spinor Superfields

Here we review from chapter five in "Superspace" [100], the discussion of the gauge transformation of the supergravity prepotential superfield $H_{\underline{a}}$, which can be written in terms of the covariant derivatives of spinor superfields.

$$\delta H_{\underline{a}} \;=\; {\rm D}_\alpha \overline{L}_{\dot\alpha} \;-\; \overline{\rm D}_{\dot\alpha} L_\alpha \quad , \tag{6.4.21}$$

where

$${\rm D}_\alpha \;=\; \partial_\alpha \;+\; i\,\tfrac{1}{2}\,\overline{\theta}^{\dot\alpha}\partial_{\alpha\dot\alpha} \quad , \quad \overline{\rm D}_{\dot\alpha} \;=\; \partial_{\dot\alpha} \;+\; i\,\tfrac{1}{2}\,\theta^{\alpha}\partial_{\alpha\dot\alpha} \quad , \tag{6.4.22}$$

The component fields of $\overline{\rm D}_{\dot\alpha} L_\alpha$ are defined by

$$\begin{gathered}
\xi_{\underline{a}} \;=\; \overline{\rm D}_{\dot\alpha} L_\alpha| \quad , \\
L^{(1)}{}_{\alpha\beta\dot\beta} \;=\; {\rm D}_\alpha \overline{\rm D}_{\dot\beta} L_\beta| \quad , \quad \epsilon_\alpha \;=\; \overline{\rm D}^2 L_\alpha| \quad , \\
L^{(2)}{}_{\underline{a}} \;=\; {\rm D}^2 \overline{\rm D}_{\dot\alpha} L_\alpha| \quad , \quad \sigma \;=\; {\rm D}^\alpha \overline{\rm D}^2 L_\alpha| \quad , \quad \omega_{\alpha\beta} \;=\; \tfrac{1}{2}\,{\rm D}_{(\alpha} \overline{\rm D}^2 L_{\beta)}| \quad , \\
\eta_\alpha \;=\; {\rm D}^2 \overline{\rm D}^2 L_\alpha| \quad ,
\end{gathered} \tag{6.4.23}$$

and similarly for the complex conjugates.

Finally, one obtains for the transformation law of the various component fields the expressions

$$
\begin{aligned}
\delta h_{\underline{a}} &= -\ 2\,\mathrm{Re}\,\xi_{\underline{a}} \quad , \\
\delta h_{\alpha\beta\dot{\beta}} &= C_{\alpha\beta}\,\bar{\epsilon}_{\dot{\beta}} \ - \ L^{(1)}{}_{\beta\alpha\dot{\beta}} \quad , \\
\delta \overline{h}_{\dot{\alpha}\beta\dot{\beta}} &= -\ C_{\dot{\alpha}\dot{\beta}}\,\epsilon_{\beta} \ - \ \overline{L}^{(1)}{}_{\dot{\beta}\beta\dot{\alpha}} \quad , \\
\delta h^{(2)}{}_{\underline{b}} &= -\ L^{(2)}{}_{\underline{b}} \quad , \\
\delta \overline{h}^{(2)}{}_{\underline{b}} &= -\ \overline{L}^{(2)}{}_{\underline{b}} \quad , \\
\delta h_{\underline{a}\underline{b}} &= -\ \big(C_{\dot{\alpha}\dot{\beta}}\,\omega_{\alpha\beta} \ - \ C_{\alpha\beta}\,\overline{\omega}_{\dot{\alpha}\dot{\beta}} \big) \ + \ C_{\alpha\beta}C_{\dot{\alpha}\dot{\beta}}\,\mathrm{Re}\,\sigma \ + \ \partial_{\underline{a}}\,\mathrm{Im}\,\xi_{\underline{b}} \quad , \\
\delta \psi_{\underline{b}\alpha} &= \partial_{\underline{b}}\epsilon_{\alpha} \ + \ i\,C_{\alpha\beta}\,\overline{\eta}_{\dot{\beta}} \quad , \\
\delta \overline{\psi}_{\underline{b}\dot{\alpha}} &= \partial_{\underline{b}}\bar{\epsilon}_{\dot{\alpha}} \ + \ i\,C_{\dot{\alpha}\dot{\beta}}\,\eta_{\beta} \quad , \\
\delta A_{\underline{b}} &= -\ \tfrac{2}{3}\,\partial_{\underline{b}}\,\mathrm{Im}\,\sigma \quad ,
\end{aligned}
\tag{6.4.24}
$$

and it should be noted these results define both the 4D, $\mathcal{N} = 1$ supergravity conformal fields and their transformations under the action of the superconformal group. The real part of $\xi_{\underline{a}}$ can be used to set $h_{\underline{a}} = 0$. The fermionic gauge parameter $L^{(1)}{}_{\beta\alpha\dot{\beta}}$ can be used to set $h_{\alpha\,\beta\dot{\beta}} = 0$, Finally, the bosonic gauge parameter $L^{(2)}{}_{\underline{a}}$ can be used to set $h^{(2)}{}_{\underline{a}} = 0$. Thus, in a Wess-Zumino gauge one is left with the component fields $h_{\underline{a}\underline{b}}$, $\psi_{\underline{a}}{}^{\beta}$, and $A_{\underline{a}}$ where their gauge transformation laws in this Wess-Zumino gauge are given by the last four lines of Eq. (6.4.24).

Consider the spinor superfield L_{α}, its component content is actually $\mathcal{V} \otimes [1, 0]$. The subscript α ($\dot{\alpha}$) and the superscript α ($\dot{\alpha}$) correspond to the same irrep $[1, 0]$ ($[0, 1]$), since they can be lowered or raised by the spinor metric which corresponds to the singlet.

Explicitly,

$$
L_\alpha \;=\; \begin{cases}
\text{level} - 0: & [1,0] \quad , \\[2ex]
\text{level} - 1: & [0,0] \oplus [2,0] \oplus [1,1] \quad , \\[2ex]
\text{level} - 2: & [0,1] \oplus (2)[1,0] \oplus [2,1] \quad , \\[2ex]
\text{level} - 3: & [0,0] \oplus [2,0] \oplus [1,1] \quad , \\[2ex]
\text{level} - 4: & [1,0] \quad .
\end{cases}
\tag{6.4.25}
$$

As we did in the vector superfield section, in order to understand the Dynkin Label content, first, we translate Dynkin Labels to YT notations.

$$
L_\alpha \;=\; \begin{cases}
\text{level} - 0: & \boxed{2} \quad , \\[2ex]
\text{level} - 1: & \cdot \;\oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}_{\text{IR},-} \oplus \; \boxed{}_{\text{IR}} \quad , \\[2ex]
\text{level} - 2: & (2)\boxed{2} \oplus \boxed{} \otimes \boxed{2} \quad , \\[2ex]
\text{level} - 3: & \cdot \;\oplus \begin{array}{|c|}\hline \; \\ \hline \; \\ \hline \end{array}_{\text{IR},-} \oplus \; \boxed{}_{\text{IR}} \quad , \\[2ex]
\text{level} - 4: & \boxed{2} \quad .
\end{cases}
\tag{6.4.26}
$$

Then we can translate YT to index notation. Note that the two-form $A_{\underline{ab}}$ can be transformed into spinor notation by using spinor metrics (which is purely imaginary in our convention),

$$
A_{\underline{ab}} \;=\; i\,C_{\alpha\beta} A_{\dot\alpha\dot\beta} \;+\; i\,C_{\dot\alpha\dot\beta} A_{\alpha\beta} \quad . \tag{6.4.27}
$$

Define

$$
A^{(+)}_{\underline{ab}} \;=\; i\,C_{\alpha\beta} A_{\dot\alpha\dot\beta} \quad , \quad A^{(-)}_{\underline{ab}} \;=\; i\,C_{\dot\alpha\dot\beta} A_{\alpha\beta} \quad , \tag{6.4.28}
$$

which are equivalent as

$$
A_{\alpha\beta} \;=\; -i\,\tfrac{1}{2}\, C^{\dot\alpha\dot\beta} A^{(-)}_{\underline{ab}} \quad , \quad A_{\dot\alpha\dot\beta} \;=\; -i\,\tfrac{1}{2}\, C^{\alpha\beta} A^{(+)}_{\underline{ab}} \quad , \tag{6.4.29}
$$

and

$$
A_{\underline{ab}} \;=\; A^{(+)}_{\underline{ab}} \;+\; A^{(-)}_{\underline{ab}} \quad . \tag{6.4.30}
$$

Here since spinor metrics satisfy $C_{\alpha\beta} = -\,C_{\beta\alpha}$ and $C_{\dot{\alpha}\dot{\beta}} = -\,C_{\dot{\beta}\dot{\alpha}}$, $A_{\alpha\beta}$ and $A_{\dot{\alpha}\dot{\beta}}$ have to be symmetric tensors and their independent components are both $\{3\}$. The SO(4) group tells us the same story. In terms of YTs, we have

$$\ydiagram{1,1} = \ydiagram{1,1}_{\text{IR},+} \oplus \ydiagram{1,1}_{\text{IR},-} = \{\bar{3}\} \oplus \{3\} = [0,2] \oplus [2,0] \quad . \tag{6.4.31}$$

Moreover, an application of Klimyk's formula reads

$$\begin{aligned} [0,1] \otimes [0,1] &= [0,0] \oplus [0,2] \quad , \\ [1,0] \otimes [1,0] &= [0,0] \oplus [2,0] \quad . \end{aligned} \tag{6.4.32}$$

Therefore, we can assert the following translations

$$\begin{aligned} \ydiagram{1,1}_{\text{IR},+} &= A^{(+)}_{\underline{ab}} \text{ or } A_{\dot{\alpha}\dot{\beta}} \quad , \\ \ydiagram{1,1}_{\text{IR},-} &= A^{(-)}_{\underline{ab}} \text{ or } A_{\alpha\beta} \quad . \end{aligned} \tag{6.4.33}$$

Further discussions of two-forms can be found in Appendix D.

At this point, it is useful to discuss multiplication of the red boxes and tensor products needed to obtain results:

$$\boxed{2} \otimes \boxed{2} = \cdot \oplus \ydiagram{1,1}_{\text{IR},-} \quad , \tag{6.4.34}$$

$$\boxed{\bar{2}} \otimes \boxed{\bar{2}} = \cdot \oplus \ydiagram{1,1}_{\text{IR},+} \quad , \tag{6.4.35}$$

$$\boxed{2} \otimes \boxed{\bar{2}} = \ydiagram{1}_{\text{IR}} \quad , \tag{6.4.36}$$

$$\ydiagram{1} \otimes \boxed{2} = \begin{ytableau} \none[] & 2 \end{ytableau}_{\text{IR}} \oplus \boxed{\bar{2}} \quad , \tag{6.4.37}$$

$$\ydiagram{1} \otimes \boxed{\bar{2}} = \begin{ytableau} \none[] & \bar{2} \end{ytableau}_{\text{IR}} \oplus \boxed{2} \quad . \tag{6.4.38}$$

Relevant results for wedge product $\wedge$ calculations include: first we know

$$\boxed{2} \wedge \boxed{2} = \cdot \quad , \quad \boxed{\bar{2}} \wedge \boxed{\bar{2}} = \cdot \quad . \tag{6.4.39}$$

Then look at the quadratic theta-monomial decompositions occuring in the scalar superfield. We can construct three different quadratic theta-monomials, $\theta^\alpha\theta^\beta$, $\bar\theta^{\dot\alpha}\bar\theta^{\dot\beta}$, and $\theta^\alpha\bar\theta^{\dot\beta}$, which correspond to $\boxed{2} \wedge \boxed{2}$, $\boxed{\bar{2}} \wedge \boxed{\bar{2}}$, and $\boxed{2} \otimes \boxed{\bar{2}}$ respectively. On the other hand, we can consider the quadratic theta-monomial decompositions as $[\,\theta^\alpha + \bar\theta^{\dot\alpha}\,]\,[\,\theta^\beta + \bar\theta^{\dot\beta}\,]$ and in YT language

$$
[\,\boxed{2} \oplus \boxed{\bar{2}}\,] \wedge [\,\boxed{2} \oplus \boxed{\bar{2}}\,] \;=\; \boxed{2} \wedge \boxed{2} \;\oplus\; \boxed{\bar{2}} \wedge \boxed{\bar{2}} \;\oplus (2)\, \boxed{2} \wedge \boxed{\bar{2}} \quad . \tag{6.4.40}
$$

Therefore we obtain

$$
\boxed{2} \wedge \boxed{\bar{2}} \;=\; \boxed{2} \;\otimes\; \boxed{\bar{2}} \quad . \tag{6.4.41}
$$

In summary, the wedge product between the same two irreps/YTs is well defined and one can use the Plethysm function to carry out the calculation; the wedge product between two different irreps/YTs can be treated as the tensor product.

Gathering all discussions made above, the component content of L_α is

$$
L_\alpha \;=\; \begin{cases}
\text{level} - 0: & l_\alpha \quad , \\
\text{level} - 1: & l \oplus \tau_{\alpha\beta} \oplus \xi_{\underline{a}} \quad , \\
\text{level} - 2: & \psi_\alpha \oplus \epsilon_\alpha \oplus L^{(1)}{}_{\alpha \underline{b}} \quad , \\
\text{level} - 3: & \sigma \oplus \omega_{\alpha\beta} \oplus L^{(2)}{}_{\underline{a}} \quad , \\
\text{level} - 4: & \eta_\alpha \quad ,
\end{cases} \tag{6.4.42}
$$

which is consistent with Equation (6.4.23) as considering (below are the fields set to zero in the gauge $\bar{\mathrm{D}}_{\dot\alpha} L_\alpha = 0$)

$$
\begin{gathered}
l_\alpha \;=\; L_\alpha| \quad , \\
\tau_{\alpha\beta} \;=\; \tfrac{1}{2}\,\mathrm{D}_{(\alpha} L_{\beta)}| \quad , \quad l \;=\; \mathrm{D}^\alpha L_\alpha| \quad , \\
\psi_\alpha \;=\; \mathrm{D}^2 L_\alpha| \quad .
\end{gathered} \tag{6.4.43}
$$

Similarly, for the superfield $\bar{L}_{\dot\alpha}$, the Dynkin label and YT descriptions of its component fields are as below. Its component content is actually $\mathcal{V} \otimes [0, 1]$.

$$
\overline{L}_{\dot{\alpha}} \;=\; \begin{cases} \text{level} - 0: & [0,1] \quad , \\ \text{level} - 1: & [0,0] \oplus [0,2] \oplus [1,1] \quad , \\ \text{level} - 2: & [1,0] \oplus (2)[0,1] \oplus [1,2] \quad , \\ \text{level} - 3: & [0,0] \oplus [0,2] \oplus [1,1] \quad , \\ \text{level} - 4: & [0,1] \quad . \end{cases} \tag{6.4.44}
$$

$$
\overline{L}_{\dot{\alpha}} \;=\; \begin{cases} \text{level} - 0: & \boxed{\bar{2}} \quad , \\ \text{level} - 1: & \cdot \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},+} \oplus \boxed{}_{\text{IR}} \quad , \\ \text{level} - 2: & (2)\boxed{\bar{2}} \oplus \boxed{} \otimes \boxed{\bar{2}} \quad , \\ \text{level} - 3: & \cdot \oplus \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array}_{\text{IR},+} \oplus \boxed{}_{\text{IR}} \quad , \\ \text{level} - 4: & \boxed{\bar{2}} \quad . \end{cases} \tag{6.4.45}
$$

Finally, the index notations of its component fields are presented below.

$$
\overline{L}_{\dot{\alpha}} \;=\; \begin{cases} \text{level} - 0: & \bar{l}_{\dot{\alpha}} \quad , \\ \text{level} - 1: & \bar{l} \oplus \overline{\tau}_{\dot{\alpha}\dot{\beta}} \oplus \overline{\xi}_{\underline{a}} \quad , \\ \text{level} - 2: & \overline{\psi}_{\dot{\alpha}} \oplus \bar{\epsilon}_{\dot{\alpha}} \oplus \overline{L}^{(1)}{}_{\dot{\alpha}\underline{b}} \quad , \\ \text{level} - 3: & \overline{\sigma} \oplus \overline{\omega}_{\dot{\alpha}\dot{\beta}} \oplus \overline{L}^{(2)}{}_{\underline{a}} \quad , \\ \text{level} - 4: & \overline{\eta}_{\dot{\alpha}} \quad . \end{cases} \tag{6.4.46}
$$

6.4.3 Adynkra Genomes

As mentioned earlier, adynkras correspond to a "2.0 version" of adinkras. Thus, it is natural that adinkras and adynkras share some properties. Many years ago in the study of the adinkra adjacency matrices $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$ [84], it was shown that adinkra nodes could be raised and/lowered to connect distinct supermultiplets. This gives rise to the concept of a "level parameter" whose change connects the different supermultiplets. Multiple level parameters also account for the compatibility of component fields with distinct engineering dimensions to occur in the same supermultiplet. In the context of adynkras, multiple level parameters allow for the same YT to occur within a single level a multiple number of times.

There are two advantages, coding, and analytical computation, from using adynkras that this approach possesses with respect to the convention superfield approach:

(a.) as shown in the works of [57–59, 62], codes can be written and practically executed on existing IT platforms to derive component level results, and

(b.) these methods can determine the non-vanishing terms in Fierz identities, although not necessarily their normalization constants.

Both of these advantages are derived from the same source. Restricting the domain of exploration in terms of either YT or equivalently DL's, permits the harnessing of powerful mathematical results of long-standing.

At this juncture, some displays of examples seem in order to show the calculational efficacy and efficiency of the AG concept.

We shall first review the prescription given in [59] for how to construct an adynkrafield. The first operator required is an "Adynkra Genome" (AG). We will use the symbol $\boldsymbol{\mathcal{G}}$ to denote such a AG that will act on the space of YT. There are several parts to its construction.

We introduce "level parameters" ℓ and $\bar{\ell}$ where the former is associated with the $[1,0]$ irrep and the latter with $[0,1]$ irrep. Recalling the results of Eq. (6.4.8), in a respective graphical manner, the DL's are associated with $\boxed{2}$ and $\boxed{\bar{2}}$.

The next step is to form the products $\ell\,\boxed{2}$ and $\bar{\ell}\,\boxed{\bar{2}}$ and exponentiate them to obtain

$$
\begin{aligned}
\boldsymbol{\mathcal{G}}^{(L)}\big[\,\ell\,\boxed{2}\,\big] &= \exp\big[\,\ell\,\boxed{2}\,\big] \quad , \\
\boldsymbol{\mathcal{G}}^{(R)}\big[\,\bar{\ell}\,\boxed{\bar{2}}\,\big] &= \exp\big[\,\bar{\ell}\,\boxed{\bar{2}}\,\big] \quad ,
\end{aligned}
\tag{6.4.47}
$$

which permit the definition of the AG's via the definitions

$$
\boldsymbol{\mathcal{G}}\big[\,\ell,\bar{\ell},\boxed{2},\boxed{\bar{2}}\,\big]\;[\mathsf{YT}] \;\equiv\; \boldsymbol{\mathcal{G}}^{(L)}\big[\,\ell\,\boxed{2}\,\big]\;\boldsymbol{\mathcal{G}}^{(R)}\big[\,\bar{\ell}\,\boxed{\bar{2}}\,\big]\;[\mathsf{YT}] \quad , \tag{6.4.48}
$$

for a bosonic YT [YT] and

$$\mathcal{G}[\,\ell,\bar{\ell},\boxed{2},\boxed{\bar{2}}\,]\ [\mathrm{YT}] \ \equiv\ \mathcal{G}^{(L)}[\,\ell\boxed{2}\,]\ \mathcal{G}^{(R)}[\,\bar{\ell}\boxed{\bar{2}}\,]\ [\mathrm{YT}] \quad , \tag{6.4.49}$$

for a fermionic YT [YT].

There are a few points about these equations to note:

(a.) the rules for construction of [YT] and [YT] have been stated in cases for higher dimensional theories in the works of [40, 57–59, 62] and may be adapted to the present case in the most obvious ways, and

(b.) the rules of multiplications for $\boxed{2}$ and $\boxed{\bar{2}}$ needed for the exponential expansions of the fermionic YT are *not* the usual one that corresponds to

$$\begin{aligned} \boxed{2} \otimes \boxed{2} &= \begin{array}{|c|}\hline 2\\ \hline 2\\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline 2 & 2\\ \hline\end{array} = \cdot \ \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_{\mathrm{IR},-} \quad , \\ \boxed{\bar{2}} \otimes \boxed{\bar{2}} &= \begin{array}{|c|}\hline \bar{2}\\ \hline \bar{2}\\ \hline\end{array} \oplus \begin{array}{|c|c|}\hline \bar{2} & \bar{2}\\ \hline\end{array} = \cdot \ \oplus \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array}_{\mathrm{IR},+} \quad , \end{aligned} \tag{6.4.50}$$

instead we have

$$\boxed{} \wedge \boxed{} = \begin{array}{|c|}\hline \ \\ \hline \ \\ \hline\end{array} = \cdot \quad , \tag{6.4.51}$$

which is valid whether both the red tableaux in Eq. (6.4.51) correspond to $\boxed{2}$ or to $\boxed{\bar{2}}$.

There is another important feature to observe in these constructions. The presence of the "level parameters" (i.e. ℓ and $\widetilde{\ell}$) is required from two distinct viewpoints.

6.4.4 Adynkra Genome Examples

In this current section, we will use the same six supermultiplets to be discussed in section 6.4.5 and explicitly demonstrate the corresponding adynkra genomes in the same order of presentation.

$$\mathcal{G}[\,\ell,0,\boxed{2},\boxed{\bar{2}}\,]\ \cdot \ = \mathcal{G}^{(L)}[\,\ell\boxed{2}\,]\ \cdot \ = \ \cdot \ \oplus\ \ell\boxed{2} \ \oplus\ \tfrac{1}{2!}\,(\ell)^2\ \cdot \quad , \tag{6.4.52}$$

$$\mathscr{G}[\,\ell, 0, \boxed{2}, \boxed{\bar{2}}\,]\;\boxed{2} = \mathscr{G}^{(L)}[\,\ell\,\boxed{2}\,]\;\boxed{2} = \boxed{2} \oplus \ell\,\Big(\,\cdot\, \oplus \begin{array}{|c|}\hline \\\hline \\\hline\end{array}_{\,\mathrm{IR},-}\Big) \oplus \tfrac{1}{2!}\,(\ell)^2\;\boxed{2} \quad , \tag{6.4.53}$$

$$\mathscr{G}[\,\ell, 0, \boxed{2}, \boxed{\bar{2}}\,]\;\boxed{\bar{2}} = \mathscr{G}^{(L)}[\,\ell\,\boxed{2}\,]\;\boxed{\bar{2}} = \boxed{\bar{2}} \oplus \ell\,\Box_{\mathrm{IR}} \oplus \tfrac{1}{2!}\,(\ell)^2\;\boxed{\bar{2}} \quad , \tag{6.4.54}$$

$$\begin{aligned}\mathscr{G}[\,\ell, \bar{\ell}, \boxed{2}, \boxed{\bar{2}}\,]\;\cdot\; = {}& \cdot \oplus \ell\;\boxed{2} \oplus \bar{\ell}\;\boxed{\bar{2}} \oplus \tfrac{1}{2!}\,(\ell)^2\;\cdot \oplus \tfrac{1}{2!}\,(\bar{\ell})^2\;\cdot \oplus \ell\,\bar{\ell}\;\Box_{\mathrm{IR}} \\ &\oplus \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\;\boxed{\bar{2}} \oplus \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\;\boxed{2} \oplus \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\;\cdot \quad ,\end{aligned} \tag{6.4.55}$$

$$\begin{aligned}\mathscr{G}[\,\ell, \bar{\ell}, \boxed{2}, \boxed{\bar{2}}\,]\;\boxed{2} = {}& \boxed{2} \oplus \ell\,\Big(\,\cdot\, \oplus \begin{array}{|c|}\hline \\\hline \\\hline\end{array}_{\,\mathrm{IR},-}\Big) \oplus \bar{\ell}\;\Box_{\mathrm{IR}} \oplus \tfrac{1}{2!}\,(\ell)^2\;\boxed{2} \oplus \tfrac{1}{2!}\,(\bar{\ell})^2\;\boxed{2} \\ &\oplus \ell\,\bar{\ell}\,\Big(\boxed{\bar{2}} \oplus \begin{array}{|c|c|}\hline & 2\\\hline\end{array}_{\,\mathrm{IR}}\Big) \oplus \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\;\Box_{\mathrm{IR}} \\ &\oplus \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\,\Big(\,\cdot\, \oplus \begin{array}{|c|}\hline \\\hline \\\hline\end{array}_{\,\mathrm{IR},-}\Big) \oplus \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\;\boxed{2} \quad ,\end{aligned} \tag{6.4.56}$$

$$\begin{aligned}\mathscr{G}[\,\ell, \bar{\ell}, \boxed{2}, \boxed{\bar{2}}\,]\;\Box = {}& \Box \oplus \ell\,\Big(\boxed{\bar{2}} \oplus \begin{array}{|c|c|}\hline & 2\\\hline\end{array}_{\,\mathrm{IR}}\Big) \oplus \bar{\ell}\,\Big(\boxed{2} \oplus \begin{array}{|c|c|}\hline & \bar{2}\\\hline\end{array}_{\,\mathrm{IR}}\Big) \oplus \tfrac{1}{2!}\,(\ell)^2\;\Box \\ &\oplus \tfrac{1}{2!}\,(\bar{\ell})^2\;\Box \oplus \ell\,\bar{\ell}\,\Big(\begin{array}{|c|c|}\hline & \\\hline\end{array}_{\,\mathrm{IR}} \oplus \begin{array}{|c|}\hline \\\hline \\\hline\end{array}_{\,\mathrm{IR},-} \oplus \begin{array}{|c|}\hline \\\hline \\\hline\end{array}_{\,\mathrm{IR},+} \oplus \,\cdot\,\Big) \\ &\oplus \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\,\Big(\boxed{2} \oplus \begin{array}{|c|c|}\hline & \bar{2}\\\hline\end{array}_{\,\mathrm{IR}}\Big) \oplus \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\,\Big(\boxed{\bar{2}} \oplus \begin{array}{|c|c|}\hline & 2\\\hline\end{array}_{\,\mathrm{IR}}\Big) \\ &\oplus \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\;\Box \quad .\end{aligned} \tag{6.4.57}$$

Each of these adynkra genomes will be shortly shown to correspond to a previously identified 4D, $\mathcal{N} = 1$ supermultiplet in the literature where the correspondence looks as:

(1.) chiral supermultiplet - (6.4.52),

(2.) 2-form gauge field supermultiplet - (6.4.53),

(3.) 1-form variant gauge field supermultiplet - (6.4.54),

(4.) 1-form gauge field supermultiplet - (6.4.55),

(5.) matter gravitino - (6.4.56), and

(6.) supergravity supermultiplet - (6.4.57).

In the cases of (6.4.55) and (6.4.57), we can respectively look back at equations in comparisons to (6.4.2) and (6.4.5) and recognize that the adynkra genomes are expansions in the level parameter where the latter take the places of the usual θ-expansions. In both cases, exactly identical sets of representations of the Lorentz Groups show up at corresponding orders in the respective expansions.

6.4.5 Descending from Adynkra Genomes to 4D, $\mathcal{N} = 1$ Adynkrafields

Clearly, the definitions of adynkra genomes in the equations (6.4.48) and (6.4.49) imply these mathematical object may be regarded as elements in the space of expansions over YT or alternately the space of DL's. Passing from an adynkra genome to an adinkrafield amounts to taking an "overlap" between an adynkra genome $\boldsymbol{\mathcal{G}}[\,\ell, \bar{\ell}, \boxed{2}, \boxed{\bar{2}}\,]$, specified by [YT] and [YT], and the space of field variables denoted by $\{\mathcal{F}\}$. It is well known the space of fields $\{\mathcal{F}\}$ describing all representations of the Lorentz group is bisected into two parts and can be described by the equation

$$\boldsymbol{\{\mathcal{F}\}} \;=\; \boldsymbol{\{\mathcal{F}\}}_b \,\oplus\, \boldsymbol{\{\mathcal{F}\}}_f \quad , \tag{6.4.58}$$

where $\boldsymbol{\{\mathcal{F}\}}_b$ and $\boldsymbol{\{\mathcal{F}\}}_f$ respectively denote the space of bosonic reps and fermionic reps. For the convenience of the reader, we have appropriately adapted Eq. (6.4.59) (showing conventional displays of fields) and Eq. (6.4.60) (showing YT-field pairings)

$$\begin{array}{cccccccccc} & \text{scalar} & \text{photon} & \text{graviton} & & \text{spinor} & \text{spinor} & \text{gravitino} & \text{gravitino} & \\ \boldsymbol{\{\mathcal{F}\}} = \{ & \phi(x) \;, & A_{\underline{a}}(x) \;, & h_{\underline{a}\underline{b}}(x) & ,\dots\}\oplus\{ & \lambda_\alpha(x) \;, & \lambda_{\dot\alpha}(x) \;, & \psi_{\underline{a}}{}^{\beta}(x) \;, & \psi_{\underline{a}}{}^{\dot\beta}(x) & ,\dots\}, \\ & \text{spin}-0, & \text{spin}-1, & \text{spin}-2,\dots & & \text{spin}-\frac{1}{2}, & \text{spin}-\frac{1}{2}, & \text{spin}-\frac{3}{2}, & \text{spin}-\frac{3}{2},\dots & \end{array} \tag{6.4.59}$$

and our mapping to YT

$$\begin{array}{cccccccccc} & \text{scalar} & \text{photon} & \text{graviton} & & \text{spinor} & \text{spinor} & \text{gravitino} & \text{gravitino} & \\ \boldsymbol{\{\mathcal{F}\}} = \{ & \cdot\,(x) \;, & \square(x) \;, & \square\square(x) & ,\dots\}\oplus\{ & \boxed{2}(x) \;, & \boxed{\bar{2}}(x) \;, & \square\boxed{2}(x), & \square\boxed{\bar{2}}(x) & ,\dots\}, \\ & \text{spin}-0, & \text{spin}-1, & \text{spin}-2,\dots & & \text{spin}-\frac{1}{2}, & \text{spin}-\frac{1}{2}, & \text{spin}-\frac{3}{2}, & \text{spin}-\frac{3}{2},\dots & \end{array} \tag{6.4.60}$$

motivated by the work seen in [59] and to be more explicit in the contents of $\boldsymbol{\{\mathcal{F}\}}_b$ and $\boldsymbol{\{\mathcal{F}\}}_f$.

After making a choice of YT (indicated by a subscript [YT]/[YT]), it becomes possible to define an adynkrafield denoted by $\widehat{\mathcal{F}}(x)$ via the definition

$$\widehat{\mathcal{F}}(x) \;=\; \Big\langle\, \boldsymbol{\mathcal{G}}[\,\ell,\bar{\ell},\boxed{2},\boxed{\bar{2}}\,]\;,\;\{\boldsymbol{\mathcal{F}}\}\,\Big\rangle\Big|_{[\mathsf{YT}]/[\mathsf{YT}]} \quad . \tag{6.4.61}$$

Due to the expansions seen in the previous chapters and the pairing rule indicated immediately above, the adynkrafield becomes a specified expansion with coefficients that are elements of $\{\boldsymbol{\mathcal{F}}\}$.

In the following the presentation turns to examples.

Let us provide a notational note before we dive into adynkrafields. One notice that in Section 6.4.4, when we talk about adynkra genomes, we use $\oplus$ throughout. However, when we translate to adynkrafield notation, there involve contractions between "hidden" indices carried by YT and explicit indices carried by field variables. In the following, $+$ signs are used when these contractions occur.

Eq. (6.4.52) yields

$$\widehat{\boldsymbol{\mathcal{G}}}\;[\,\ell,0,\boxed{2},\boxed{\bar{2}}\,]\quad\cdot\quad=\;\Phi(x)\;+\;\ell\,\boxed{2}\,\Psi_{\alpha}(x)\;+\;\tfrac{1}{2!}\,(\ell)^2\Phi^{(2)}(x)\quad . \tag{6.4.62}$$

Equivalently we have

$$\widehat{\Phi}(x)\;=\;\Phi(x)\;+\;\ell\,\boxed{2}\,\Psi_{\alpha}(x)\;+\;\tfrac{1}{2!}\,(\ell)^2\Phi^{(2)}(x)\quad , \tag{6.4.63}$$

where the adynkrafield $\widehat{\Phi}$ corresponds to a chiral superfield.

Eq. (6.4.53) yields

$$\widehat{\boldsymbol{\mathcal{G}}}\;[\,\ell,0,\boxed{2},\boxed{\bar{2}}\,]\;\boxed{2}\;=\;\boxed{2}\,\Phi_{\alpha}(x)\;+\;\ell\,\Phi(x)\;+\;\ell\,\begin{array}{|c|}\hline \\ \hline \\ \hline\end{array}_{\,\mathrm{IR},-}\;\Phi^{(-)}_{\underline{ab}}(x)\;+\;\tfrac{1}{2!}\,(\ell)^2\,\boxed{2}\,\varphi_{\alpha}(x)\quad . \tag{6.4.64}$$

Equivalently we have

$$\widehat{\Phi}_{\alpha}(x)\;=\;\Phi_{\alpha}(x)\;+\;\ell\,\boxed{2}\,\Phi(x)\;+\;\ell\,\boxed{2}\,\Phi_{\alpha\beta}(x)\;+\;\tfrac{1}{2!}\,(\ell)^2\varphi_{\alpha}(x)\quad . \tag{6.4.65}$$

Now we look back at the result shown in the equations (6.4.27) - (6.4.30) to rewrite this in the form

$$\widehat{\Phi}_{\alpha}(x) \;=\; \Phi_{\alpha}(x) \;+\; \ell \boxed{2} \Phi(x) \;-\; i\tfrac{1}{2}\ell \boxed{2} C^{\dot{\alpha}\dot{\beta}} \Phi^{(-)}_{\underline{ab}}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2 \varphi_{\alpha}(x) \quad . \qquad (6.4.66)$$

Eq. (6.4.54) yields

$$\widehat{\mathcal{G}}\,[\,\ell, 0, \boxed{2}, \boxed{\bar{2}}\,]\; \boxed{\bar{2}} \;=\; \boxed{\bar{2}}\, \Phi_{\dot{\alpha}}(x) \;+\; \ell \boxed{}\, \Phi_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2 \boxed{\bar{2}}\, \rho_{\dot{\alpha}}(x) \quad . \qquad (6.4.67)$$

Equivalently we have

$$\widehat{\Phi}_{\dot{\alpha}}(x) \;=\; \Phi_{\dot{\alpha}}(x) \;+\; \ell \boxed{2}\, \Phi_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2 \rho_{\dot{\alpha}}(x) \quad . \qquad (6.4.68)$$

Eq. (6.4.2) yields an adynkrafield

$$\begin{aligned} \widehat{\mathcal{V}}(x) \;&=V(x) \;+\; \ell \boxed{2}\, \Psi_{\alpha}(x) \;+\; \bar{\ell} \boxed{\bar{2}}\, \Psi_{\dot{\alpha}}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2\, \Phi(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \widetilde{\Phi}(x) \\ &\;+\; \ell\,\bar{\ell} \boxed{}_{\rm IR}\, \Phi_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2\, \bar{\ell} \boxed{\bar{2}}\, \widehat{\Psi}_{\dot{\alpha}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \ell \boxed{2}\, \widehat{\Psi}_{\alpha}(x) \\ &\;+\; \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\, \widehat{V}(x) \quad . \end{aligned} \qquad (6.4.69)$$

Eq. (6.4.15) yields another adynkrafield

$$\begin{aligned} \widehat{H}_{\underline{a}}(x) \;&=h_{\underline{a}}(x) \;+\; \ell \boxed{2}\, h_{\beta\alpha\dot{\alpha}}(x) \;+\; \bar{\ell} \boxed{\bar{2}}\, \overline{h}_{\dot{\beta}\alpha\dot{\alpha}}(x) \\ &\;+\; \tfrac{1}{2!}\,(\ell)^2\, h^{(2)}{}_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \overline{h}^{(2)}{}_{\underline{a}}(x) \;+\; \ell\,\bar{\ell} \boxed{}_{\rm IR}\, h_{\underline{ab}}(x) \\ &\;+\; \tfrac{1}{2!}\,(\ell)^2\, \bar{\ell} \boxed{\bar{2}}\, \overline{\psi}_{\underline{a}\,\dot{\beta}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \ell \boxed{2}\, \psi_{\underline{a}\,\beta}(x) \\ &\;+\; \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\, A_{\underline{a}}(x) \quad , \end{aligned} \qquad (6.4.70)$$

$$
\begin{aligned}
\square_{\rm IR}\, \widehat{H}_{\underline{a}}(x) \;=&\; \square_{\rm IR}\, h_{\underline{a}}(x) \;+\; \ell \left[\square\boxed{2}_{\rm IR} \oplus \boxed{\bar{2}} \right] h_{\beta\alpha\dot{\alpha}}(x) \;+\; \bar{\ell} \left[\square\boxed{\bar{2}}_{\rm IR} \oplus \boxed{2} \right] \bar{h}_{\dot{\beta}\alpha\dot{\alpha}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\, \square_{\rm IR}\, h^{(2)}{}_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \square_{\rm IR}\, \bar{h}^{(2)}{}_{\underline{a}}(x) \\
&+\; \ell\,\bar{\ell} \left[\square\square_{\rm IR} \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}_{{\rm IR},+} \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}_{{\rm IR},-} \oplus\; \cdot \;\right] h_{\underline{ab}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell} \left[\square\boxed{\bar{2}}_{\rm IR} \oplus \boxed{2} \right] \bar{\psi}_{\underline{a}\,\dot{\beta}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell \left[\square\boxed{2}_{\rm IR} \oplus \boxed{\bar{2}} \right] \psi_{\underline{a}\,\beta}(x) \\
&+\; \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\, \square_{\rm IR}\, A_{\underline{a}}(x) \quad ,
\end{aligned}
\tag{6.4.71}
$$

where $h_{\alpha\beta\dot{\beta}}(x)$ and $\psi_{\underline{a}\,\beta}(x)$ correspond to $\square \otimes \boxed{2} = \square\boxed{2}_{\rm IR} \oplus \boxed{\bar{2}}$, $h_{\underline{a}}(x)$, $h^{(2)}{}_{\underline{a}}(x)$, $\bar{h}^{(2)}{}_{\underline{a}}(x)$, and $A_{\underline{a}}(x)$ corresponds to $\square_{\rm IR}$, $\bar{h}_{\dot{\beta}\alpha\dot{\alpha}}(x)$ and $\bar{\psi}_{\underline{a}\,\dot{\beta}}(x)$ corresponds to $\square \otimes \boxed{\bar{2}} = \square\boxed{\bar{2}}_{\rm IR} \oplus \boxed{2}$, and $h_{\underline{ab}}(x)$ corresponds to $\square \otimes \square = \square\square_{\rm IR} \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}_{{\rm IR},+} \oplus \begin{array}{c}\square\\[-4pt]\square\end{array}_{{\rm IR},-} \oplus\; \cdot$.

Eq. (6.4.26) yields an adynkrafield

$$
\begin{aligned}
\widehat{L}_{\alpha}(x) \;=&\; l_{\alpha}(x) \;+\; \ell\, \boxed{2}\, l(x) \;+\; \ell\, \boxed{2}\, \tau_{\alpha\beta}(x) \;+\; \bar{\ell}\, \boxed{\bar{2}}\, \xi_{\underline{a}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\, \psi_{\alpha}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \epsilon_{\alpha}(x) \;+\; \ell\,\bar{\ell}\, \square_{\rm IR}\, L^{(1)}{}_{\alpha\underline{b}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\, \boxed{\bar{2}}\, L^{(2)}{}_{\underline{a}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\, \boxed{2}\, \sigma(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\, \boxed{2}\, \omega_{\alpha\beta}(x) \\
&+\; \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\, \eta_{\alpha}(x) \quad ,
\end{aligned}
\tag{6.4.72}
$$

where $L^{(1)}{}_{\alpha\underline{b}}(x)$ corresponds to $\square \otimes \boxed{2} = \square\boxed{2}_{\rm IR} \oplus \boxed{\bar{2}}$. Also $\boxed{2}\, \omega_{\alpha\beta}(x) = -\frac{i}{2}\, \boxed{2}\, C^{\dot{\alpha}\dot{\beta}}\, \omega^{(-)}_{\underline{ab}}(x)$. If we impose the irreducible condition, $L^{(1)}{}_{\alpha\underline{b}}(x)$ can be split into two irreducible component fields $\Psi_{\alpha\underline{b}}(x)$ and $\bar{\Psi}_{\dot{\alpha}}(x)$.

Eq. (6.4.45) yields an "adynkrafield"

$$
\begin{aligned}
\widehat{\bar{L}}_{\dot{\alpha}}(x) \;=&\; \bar{l}_{\dot{\alpha}}(x) \;+\; \ell\, \boxed{2}\, \bar{\xi}_{\underline{a}}(x) \;+\; \bar{\ell}\, \boxed{\bar{2}}\, \bar{l}(x) \;+\; \bar{\ell}\, \boxed{\bar{2}}\, \bar{\tau}_{\dot{\alpha}\dot{\beta}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\, \bar{\epsilon}_{\dot{\alpha}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\, \bar{\psi}_{\dot{\alpha}}(x) \;+\; \ell\,\bar{\ell}\, \square_{\rm IR}\, \bar{L}^{(1)}{}_{\dot{\alpha}\underline{b}}(x) \\
&+\; \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\, \boxed{\bar{2}}\, \bar{\sigma}(x) \;+\; \tfrac{1}{2!}\,(\ell)^2\,\bar{\ell}\, \boxed{\bar{2}}\, \bar{\omega}_{\dot{\alpha}\dot{\beta}}(x) \;+\; \tfrac{1}{2!}\,(\bar{\ell})^2\,\ell\, \boxed{2}\, \bar{L}^{(2)}{}_{\underline{a}}(x) \\
&+\; \tfrac{1}{2!2!}\,(\ell)^2\,(\bar{\ell})^2\, \bar{\eta}_{\dot{\alpha}}(x) \quad ,
\end{aligned}
\tag{6.4.73}
$$

where $\bar{L}^{(1)}{}_{\dot{\alpha}\underline{b}}(x)$ corresponds to $\square \otimes \boxed{\bar{2}} = \square\boxed{\bar{2}}_{\rm IR} \oplus \boxed{2}$. Also $\boxed{\bar{2}}\, \bar{\omega}_{\dot{\alpha}\dot{\beta}}(x) = -\frac{i}{2}\, \boxed{\bar{2}}\, C^{\alpha\beta}\, \omega^{(+)}_{\underline{ab}}(x)$.

If we impose the irreducible condition, $\overline{L}^{(1)}{}_{\dot{\alpha}\underline{b}}(x)$ can be split into two irreducible component fields $\overline{\Psi}_{\dot{\alpha}\underline{b}}(x)$ and $\Psi_{\alpha}(x)$.

6.4.6 A brief gauge group discussion via branching rules

For a vector field $h_{\underline{a}}$, we consider its gauge transformation as

$$\delta_G\, h_{\underline{a}} \;=\; \partial_{\underline{a}}\, \xi \quad . \tag{6.4.74}$$

Then when count the off-shell degree of freedom of $h_{\underline{a}}$, one would get the equation

$$\{4\} \;=\; \{3\} + \{1\} \tag{6.4.75}$$

where $\{3\}$ is the off-shell degree of freedom and $\{1\}$ comes from the gauge parameter. Meanwhile, equation (6.4.75) is exactly the one if you consider the branching rules for $\mathfrak{so}(4) \supset \mathfrak{so}(3)$. The left hand side is an irrep in $\mathfrak{so}(4)$ and irreps in the right hand side are in $\mathfrak{so}(3)$. The weight systems for both sides of (6.4.75) are as below.

$$\begin{array}{ccccccc} & [1,1] & & & & [2] & \\ [-1,1] & & [1,-1] & = & [0] & & [0] \\ & [-1,-1] & & & & [-2] & \end{array} \tag{6.4.76}$$

One can read off the projection matrix as

$$P_{\mathfrak{so}(4)\supset\mathfrak{so}(3)} \;=\; \begin{pmatrix} 1 & 1 \end{pmatrix} \tag{6.4.77}$$

Using this projection matrix, let's examine how some examples component fields split under $\mathfrak{so}(4) \supset \mathfrak{so}(3)$. For 4D off-shell graviton, we have

$$\begin{array}{ccccc} [00] \oplus [22] & \overset{\mathfrak{so}(4)\supset\mathfrak{so}(3)}{=} & [0] \oplus [4] & \oplus & [2] \oplus [0] \\ \{1\} \oplus \{9\} & & \{1\} \oplus \{5\} & & \{3\} \oplus \{1\} \\ h + h_{(\underline{a}\underline{b})} & & \text{off-shell 4D graviton counting} & & \xi_{\underline{a}}-\text{ gauge parameter} \end{array} \tag{6.4.78}$$

where the branching rule exactly tells us how to count the off-shell degree of freedom. For 4D off-shell gravitino, we can check

$$
\begin{array}{llll}
[01] \oplus [21] & \overset{\mathfrak{so}(4) \supset \mathfrak{so}(3)}{=} & [1] \oplus [3] & \oplus \qquad [1] \\
\{\bar{2}\} \oplus \{6\} & & \{2\} \oplus \{5\} & \qquad \{4\} \oplus \{2\} \\
\psi_{\dot{\alpha}} + \psi_{\underline{a}}{}^{\alpha} & & \text{off-shell 4D gravitino counting} & \xi^{\alpha} \text{ – gauge parameter}
\end{array}
\tag{6.4.79}
$$

Chapter 7

Supergravity Derivatives & Scale Transformations

This chapter is based on [40].

In this chapter, proposals are made to describe the Weyl scaling transformation laws of supercovariant derivatives $\nabla_{\underline{A}}$, the torsion supertensors $T_{\underline{A}\,\underline{B}}{}^{\underline{C}}$, and curvature supertensors $R_{\underline{A}\,\underline{B}\,\underline{c}}{}^{\underline{d}}$ in 10D superspaces.

7.1 Review of 11D, $\mathcal{N}=1$ Supergravity Derivatives & Scale Transformations

The spinorial 11D frame operator may be parametrized in the following equations[1],

$$
\begin{aligned}
\mathrm{E}_{\alpha} &= \Psi^{1/2}\left[\,\exp\left(\tfrac{1}{2}\mathcal{A}^{\underline{a}\,\underline{b}}\gamma_{\underline{a}\,\underline{b}}\right)\right]_{\alpha}{}^{\beta}\left[\mathcal{N}_{\beta}{}^{\gamma}\right]\left[\mathrm{D}_{\gamma}+\widehat{\mathrm{H}}_{\gamma}{}^{\underline{b}}\partial_{\underline{b}}\right] \quad , \\
\mathcal{N}_{\alpha}{}^{\beta} &\equiv \left[\mathbf{I}+\mathcal{A}^{\underline{a}}\gamma_{\underline{a}}+\tfrac{1}{3!}\mathcal{A}^{[\underline{3}]}\gamma_{[\underline{3}]}+\tfrac{1}{4!}\mathcal{A}^{[\underline{4}]}\gamma_{[\underline{4}]}+\tfrac{1}{5!}\mathcal{A}^{[\underline{5}]}\gamma_{[\underline{5}]}\right]_{\alpha}{}^{\beta} \quad ,
\end{aligned}
\tag{7.1.1}
$$

where Ψ is the "scale compensator," $\mathcal{A}^{\underline{a}\,\underline{b}}$ is the "Lorentz compensator," $\mathcal{N}_{\alpha}{}^{\beta}$ is a "coset factor," and $\widehat{\mathrm{H}}_{\beta}{}^{\underline{b}}$ is the "conformal graviton semi-prepotential." The parametrization of $\mathcal{N}_{\alpha}{}^{\beta}$ shown in (7.1.1) is strongly dependent on the dimensionality of the superspace. For example, in simple supergravity theory in two dimensions[2], only the first two terms are

[1] Roughly speaking, this is the superspace supergravity equivalent to the ADM [97–99] formulation but for the spinorial superframe.

[2] To our knowledge, the case of D = 2 is the only one in existence where the dependence [134] of $\mathcal{N}_{\alpha}{}^{\beta}$ on $\mathrm{H}_{\alpha}{}^{\underline{a}}$ has been explicitly presented. However, other complete solutions for constaints implicitly contain such information.

present [134].

A slightly different, but equivalent parametrization was introduced in a previous work [135]. Without loss of generality, a slightly different semi-prepotential can be written in the form

$$\widehat{\mathrm{H}}_\beta{}^{\underline{b}} \;=\; \mathrm{H}_\beta{}^{\underline{b}} \;-\; \tfrac{1}{\mathrm{D}}\,(\,\gamma^{\underline{b}}\gamma_{\underline{d}}\,)_\beta{}^\delta\,\mathrm{H}_\delta{}^{\underline{d}} \quad , \tag{7.1.2}$$

where $\mathrm{D} = 11$ for this eleven dimensional case. This also implies there is a local symmetry of the form

$$\mathrm{H}_\beta{}^{\underline{b}} \;\to\; \mathrm{H}_\beta{}^{\underline{b}} \;+\; (\gamma^{\underline{b}})_\beta{}^\alpha\Lambda_\alpha \quad , \tag{7.1.3}$$

that applies to (7.1.2).

All previous experience in superspace supergravity implies the vectorial 11D frame operator $\mathrm{E}_{\underline{a}}$, the spinorial spin connection $\omega_{\alpha\,\underline{cd}}$, and the vectorial spin connection $\omega_{\underline{a}\,\underline{cd}}$ must be determined in terms of the content of the spinorial 11D frame field. This is done by the imposition of "conventional constraints" on the torsion and curvature supertensors defined in (7.1.5). However, as first noted in the work [134], "conventional constraints" must also be imposed to determine $\mathcal{N}_\alpha{}^\beta$ <u>*solely*</u> in terms of $\mathrm{H}_\beta{}^{\underline{b}}$, but totally <u>*independent*</u> of Ψ. This has powerful implications for constraints in superfield supergravity.

A Poincaré supergeometry of the 11D theory demands the introduction of superspace supergravity covariant derivatives ∇_α and $\nabla_{\underline{a}}$ defined by

$$\begin{aligned}
\nabla_\alpha \;&=\mathrm{E}_\alpha \;+\; \tfrac{1}{2}\,\omega_{\alpha\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
\nabla_{\underline{a}} \;&=\mathrm{E}_{\underline{a}} \;+\; \tfrac{1}{2}\,\omega_{\underline{a}\,\underline{d}}{}^{\underline{e}}\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad .
\end{aligned} \tag{7.1.4}$$

Upon calculating the graded commutator of these leads to superspace torsions and curvatures according to,

$$\begin{aligned}
[\,\nabla_\alpha\;,\;\nabla_\beta\,\} \;&= T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\alpha\beta}{}^\gamma\,\nabla_\gamma \;+\; \tfrac{1}{2}R_{\alpha\beta\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\,\nabla_\alpha\;,\;\nabla_{\underline{b}}\,\} \;&= T_{\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\alpha\underline{b}}{}^\gamma\,\nabla_\gamma \;+\; \tfrac{1}{2}R_{\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,\\
[\,\nabla_{\underline{a}}\;,\;\nabla_{\underline{b}}\,\} \;&= T_{\underline{a}\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\underline{a}\underline{b}}{}^\gamma\,\nabla_\gamma \;+\; \tfrac{1}{2}R_{\underline{a}\underline{b}\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad .
\end{aligned} \tag{7.1.5}$$

Based on an analysis of constraints for the 11D vielbein, the work in [56] concluded the set

$$
\begin{aligned}
i\,\tfrac{1}{32}\,(\gamma_{\underline{a}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{b}} &= \delta_{\underline{a}}{}^{\underline{b}} \quad , & (\gamma_{\underline{a}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\gamma} &= 0 \quad , \\
T_{\alpha\,[\underline{de}]} \;-\; \tfrac{2}{55}\,(\gamma_{\underline{de}})_{\alpha}{}^{\gamma}\,T_{\gamma\underline{b}}{}^{\underline{b}} &= 0 \quad , & (\gamma_{\underline{a}})^{\alpha\beta}\,R_{\alpha\beta}{}^{\underline{de}} &= 0 \quad , \\
(\gamma_{\underline{abcde}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{e}} &= 0 \quad , & (\gamma_{[\underline{abcde}|})^{\alpha\beta}\,T_{\alpha\beta|\underline{f}]} &= 0 \quad , \\
(\gamma_{\underline{ab}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{b}} &= 0 \quad , & (\gamma_{[\underline{ab}|})^{\alpha\beta}\,T_{\alpha\beta|\underline{c}]} &= 0 \quad ,
\end{aligned}
\tag{7.1.6}
$$

can be enforced without implying any dynamical consequences on the components fields that remain with the vielbein. The first two constraints in (7.1.6) ensure that the graviton and gravitino, respectively, that appear at first order in the θ-expansion of in ∇_α are identified with the self-same fields that appear at zeroth order in $\nabla_{\underline{a}}$. The next two constraints have the same effects by removing the Lorentz connection of the theory (as an independent variable) and defining it in terms of the anholonomy associated with the component frame field. These second two constraints of (7.1.6) also ensure that the spin-connection that appears at first order in the θ-expansion of in ∇_α is identified with the self-same field appearing at zeroth order in $\nabla_{\underline{a}}$. The last four constraints in (7.1.6) remove all the superfields that appear in $\mathcal{N}_\alpha{}^\beta$ as independent quantities and makes them non-trivially dependent only on $\mathrm{H}_\alpha{}^{\underline{a}}$. It is directly possible to prove at the linearized order. However, beyond this order the process is expected to be horribly non-linear as in the 2D, $\mathcal{N}=1$ supergravity theory.

Calculations in [58] inspired by adinkras have yielded IT-based techniques providing unprecedented access to the component field composition of superfields in 11D. Based on this, we have proposed the set of constraints above may be strengthened by the replacement as indicated below,

$$
\begin{aligned}
& i\,\tfrac{1}{32}\,(\gamma_{\underline{a}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{b}} = \delta_{\underline{a}}{}^{\underline{b}} \quad , \qquad && (\gamma_{\underline{a}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\gamma} = 0 \quad , \\
T_{\alpha\,[\underline{de}]} \; - \; & \tfrac{2}{55}\,(\gamma_{\underline{de}})_{\alpha}{}^{\gamma}\,T_{\gamma\underline{b}}{}^{\underline{b}} = 0 \quad , \qquad && (\gamma_{\underline{a}})^{\alpha\beta}\,R_{\alpha\beta}{}^{\underline{de}} = 0 \quad , \\
& (\gamma_{\underline{abcde}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{e}} = 0 \quad , \qquad && (\gamma_{[\underline{abcde}|})^{\alpha\beta}\,T_{\alpha\beta|\underline{f}]} = 0 \quad , \\
& \qquad (\gamma_{\underline{ab}})^{\alpha\beta}\,T_{\alpha\beta}{}^{\underline{c}} = 0 \quad ,
\end{aligned}
\tag{7.1.7}
$$

as this set, "CGNN constraints" [26], is consistent with the choice of a scalar superfield $\mathcal{V}$ as the supergravity prepotential. This implies the superfunction $\mathrm{H}_{\beta}{}^{\underline{c}} = \mathrm{H}_{\beta}{}^{\underline{c}}(\mathcal{V})$ must admit a functional dependence that satisfies this equation. This functional dependence must involve fifteen powers of D_α acting on $\mathcal{V}$.

A solution to (7.1.7) is provided by making the "shift" $\Psi \to 1 + \Psi$ (Equations (12) and (18) in [56]) so that

$$
\begin{aligned}
\nabla_{\alpha} &= \mathrm{D}_{\alpha} \; + \; \tfrac{1}{2}\,\Psi\,\mathrm{D}_{\alpha} \; + \; \tfrac{1}{10}\,(\mathrm{D}_{\beta}\Psi)(\gamma^{\underline{de}})_{\alpha}{}^{\beta}\mathcal{M}_{\underline{de}} \quad , \\
\nabla_{\underline{a}} &= \partial_{\underline{a}} \; + \; \Psi\,\partial_{\underline{a}} \; + \; i\,\tfrac{1}{4}\,(\gamma_{\underline{a}})^{\alpha\beta}\,(\mathrm{D}_{\alpha}\Psi)\,\mathrm{D}_{\beta} \; + \; \tfrac{1}{5}\,(\partial_{\underline{c}}\Psi)\,\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\
& \qquad + \; i\,\tfrac{1}{160}\,(\gamma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta}\,(\mathrm{D}_{\alpha}\mathrm{D}_{\beta}\Psi)\,\mathcal{M}_{\underline{de}} \quad ,
\end{aligned}
\tag{7.1.8}
$$

where Ψ in (7.1.8) is an infinitesimal superfield and we set the superfield $\mathrm{H}_{\beta}{}^{\underline{c}} = 0$. It should also be noted we can regard the supervector fields here as describing a Nordström supergravity theory as in chapter 2, although we used a different set of constraints in chapter 2.

In turn, (7.1.8) suggests the definition of a set of Weyl scaling transformations of the full superspace covariant derivatives given by

$$
\begin{aligned}
\delta_S \nabla_{\alpha} &= \tfrac{1}{2}\,L\,\nabla_{\alpha} \; + \; \tfrac{1}{10}\,(\nabla_{\beta}L)(\gamma^{\underline{de}})_{\alpha}{}^{\beta}\mathcal{M}_{\underline{de}} \quad , \\
\delta_S \nabla_{\underline{a}} &= L\,\nabla_{\underline{a}} \; + \; i\,\tfrac{1}{4}\,(\gamma_{\underline{a}})^{\alpha\beta}\,(\nabla_{\alpha}L)\,\nabla_{\beta} \; + \; \tfrac{1}{5}\,(\nabla_{\underline{c}}L)\,\mathcal{M}_{\underline{a}}{}^{\underline{c}} \\
& \qquad + \; i\,\tfrac{1}{160}\,(\gamma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta}\,(\nabla_{\alpha}\nabla_{\beta}L)\,\mathcal{M}_{\underline{de}} \quad .
\end{aligned}
\tag{7.1.9}
$$

In the results shown in (7.1.9), the superspace covariant derivatives ∇_α and $\nabla_{\underline{a}}$ are "full" superspace covariant derivatives where the only remaining independent superfield variables are Ψ and $\mathrm{H}_{\beta}{}^{\underline{c}}(\mathcal{V})$.

It should be noted there is a lesson to learn from (7.1.8) and (7.1.9). A set of full Weyl scale transformation laws on the superspace supergravity supercovariant derivative operators can be obtained by starting from the formulation of a Nordström superspace supergravity theory, acting with a superspace scaling operation δ_S on the Nordström scalar field in the theory, calling $\delta_S \Psi = L$, and finally replaces all "bare" derivative operators by full superspace supergravity supercovariant derivative operators.

In turn, this implies we are able to analyze the Weyl scaling properties of all the superspace torsion and curvature supertensors with engineering dimensions of less than three-halves to find,

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0 \quad , \tag{7.1.10}$$

$$\begin{aligned}\delta_S T_{\alpha\beta}{}^{\gamma} =\ & \tfrac{1}{2} L\, T_{\alpha\beta}{}^{\gamma} - i\tfrac{1}{4}\, T_{\alpha\beta}{}^{\underline{c}}\, (\gamma_{\underline{c}})^{\delta\gamma}\, (\nabla_\delta L) + \tfrac{1}{2} (\nabla_{(\alpha} L)\, \delta_{\beta)}{}^{\gamma} \\ & + \tfrac{1}{20} (\nabla_\delta L)\, (\gamma^{[2]})_{(\alpha}{}^{\delta}\, (\gamma_{[2]})_{\beta)}{}^{\gamma} \quad ,\end{aligned} \tag{7.1.11}$$

$$\begin{aligned}\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} =\ & \tfrac{1}{2} L\, T_{\alpha\underline{b}}{}^{\underline{c}} - i\tfrac{1}{4}\, (\gamma_{\underline{b}})^{\gamma\delta}\, (\nabla_\gamma L)\, T_{\alpha\delta}{}^{\underline{c}} + (\nabla_\alpha L)\, \delta_{\underline{b}}{}^{\underline{c}} \\ & + \tfrac{1}{5}\, (\nabla_\gamma L)\, (\gamma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma} \quad ,\end{aligned} \tag{7.1.12}$$

$$\begin{aligned}\delta_S T_{\alpha\underline{b}}{}^{\gamma} =\ & L\, T_{\alpha\underline{b}}{}^{\gamma} - i\tfrac{1}{4}\, (\gamma_{\underline{b}})^{\delta\epsilon}\, (\nabla_\delta L)\, T_{\alpha\epsilon}{}^{\gamma} - i\tfrac{1}{4}\, T_{\alpha\underline{b}}{}^{\underline{c}}\, (\gamma_{\underline{c}})^{\delta\gamma}\, (\nabla_\delta L) \\ & - \tfrac{1}{2}\, (\nabla_{\underline{b}} L)\, \delta_\alpha{}^{\gamma} - \tfrac{1}{10}\, (\nabla_{\underline{d}} L)\, (\gamma_{\underline{b}}{}^{\underline{d}})_\alpha{}^{\gamma} \\ & + i\tfrac{1}{4}\, (\gamma_{\underline{b}})^{\delta\gamma}\, (\nabla_\alpha \nabla_\delta L) - i\tfrac{1}{320}\, (\gamma_{\underline{b}}{}^{\underline{de}})^{\delta\epsilon}\, (\nabla_\delta \nabla_\epsilon L)\, (\gamma_{\underline{de}})_\alpha{}^{\gamma} \quad ,\end{aligned} \tag{7.1.13}$$

$$\begin{aligned}\delta_S T_{\underline{ab}}{}^{\underline{c}} =\ & L\, T_{\underline{ab}}{}^{\underline{c}} + i\tfrac{1}{4} (\gamma_{[\underline{a}|})^{\delta\epsilon}\, (\nabla_\delta L)\, T_{\epsilon|\underline{b}]}{}^{\underline{c}} + \tfrac{6}{5}\, (\nabla_{[\underline{a}|} L)\, \delta_{|\underline{b}]}{}^{\underline{c}} \\ & + i\tfrac{1}{40}\, (\gamma_{\underline{ab}}{}^{\underline{c}})^{\delta\epsilon} (\nabla_\delta \nabla_\epsilon L) \quad ,\end{aligned} \tag{7.1.14}$$

$$\begin{aligned}\delta_S R_{\alpha\beta}{}^{\underline{de}} \;=\; & L\, R_{\alpha\beta}{}^{\underline{de}} \;+\; \tfrac{1}{5}\, T_{\alpha\beta}{}^{[\underline{d}}\, (\nabla^{\underline{e}]} L) \;+\; \tfrac{1}{5}\, T_{\alpha\beta}{}^{\delta}\, (\nabla_\gamma L)\, (\gamma^{\underline{de}})_\delta{}^\gamma \\ & -\; \tfrac{1}{5} (\nabla_{(\alpha|} \nabla_\gamma L)\, (\gamma^{\underline{de}})_{|\beta)}{}^\gamma \;+\; i \tfrac{1}{80}\, T_{\alpha\beta}{}^{\underline{c}} (\gamma_{\underline{c}}{}^{\underline{de}})^{\delta\epsilon}\, (\nabla_\delta \nabla_\epsilon L) \quad .\end{aligned} \tag{7.1.15}$$

The constraints on the last three lines of (7.1.7) have a solution given by [26][3]

$$T_{\alpha\beta}{}^{\underline{c}} \;=\; i\, (\gamma^{\underline{c}})_{\alpha\beta} \;+\; i\, \tfrac{1}{32\cdot 5!}\, (\gamma^{\underline{b}_1 \cdots \underline{b}_5})_{\alpha\beta}\, X_{\underline{b}_1 \cdots \underline{b}_5}{}^{\underline{c}} \quad , \quad X_{\underline{abcde}}{}^{\underline{e}} \;=\; X_{[\underline{abcde}\, \underline{f}]} \;=\; 0 \quad , \tag{7.1.16}$$

where $X_{\underline{a}_1 \cdots \underline{a}_5}{}^{\underline{b}} \;\equiv\; i\, (\gamma_{\underline{a}_1 \cdots \underline{a}_5})^{\alpha\beta}\, T_{\alpha\beta}{}^{\underline{b}}$ alternatively[4]. The expression of $T_{\alpha\beta}{}^{\underline{c}}$ may be substituted in (7.1.11) and (7.1.12). Contraction of the β and γ indices in the former and the $\underline{b}$ and $\underline{c}$ indices in the latter yields respectively

$$\begin{aligned}\delta_S T_{\alpha\beta}{}^{\beta} \;=\; & \tfrac{1}{2} L\, T_{\alpha\beta}{}^\beta \;-\; i\, \tfrac{1}{4} \left[i\, (\gamma^{\underline{c}})_{\alpha\beta} \;+\; i\, \tfrac{1}{32\cdot 5!}\, (\gamma^{\underline{b}_1 \cdots \underline{b}_5})_{\alpha\beta}\, X_{\underline{b}_1 \cdots \underline{b}_5}{}^{\underline{c}} \right] (\gamma_{\underline{c}})^{\delta\beta}\, (\nabla_\delta L) \\ & +\; \tfrac{1}{2} (\nabla_{(\alpha} L)\, \delta_{\beta)}{}^\beta \;+\; \tfrac{1}{20} (\nabla_\delta L)\, (\gamma^{[2]})_{(\alpha}{}^\delta\, (\gamma_{[2]})_{\beta)}{}^\beta \\ =\; & \tfrac{1}{2} L\, T_{\alpha\beta}{}^\beta \;+\; \tfrac{1}{4}\, (\gamma^{\underline{c}})_{\alpha\beta}\, (\gamma_{\underline{c}})^{\delta\beta}\, (\nabla_\delta L) \;+\; \tfrac{1}{2} (\nabla_{(\alpha} L)\, \delta_{\beta)}{}^\beta \\ & +\; \tfrac{1}{20} (\nabla_\delta L)\, (\gamma^{[2]})_{(\alpha}{}^\delta\, (\gamma_{[2]})_{\beta)}{}^\beta \\ =\; & \tfrac{1}{2} L\, T_{\alpha\beta}{}^\beta \;+\; \tfrac{33}{4}\, (\nabla_\alpha L) \quad ,\end{aligned} \tag{7.1.17}$$

$$\begin{aligned}\delta_S T_{\alpha\underline{b}}{}^{\underline{b}} \;=\; & \tfrac{1}{2} L\, T_{\alpha\underline{b}}{}^{\underline{b}} \;+\; \tfrac{1}{4}\, (\gamma_{\underline{b}})^{\gamma\delta}\, (\nabla_\gamma L) \left[(\gamma^{\underline{b}})_{\alpha\delta} \;+\; \tfrac{1}{32\cdot 5!}\, (\gamma^{\underline{b}_1 \cdots \underline{b}_5})_{\alpha\delta}\, X_{\underline{b}_1 \cdots \underline{b}_5}{}^{\underline{b}} \right] \;+\; (\nabla_\alpha L)\, \delta_{\underline{b}}{}^{\underline{b}} \\ =\; & \tfrac{1}{2} L\, T_{\alpha\underline{b}}{}^{\underline{b}} \;+\; \tfrac{1}{4}\, (\gamma_{\underline{b}})^{\gamma\delta}\, (\nabla_\gamma L)\, (\gamma^{\underline{b}})_{\alpha\delta} \;+\; (\nabla_\alpha L)\, \delta_{\underline{b}}{}^{\underline{b}} \\ =\; & \tfrac{1}{2} L\, T_{\alpha\underline{b}}{}^{\underline{b}} \;+\; \tfrac{33}{4}\, (\nabla_\alpha L) \quad .\end{aligned} \tag{7.1.18}$$

The equation in the final portion of the results shown in (7.1.16) is responsible for the disappearance of the X-tensors from the second lines in (7.1.17) and (7.1.18).

If we make the definitions

$$\mathcal{J}_\alpha{}^{(1)} \;=\; \tfrac{4}{33}\, T_{\alpha\beta}{}^\beta \quad , \quad \mathcal{J}_\alpha{}^{(2)} \;=\; \tfrac{4}{33}\, T_{\alpha\underline{b}}{}^{\underline{b}} \quad , \tag{7.1.19}$$

---

[3] The first indication of the need for modification to the superspace torsion proportional to γ/σ matrices was noted in 1983 [136] in the context of 4D, $\mathcal{N} > 4$ conformal supergravity in superspace.

[4] Notice that the normalization of the $X_{[5]}{}^{\underline{b}}$ differs by a factor of 32 with that in the work of [56], i.e. the factor $\frac{1}{32}$ is put in Equation (7.1.16) instead of the definition of $X_{[5]}{}^{\underline{b}}$.

then we see

$$\delta_S \mathcal{J}_\alpha{}^{(1)} ~=~ \tfrac{1}{2} L\, \mathcal{J}_\alpha{}^{(1)} ~+~ (\nabla_\alpha L) ~~~,~~~ \delta_S \mathcal{J}_\alpha{}^{(2)} ~=~ \tfrac{1}{2} L\, \mathcal{J}_\alpha{}^{(2)} ~+~ (\nabla_\alpha L) ~~~. \tag{7.1.20}$$

Upon changing basis for these superfields by defining

$$\mathcal{J}_\alpha{}^{(\pm)} ~=~ \tfrac{1}{2} \left[\mathcal{J}_\alpha{}^{(1)} ~\pm~ \mathcal{J}_\alpha{}^{(2)} \right] ~~~, \tag{7.1.21}$$

we observe that

$$\delta_S \mathcal{J}_\alpha{}^{(+)} ~=~ \tfrac{1}{2} L\, \mathcal{J}_\alpha{}^{(+)} ~+~ (\nabla_\alpha L) ~~~,~~~ \delta_S \mathcal{J}_\alpha{}^{(-)} ~=~ \tfrac{1}{2} L\, \mathcal{J}_\alpha{}^{(-)} ~~~. \tag{7.1.22}$$

While $\mathcal{J}_\alpha{}^{(-)}$ transforms like a scale covariant tensor of weight $\frac{1}{2}$, the quantity $\mathcal{J}_\alpha{}^{(+)}$ transforms with a scale weight of $\frac{1}{2}$ while being a spinorial gauge connection under a scaling transformation! The work of [55] indicated that such a dimension one-half supertorsion tensor was required for an off-shell description of 11D, $\mathcal{N} = 1$ Poincaré supergravity.

According to the analysis in [56], there is also necessarily an engineering dimension one superfield $\mathcal{W}_{\underline{abcd}}$ with Weyl weight one (i.e. $\delta_S \mathcal{W}_{\underline{abcd}} ~=~ L\, \mathcal{W}_{\underline{abcd}}$) defined by[5]

$$\mathcal{W}_{\underline{abcd}} \equiv ~ \tfrac{1}{32} \Big[(\gamma^{\underline{e}} \gamma_{\underline{abcd}})_\gamma{}^\alpha \, T_{\alpha\, \underline{e}}{}^\gamma ~+~ i\, \tfrac{11}{4} \, (\gamma_{\underline{abcd}})^{\alpha\beta} \left(\nabla_\alpha \mathcal{J}_\beta{}^{(+)} ~-~ \tfrac{23}{220} \, \mathcal{J}_\alpha{}^{(+)} \mathcal{J}_\beta{}^{(+)} \right) \Big] ~~~, \tag{7.1.23}$$

and which contains all the on-shell degrees of freedom. In particular, the lowest order term in this superfield (i.e. setting $\theta = 0$) is the supercovariantized field strength of the component level 3-form gauge field. Moreover, the 11D supercovarinatized Weyl-gravitino field strength is contained at the first order in the θ-expansion, and the super-covariantized Weyl tensor of the 11D bosonic spacetime is contained at the second order in the θ-expansion of $\mathcal{W}_{\underline{abcd}}$.

Thus, we conclude the most likely path forward for an off-shell 11D, $\mathcal{N} = 1$ superspace

[5] Notice that the Weyl tensor $\mathcal{W}_{\underline{abcd}}$ here differs by an overall factor of i with that in the work of [56]. It is because in our convention here, all γ-matrices are real, while in [56], $(\gamma^{[1]})_{\alpha\beta}$ is real but $(\gamma^{[4]})_{\alpha\beta}$ is imaginary. Thus $\mathcal{W}_{\underline{abcd}}$ is real in both cases.

supergravity theory must be the construction, as we suspected in 1996, on the basis of three superfields now known to be $\mathcal{W}_{\underline{abcd}}$, $X_{\underline{a}_1 \cdots \underline{a}_5}{}^{\underline{b}}$, and $\mathcal{J}_\alpha$. The ultimate deepest reason to expect this decomposition is the structure of gravity itself. When one considers the Riemann Tensor of ordinary gravity (which must be embedded within any supergravity theory), it can be decomposed into the:

(a.) Weyl,

(b.) Ricci, and

(c.) Scalar Curvature

portions. In our opinion, any superspace construction that ignores this tripartite division is unlikely to be adequate.

7.2 Supergravity Derivatives & Scale Transformations in 10D

The content of the previous section about the superspace supergravity supercovariant derivatives $\nabla_{\underline{A}} = (\nabla_\alpha, \nabla_{\underline{a}})$, the torsion supertensors $T_{\underline{A}\,\underline{B}}{}^{\underline{C}}$, and curvature supertensors $R_{\underline{A}\,\underline{B}\,\underline{c}}{}^{\underline{d}}$ can be directly used to derive similar results for the superspaces associated with Type-IIA and Type-I superstring theories. However, there is a much more efficient and rapid way in which this can be done. This alternative route is also directly applicable to the Type-IIB case. So rather than going the route of the dimensional reduction of these geometrical quantities, we choose to take a less obvious pathway.

The bottom-line message from the last section is that the scalar compensator, which appears in the parametrization of the framefield (7.1.1), is isomorphic to the concept of a scalar theory (i.e. Nordström gravitation). This means explorations of Nordström supergravity are equivalent to the exploration of the limit of a full superframe where only the compensator is retained. This has the implication that investigations of Nordström supergravity theories fix the dependences of the superspace supergravity supercovariant derivatives $\nabla_{\underline{A}}$, the torsion supertensors $T_{\underline{A}\,\underline{B}}{}^{\underline{C}}$ and curvature supertensors $R_{\underline{A}\,\underline{B}\,\underline{c}}{}^{\underline{d}}$ on the scale compensating superfield.

The constraints used in the following results can be derived from equations found in [36]. Take the following indicated equations from that work which are linear in the first derivative of the scale compensator, use appropriate algebraic operations to express the first derivatives in terms of torsion, and substitute those expressions back into the equations. For the $\mathcal{N}$ = IIA supergeometry, the appropriate equations from which to start the derivation are given in (6.10) - (6.19), (6.22), and (6.25). For the $\mathcal{N}$ =IIB supergeometry, the appropriate equations from which to start the derivation are given in (7.11) - (7.20), (7.23), and (7.26). Finally, for the $\mathcal{N}$ =I supergeometry, the appropriate equations from which to start the derivation are given in (5.8), (5.9), (5.10), and (5.12). These are found to be equivalent to results given previously in [45].

7.2.1 10D, $\mathcal{N}$ = IIA Supergravity Derivatives & Scale Transformations

The covariant derivatives (Equations (8.51) - (8.53) in [36]) linear in the real conformal compensator Ψ are given by

$$\nabla_\alpha = \mathrm{D}_\alpha + \tfrac{1}{2}\Psi\mathrm{D}_\alpha + \tfrac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.1}$$

$$\nabla_{\dot\alpha} = \mathrm{D}_{\dot\alpha} + \tfrac{1}{2}\Psi\mathrm{D}_{\dot\alpha} + \tfrac{1}{10}(\sigma^{\underline{ab}})_{\dot\alpha}{}^{\dot\beta}(\mathrm{D}_{\dot\beta}\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.2}$$

$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\tfrac{1}{5}(\sigma_{\underline{a}})^{\delta\gamma}(\mathrm{D}_\delta\Psi)\mathrm{D}_\gamma - i\tfrac{1}{5}(\sigma_{\underline{a}})^{\dot\delta\dot\gamma}(\mathrm{D}_{\dot\delta}\Psi)\mathrm{D}_{\dot\gamma} - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{7.2.3}$$

where the “bare” algebra of the supersymmetry covariant derivatives and the spacetime partial derivative takes the forms

$$\{\mathrm{D}_\alpha\,,\,\mathrm{D}_\beta\} = i\,(\sigma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad , \quad \{\mathrm{D}_{\dot\alpha}\,,\,\mathrm{D}_{\dot\beta}\} = i\,(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}\,\partial_{\underline{a}} \quad , \quad \{\mathrm{D}_\alpha\,,\,\mathrm{D}_{\dot\beta}\} = 0 \quad , \tag{7.2.4}$$

which indicate the non-vanishing values of the torsion supertensors must be given by

$$T_{\alpha\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\alpha\beta} \quad , \quad T_{\dot\alpha\dot\beta}{}^{\underline{c}} = i(\sigma^{\underline{c}})_{\dot\alpha\dot\beta} \quad , \quad T_{\alpha\dot\beta}{}^{\underline{c}} = 0 \quad , \tag{7.2.5}$$

and noting $[\partial_{\underline{a}}\,,\,\partial_{\underline{b}}] = 0$.

Mimicking results shown in going from (7.1.8) to (7.1.9), we conclude the super scale transformation laws for the full superspace supercovariant supergravity derivative operators must take the forms which follow from the results in (7.2.1), (7.2.2), and (7.2.3).

Thus, we write

$$\delta_S \nabla_\alpha \; = \tfrac{1}{2}\, L\, \nabla_\alpha \;+\; \tfrac{1}{10}\,(\sigma^{\underline{ab}})_\alpha{}^\beta (\nabla_\beta L)\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.6}$$

$$\delta_S \nabla_{\dot\alpha} \; = \tfrac{1}{2}\, L\, \nabla_{\dot\alpha} \;+\; \tfrac{1}{10}\,(\sigma^{\underline{ab}})_{\dot\alpha}{}^{\dot\beta} (\nabla_{\dot\beta} L)\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.7}$$

$$\delta_S \nabla_{\underline{a}} \; = L\, \nabla_{\underline{a}} \;-\; i\tfrac{1}{5}(\sigma_{\underline{a}})^{\delta\gamma}(\nabla_\delta L)\nabla_\gamma \;-\; i\tfrac{1}{5}(\sigma_{\underline{a}})^{\dot\delta\dot\gamma}(\nabla_{\dot\delta} L)\nabla_{\dot\gamma} \;-\; (\nabla_{\underline{c}} L)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{7.2.8}$$

as the super Weyl transformation laws of superspace supergravity derivatives for 10D, $\mathcal{N}$ = IIA theories.

Upon using the definitions of torsion and curvature tensors

$$\begin{aligned}
[\,\nabla_\alpha \,,\, \nabla_\beta\,\} \; &= T_{\alpha\beta}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\alpha\beta}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\alpha\beta}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\alpha\beta\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\
[\,\nabla_\alpha \,,\, \nabla_{\dot\beta}\,\} \; &= T_{\alpha\dot\beta}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\alpha\dot\beta}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\alpha\dot\beta}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\alpha\dot\beta\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\
[\,\nabla_{\dot\alpha} \,,\, \nabla_{\dot\beta}\,\} \; &= T_{\dot\alpha\dot\beta}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\dot\alpha\dot\beta}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\dot\alpha\dot\beta}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\dot\alpha\dot\beta\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\
[\,\nabla_\alpha \,,\, \nabla_{\underline{b}}\,\} \; &= T_{\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\alpha\underline{b}}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\alpha\underline{b}}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\
[\,\nabla_{\dot\alpha} \,,\, \nabla_{\underline{b}}\,\} \; &= T_{\dot\alpha\underline{b}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\dot\alpha\underline{b}}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\dot\alpha\underline{b}}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\dot\alpha\underline{b}\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\
[\,\nabla_{\underline{a}} \,,\, \nabla_{\underline{b}}\,\} \; &= T_{\underline{ab}}{}^{\underline{c}}\nabla_{\underline{c}} \;+\; T_{\underline{ab}}{}^{\gamma}\,\nabla_\gamma \;+\; T_{\underline{ab}}{}^{\dot\gamma}\,\nabla_{\dot\gamma} \;+\; \tfrac{1}{2}R_{\underline{ab}\,\underline{d}}{}^{\underline{e}}\,\mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad ,
\end{aligned} \tag{7.2.9}$$

appropriate to the case of the 10, $\mathcal{N}$ = IIA theory, we find the Weyl scaling properties of all the superspace torsion and curvature supertensors with weights of less than three-halves as below. Weyl scaling properties of all other superspace torsion and curvature supertensors with weights of equal and larger than three-halves will be presented in Appendix E.1. Grouping according to scale. we find,

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0 \quad , \tag{7.2.10}$$

$$\delta_S T_{\alpha\dot\beta}{}^{\underline{c}} = 0 \quad , \tag{7.2.11}$$

$$\delta_S T_{\dot\alpha\dot\beta}{}^{\underline{c}} = 0 \quad , \tag{7.2.12}$$

$$\delta_S T_{\alpha\beta}{}^{\gamma} = \tfrac{1}{2} L T_{\alpha\beta}{}^{\gamma} + i\tfrac{1}{5} T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) + \tfrac{1}{2} (\nabla_{(\alpha} L) \delta_{\beta)}{}^{\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{(\alpha}{}^{\delta} (\sigma_{[2]})_{\beta)}{}^{\gamma} (\nabla_\delta L) \quad , \tag{7.2.13}$$

$$\delta_S T_{\alpha\beta}{}^{\dot\gamma} = \tfrac{1}{2} L T_{\alpha\beta}{}^{\dot\gamma} + i\tfrac{1}{5} T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot\delta\dot\gamma} (\nabla_{\dot\delta} L) \quad , \tag{7.2.14}$$

$$\delta_S T_{\dot\alpha\dot\beta}{}^{\dot\gamma} = \tfrac{1}{2} L T_{\dot\alpha\dot\beta}{}^{\dot\gamma} + i\tfrac{1}{5} T_{\dot\alpha\dot\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot\gamma\dot\delta} (\nabla_{\dot\delta} L) + \tfrac{1}{2} (\nabla_{(\dot\alpha} L) \delta_{\dot\beta)}{}^{\dot\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{(\dot\alpha}{}^{\dot\delta} (\sigma_{[2]})_{\dot\beta)}{}^{\dot\gamma} (\nabla_{\dot\delta} L) \quad , \tag{7.2.15}$$

$$\delta_S T_{\dot\alpha\dot\beta}{}^{\gamma} = \tfrac{1}{2} L T_{\dot\alpha\dot\beta}{}^{\gamma} + i\tfrac{1}{5} T_{\dot\alpha\dot\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) \quad , \tag{7.2.16}$$

$$\delta_S T_{\alpha\dot\beta}{}^{\gamma} = \tfrac{1}{2} L T_{\alpha\dot\beta}{}^{\gamma} + i\tfrac{1}{5} T_{\alpha\dot\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L) + \tfrac{1}{2} (\nabla_{\dot\beta} L) \delta_\alpha{}^{\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{\dot\beta}{}^{\dot\delta} (\sigma_{[2]})_\alpha{}^{\gamma} (\nabla_{\dot\delta} L) \quad , \tag{7.2.17}$$

$$\delta_S T_{\alpha\dot\beta}{}^{\dot\gamma} = \tfrac{1}{2} L T_{\alpha\dot\beta}{}^{\dot\gamma} + i\tfrac{1}{5} T_{\alpha\dot\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot\delta\dot\gamma} (\nabla_{\dot\delta} L) + \tfrac{1}{2} (\nabla_\alpha L) \delta_{\dot\beta}{}^{\dot\gamma} + \tfrac{1}{20} (\sigma^{[2]})_{\dot\beta}{}^{\dot\gamma} (\sigma_{[2]})_\alpha{}^{\delta} (\nabla_\delta L) \quad , \tag{7.2.18}$$

$$\begin{aligned} \delta_S T_{\alpha\underline{b}}{}^{\underline{c}} = {} & \tfrac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{c}} + (\nabla_\alpha L) \delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5} (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\beta} (\nabla_\beta L) + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\beta\delta} (\nabla_\delta L) T_{\alpha\beta}{}^{\underline{c}} \\ & + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\beta\dot\delta} (\nabla_{\dot\delta} L) T_{\alpha\dot\beta}{}^{\underline{c}} \quad , \end{aligned} \tag{7.2.19}$$

$$\begin{aligned} \delta_S T_{\dot\alpha\underline{b}}{}^{\underline{c}} = {} & \tfrac{1}{2} L T_{\dot\alpha\underline{b}}{}^{\underline{c}} + (\nabla_{\dot\alpha} L) \delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5} (\sigma_{\underline{b}}{}^{\underline{c}})_{\dot\alpha}{}^{\dot\beta} (\nabla_{\dot\beta} L) + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\beta\delta} (\nabla_\delta L) T_{\dot\alpha\beta}{}^{\underline{c}} \\ & + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\beta\dot\delta} (\nabla_{\dot\delta} L) T_{\dot\alpha\dot\beta}{}^{\underline{c}} \quad , \end{aligned} \tag{7.2.20}$$

$$\begin{aligned} \delta_S T_{\alpha\underline{b}}{}^{\gamma} = {} & L T_{\alpha\underline{b}}{}^{\gamma} + i\tfrac{1}{5} T_{\alpha\underline{b}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) - \tfrac{1}{2} (\nabla_{\underline{b}} L) \delta_\alpha{}^{\gamma} - i\tfrac{1}{5} (\sigma_{\underline{b}})^{\gamma\beta} (\nabla_\alpha \nabla_\beta L) \\ & + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\alpha\beta}{}^{\gamma} + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\delta\dot\beta} (\nabla_{\dot\delta} L) T_{\alpha\dot\beta}{}^{\gamma} + \tfrac{1}{2} (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma} (\nabla_{\underline{c}} L) \quad , \end{aligned} \tag{7.2.21}$$

$$\begin{aligned} \delta_S T_{\alpha\underline{b}}{}^{\dot\gamma} = {} & L T_{\alpha\underline{b}}{}^{\dot\gamma} - i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\gamma\dot\beta} (\nabla_\alpha \nabla_{\dot\beta} L) + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\alpha\beta}{}^{\dot\gamma} + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\delta\dot\beta} (\nabla_{\dot\delta} L) T_{\alpha\dot\beta}{}^{\dot\gamma} \\ & - i\tfrac{1}{5} (\sigma_{\underline{c}})^{\dot\delta\dot\gamma} (\nabla_{\dot\delta} L) T_{\alpha\underline{b}}{}^{\underline{c}} \quad , \end{aligned} \tag{7.2.22}$$

$$\begin{aligned} \delta_S T_{\dot\alpha\underline{b}}{}^{\gamma} = {} & L T_{\dot\alpha\underline{b}}{}^{\gamma} - i\tfrac{1}{5} (\sigma_{\underline{b}})^{\gamma\beta} (\nabla_{\dot\alpha} \nabla_\beta L) + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\dot\alpha\beta}{}^{\gamma} + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\delta\dot\beta} (\nabla_{\dot\delta} L) T_{\dot\alpha\dot\beta}{}^{\gamma} \\ & - i\tfrac{1}{5} (\sigma_{\underline{c}})^{\delta\gamma} (\nabla_\delta L) T_{\dot\alpha\underline{b}}{}^{\underline{c}} \quad , \end{aligned} \tag{7.2.23}$$

$$\begin{aligned} \delta_S T_{\dot\alpha\underline{b}}{}^{\dot\gamma} = {} & L T_{\dot\alpha\underline{b}}{}^{\dot\gamma} + i\tfrac{1}{5} T_{\dot\alpha\underline{b}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\dot\gamma\dot\delta} (\nabla_{\dot\delta} L) - \tfrac{1}{2} (\nabla_{\underline{b}} L) \delta_{\dot\alpha}{}^{\dot\gamma} - i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\gamma\dot\beta} (\nabla_{\dot\alpha} \nabla_{\dot\beta} L) \\ & + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\dot\alpha\beta}{}^{\dot\gamma} + i\tfrac{1}{5} (\sigma_{\underline{b}})^{\dot\delta\dot\beta} (\nabla_{\dot\delta} L) T_{\dot\alpha\dot\beta}{}^{\dot\gamma} + \tfrac{1}{2} (\sigma_{\underline{b}}{}^{\underline{c}})_{\dot\alpha}{}^{\dot\gamma} (\nabla_{\underline{c}} L) \quad , \end{aligned} \tag{7.2.24}$$

$$\delta_S T_{\underline{ab}}{}^{\underline{c}} = L T_{\underline{ab}}{}^{\underline{c}} - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_\alpha L)T_{\beta|\underline{b}]}{}^{\underline{c}} - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\beta}}(\nabla_{\dot{\alpha}} L)T_{\dot{\beta}|\underline{b}]}{}^{\underline{c}} \quad , \tag{7.2.25}$$

$$\begin{aligned}\delta_S R_{\alpha\beta}{}^{\underline{de}} &= L R_{\alpha\beta}{}^{\underline{de}} - T_{\alpha\beta}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) + \tfrac{1}{5}T_{\alpha\beta}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) \ + \tfrac{1}{5}T_{\alpha\beta}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\ &\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\nabla_{\beta)}\nabla_\delta L) \quad ,\end{aligned} \tag{7.2.26}$$

$$\begin{aligned}\delta_S R_{\alpha\dot{\beta}}{}^{\underline{de}} &= L R_{\alpha\dot{\beta}}{}^{\underline{de}} - T_{\alpha\dot{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) + \tfrac{1}{5}T_{\alpha\dot{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) \ + \tfrac{1}{5}T_{\alpha\dot{\beta}}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\ &\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\delta}(\nabla_{\dot{\beta}}\nabla_\delta L) \ - \tfrac{1}{5}(\sigma^{\underline{de}})_{\dot{\beta}}{}^{\dot{\gamma}}(\nabla_\alpha\nabla_{\dot{\gamma}} L) \quad ,\end{aligned} \tag{7.2.27}$$

$$\begin{aligned}\delta_S R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} &= L R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} - T_{\dot{\alpha}\dot{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) + \tfrac{1}{5}T_{\dot{\alpha}\dot{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^\delta(\nabla_\delta L) \ + \tfrac{1}{5}T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\ &\quad - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\dot{\alpha}}{}^{\dot{\delta}}(\nabla_{\dot{\beta})}\nabla_{\dot{\delta}} L) \quad .\end{aligned} \tag{7.2.28}$$

Similar to 11D, we define $\mathcal{J}$-tensors and use them to construct the Weyl tensor superfield. The $\mathcal{J}$-tensors constructed are the following,

$$\begin{aligned}\mathcal{J}^{(1)}_\alpha &= \tfrac{1}{2}T_{\alpha\beta}{}^{\beta} \quad , \quad & \mathcal{J}^{(1)}_{\dot{\alpha}} &= \tfrac{1}{2}T_{\dot{\alpha}\dot{\beta}}{}^{\dot{\beta}} \quad , \\ \mathcal{J}^{(2)}_\alpha &= \tfrac{1}{8}T_{\alpha\dot{\beta}}{}^{\dot{\beta}} \quad , \quad & \mathcal{J}^{(2)}_{\dot{\alpha}} &= \tfrac{1}{8}T_{\dot{\alpha}\beta}{}^{\beta} \quad , \\ \mathcal{J}^{(3)}_\alpha &= \tfrac{1}{8}T_{\alpha\underline{b}}{}^{\underline{b}} \quad , \quad & \mathcal{J}^{(3)}_{\dot{\alpha}} &= \tfrac{1}{8}T_{\dot{\alpha}\underline{b}}{}^{\underline{b}} \quad .\end{aligned} \tag{7.2.29}$$

Thus we see

$$\begin{aligned}\delta_S\mathcal{J}^{(1)}_\alpha &= \tfrac{1}{2}L\mathcal{J}^{(1)}_\alpha \ + \ (\nabla_\alpha L) \quad , \quad & \delta_S\mathcal{J}^{(1)}_{\dot{\alpha}} &= \tfrac{1}{2}L\mathcal{J}^{(1)}_{\dot{\alpha}} \ + \ (\nabla_{\dot{\alpha}} L) \quad , \\ \delta_S\mathcal{J}^{(2)}_\alpha &= \tfrac{1}{2}L\mathcal{J}^{(2)}_\alpha \ + \ (\nabla_\alpha L) \quad , \quad & \delta_S\mathcal{J}^{(2)}_{\dot{\alpha}} &= \tfrac{1}{2}L\mathcal{J}^{(2)}_{\dot{\alpha}} \ + \ (\nabla_{\dot{\alpha}} L) \quad , \\ \delta_S\mathcal{J}^{(3)}_\alpha &= \tfrac{1}{2}L\mathcal{J}^{(3)}_\alpha \ + \ (\nabla_\alpha L) \quad , \quad & \delta_S\mathcal{J}^{(3)}_{\dot{\alpha}} &= \tfrac{1}{2}L\mathcal{J}^{(3)}_{\dot{\alpha}} \ + \ (\nabla_{\dot{\alpha}} L) \quad .\end{aligned} \tag{7.2.30}$$

One can see there are only two independent objects (one from each copy of Grassmann coordinates) by changing the basis for these superfields. Define

$$\begin{aligned}\mathcal{J}^{(+)}_\alpha &= \tfrac{1}{3}\big(\mathcal{J}^{(3)}_\alpha \ + \ \mathcal{J}^{(1)}_\alpha \ + \ \mathcal{J}^{(2)}_\alpha\big) \quad , \quad & \mathcal{J}^{(+)}_{\dot{\alpha}} &= \tfrac{1}{3}\big(\mathcal{J}^{(3)}_{\dot{\alpha}} \ + \ \mathcal{J}^{(1)}_{\dot{\alpha}} \ + \ \mathcal{J}^{(2)}_{\dot{\alpha}}\big) \quad , \\ \mathcal{J}^{(-1)}_\alpha &= \tfrac{1}{2}\big(\mathcal{J}^{(3)}_\alpha \ - \ \mathcal{J}^{(1)}_\alpha\big) \quad , \quad & \mathcal{J}^{(-1)}_{\dot{\alpha}} &= \tfrac{1}{2}\big(\mathcal{J}^{(3)}_{\dot{\alpha}} \ - \ \mathcal{J}^{(1)}_{\dot{\alpha}}\big) \quad , \\ \mathcal{J}^{(-2)}_\alpha &= \tfrac{1}{6}\big(\mathcal{J}^{(3)}_\alpha \ + \ \mathcal{J}^{(1)}_\alpha \ - \ 2\mathcal{J}^{(2)}_\alpha\big) \quad , \quad & \mathcal{J}^{(-2)}_{\dot{\alpha}} &= \tfrac{1}{6}\big(\mathcal{J}^{(3)}_{\dot{\alpha}} \ + \ \mathcal{J}^{(1)}_{\dot{\alpha}} \ - \ 2\mathcal{J}^{(2)}_{\dot{\alpha}}\big) \quad .\end{aligned} \tag{7.2.31}$$

The variations of these $\mathcal{J}$-tensors under the scale variation take the forms

$$\begin{aligned} \delta_S \mathcal{J}_\alpha^{(+)} &= \tfrac{1}{2} L \mathcal{J}_\alpha^{(+)} + (\nabla_\alpha L) \quad , \qquad & \delta_S \mathcal{J}_{\dot\alpha}^{(+)} &= \tfrac{1}{2} L \mathcal{J}_{\dot\alpha}^{(+)} + (\nabla_{\dot\alpha} L) \quad , \\ \delta_S \mathcal{J}_\alpha^{(-1)} &= \tfrac{1}{2} L \mathcal{J}_\alpha^{(-1)} \quad , & \delta_S \mathcal{J}_{\dot\alpha}^{(-1)} &= \tfrac{1}{2} L \mathcal{J}_{\dot\alpha}^{(-1)} \quad , \\ \delta_S \mathcal{J}_\alpha^{(-2)} &= \tfrac{1}{2} L \mathcal{J}_\alpha^{(-2)} \quad , & \delta_S \mathcal{J}_{\dot\alpha}^{(-2)} &= \tfrac{1}{2} L \mathcal{J}_{\dot\alpha}^{(-2)} \quad . \end{aligned} \tag{7.2.32}$$

In this basis, only $\mathcal{J}_\alpha^{(+)}$ and $\mathcal{J}_{\dot\alpha}^{(+)}$ serve as spinorial gauge connections. The Weyl tensor is then constructed using conformal weight 1 torsion supertensors and conformal weight $\frac{1}{2}$ spinorial gauge connections. By requiring $\delta_S \mathcal{W}_{\underline{abc}} = L \mathcal{W}_{\underline{abc}}$, we obtain

$$\begin{aligned} \mathcal{W}_{\underline{abc}} = \tfrac{1}{32} \Big\{ & (\sigma^{\underline{d}})_{\gamma\beta} (\sigma_{\underline{abc}})^{\beta\alpha} \, T_{\alpha\underline{d}}{}^{\gamma} - i2 (\sigma_{\underline{abc}})^{\alpha\beta} \left[\nabla_\alpha \mathcal{J}_\beta^{(+)} - \tfrac{6}{5} \mathcal{J}_\alpha^{(+)} \mathcal{J}_\beta^{(+)} \right] \\ & + (\sigma^{\underline{d}})_{\dot\gamma\dot\beta} (\sigma_{\underline{abc}})^{\dot\beta\dot\alpha} \, T_{\dot\alpha\underline{d}}{}^{\dot\gamma} - i2 (\sigma_{\underline{abc}})^{\dot\alpha\dot\beta} \left[\nabla_{\dot\alpha} \mathcal{J}_{\dot\beta}^{(+)} - \tfrac{6}{5} \mathcal{J}_{\dot\alpha}^{(+)} \mathcal{J}_{\dot\beta}^{(+)} \right] \Big\} \quad . \end{aligned} \tag{7.2.33}$$

7.2.2 10D, $\mathcal{N}$ = IIB Supergravity Derivatives & Scale Transformations

Now, the covariant derivative operators linear in the complex conformal compensator Ψ (appropriate for the IIB theory) and necessary for a Nordström theory (Equations (8.92) - (8.94) in [36]) may be given by

$$\nabla_\alpha = \mathrm{D}_\alpha + \tfrac{1}{2} \Psi \mathrm{D}_\alpha + \tfrac{1}{10} (\sigma^{\underline{ab}})_\alpha{}^\beta (\mathrm{D}_\beta \Psi) \mathcal{M}_{\underline{ab}} \quad , \tag{7.2.34}$$

$$\overline{\nabla}_\alpha = \overline{\mathrm{D}}_\alpha + \tfrac{1}{2} \overline{\Psi} \overline{\mathrm{D}}_\alpha + \tfrac{1}{10} (\sigma^{\underline{ab}})_\alpha{}^\beta (\overline{\mathrm{D}}_\beta \overline{\Psi}) \mathcal{M}_{\underline{ab}} \quad , \tag{7.2.35}$$

$$\begin{aligned} \nabla_{\underline{a}} = {} & \partial_{\underline{a}} + \tfrac{1}{2} (\Psi + \overline{\Psi}) \partial_{\underline{a}} - i \tfrac{1}{32} (\sigma_{\underline{a}})^{\alpha\beta} \left[\overline{\mathrm{D}}_\alpha (\Psi + \tfrac{27}{5} \overline{\Psi}) \right] \mathrm{D}_\beta \\ & - i \tfrac{1}{32} (\sigma_{\underline{a}})^{\alpha\beta} \left[\mathrm{D}_\alpha (\overline{\Psi} + \tfrac{27}{5} \Psi) \right] \overline{\mathrm{D}}_\beta - \tfrac{1}{2} \left[\partial_{\underline{c}} (\Psi + \overline{\Psi}) \right] \mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \end{aligned} \tag{7.2.36}$$

here the "bare" algebra of the supersymmetry covariant derivatives and the spacetime partial derivative takes the forms

$$\{ \mathrm{D}_\alpha \, , \, \mathrm{D}_\beta \} = 0 \quad , \quad \{ \overline{\mathrm{D}}_\alpha \, , \, \overline{\mathrm{D}}_\beta \} = 0 \quad , \quad \{ \mathrm{D}_\alpha \, , \, \overline{\mathrm{D}}_\beta \} = i (\sigma^{\underline{a}})_{\alpha\beta} \, \partial_{\underline{a}} \quad , \tag{7.2.37}$$

which indicate the non-vanishing torsion supertensors must be given by

$$T_{\alpha\beta}{}^{\underline{c}} = 0 \quad , \quad T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} = 0 \quad , \quad T_{\alpha\overline{\beta}}{}^{\underline{c}} = i (\sigma^{\underline{c}})_{\alpha\beta} \quad . \tag{7.2.38}$$

By repeating the steps used to obtain (7.2.6), (7.2.7), and (7.2.8) in the last section, here we find,

$$\delta_S \nabla_\alpha = \tfrac{1}{2} L \nabla_\alpha + \tfrac{1}{10} (\sigma^{\underline{ab}})_\alpha{}^\beta (\nabla_\beta L) \mathcal{M}_{\underline{ab}} \quad , \tag{7.2.39}$$

$$\delta_S \overline{\nabla}_\alpha = \tfrac{1}{2} \overline{L} \overline{\nabla}_\alpha + \tfrac{1}{10} (\sigma^{\underline{ab}})_\alpha{}^\beta (\overline{\nabla}_\beta \overline{L}) \mathcal{M}_{\underline{ab}} \quad , \tag{7.2.40}$$

$$\begin{aligned} \delta_S \nabla_{\underline{a}} = & \tfrac{1}{2} (L + \overline{L}) \nabla_{\underline{a}} - i \tfrac{1}{32} (\sigma_{\underline{a}})^{\alpha\beta} \left[\overline{\nabla}_\alpha (L + \tfrac{27}{5} \overline{L}) \right] \nabla_\beta \\ & - i \tfrac{1}{32} (\sigma_{\underline{a}})^{\alpha\beta} \left[\nabla_\alpha (\overline{L} + \tfrac{27}{5} L) \right] \overline{\nabla}_\beta - \tfrac{1}{2} \left[\nabla_{\underline{c}} (L + \overline{L}) \right] \mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad . \end{aligned} \tag{7.2.41}$$

The super-Weyl parameter in (7.2.6), (7.2.7), and (7.2.8) is distinguished from that in (7.2.39), (7.2.40), and (7.2.41). The former involve a real scaling parameter superfield $L = \overline{L}$, whereas the latter involve a complex scaling parameter superfield $L \neq \overline{L}$. This means the transformation laws in (7.2.39), (7.2.40), and (7.2.41) imply both a real scaling parametrized by $\frac{1}{2}(L + \overline{L})$ and also a U(1) rotation parametrized by $i\frac{1}{2}(L - \overline{L})$. The terms proportional to $\mathcal{M}$ in (7.2.39), and (7.2.40) as well as the ones proportional to the spinorial derivatives in (7.2.41) depend on both the real superspace Weyl scaling parameter $\mathrm{u} = \frac{1}{2}(L + \overline{L})$ and the real U(1) rotation parameter $\mathrm{v} = -i\frac{1}{2}(L - \overline{L})$. To see this, one need only note $L = (\mathrm{u} + i\mathrm{v})$ and $\overline{L} = (\mathrm{u} - i\mathrm{v})$ in the latter equations.

Upon using the definitions of torsion and curvature tensors

$$\begin{aligned} [\nabla_\alpha \, , \, \nabla_\beta \} &= T_{\alpha\beta}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\alpha\beta}{}^\gamma \nabla_\gamma + T_{\alpha\beta}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\alpha\beta\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\nabla_\alpha \, , \, \overline{\nabla}_\beta \} &= T_{\alpha\overline{\beta}}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\alpha\overline{\beta}}{}^\gamma \nabla_\gamma + T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\alpha\overline{\beta}\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\overline{\nabla}_\alpha \, , \, \overline{\nabla}_\beta \} &= T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\overline{\alpha}\overline{\beta}}{}^\gamma \nabla_\gamma + T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\overline{\alpha}\overline{\beta}\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\nabla_\alpha \, , \, \nabla_{\underline{b}} \} &= T_{\alpha\underline{b}}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\alpha\underline{b}}{}^\gamma \nabla_\gamma + T_{\alpha\underline{b}}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\alpha\underline{b}\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\overline{\nabla}_\alpha \, , \, \nabla_{\underline{b}} \} &= T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\overline{\alpha}\underline{b}}{}^\gamma \nabla_\gamma + T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\overline{\alpha}\underline{b}\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \\ [\nabla_{\underline{a}} \, , \, \nabla_{\underline{b}} \} &= T_{\underline{ab}}{}^{\underline{c}} \nabla_{\underline{c}} + T_{\underline{ab}}{}^\gamma \nabla_\gamma + T_{\underline{ab}}{}^{\overline{\gamma}} \overline{\nabla}_\gamma + \tfrac{1}{2} R_{\underline{ab}\,\underline{d}}{}^{\underline{e}} \mathcal{M}_{\underline{e}}{}^{\underline{d}} \quad , \end{aligned} \tag{7.2.42}$$

appropriate to the case of the 10D, $\mathcal{N}$ = IIB theory, we find the Weyl scaling properties of all the superspace torsion and curvature supertensors with weights of less than three-halves as below. Weyl scaling properties of all other superspace torsion and curvature supertensors with weights of equal and larger than three-halves will be presented in

Appendix E.2.

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = \tfrac{1}{2}(L-\overline{L})T_{\alpha\beta}{}^{\underline{c}} \quad , \tag{7.2.43}$$

$$\delta_S T_{\alpha\overline{\beta}}{}^{\underline{c}} = 0 \quad , \tag{7.2.44}$$

$$\delta_S T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} = -\tfrac{1}{2}(L-\overline{L})T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} \quad , \tag{7.2.45}$$

$$\delta_S T_{\alpha\beta}{}^{\overline{\gamma}} = (L-\tfrac{1}{2}\overline{L})T_{\alpha\beta}{}^{\overline{\gamma}} + iT_{\alpha\beta}{}^{\underline{c}}(\sigma_{\underline{c}})^{\delta\gamma}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L)) \quad , \tag{7.2.46}$$

$$\begin{aligned}\delta_S T_{\alpha\beta}{}^{\gamma} = {}& \tfrac{1}{2}LT_{\alpha\beta}{}^{\gamma} + iT_{\alpha\beta}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L})) \\ & + \tfrac{1}{2}(\nabla_{(\alpha}L)\delta_{\beta)}{}^{\gamma} + \tfrac{1}{20}(\sigma^{[2]})_{(\alpha}{}^{\delta}(\sigma_{[2]})_{\beta)}{}^{\gamma}(\nabla_\delta L) \quad ,\end{aligned} \tag{7.2.47}$$

$$\begin{aligned}\delta_S T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} = {}& \tfrac{1}{2}LT_{\alpha\overline{\beta}}{}^{\overline{\gamma}} + iT_{\alpha\overline{\beta}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\delta\gamma}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L)) + \tfrac{1}{2}(\nabla_\alpha\overline{L})\delta_\beta{}^{\gamma} \\ & + \tfrac{1}{20}(\sigma^{[2]})_{\beta}{}^{\gamma}(\sigma_{[2]})_{\alpha}{}^{\delta}(\nabla_\delta L) \quad ,\end{aligned} \tag{7.2.48}$$

$$\begin{aligned}\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} = {}& \tfrac{1}{2}LT_{\alpha\underline{b}}{}^{\underline{c}} + \tfrac{1}{2}(\nabla_\alpha(L+\overline{L}))\delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\beta}(\nabla_\beta L) \\ & + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L))T_{\alpha\overline{\beta}}{}^{\underline{c}} + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L}))T_{\alpha\beta}{}^{\underline{c}} \quad ,\end{aligned} \tag{7.2.49}$$

$$\begin{aligned}\delta_S T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} = {}& \tfrac{1}{2}\overline{L}T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} + \tfrac{1}{2}(\overline{\nabla}_\alpha(L+\overline{L}))\delta_{\underline{b}}{}^{\underline{c}} + \tfrac{1}{5}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\beta}(\overline{\nabla}_\beta\overline{L}) \\ & + i(\sigma_{\underline{b}})^{\beta\delta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\beta}{}^{\underline{c}} + i(\sigma_{\underline{b}})^{\beta\delta}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L))T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}} \quad ,\end{aligned} \tag{7.2.50}$$

$$\begin{aligned}\delta_S T_{\alpha\overline{\beta}}{}^{\gamma} = {}& \tfrac{1}{2}\overline{L}T_{\alpha\overline{\beta}}{}^{\gamma} + iT_{\alpha\overline{\beta}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\delta\gamma}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L})) + \tfrac{1}{2}(\overline{\nabla}_\beta L)\delta_\alpha{}^{\gamma} \\ & + \tfrac{1}{20}(\sigma^{[2]})_{\beta}{}^{\delta}(\sigma_{[2]})_{\alpha}{}^{\gamma}(\overline{\nabla}_\delta\overline{L}) \quad ,\end{aligned} \tag{7.2.51}$$

$$\begin{aligned}\delta_S T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} = {}& \tfrac{1}{2}\overline{L}T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + iT_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L)) \\ & + \tfrac{1}{2}(\overline{\nabla}_{(\alpha}\overline{L})\delta_{\beta)}{}^{\gamma} + \tfrac{1}{20}(\sigma^{[2]})_{(\alpha}{}^{\delta}(\sigma_{[2]})_{\beta)}{}^{\gamma}(\overline{\nabla}_\delta\overline{L}) \quad ,\end{aligned} \tag{7.2.52}$$

$$\delta_S T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} = (\overline{L}-\tfrac{1}{2}L)T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} + iT_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L})) \quad , \tag{7.2.53}$$

$$\begin{aligned}\delta_S T_{\alpha\underline{b}}{}^{\overline{\gamma}} = {}& LT_{\alpha\underline{b}}{}^{\overline{\gamma}} - i(\sigma_{\underline{b}})^{\gamma\beta}(\nabla_\alpha\nabla_\beta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L)) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L}))T_{\alpha\beta}{}^{\overline{\gamma}} \\ & + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L))T_{\alpha\overline{\beta}}{}^{\overline{\gamma}} - i(\sigma_{\underline{c}})^{\delta\gamma}(\nabla_\delta(\tfrac{1}{32}\overline{L}+\tfrac{27}{160}L))T_{\alpha\underline{b}}{}^{\underline{c}} \quad ,\end{aligned} \tag{7.2.54}$$

$$\delta_S T_{\alpha\underline{b}}{}^{\gamma} = \tfrac{1}{2}(L+\overline{L})T_{\alpha\underline{b}}{}^{\gamma} + iT_{\alpha\underline{b}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_\delta(\tfrac{1}{32}L+\tfrac{27}{160}\overline{L})) - \tfrac{1}{2}(\nabla_{\underline{b}}L)\delta_\alpha{}^{\gamma}$$

$$- i(\sigma_{\underline{b}})^{\gamma\beta}(\nabla_\alpha\overline{\nabla}_\beta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\alpha\overline{\beta}}{}^{\gamma}$$
$$+ i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\alpha\beta}{}^{\gamma} + \tfrac{1}{4}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma}(\nabla_{\underline{c}}(L + \overline{L})) \quad , \tag{7.2.55}$$

$$\delta_S T_{\underline{a}\underline{b}}{}^{\underline{c}} = \tfrac{1}{2}(L + \overline{L})T_{\underline{a}\underline{b}}{}^{\underline{c}} - i(\sigma_{[\underline{a}})^{\alpha\beta}\left[(\overline{\nabla}_\alpha(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\beta|\underline{b}]}{}^{\underline{c}}(\nabla_\alpha(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\beta}|\underline{b}]}{}^{\underline{c}}\right] \quad , \tag{7.2.56}$$

$$\delta_S T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}} = \tfrac{1}{2}(L + \overline{L})T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}} + iT_{\overline{\alpha}\underline{b}}{}^{\underline{c}}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) - \tfrac{1}{2}(\nabla_{\underline{b}}\overline{L})\delta_\alpha{}^{\gamma}$$
$$- i(\sigma_{\underline{b}})^{\gamma\beta}(\overline{\nabla}_\alpha\nabla_\beta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\beta}{}^{\overline{\gamma}}$$
$$+ i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}} + \tfrac{1}{4}(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma}(\nabla_{\underline{c}}(L + \overline{L})) \quad , \tag{7.2.57}$$

$$\delta_S T_{\overline{\alpha}\underline{b}}{}^{\gamma} = \overline{L}T_{\overline{\alpha}\underline{b}}{}^{\gamma} - i(\sigma_{\underline{b}})^{\gamma\beta}(\overline{\nabla}_\alpha\overline{\nabla}_\beta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\beta}{}^{\gamma}$$
$$+ i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_\delta(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\alpha}\overline{\beta}}{}^{\gamma} - i(\sigma_{\underline{c}})^{\delta\gamma}(\overline{\nabla}_\delta(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\overline{\alpha}\underline{b}}{}^{\underline{c}} \quad , \tag{7.2.58}$$

$$\delta_S R_{\alpha\beta}{}^{\underline{de}} = LR_{\alpha\beta}{}^{\underline{de}} - \tfrac{1}{2}T_{\alpha\beta}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\alpha\beta}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L)$$
$$+ \tfrac{1}{5}T_{\alpha\beta}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\nabla_{\beta)}\nabla_\delta L) \quad , \tag{7.2.59}$$

$$\delta_S R_{\alpha\overline{\beta}}{}^{\underline{de}} = \tfrac{1}{2}(L + \overline{L})R_{\alpha\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}T_{\alpha\overline{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\alpha\overline{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L)$$
$$+ \tfrac{1}{5}T_{\alpha\overline{\beta}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_\alpha{}^{\delta}(\overline{\nabla}_\beta\nabla_\delta L) - \tfrac{1}{5}(\sigma^{\underline{de}})_\beta{}^{\gamma}(\nabla_\alpha\overline{\nabla}_\gamma\overline{L}) \quad , \tag{7.2.60}$$

$$\delta_S R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}} = \overline{L}R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}T_{\overline{\alpha}\overline{\beta}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\overline{\alpha}\overline{\beta}}{}^{\gamma}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\nabla_\delta L)$$
$$+ \tfrac{1}{5}T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_\gamma{}^{\delta}(\overline{\nabla}_\delta\overline{L}) - \tfrac{1}{5}(\sigma^{\underline{de}})_{(\alpha}{}^{\delta}(\overline{\nabla}_{\beta)}\overline{\nabla}_\delta\overline{L}) \quad . \tag{7.2.61}$$

The grouping of terms in (7.2.43) - (7.2.61) is according to the real scale weight and U(1) charge. Their real scale weights and U(1) charges can be read off by expressing all the L and $\overline{L}$ super parameters in terms of u and v. All this information is summarized in Table 7.1.

Again, we can construct $\mathcal{J}$-tensors from linear combinations of torsions with indices contracted in Type IIB theory. There are two independent ones (and their complex conjugates). This is because our scaling parameter superfield is complex and thus we

Torsion / Curvature	Scaling Coefficient	Real Scale Weight (u)	U(1) Charge (v)
$T_{\alpha\beta}{}^{\underline{c}}$	$\frac{1}{2}(L-\overline{L})$	0	$+1$
$T_{\alpha\overline{\beta}}{}^{\underline{c}}$	0	0	0
$T_{\overline{\alpha}\overline{\beta}}{}^{\underline{c}}$	$-\frac{1}{2}(L-\overline{L})$	0	-1
$T_{\alpha\beta}{}^{\overline{\gamma}}$	$L-\frac{1}{2}\overline{L}$	$\frac{1}{2}$	$+\frac{3}{2}$
$T_{\alpha\beta}{}^{\gamma}$	$\frac{1}{2}L$	$\frac{1}{2}$	$+\frac{1}{2}$
$T_{\alpha\overline{\beta}}{}^{\overline{\gamma}}$	$\frac{1}{2}L$	$\frac{1}{2}$	$+\frac{1}{2}$
$T_{\alpha\underline{b}}{}^{\underline{c}}$	$\frac{1}{2}L$	$\frac{1}{2}$	$+\frac{1}{2}$
$T_{\overline{\alpha}\underline{b}}{}^{\underline{c}}$	$\frac{1}{2}\overline{L}$	$\frac{1}{2}$	$-\frac{1}{2}$
$T_{\alpha\overline{\beta}}{}^{\gamma}$	$\frac{1}{2}\overline{L}$	$\frac{1}{2}$	$-\frac{1}{2}$
$T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\gamma}}$	$\frac{1}{2}\overline{L}$	$\frac{1}{2}$	$-\frac{1}{2}$
$T_{\overline{\alpha}\overline{\beta}}{}^{\gamma}$	$\overline{L}-\frac{1}{2}L$	$\frac{1}{2}$	$-\frac{3}{2}$
$T_{\alpha\underline{b}}{}^{\overline{\gamma}}$	L	1	$+1$
$T_{\alpha\underline{b}}{}^{\gamma}$	$\frac{1}{2}(L+\overline{L})$	1	0
$T_{\underline{a}\underline{b}}{}^{\underline{c}}$	$\frac{1}{2}(L+\overline{L})$	1	0
$T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}}$	$\frac{1}{2}(L+\overline{L})$	1	0
$T_{\overline{\alpha}\underline{b}}{}^{\gamma}$	$\overline{L}$	1	-1
$R_{\alpha\beta}{}^{\underline{de}}$	L	1	$+1$
$R_{\alpha\overline{\beta}}{}^{\underline{de}}$	$\frac{1}{2}(L+\overline{L})$	1	0
$R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}}$	$\overline{L}$	1	-1

Table 7.1: Summary of Weyl complex scaling properties of 10D, $\mathcal{N}$ = 2B torsions and curvatures

have two degrees of freedom. The $\mathcal{J}$-tensors are defined by

$$
\begin{aligned}
\mathcal{J}^{(1)}_{\alpha} &= \tfrac{1}{4}T_{\alpha\beta}{}^{\beta} \quad , &\quad \overline{\mathcal{J}}^{(1)}_{\alpha} &= \tfrac{1}{4}T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\beta}} \quad , \\
\mathcal{J}^{(2)}_{\alpha} &= \tfrac{5}{12}T_{\alpha\beta}{}^{\beta} + \tfrac{1}{3}T_{\alpha\overline{\beta}}{}^{\overline{\beta}} - \tfrac{1}{3}T_{\alpha\underline{b}}{}^{\underline{b}} \quad , &\quad \overline{\mathcal{J}}^{(2)}_{\alpha} &= \tfrac{5}{12}T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\beta}} + \tfrac{1}{3}T_{\overline{\alpha}\beta}{}^{\beta} - \tfrac{1}{3}T_{\overline{\alpha}\underline{b}}{}^{\underline{b}} \quad , \\
\mathcal{J}^{(3)}_{\alpha} &= T_{\alpha\beta}{}^{\beta} + \tfrac{50}{89}T_{\alpha\overline{\beta}}{}^{\overline{\beta}} - \tfrac{82}{89}T_{\alpha\underline{b}}{}^{\underline{b}} \quad , &\quad \overline{\mathcal{J}}^{(3)}_{\alpha} &= T_{\overline{\alpha}\overline{\beta}}{}^{\overline{\beta}} + \tfrac{50}{89}T_{\overline{\alpha}\beta}{}^{\beta} - \tfrac{82}{89}T_{\overline{\alpha}\underline{b}}{}^{\underline{b}} \quad .
\end{aligned}
\tag{7.2.62}
$$

Their scale transformations are

$$
\begin{aligned}
\delta_S\mathcal{J}^{(1)}_{\alpha} &= \tfrac{1}{2}L\,\mathcal{J}^{(1)}_{\alpha} + (\nabla_{\alpha}L) \quad , &\quad \delta_S\overline{\mathcal{J}}^{(1)}_{\alpha} &= \tfrac{1}{2}\overline{L}\,\overline{\mathcal{J}}^{(1)}_{\alpha} + (\overline{\nabla}_{\alpha}\overline{L}) \quad , \\
\delta_S\mathcal{J}^{(2)}_{\alpha} &= \tfrac{1}{2}L\,\mathcal{J}^{(2)}_{\alpha} + (\nabla_{\alpha}\overline{L}) \quad , &\quad \delta_S\overline{\mathcal{J}}^{(2)}_{\alpha} &= \tfrac{1}{2}\overline{L}\,\overline{\mathcal{J}}^{(2)}_{\alpha} + (\overline{\nabla}_{\alpha}L) \quad , \\
\delta_S\mathcal{J}^{(3)}_{\alpha} &= \tfrac{1}{2}L\,\mathcal{J}^{(3)}_{\alpha} \quad , &\quad \delta_S\overline{\mathcal{J}}^{(3)}_{\alpha} &= \tfrac{1}{2}\overline{L}\,\overline{\mathcal{J}}^{(3)}_{\alpha} \quad .
\end{aligned}
\tag{7.2.63}
$$

As we can see from the scale transformations, $\mathcal{J}_\alpha^{(1)}$ and $\mathcal{J}_\alpha^{(2)}$ (and their complex conjugates) are spinorial gauge connections, while $\mathcal{J}_\alpha^{(3)}$ (and its complex conjugate) only transforms as a scale covariant tensor of weight $\frac{1}{2}$ without any gauge term. The Weyl tensor can now be constructed from various (u,v) = (1,0) torsions, and combinations of (u,v) = $(\frac{1}{2},+\frac{1}{2})$ and (u,v) = $(\frac{1}{2},-\frac{1}{2})$ spinorial covariant derivatives and spinorial gauge connections such that each term has scaling properties (u,v) = $(1,0)$. The final result is

$$
\begin{aligned}
\mathcal{W}_{\underline{abc}} = \tfrac{1}{32}\Big\{ & (\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha}\, T_{\alpha\underline{d}}{}^{\gamma} + (\sigma^{\underline{d}})_{\gamma\beta}(\sigma_{\underline{abc}})^{\beta\alpha}\, T_{\overline{\alpha}\underline{d}}{}^{\overline{\gamma}} - 4T_{[\underline{abc}]} \\
& - i(\sigma_{\underline{abc}})^{\alpha\beta}\Big[\tfrac{27}{16}\big(\nabla_\alpha \overline{\mathcal{J}}_\beta^{(1)} + \overline{\nabla}_\alpha \mathcal{J}_\beta^{(1)}\big) + \tfrac{5}{16}\big(\nabla_\alpha \overline{\mathcal{J}}_\beta^{(2)} + \overline{\nabla}_\alpha \mathcal{J}_\beta^{(2)}\big) \\
& + \tfrac{1{,}863}{3{,}200}\mathcal{J}_\alpha^{(1)}\overline{\mathcal{J}}_\beta^{(1)} - \tfrac{33}{128}\mathcal{J}_\alpha^{(2)}\overline{\mathcal{J}}_\beta^{(2)} - \tfrac{411}{640}\big(\mathcal{J}_\alpha^{(1)}\overline{\mathcal{J}}_\beta^{(2)} + \mathcal{J}_\alpha^{(2)}\overline{\mathcal{J}}_\beta^{(1)}\big)\Big\} \quad .
\end{aligned}
\tag{7.2.64}
$$

7.2.3 10D, $\mathcal{N} = 1$ Supergravity Derivatives & Scale Transformations

The covariant derivatives linear in the conformal compensator Ψ are given by (Equations (8.27) - (8.28) in [36])

$$\nabla_\alpha = \mathrm{D}_\alpha + \tfrac{1}{2}\Psi \mathrm{D}_\alpha + \tfrac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\mathrm{D}_\beta\Psi)\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.65}$$

$$\nabla_{\underline{a}} = \partial_{\underline{a}} + \Psi\partial_{\underline{a}} - i\,\tfrac{2}{5}\,(\sigma_{\underline{a}})^{\alpha\beta}(\mathrm{D}_\alpha\Psi)\,\mathrm{D}_\beta - (\partial_{\underline{c}}\Psi)\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad , \tag{7.2.66}$$

where the $\mathcal{N} = 1$ "D-operators" satisfy

$$\{\,\mathrm{D}_\alpha \,,\, \mathrm{D}_\beta\,\} = i\,(\sigma^{\underline{a}})_{\alpha\beta}\,\partial_{\underline{a}} \quad . \tag{7.2.67}$$

As the reader has seen the argument that follows from (7.2.65) and (7.2.66), thus we find

$$\delta_S\nabla_\alpha = \tfrac{1}{2}L\nabla_\alpha + \tfrac{1}{10}(\sigma^{\underline{ab}})_\alpha{}^\beta(\nabla_\beta L)\,\mathcal{M}_{\underline{ab}} \quad , \tag{7.2.68}$$

$$\delta_S\nabla_{\underline{a}} = L\nabla_{\underline{a}} - i\,\tfrac{2}{5}\,(\sigma_{\underline{a}})^{\alpha\beta}(\nabla_\alpha L)\nabla_\beta - (\nabla_{\underline{c}}L)\,\mathcal{M}_{\underline{a}}{}^{\underline{c}} \quad . \tag{7.2.69}$$

These results in (7.2.68) and (7.2.69) may be compared with the results derived in [45]. The two sets of Weyl scaling laws are different. However, this is due to the different choices of constraints that were chosen. Once this is taken into account, the scaling laws agree. The scaling laws used in [45] were presented as ansatzë, whereas the ones in (7.2.68) and (7.2.69) were derived from the Nordström supergravity results [36].

The definition of torsion and curvature tensors for the 10D, $\mathcal{N}=1$ theory are identical in form to the equations given in (7.1.5). Thus, here we find repeating the series of calculations for 10D, $\mathcal{N}=1$ superspace as was done for 11D, $\mathcal{N}=1$ superspace yields the results that follow. Weyl scaling properties of all other superspace torsion and curvature supertensors with weights of equal and larger than three halves will be presented in Appendix E.3.

$$\delta_S T_{\alpha\beta}{}^{\underline{c}} = 0 \quad , \tag{7.2.70}$$

$$\begin{aligned} \delta_S T_{\alpha\beta}{}^{\gamma} =& \tfrac{1}{2} L\, T_{\alpha\beta}{}^{\gamma} \;+\; i\,\tfrac{2}{5}\, T_{\alpha\beta}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) \;+\; \tfrac{1}{2}\, (\nabla_{(\alpha} L) \delta_{\beta)}{}^{\gamma} \\ &+\; \tfrac{1}{20}\, (\sigma^{[2]})_{(\alpha}{}^{\delta} (\sigma_{[2]})_{\beta)}{}^{\gamma} (\nabla_\delta L) \quad , \end{aligned} \tag{7.2.71}$$

$$\delta_S T_{\alpha\underline{b}}{}^{\underline{c}} = \tfrac{1}{2} L T_{\alpha\underline{b}}{}^{\underline{c}} \;+\; (\nabla_\alpha L) \delta_{\underline{b}}{}^{\underline{c}} \;+\; \tfrac{1}{5}\, (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\beta} (\nabla_\beta L) \;+\; i\,\tfrac{2}{5}\, (\sigma_{\underline{b}})^{\beta\delta} (\nabla_\delta L) T_{\alpha\beta}{}^{\underline{c}} \quad , \tag{7.2.72}$$

$$\begin{aligned} \delta_S T_{\alpha\underline{b}}{}^{\gamma} =& L\, T_{\alpha\underline{b}}{}^{\gamma} \;+\; i\,\tfrac{2}{5}\, T_{\alpha\underline{b}}{}^{\underline{c}} (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_\delta L) \;-\; \tfrac{1}{2}\, (\nabla_{\underline{b}} L) \delta_\alpha{}^{\gamma} \\ &-\, i\,\tfrac{2}{5}\, (\sigma_{\underline{b}})^{\gamma\beta} (\nabla_\alpha \nabla_\beta L) \;+\; i\,\tfrac{2}{5}\, (\sigma_{\underline{b}})^{\delta\beta} (\nabla_\delta L) T_{\alpha\beta}{}^{\gamma} \;+\; \tfrac{1}{2}\, (\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^{\gamma} (\nabla_{\underline{c}} L) \quad , \end{aligned} \tag{7.2.73}$$

$$\delta_S T_{\underline{a}\underline{b}}{}^{\underline{c}} = L\, T_{\underline{a}\underline{b}}{}^{\underline{c}} \;-\; i\,\tfrac{2}{5}\, (\sigma_{[\underline{a}})^{\alpha\beta} (\nabla_\alpha L)\, T_{\beta|\underline{b}]}{}^{\underline{c}} \quad , \tag{7.2.74}$$

$$\begin{aligned} \delta_S R_{\alpha\beta}{}^{\underline{de}} =& L\, R_{\alpha\beta}{}^{\underline{de}} \;-\; T_{\alpha\beta}{}^{[\underline{d}} (\nabla^{\underline{e}]} L) \;+\; \tfrac{1}{5}\, T_{\alpha\beta}{}^{\gamma} (\sigma^{\underline{de}})_\gamma{}^{\delta} (\nabla_\delta L) \\ &-\, \tfrac{1}{5}\, (\sigma^{\underline{de}})_{(\alpha}{}^{\delta} (\nabla_{\beta)} \nabla_\delta L) \quad . \end{aligned} \tag{7.2.75}$$

Here we define the $\mathcal{J}$-tensors

$$\mathcal{J}_\alpha^{(+)} = \tfrac{1}{6} T_{\alpha\underline{b}}{}^{\underline{b}} \quad , \quad \mathcal{J}_\alpha^{(-)} = T_{\alpha\beta}{}^{\beta} \quad , \tag{7.2.76}$$

and they transform as

$$\delta_S \mathcal{J}_\alpha^{(+)} = \tfrac{1}{2} L \mathcal{J}_\alpha^{(+)} + (\nabla_\alpha L) \quad , \quad \delta_S \mathcal{J}_\alpha^{(-)} = \tfrac{1}{2} L \mathcal{J}_\alpha^{(-)} \quad . \tag{7.2.77}$$

Again, similar as in 11D, $\mathcal{N}=1$ case, $\mathcal{J}_\alpha^{(-)}$ transforms like a scale covariant tensor of weight $\frac{1}{2}$, the quantity $\mathcal{J}_\alpha^{(+)}$ transforms with a scale weight of $\frac{1}{2}$ while being a spinorial gauge connection under a scaling transformation!

The necessary engineering dimension one superfield $\mathcal{W}_{\underline{abc}}$ with Weyl weight one can be defined by

$$\mathcal{W}_{\underline{abc}} = \tfrac{1}{16} \Big\{ (\sigma^{\underline{d}})_{\gamma\beta} (\sigma_{\underline{abc}})^{\beta\alpha} \, T_{\alpha\underline{d}}{}^{\gamma} - i4 (\sigma_{\underline{abc}})^{\alpha\beta} \left[\nabla_\alpha \mathcal{J}_\beta^{(+)} - \tfrac{22}{25} \mathcal{J}_\alpha^{(+)} \mathcal{J}_\beta^{(+)} \right] \Big\} \quad . \tag{7.2.78}$$

Chapter 8

Adyndra Digital Analysis Scans

This chapter is based on [40].

In [57] and [58], we have established a breakthrough approach that substantially lowered computational costs of determining how to embed a set of component fields within a superfield. Just as MRI (magnetic resonance imaging - based on the phenomenon of nuclear magnetic resonance) has brought amazing progress in creating high definition images of internal structures of bodies in biological domains, the approach in [57, 58] (which can be called "Adyndra Digital Analysis" or "ADA") permits the rapid assay of the Lorentz spectrum of component fields within superfields.

There are a number of ways to define an "adynkra[1]." One such meaning is as the name for a certain class of graphs. However, for our purposes here, we will use a definition tailored to 10D, $\mathcal{N} = 1$ SUSY where an adynkra is a collection of sets of Dynkin Labels that can be broken into subsets associated with a "Level number." The latter of these is an integer that takes on values from zero to sixteen. There exists a "generator" for such lists that takes the form

$$\mathcal{G} = 1 \oplus \ell \, \{ (\Box) \times [a_1, b_1, c_1, d_1, e_1] \} \oplus \bigoplus_{p=2}^{16} \tfrac{1}{p!} \, (\ell)^p \, \{ \big(\Box \, (\wedge \Box)^{p-2} \wedge \Box \big) \times [a_p, b_p, c_p, d_p, e_p] \} \quad , \tag{8.0.1}$$

[1] This is a word created from the concatenation of the works "adinkra" and "Dynkin" in an effort to denote the fusion of these two concepts.

which can be expanded to,

$$
\begin{aligned}
\mathcal{G} \; &= \ell \, \{(\square) \times [a_1, b_1, c_1, d_1, e_1]\} \\
&\quad \oplus \bigoplus_{p=1}^{7} \tfrac{1}{p!} \, (\ell)^{2p+1} \left\{ \left(\square \, (\wedge \square)^{2p-1} \wedge \square \right) \times [a_{2p+1}, b_{2p+1}, c_{2p+1}, d_{2p+1}, e_{2p+1}] \right\} \\
&\quad \oplus \, 1 \, \oplus \bigoplus_{p=1}^{8} \tfrac{1}{(2p)!} \, (\ell)^{2p} \left\{ \left(\square \, (\wedge \square)^{2(p-1)} \wedge \square \right) \times [a_{2p}, b_{2p}, c_{2p}, d_{2p}, e_{2p}] \right\} \quad ,
\end{aligned}
\tag{8.0.2}
$$

where $[a_p, b_p, c_p, d_p, e_p]$ (with $p = 1, \ldots, 16$) denote Dynkin Labels appropriate for the fields over the 10-dimensional manifold. The "Level number" is the exponent seen associated with various powers of the level parameter ℓ in this expression. Finally $\square$ is a spinorial Young Tableau and $\wedge$ denotes the "wedge" product of the tableau. A more complete discussion of conventions and notation used in writing (8.0.1) can be found in [59]. All the terms on the first two lines of (8.0.2) are fermionic representations as the Young Tableau associated with $\square$ corresponds to the fundamental spinor representation of $\mathfrak{so}(1, 9)$. As the final terms have only even powers of $\square$ and due to the identity

$$
\square \wedge \square \; = \; \begin{array}{|c|}\hline \\ \hline \\ \hline \end{array} \; = \; \begin{array}{|c|}\hline \\ \hline \\ \hline \\ \hline \end{array} \quad ,
\tag{8.0.3}
$$

these can be expressed *solely* in terms of YT's involving $\square$ (bosonic Young Tableaux), i. e. bosonic representations of $\mathfrak{so}(1, 9)$.

The generator (8.0.1) provides a basis for the creation of algorithms that are extraordinarily efficient at encoding representations of component fields contained within superfields. It is this efficiency that enabled unprecedented clarity about the component field contents of superfields in ten and eleven dimensions [57–59] and the computational tools we use are noted in [72, 73, 137].

The discussion in [58] is devoted to using the concepts of Branching Rules, Dynkin Labels, Plethysm, and Young Tableaux to calculate $\mathcal{G}$. In [57, 58] the basic "scalar" adynkra, denoted by $\mathcal{V}$, was calculated and described in detail. It is the foundational quantity expressed in terms of YT's. The computation expense to find all related quantities is substantially less than that needed to explicitly determine $\mathcal{V}$. This follows as higher representations are simply found from multiplication of $\mathcal{V}$ by Dynkin Labels when the latter are represented by YT's. Thus, we have

$$\begin{aligned}\mathcal{V}_{[0,0,1,0,0]} &= [0,0,1,0,0] \otimes \mathcal{V} \quad , \quad \mathcal{V}_{[1,0,1,0,1]} = [1,0,1,0,1] \otimes \mathcal{V} \quad , \\ \mathcal{V}_{[3,0,0,0,1]} &= [3,0,0,0,1] \otimes \mathcal{V} \quad , \quad \mathcal{V}_{[4,0,0,0,0]} = [4,0,0,0,0] \otimes \mathcal{V} \quad ,\end{aligned} \tag{8.0.4}$$

and these multiplications are straightforward by the use of the tools indicated in the references [72, 73, 137]. For any bosonic irrep $\mathcal{R}$ or fermionic irrep $\mathcal{R}$, (8.0.4) implies $\dim(\mathcal{V}_{\mathcal{R}}) = \dim(\mathcal{R})\dim(\mathcal{V})$ or $\dim(\mathcal{V}_{\mathcal{R}}) = \dim(\mathcal{R})\dim(\mathcal{V})$, where $\dim(\mathcal{V}) = 65{,}536$. The Dynkin Label Library of the $\mathcal{V}$-superfield is explicitly shown below and similar results are shown in the Appendix F for all superfields in (8.0.4).

- Level-0: [0, 0, 0, 0, 0]
- Level-1: [0, 0, 0, 1, 0]
- Level-2: [0, 0, 1, 0, 0]
- Level-3: [0, 1, 0, 0, 1]
- Level-4: $[0,2,0,0,0] \oplus [1,0,0,0,2]$
- Level-5: $[0,0,0,0,3] \oplus [1,1,0,0,1]$
- Level-6: $[0,1,0,0,2] \oplus [2,0,1,0,0]$
- Level-7: $[3,0,0,1,0] \oplus [1,0,1,0,1]$
- Level-8: $[4,0,0,0,0] \oplus [0,0,2,0,0] \oplus [2,0,0,1,1]$
- Level-9: $[3,0,0,0,1] \oplus [1,0,1,1,0]$
- Level-10: $[0,1,0,2,0] \oplus [2,0,1,0,0]$
- Level-11: $[0,0,0,3,0] \oplus [1,1,0,1,0]$
- Level-12: $[0,2,0,0,0] \oplus [1,0,0,2,0]$
- Level-13: [0, 1, 0, 1, 0]

- Level-14: [0, 0, 1, 0, 0]
- Level-15: [0, 0, 0, 0, 1]
- Level-16: [0, 0, 0, 0, 0]

The utility of such a library is straightforward.

For example, one could ask whether the component field corresponding to the bosonic irrep [4, 0, 1, 0, 0] occurs within the θ expansion of $\mathcal{V}$? Upon examination of the library above, it is seen that the answer is, "No." In a similar manner, one can ask if the pair of representations given by [0, 2, 0, 0, 0] and [3, 0, 0, 0, 1] occur in the superfield $\mathcal{V}$ at adjacent levels with the fermionic one higher than the lower one? Once more a quick consultation of the library above returns the answer, "no." Clearly, more complicated questions of this nature can be pursued and we turn to this next.

8.1 Component Dynkin Label Examples & ADA Scans

The component fermionic and bosonic representation content of the on-shell 10D, $\mathcal{N}$ = 1 supergravity multiplet [138, 139] is given by

$$\begin{aligned} \{\mathcal{F}\}_{\rm SGI} &= (\,\chi_\alpha,\, \psi_{\underline{a}}{}^{\alpha}\,) &&= ([0,0,0,1,0],\, [1,0,0,0,1]) \quad , \\ \{\mathcal{B}\}_{\rm SGI} &= (\,\varphi,\, B_{\underline{a}\,\underline{b}},\, {\rm e}_{\underline{a}}{}^{m}\,) &&= ([0,0,0,0,0],\, [0,1,0,0,0],\, [2,0,0,0,0]) \quad , \end{aligned} \tag{8.1.1}$$

in its most common form. However, as first emphasized in the work of [140], there is an alternate version where the replacement $B_{\underline{a}\,\underline{b}} \to M_{\underline{a}_1 \cdots \underline{a}_6}$ is compatible with the absence of anomalies [141, 142] as a realization of the LEEA of the heterotic string [143].

To complete 10D, $\mathcal{N}$ = 1 supergravity into either the 10D, $\mathcal{N}$ = IIA or IIB supergravity multiplets, it is necessary to find two other 10D, $\mathcal{N}$ = 1 multiplets that contain the matter gravitino multiplets (MGM) discussed in Appendix C of the work [102] and that indicated the existence of two such 10D, $\mathcal{N}$ = 1 MGM systems. We may call one of them the "IIAMGM" system and the "IIBMGM" system. In terms of their component fermionic and bosonic representation contents, these look as

$$\begin{aligned}
\{\mathcal{F}\}_{\text{IIAMGM}} &= (\,\chi_{\dot{\alpha}},\, \psi_{\underline{a}}{}^{\dot{\alpha}}\,) &&= ([0,0,0,0,1],\, [1,0,0,1,0]) \quad , \\
\{\mathcal{B}\}_{\text{IIAMGM}} &= (\,B_{\underline{a}},\, A_{\underline{a}\,\underline{b}\,\underline{c}}\,) &&= ([1,0,0,0,0],\, [0,0,1,0,0]) \quad ,
\end{aligned} \tag{8.1.2}$$

and

$$\begin{aligned}
\{\mathcal{F}\}_{\text{IIBMGM}} &= (\,\chi'_{\alpha},\, \psi'_{\underline{a}}{}^{\alpha}\,) &&= ([0,0,0,1,0],\, [1,0,0,0,1]) \quad , \\
\{\mathcal{B}\}_{\text{IIBMGM}} &= (\,A,\, B'{}_{\underline{a}\,\underline{b}},\, A_{\underline{a}\,\underline{b}\,\underline{c}\,\underline{d}}\,) &&= ([0,0,0,0,0],\, [0,1,0,0,0],\, [0,0,0,1,1]) \quad ,
\end{aligned} \tag{8.1.3}$$

respectively. It can be seen that the fermionic representations are presented in a manner where they are "higher" than the bosonic ones. This is due to the fact that dynamical fermionic fields possess higher engineering dimensions than dynamical bosonic fields. In terms of the "Level" numbers, the fermions are higher than the bosons.

Both the engineering dimensions and the Lorentz representations of all fields are key data inputs in the construction of Adynkra Digital Analysis (ADA) scans.

The adynkras for $\mathcal{V}$, $\mathcal{V}_{[0,0,1,0,0]}$, $\mathcal{V}_{[1,0,1,0,1]}$, $\mathcal{V}_{[3,0,0,0,1]}$, and $\mathcal{V}_{[4,0,0,0,0]}$ can be regarded as a set of "libraries." An ADA scan asks simple questions as any number of such queries can be asked. One is, "Given the data of Level difference and the Dynkin Labels demonstrated in (8.1.2) does such a data pattern occur in the $\mathcal{V}$-library?" Another might be, "How many times does such a pattern occur?"

Going back to the work of [144], we have long asserted the interpretation of adinkras as being the analogs of genetic sequence content where superfields play the roles of biological bodies. An alternative would have been to analogize adinkras to quarks. However, this analogy suffers when one realizes the numbers of degrees of freedom of systems in ten and eleven dimensions (e. g. 11D SG possesses 2,147,483,648 bosonic and the same number of fermionic degrees). So the number of possible targets in searches are of more "biological" in order of magnitude than the numbers encountered in determining the quark content in hadronic spectroscopy problems.

Thus, the process of querying adynkra libraries more closely favors the challenges encountered in DNA analysis than the analysis of the quark spectra of hadronic matter.

With modern IT platforms, both hardware and software, it is a straightforward matter to meet the challenges of writing codes to query such libraries... after they have been constructed. To focus this more accurately, the analogy is to regard the data (as given in (8.1.1), (8.1.2), and (8.1.3)) as primary biological sequence content, (e. g. similar in spirit to nucleotides of DNA/RNA or protein amino-acids). The adynkra "libraries" play the roles of genetic sequence databases/libraries[2].

8.2 Toward the Rest of the Story

The efforts we have described in chapters 7.1 and 7.2 follow the "traditional" routes to understanding supergravity in superspace where attention is focused on the "outside" variables, i. e. $\nabla_{\underline{A}}$, $T_{\underline{A}\,\underline{B}}{}^{\underline{C}}$, and $R_{\underline{A}\,\underline{B}\,\underline{c}}{}^{\underline{d}}$. However, there is also a less trod pathway based on the study of the prepotentials, i. e. Ψ, $\mathcal{A}^{\underline{a}\,\underline{b}}$, $\mathcal{N}_{\alpha}{}^{\beta}$, and $\mathrm{H}_{\beta}{}^{\underline{b}}$. Really, due to the presence of constraints and covariance, only Ψ and $\mathrm{H}_{\beta}{}^{\underline{b}}$ require deeper study. Heretofore, mostly the literature [38,56], has focused on the "outside" superfields rather than on the "inside" ones, Ψ, and $\mathrm{H}_{\beta}{}^{\underline{b}}$. The most obvious reason for this is the traditional approach requires a high computational price be paid to elucidate the θ-expansion for component fields residing within the "inside" superfields.

As we wish to treat the cases of the $\mathcal{N}=1$, $\mathcal{N}=2$A, and $\mathcal{N}=2$B uniformly, there is a conceptual approach available as an effective enabling strategy. We now turn to a discussion of this.

In the works of [146–150] there was initiated an approach where superfields with a higher realization of supersymmetry were formulated in terms of superfields that provide a lower realization of supersymmetry. In these cases, mostly superfields with 4D, $\mathcal{N}=2$ SUSY were expressed in terms of superfields with 4D, $\mathcal{N}=1$ SUSY. More recently [151–154], this approach has been implemented in the context of superfields with 11D, $\mathcal{N}=1$ SUSY expressed in terms of superfields with 4D, $\mathcal{N}=1$ SUSY. Since we are working in the arena of 10D, $\mathcal{N}=1$, $\mathcal{N}=2$A, and $\mathcal{N}=2$B SUSY, the lower dimensional superfields in which to conduct our investigations are ones that realize 10D, $\mathcal{N}=1$ SUSY.

[2] One example of such a bioinformatics IT tool is the "basis local alignment search tool" (BLAST) [145] that serves this function.

To create a set of notational conventions that are graphically easy to follow in our subsequent discussion, we will denote the coordinates of 10D, $\mathcal{N} = 1$ superspace by $(x^{\underline{a}}, \theta^{\alpha})$. In order to describe the superfields with $\mathcal{N} = 2$A, and $\mathcal{N} = 2$B SUSY, we introduce a second Grassmann coordinate of the form $\theta^{\dot{\alpha}}$ in the case of the $\mathcal{N} = 2$A superfields or of the form θ^{α} in the case of the $\mathcal{N} = 2$B superfields. Now we can continue to study the superfields $\mathcal{V}(x^{\underline{a}}, \theta^{\alpha})$, $\mathcal{V}_{IIA}(x^{\underline{a}}, \theta^{\alpha}, \theta^{\dot{\alpha}})$, or $\mathcal{V}_{IIB}(x^{\underline{a}}, \theta^{\alpha}, \theta^{\alpha})$ in each respective case. Next we treat these superfields, but in the latter two cases we use respective explicit expansions in terms of "green θ's,"

$$\mathcal{V}_{IIA}(x^{\underline{a}}, \theta^{\alpha}, \theta^{\dot{\alpha}}) \;=\; \mathcal{V}^{(0)}(x^{\underline{a}}, \theta^{\alpha}) + \theta^{\dot{\alpha}}\, \mathcal{V}^{(1)}_{\dot{\alpha}}(x^{\underline{a}}, \theta^{\alpha}) + \theta^{\dot{\alpha}}\theta^{\dot{\beta}}\, \mathcal{V}^{(2)}_{\dot{\alpha}\dot{\beta}}(x^{\underline{a}}, \theta^{\alpha}) + \dots \quad , \tag{8.2.1}$$

$$\mathcal{V}_{IIB}(x^{\underline{a}}, \theta^{\alpha}, \theta^{\alpha}) \;=\; \mathcal{V}^{(0)}(x^{\underline{a}}, \theta^{\alpha}) + \theta^{\alpha}\, \mathcal{V}^{(1)}_{\alpha}(x^{\underline{a}}, \theta^{\alpha}) + \theta^{\alpha}\theta^{\beta}\, \mathcal{V}^{(2)}_{\alpha\beta}(x^{\underline{a}}, \theta^{\alpha}) + \dots \quad , \tag{8.2.2}$$

and where these expansions terminate at the sixteenth order of the "green θ's." Clearly, these indicate respective sets of sixteen distinct 10D, $\mathcal{N} = 1$ superfields within the two types of 10D, $\mathcal{N} = 2$ superfields.

Although the two expansions in (8.2.1) and (8.2.2) appear rather similar, an interesting dichotomy emerges when ADA algorithms are applied to them. In the case of (8.2.1), it is seen for all the fermionic superfields, *none* contributes the conformal graviton representation to level-16 of Type IIA and IIB scalar superfields. In the case of (8.2.2), it is seen for all of the fermionic superfields except for $\mathcal{V}_{\alpha}$, $\mathcal{V}^{\alpha}$, $\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}{}^{\alpha}$, and $\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-\alpha}$, *all* contain the conformal graviton representation. This result suggests if we wish to find formulations of the Type-IIA and Type-IIB theories that possess a common truncation to the Type-I theory, we should eliminate the fermionic superfields in both expansions as subjects of additional study in this regard.

Now concentrating solely on the bosonic superfields in (8.2.1) and (8.2.2), we apply ADA algorithms to search within the bosonic 10D, $\mathcal{N} = 1$ superfields for the number $b_{\{1\}}$ of singlet irreps, the number $b_{\{45\}}$ of $\{45\}$ irreps, the number $b_{\{54\}}$ of $\{54\}$ irreps, the number $b_{\{210\}}$[3] of $\{210\}$ irreps, the number $b_{\{\overline{16}\}}$ of $\{\overline{16}\}$ irreps, and the number $b_{\{\overline{144}\}}$ of $\{\overline{144}\}$ irreps. Furthermore, the ADA scans for these irreps within the bosonic 10D, $\mathcal{N}$ = 1 superfields were designed additionally so that all bosonic irreps should occur at the same level in the superfield and also to fulfill the requirement that the $\{\overline{16}\}$ irrep and the $\{\overline{144}\}$ irrep should occur at one level higher than the bosonic irreps. The 10D, $\mathcal{N}$ = 2A results are shown in the Table 8.1 below.

Type-I Superfield	Dynkin Label	$b_{\{1\}}$	$b_{\{45\}}$	$b_{\{54\}}$	$b_{\{210\}}$	$b_{\{\overline{16}\}}$	$b_{\{\overline{144}\}}$	Level of $\{54\}$ in Type-I Superfield
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}$	$[0,0,1,0,0]$	1	1	1	2	1	1	14
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}$	$[0,2,0,0,0]$	1	1	1	2	1	1	12
$\mathcal{V}_{\{\underline{a}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$	$[1,0,0,2,0]$	0	0	1	1	0	1	12
$\mathcal{V}_{\{\underline{a}_2\underline{b}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$	$[0,1,0,2,0]$	0	0	1	1	0	1	10
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\}}$	$[2,0,1,0,0]$	1	2	2	4	1	3	10
$\mathcal{V}_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}$	$[4,0,0,0,0]$	1	1	1	1	1	1	8
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}$	$[0,0,2,0,0]$	1	1	1	3	1	2	8
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}$	$[2,0,0,1,1]$	1	3	2	6	2	5	8
$\mathcal{V}_{\{\underline{a}_2\underline{b}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$	$[0,1,0,0,2]$	0	0	1	1	0	1	6
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\}}$	$[2,0,1,0,0]$	1	2	2	4	2	4	6
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}$	$[0,2,0,0,0]$	1	1	1	2	1	2	4
$\mathcal{V}_{\{\underline{a}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$	$[1,0,0,0,2]$	0	0	1	1	0	1	4
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}$	$[0,0,1,0,0]$	1	1	1	2	1	2	2

Table 8.1: Contributions of Each Type-I Superfield to the component bosonic and fermionic representation content of the 10D, $\mathcal{N}$ = 1 supergravity multiplet at Level-16/17 of Type-IIA Scalar Superfield Decomposition

Looking at this table, the reader's attention is directed to the Dynkin Labels. The distinct ones are $[0,0,1,0,0]$, $[0,2,0,0,0]$, $[1,0,0,2,0]$, $[0,1,0,2,0]$, $[2,0,1,0,0]$, $[4,0,0,0,0]$, $[0,0,2,0,0]$, $[2,0,0,1,1]$, $[0,1,0,0,2]$, and $[1,0,0,0,2]$.

In a similar manner, ADA scans can be applied to the bosonic 10D, $\mathcal{N}$ = 1 superfields in the expansion shown in (8.2.2). In this case the sets of members of bosonic 10D, $\mathcal{N}$ = 1 superfields that are found are identical to the ones found by starting from the expansion in (8.2.1). The 10D, $\mathcal{N}$ = 2B results are shown in Table 8.2.

For all Type-I superfields shown in the two tables above, if there is $\{54\}$ appears in level-n, there exits $\{120\}$ irrep in level-$(n+2)$, which is the three(seven)-form (i.e. the

[3] The reason this irrep is considered in the search is due to the fact that it is known there is a duality "flip" [155] that exchanges the $b_{\{54\}}$ and $b_{\{210\}}$ irreps in 10D, $\mathcal{N}$ = 1 supergravity and which is consistent with the anomaly-freedom conditions and Green-Schwarz σ-models associated with the low-energy heterotic string [140, 156] effective action.

Type-I Superfield	Dynkin Label	$b_{\{1\}}$	$b_{\{45\}}$	$b_{\{54\}}$	$b_{\{210\}}$	$b_{\{\overline{16}\}}$	$b_{\{\overline{144}\}}$	Level of {54} in Type-I Superfield
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}$	[0, 0, 1, 0, 0]	1	1	1	2	1	1	14
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}$	[0, 2, 0, 0, 0]	1	1	1	2	1	1	12
$\mathcal{V}_{\{\underline{a}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$	[1, 0, 0, 0, 2]	1	2	1	3	1	2	12
$\mathcal{V}_{\{\underline{a}_2\underline{b}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$	[0, 1, 0, 0, 2]	1	2	1	4	2	3	10
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\}}$	[2, 0, 1, 0, 0]	1	2	2	4	1	3	10
$\mathcal{V}_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}$	[4, 0, 0, 0, 0]	1	1	1	1	1	1	8
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2\underline{b}_2\underline{c}_2\}}$	[0, 0, 2, 0, 0]	1	1	1	3	1	2	8
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}$	[2, 0, 0, 1, 1]	1	3	2	6	2	5	8
$\mathcal{V}_{\{\underline{a}_2\underline{b}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$	[0, 1, 0, 2, 0]	1	2	1	4	1	3	6
$\mathcal{V}_{\{\underline{a}_2,\underline{a}_3\vert\underline{a}_1\underline{b}_1\underline{c}_1\}}$	[2, 0, 1, 0, 0]	1	2	2	4	2	4	6
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}$	[0, 2, 0, 0, 0]	1	1	1	2	1	2	4
$\mathcal{V}_{\{\underline{a}_2\vert\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$	[1, 0, 0, 2, 0]	1	2	1	3	2	3	4
$\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}$	[0, 0, 1, 0, 0]	1	1	1	2	1	2	2

Table 8.2: Contributions of Each Type-I Superfield to the component bosonic and fermionic representation content of the 10D, $\mathcal{N} = 1$ supergravity multiplet at Level-16/17 of Type-IIB Scalar Superfield Decomposition

field strength for either the {45} or {210} irreps). Also upon comparing the two tables, it is clear that the same two sets of 10D, $\mathcal{N} = 1$ superfields appear in both. The only differences that occur take place when a prepotential involves a 5-form index. In these cases, the 5-form indices are "flipped" between the 10D, $\mathcal{N} =$ IIA and 10D, $\mathcal{N} =$ IIB theories.

The last column of the preceding two tables gives the height of the conformal graviton irrep in the Type-I superfield. This simultaneously gives the location of the Type-I superfield in the θ-expansion. So for example the conformal graviton irrep indicated in the first row of either of the preceding tables is found at Level number 14. This means that the type-I superfield containing those conformal graviton irreps are located at quadratic order in the θ-expansion.

We must exercise a note of caution in regards to our proposal, however. There is a long-standing alternate proposal [104] for the 10D, $\mathcal{N} = 1$ superspace supergravity prepotential. This work has suggested that the {120} irrep (the [0, 0, 1, 0, 0] irrep described by $\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1\}}$), should play this role. One argument that mitigates against this concern is the fact that the [0, 0, 1, 0, 0] irrep in the 10D theory does not connect to the unique conformal graviton seen in the 11D scalar supermultiplet.

In this chapter, we have looked at the issue of what superfield could serve as the 10D,

$\mathcal{N} = 1$ supergravity prepotential. Our investigation began from the scalar superfield for both the Type-IIA and Type-IIB superspaces. The choice of the scalar superfield is motivated by our study of the 11D, $\mathcal{N} = 1$ superspace gravity theory [58] where the scalar superfield is the simplest superfield that contains the conformal graviton representation. In this context, it was found that the conformal graviton occurs exactly at the middle level of the superfield. A truncation of 11D, $\mathcal{N} = 1$ superspace to 10D, $\mathcal{N} =$ IIA superspace must also then yield a scalar superfield as the supergravity prepotential.

Next in this line of reasoning, we may consider a truncation of 10D, $\mathcal{N} =$ IIA superspace to 10D, $\mathcal{N} = 1$ superspace. As our discussions in this chapter showed, there are choices to explore as the putative supergravity prepotential. The ADA algorithms inform us there are only a few options to explore, at least under the set of assumptions we are using. These include the ten possibilities given by $\mathcal{V}_{\{\underline{a}_1\underline{a}_2\underline{a}_3\}}$, $\mathcal{V}_{\{\underline{a}_1\underline{b}_1,\underline{a}_2\underline{b}_2\}}$, $\mathcal{V}_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}$, $\mathcal{V}_{\{\underline{a}_2,\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\}}$, $\mathcal{V}_{\{\underline{a}_1\underline{b}_1\underline{c}_1,\underline{a}_2,\underline{b}_2\underline{c}_2\}}$, $\mathcal{V}_{\{\underline{a}_2\underline{a}_3|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\}}$, $\mathcal{V}_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$, $\mathcal{V}_{\{\underline{a}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$, $\mathcal{V}_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^+}$, and $\mathcal{V}_{\{\underline{a}_2\underline{b}_2|\underline{a}_1\underline{b}_1\underline{c}_1\underline{d}_1\underline{e}_1\}^-}$.

However, among the ten choices indicated by the two tables in this chapter, there is one that is extremely interesting as it contains a single copy of each of the component irreps that were the parameters of our ADA scan. This representation is the superfield $\mathcal{V}_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}$, with Dynkin Label $[4,0,0,0,0]$ which corresponds to a 10D, totally symmetical, completely traceless fourth rank tensor superfield

$$\mathcal{V}_{[4,0,0,0,0]}(x^{\underline{a}},\, \theta^{\alpha}) \;=\; \mathcal{V}_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3,\underline{a}_4\}}(x^{\underline{a}},\, \theta^{\alpha}) \quad . \tag{8.2.3}$$

It also possesses the property that the conformal graviton irrep occurs in the middle level of the superfield. These two properties suggest to us this should be the superfield that is the most likely candidate for the 10D, $\mathcal{N} = 1$ supergravity prepotential.

The next task becomes one of finding if there are Type-I superfields (SF's) in the expansions shown in (8.2.1) and (8.2.2) that will allow the patterns of height assignments and irreps seen in (8.1.2) and (8.1.3) to be found within them? So we set up an ADA scan in (8.2.1) looking for the first pattern and one in (8.2.2) looking for the second pattern at levels compatible with the 10D, $\mathcal{N} = 1$ SG prepotential identified in (8.2.3). The task is simplified by an observation on how the non-manifest supercharges act relative to the identification of the [4, 0, 0, 0, 0] supergravity prepotential which occurs at θ-Level eight. The second non-manifest supercharge must connect quantities at different θ-Levels. The simplest example of this phenomenon is seen in the 4D, $\mathcal{N} = 1$ chiral supermultiplet. The spin-0 fields are at the lowest θ-Level and a supercharge acting on these spin-0 fields connect them to the spin-1/2 fields at one θ-Level higher. Translating this lesson for the present consideration, the [4, 0, 0, 0, 0] supergravity prepotential should be connected to a 10D, $\mathcal{N} = 1$ superfield at θ-Level nine. So the present scan can be restricted to a search at this level. The results of these scans are shown in Tables 8.3 and 8.4.

Type-I SF	Dynkin Label	$b_{\{10\}}$	$b_{\{120\}}$	$b_{\{16\}}$	$b_{\{144\}}$	θ-Level	Boson Height	Fermion Height
$\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}\alpha}$	[3, 0, 0, 1, 0]	1	3	2	3	9	7	8
$\Psi_{\{\underline{a}_2\|\underline{a}_1\underline{b}_1\underline{c}_1\}}{}^{\alpha}$	[1, 0, 1, 0, 1]	1	4	2	5	9	7	8

Table 8.3: Type-I Superfield Contributions to Bosons at Level-16 & Fermions at Level-17 in Type-IIA Scalar Superfield Decomposition

Type-I SF	Dynkin Label	$b_{\{1\}}$	$b_{\{45\}}$	$b_{\{210\}}$	$b_{\{\overline{16}\}}$	$b_{\{\overline{144}\}}$	θ-Level	Boson Height	Fermion Height
$\Psi_{\{\underline{a}_1,\underline{a}_2,\underline{a}_3\}}{}^{\alpha}$	[3, 0, 0, 0, 1]	1	2	3	2	3	9	7	8
$\Psi_{\{\underline{a}_2\|\underline{a}_1\underline{b}_1\underline{c}_1\}\alpha}$	[1, 0, 1, 1, 0]	1	3	6	2	5	9	7	8

Table 8.4: Type-I Superfield Contributions to Bosons at Level-16 & Fermions at Level-17 in Type-IIB Scalar Superfield Decomposition

Beyond this point, ADA scans provide no guidance or other means that will be required for further progress. But these scans have provided valuable insights that have "shone a spotlight" within these systems containing up to 2,147,483,648 bosonic degrees of freedom and 2,147,483,648 fermionic degrees of freedom. The results derived indicate where future investigations might show the greatest return on investment of time and energy.

Chapter 9

SUSY Holography

In this chapter, we are going to present two main works related to the SUSY holography program [83, 85].

9.1 The 300 "Correlators" Suggests 4D, $\mathcal{N} = 1$ SUSY Is a Solution to a Set of Sudoku Puzzles

This section is based on [83].

9.1.1 Introduction

A research work [157] in 2014 revealed a previously unsuspected link between $\mathcal{N} = 1$ supersymmetrical field theories in four-dimensional spacetimes and Coxeter Groups [158–160]. Prior to the 2014 work, a program had been created whereby 4D, $\mathcal{N} = 1$ SUSY theories had been reduced to 1D, $\mathcal{N} = 4$ SUSY theories and there was shown the ubiquitous appearance of certain types of matrices [75, 76] that eventually were identified with an elaboration of adjacency matrices for a class of graphs given the monikers of "adinkras" [77]. An even later development occurred with the realization [91, 92] that these graphs correspond to a special class of Grothendieck's "dessins d'enfant" in algebraic geometry.

These modified adjacency matrices were found to occur in two types indicated by the symbols $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$. The first type is associated with the transformations of bosons into fermions under the action of supercharges, while the second type is associated with the transformations of fermions into bosons under the action of supercharges.

For 4D, $\mathcal{N} = 1$ off-shell theories, the index I takes on values 1, . . . , 4 and the matrices are real 4p × 4p matrices for some integer p. The minimal non-trivial representations occur for $p = 1$. These real matrices satisfy an algebra given by

$$\mathbf{L}_{\rm I}\,\mathbf{R}_{\rm J} ~+~ \mathbf{L}_{\rm J}\,\mathbf{R}_{\rm I} ~=~ 2\,\delta_{\rm I\,J}\,\mathbb{I}_4 \quad , \quad \mathbf{R}_{\rm I}\,\mathbf{L}_{\rm J} ~+~ \mathbf{R}_{\rm J}\,\mathbf{L}_{\rm I} ~=~ 2\,\delta_{\rm I\,J}\,\mathbb{I}_4 \quad . \tag{9.1.1}$$

and which is referred to as "the Garden Algebra." Some years [74] prior to this suggested name, it was noted these conditions are closely related to the definition of a special class of real Euclidean Clifford algebras. The real matrices $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$ can be used to form $8p \times 8p$ matrices using the definition

$$\widehat{\gamma}_{\rm I} ~=~ \left[\begin{array}{cc} 0 & \mathbf{L}_{\rm I} \\ & \\ \mathbf{R}_{\rm I} & 0 \end{array} \right] \quad , \tag{9.1.2}$$

which form a Clifford Algebra, with respect to the Euclidean metric defined by the Kronecker delta symbol $\delta_{\rm IJ}$

$$\{\widehat{\gamma}_{\rm I}, \widehat{\gamma}_{\rm J}\} = 2\delta_{\rm IJ}\,\mathbb{I}_8 \quad . \tag{9.1.3}$$

Here the symbol $\mathbb{I}_8$ denotes the 8 × 8 identity matrix to be distinguished from the case of the 4×4 identity matrix $\mathbb{I}_4$.

There exists ten distinct minimal off-shell 4D, $\mathcal{N} = 1$ supermultiplets as indicated below:

(S01.) Chiral Supermultiplet : $(A,\, B,\, \psi_a,\, F,\, G)$,

(S02.) Hodge – Dual #1 Chiral Supermultiplet : $(\widehat{A},\, \widehat{B},\, \psi_a,\, {\rm f}_{\mu\,\nu\,\rho},\, \widehat{G})$,

(S03.) Hodge – Dual #2 Chiral Supermultiplet : $(\widetilde{A},\, \widetilde{B},\, \psi_a,\, \widehat{F},\, {\rm g}_{\mu\,\nu\,\rho})$,

(S04.) Hodge – Dual #3 Chiral Supermultiplet : $(\check{A},\, \check{B},\, \psi_a,\, \check{\rm f}_{\mu\,\nu\,\rho},\, \check{\rm g}_{\mu\,\nu\,\rho})$,

(S05.) Tensor Supermultiplet : $(\varphi,\, B_{\mu\,\nu},\, \chi_a)$,

(S06.) Axial – Tensor Supermultiplet : $(\widehat{\varphi},\, \widehat{B}_{\mu\,\nu},\, \widehat{\chi}_a)$,

(S07.) Vector Supermultiplet : $(A_\mu,\, \lambda_b,\, {\rm d})$,

(S08.) Axial – Vector Supermultiplet : $(U_\mu,\, \widehat{\lambda}_b,\, \widehat{\rm d})$,

(S09.) Hodge – Dual Vector Supermultiplet : $(\widetilde{A}_\mu,\, \widetilde{\lambda}_b,\, \widetilde{\rm d}_{\mu\,\nu\,\rho})$, (S10.) Hodge – Dual Axial – Vector Supermultiplet : $(\breve{U}_\mu,\ \breve{\lambda}_b,\ \breve{\mathrm{d}}_{\mu\,\nu\,\rho})$.

Each of these systems can be reduced to one dimension by simply ignoring the spatial dependence of the field variables. If we ignore the supermultiplets that are equivalent to one another via 3-form Hodge duality[1] or parity transformations, this leaves only the supermultiplets described by the fields contained in (S01.), (S05.), and (S07.).

The work in [157] showed that each of these remaining three supermultiplets possesses distinct $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$ matrices that can be factorized. However this work, (based on a search algorithm), explicitly constructed many more such sets of matrices. The twelve matrices obtained from the reduction of the three supermultiplets were shown to be only a small number of 384 matrices that satisfy the "Garden Algebra" in quartets. For this larger number, explicit expressions of the form (there is no sum over the $\widehat{\rm I}$ index implied on the RHS of this equation),

$$\mathbf{L}_{\widehat{\rm I}} ~=~ \boldsymbol{\mathcal{S}}_{\widehat{\rm I}} \boldsymbol{\mathcal{P}}_{\widehat{\rm I}} ~~~. \tag{9.1.4}$$

were shown for every matrix. Thus, the matrices found by the search algorithm constitute 96 quartets of matrices that satisfy the condition shown in Eq. (9.1.3). Here we use the index $\widehat{\rm I}$ that takes on values 1, 2, . . . , 384 in order to denote each of these matrices. Moreover, fixing the value of $\widehat{\rm I}$, the matrices $\boldsymbol{\mathcal{S}}_{\widehat{\rm I}}$ were all found to take the form of purely diagonal matrices that each square to the identity matrix. We should be clear that the entirety of these 384 matrices do *not* form a Clifford algebra. Instead, judicious choices of quartets of these do satisfy the conditions shown in Eq. (9.1.3).

In a very similar manner, for a fixed value of $\widehat{\rm I}$, all of the matrices $\boldsymbol{\mathcal{P}}_{\widehat{\rm I}}$, take the forms of elements in the permutation group ($\mathbb{S}_4$) of order four. However, in order to satisfy the condition in Eq. (9.1.3), the entirety of the permutation group was found to be dissected into six distinct *unordered* subsets. One of the main purposes of this current work is to revisit this dissection behavior and discover how this can be uncovered without the use of supersymmetry-based arguments, but only relying on properties of $\mathbb{S}_4$. This approach

[1] In the works of [85, 161] a discussion of the use of eigenvalues determined by $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$ matrices has been presented to describe the relationships among such Hodge 1-form/3-form dual supermultiplets.

will lead to a well-known algebraic polytope associated with the permutation groups, the permutohedron. The permutohedron is a polytope whose vertices represent the elements of the permutations acting on the first $\mathcal{N} - 1$ integers. For $\mathcal{N} = 1$ there are 24 elements in the set of permutations which means that the factor of the $\boldsymbol{\mathcal{S}}_{\hat{\mathrm{I}}}$ matrices that appear in eq. (9.1.4) contains sixteen copies of the permutation elements.

The dissection can be seen from this perspective to arise from the symmetries of the permutohedron and the use of weak Bruhat ordering.

This section is organized as follows.

The first subsection is dedicated to an explanatory discussion of the two sets of notational conventions we use to describe elements of the permutation group. The description of how to construct a "dictionary" (which is actually given in section 9.1.4) is provided for the "translation" between the two sets of conventions.

The second subsection contains a review of the previous discussion of how the use of projection techniques from 4D, $\mathcal{N} = 1$ SUSY to 1D, $\mathcal{N} = 4$ SUSY implied a dissection of $\mathbb{S}_4$ was required to be consistent with SUSY. Venn diagrams are used to explain how "Kevin's Pizza" (a dissection of $\mathbb{S}_4$ in six disjoint subsets) was discovered as a result of projecting 4D, $\mathcal{N} = 1$ SUSY to 1D, $\mathcal{N} = 4$ on theories that are related by Hodge Duality in four dimensions.

The third subsection provides a concise and simplified discussion of Bruhat weak ordering and how (when applied to $\mathbb{S}_4$) it acts to create a "distance measuring function" of the elements of the permutation group. This is contrasted with the Hamming distance that is defined on strings of bits. It is noted that the truncated octahedron provides a framework for being overlaid by a network created by the Bruhat weak ordering to lead to the permutohedron.

The forth subsection introduces the word "correlators" to define the minimal path length between any number of the vertices of the permutohedron. The focus of the presentation is only upon two-point correlators, though it's easily possible to define n-point correlators by analogy. An investigation of correlators for permutations from SUSY quartets is performed. The complete results for the two-point correlators associated with these are reported. This is the first part of studying all three hundred two-point correlators.

The fifth subsection contains the calculations of two-point correlators among distinct pairs of the quartets. This completes the calculation of the three hundred two-point correlators. In particular, it is found that the quartets of permutation generators that are consistent with SUSY are those with the property that the sums of their eigenvalues exceed non-quartet calculations when compared to pairs of distinct quartets.

The sixth subsection contains sets of calculations similar to those of the preceding two subsections. However, the emphasis here is upon the study of such correlators among the chromotopology of four-color adinkras. The values of twenty thousand seven hundred and thirty-six in terms of eigenvalues and traces are reported.

The seventh sets the stage for future investigations. The permutation generators for *all* values of $\mathcal{N}$ were established in papers [75,76] written in the 1995-1996 period and it is noted that this implies the opportunity to study the correlators with $\mathcal{N}$ greater than four (the focus of this work). A final subsection containing the conclusion closes out the discussion.

9.1.2 Notational Conventions

In Appendix G, an explicit matrix representation of all twenty-four elements of $\mathbb{S}_4$ is given. However, in order to more efficiently give the presentation, there are two notational conventions introduced to avoid explicitly writing the matrices.

There exists what may be called the "parenthetical convention," or "cycle convention." In this convention, parenthesis marks () surround a set of numerical entries "a_1, a_2, ...", e. g. $(a_1 a_2 a_3 \dots)$ which provides an instruction that $a_1 \to a_2$, $a_2 \to a_3$, $a_3 \to a_4$, etc. until one gets to the last element which is instructed to "move" to the position of the first element. This describes the cycles that are induced by any permutation element. The symbol "()" implies no replacements or "movements" and hence denotes the identity permutation element. This notational convention also eskews "one-cycles" (i. e. (a)) since these are equivalent to the identity permutation also. Thus, the number of elements enclosed is either zero, two, three, or four. It should be noted this is a "unidirectional" convention where the symbol is to be read from left to right. Finally, the maximum number of distinct elements that can appear is four. In Appendix G this notation is related to the explicit matrix representation.

There is also what can be called the "bra-ket convention." In this convention, the "bra-ket" symbol $\langle\rangle$ is used to always enclose four elements $\langle a_1a_2a_3a_4\rangle$. Perhaps the most efficient manner to present this is via four examples as shown in eq. (9.1.5)

$$
\begin{aligned}
&\begin{matrix}\langle 1234\rangle \\ ()\end{matrix} \Rightarrow \begin{matrix}\langle 1234\rangle \\ \langle 1234\rangle\end{matrix} \Rightarrow 1\to 1,\ 2\to 2,\ 3\to 3,\ 4\to 4, \\
&\begin{matrix}\langle 1324\rangle \\ (23)\end{matrix} \Rightarrow \begin{matrix}\langle 1234\rangle \\ \langle 1324\rangle\end{matrix} \Rightarrow 1\to 1,\ 2\to 3,\ 3\to 2,\ 4\to 4, \\
&\begin{matrix}\langle 1342\rangle \\ (234)\end{matrix} \Rightarrow \begin{matrix}\langle 1234\rangle \\ \langle 1342\rangle\end{matrix} \Rightarrow 1\to 1,\ 2\to 3,\ 3\to 4,\ 4\to 2, \\
&\begin{matrix}\langle 2143\rangle \\ (12)(34)\end{matrix} \Rightarrow \begin{matrix}\langle 1234\rangle \\ \langle 2143\rangle\end{matrix} \Rightarrow 1\to 2,\ 2\to 1,\ 3\to 4,\ 4\to 3,
\end{aligned}
\tag{9.1.5}
$$

In the leftmost column the "bra-ket" convention is written above its equivalent "parenthetical" equivalent. To the immediate right of the "bra-ket" description and in red numbers appears the fiducial bra-ket description of the identity element. By comparing the entries in the red numbered fiducial bra-ket to the black lettered entries immediately below it, one can record the actions of the "bra-keted" element on the numbers 1, 2, 3, and 4 and easily extract the cycles. The cycles are then recorded into the "parenthetical" description at the bottom leftmost entries in the first column for each example. Thus, the translations can be read vertically by looking at the leftmost column for each example.

One other convention that provides a foundation of our exploration is to use the lexicographical ordering of the permutation elements.

For the bra-ket convention, the lexicographical order is simple to "read off." Given the permutation element $\langle a_1a_2a_3a_4\rangle$ we simply map this to the numerical value of $a_1,\, a_2a_3a_4$. For two or more permutation elements, one uses the mapping and the usual magnitude of the numerical equivalents to determine the lexicographical order. One example of the mapping is provided by $\langle 1234 \rangle \rightarrow 1{,}234$ and another is provided by $\langle 2314 \rangle \rightarrow 2{,}314$. So the lexicographical ordering of $\langle 2314 \rangle$ and $\langle 1234 \rangle$ is $\langle 1234 \rangle$ first followed by $\langle 2314 \rangle$.

If one were to attempt to extend the concept of lexicographical ordering into the realm of the parenthetical basis, an unacceptable degree of arbitrariness enters. To see this a simple example suffices.

Consider the two distinct permutations which in the ()-basis are expressed as (12)(34) and (1234). The first has two distinct 2-cycles and the second has a single 4-cycle. While it is possible to introduce some additional rules over and above that which is used in the $\langle\rangle$-basis, this would also introduce arbitrariness.

9.1.3 Dissecting A Coxeter Group Into 'Kevin's Pizza'

The 384 elements of the Coxeter Group $\mathbb{BC}_4$ are represented in Eq. (9.1.4) and upon taking the absolute values of the matrix representations, one obtains sixteen copies of $\mathbb{S}_4$, the group of permutations of four objects.

Lexicographically listed in the $\langle\rangle$-basis, the twenty-four permutation elements of $\mathbb{S}_4$ are given by

$$\begin{aligned}
\{\mathbb{S}_4\} \;=\; \{ \; & \langle 1234 \rangle, \langle 1243 \rangle, \langle 1324 \rangle, \langle 1342 \rangle, \langle 1423 \rangle, \langle 1432 \rangle, \\
& \langle 2134 \rangle, \langle 2143 \rangle, \langle 2314 \rangle, \langle 2341 \rangle, \langle 2413 \rangle, \langle 2431 \rangle, \\
& \langle 3124 \rangle, \langle 3142 \rangle, \langle 3214 \rangle, \langle 3241 \rangle, \langle 3412 \rangle, \langle 3421 \rangle, \\
& \langle 4123 \rangle, \langle 4132 \rangle, \langle 4213 \rangle, \langle 4231 \rangle, \langle 4312 \rangle, \langle 4321 \rangle \; \} \quad ,
\end{aligned} \tag{9.1.6}$$

or alternately illustrated in the ()-basis, the twenty-four permutation elements of $\mathbb{S}_4$ are

$$\begin{aligned}
\{\mathbb{S}_4\} \;=\; \{ \; & (), (34), (23), (234), (243), (24), (12), (12)(34), (123), (1234), \\
& (1243), (124), (132), (1342), (13), (134), (13)(24), (1324), \\
& (1432), (142), (143), (14), (1423), (14)(23) \; \} \quad ,
\end{aligned} \tag{9.1.7}$$

with the permutation elements being written in the same order as in Eq. (9.1.6). Together

Eq. (9.1.6) and Eq. (9.1.7) constitute a complete dictionary for translating between expressing the permutation elements of $\mathbb{S}_4$ in either the $\langle\rangle$-notation or the ()-notation. A Venn diagram in the form of a circle may be used to represent $\mathbb{S}_4$ with the twenty-four permutation elements contained in its interior.

Figure 9.1: A Venn Diagram Representation of $\mathbb{BC}_4$

A fact uncovered by the work in [157] is that when the 4D, $\mathcal{N} = 1$ supermultiplets are reduced to 1D, $\mathcal{N} = 4$ supermultiplets, the associated $\mathbf{L}_{\rm I}$ and $\mathbf{R}_{\rm I}$ matrices require a "dissection" of $\mathbb{S}_4$ into six non-overlapping distinct subsets. For reasons that will be made clear shortly, we denote these subsets by $\{\mathcal{P}_{[1]}\}$, $\{\mathcal{P}_{[2]}\}$, $\{\mathcal{P}_{[3]}\}$, $\{\mathcal{P}_{[4]}\}$, $\{\mathcal{P}_{[5]}\}$, and $\{\mathcal{P}_{[6]}\}$ and write the equation

$$\{\mathbb{S}_4\} \;=\; \{\mathcal{P}_{[1]}\} \,\cup\, \{\mathcal{P}_{[2]}\} \,\cup\, \{\mathcal{P}_{[3]}\} \,\cup\, \{\mathcal{P}_{[4]}\} \,\cup\, \{\mathcal{P}_{[5]}\} \,\cup\, \{\mathcal{P}_{[6]}\} \quad , \tag{9.1.8}$$

where the elements in each subset are given by Some explanatory words are in order about this figure.

The leftmost column indicates the distinct subsets. In the middle of the figure, the quartet of permutation elements in each subset is shown. In the rightmost column, the corresponding supermultiplets associated with the subset are shown. While the left-hand column only contains data associated with $\{\mathbb{S}_4\}$, the right-hand column is the result of a specific choice of the reduction technique applied to the 4D, $\mathcal{N} = 1$ supermultiplets.

$$\{\mathcal{P}_{[1]}\} = \{\, \langle 1423\rangle, \langle 2314\rangle, \langle 3241\rangle, \langle 4132\rangle \,\} = \{CM\}$$
$$\{\mathcal{P}_{[2]}\} = \{\, \langle 1342\rangle, \langle 2431\rangle, \langle 3124\rangle, \langle 4213\rangle \,\} = \{TM\}$$
$$\{\mathcal{P}_{[3]}\} = \{\, \langle 1324\rangle, \langle 2413\rangle, \langle 3142\rangle, \langle 4231\rangle \,\} = \{VM\}$$
$$\{\mathcal{P}_{[4]}\} = \{\, \langle 1432\rangle, \langle 2341\rangle, \langle 3214\rangle, \langle 4123\rangle \,\} = \{VM_1\}$$
$$\{\mathcal{P}_{[5]}\} = \{\, \langle 1243\rangle, \langle 2134\rangle, \langle 3421\rangle, \langle 4312\rangle \,\} = \{VM_2\}$$
$$\{\mathcal{P}_{[6]}\} = \{\, \langle 1234\rangle, \langle 2143\rangle, \langle 3412\rangle, \langle 4321\rangle \,\} = \{VM_3\}$$

Figure 9.2: A Dissection of $\mathbb{BC}_4$ in Lexicographical Ordering Using "$\langle\rangle''$ Basis Elements

Finally, the $\{CM\}$, the $\{TM\}$, and the $\{VM\}$ labels correspond respectively to the (S01.), (S05.), and (S07.) supermultiplets described in the introduction.

After the dissection, the previous Venn diagram can be replaced by one that makes the subsets more obvious. The color-coding of the notations in Fig. 9.2 indicate which permutation elements belong to each subset.

Figure 9.3: "Hodge Duality" Dissection of $\mathbb{BC}_4$

The most obvious feature of the image in Fig. 9.3 is its highly asymmetrical form. This is due to two features. The first can be seen by using the ()-basis for the permutation elements. In that convention, the various subsets look as shown in Fig. 9.4. This makes it clear that the first two subsets *only* contain permutations that describe 3-cycles. All the remaining subsets contain either 0-cycles, 2-cycles, or 4-cycles. So the split of the subsets $\mathcal{P}_{[1]}$ and $\mathcal{P}_{[2]}$ away from $\mathcal{P}_{[3]}$, $\mathcal{P}_{[4]}$, $\mathcal{P}_{[5]}$, and $\mathcal{P}_{[6]}$ is actually a split between

$$
\begin{aligned}
\{\mathcal{P}_{[1]}\} &= \{\, (243),\ (123),\ (134),\ (142)\ \} \\
\{\mathcal{P}_{[2]}\} &= \{ (234),\ (124),\ (132),\ (143)\ \} \\
\{\mathcal{P}_{[3]}\} &= \{ (23),\ (1243),\ (1342),\ (14) \} \\
\{\mathcal{P}_{[4]}\} &= \{\, (24),\ (1234),\ (13),\ (1432)\ \} \\
\{\mathcal{P}_{[5]}\} &= \{ (34),\ (12),\ (1324),\ (1423)\ \} \\
\{\mathcal{P}_{[6]}\} &= \{ (),\ (12)(43),\ (13)(42),\ (14)(23) \}
\end{aligned}
$$

Figure 9.4: A Dissection of $\mathbb{BC}_4$ in Lexicographical Ordering Using "()$''$ Basis Elements

permutations with either even or odd cycles. None of this is related to the arguments that come from SUSY.

However, among the permutations that describe even cycles, there is no obvious rationale for why the groupings above occur. It is the fact that these quartets were derived from projections of the three supermultiplets (S01.), (S05.), and (S07.) that inspired the arrangements of the permutations with even cycles into the groupings shown in $\{VM\}$, $\{VM_1\}$, $\{VM_2\}$, and $\{VM_3\}$ subsets. Before we continue, it is perhaps useful to recall the arguments related to duality on the SUSY side.

If we begin with a scalar Φ, a quantity F_μ can be defined from its derivative. Contracting with F^μ then leads (with proper normalization) to the Lagrangian proportional to the kinetic energy, $\mathcal{L}_\Phi$ for a massless scalar field, i. e.

$$
\Phi \ \to\ \partial_\mu \Phi \ =\ F_\mu \quad , \quad \mathcal{L}_\Phi \ =\ -\tfrac{1}{2}\, F^\mu\, F_\mu \quad . \tag{9.1.9}
$$

There are two differential equations that can be written in terms of F_μ,

$$
Bianchi\ Identity \quad \partial_\mu F_\nu \ -\ \partial_\nu F_\mu \ =\ 0 \quad , \quad EoM:\ \partial^\mu F_\mu \ =\ 0 \quad , \tag{9.1.10}
$$

where the first of these (referred to as the "*Bianchi Identity*") follows from the definition of F_μ. The second equation (referred to as the "*EoM*" or "*Equation of Motion*") arises as the extremum of the field variation applied to $\mathcal{L}_\Phi$.

One can repeat line for line the argument above by starting with a 2-form $B_{\mu\nu}$

$$B_{\mu\nu} \ \rightarrow \ \epsilon^{\mu\nu\rho\sigma}\partial_\nu B_{\rho\sigma} \ = \ H^\mu \quad , \qquad \mathcal{L}_B \ = \ \tfrac{1}{2}\, H^\mu \, H_\mu \quad , \tag{9.1.11}$$

$$\textit{Bianchi Identity} \quad \partial_\mu H^\mu \ = \ 0 \quad , \quad \textit{EoM}: \ \partial_\mu H_\nu \ - \ \partial_\nu H_\mu \ = \ 0 \quad . \tag{9.1.12}$$

It is thus clear that the final difference in starting from Φ versus starting from $B_{\mu\nu}$ is the exchange of the "*Bianchi Identity*" and the "*EoM*".

The behavior of the 0-form Φ and the 2-form $B_{\mu\nu}$ contrast greatly with the behavior of the 1-form A_μ. Following the same reasoning as above, one is led to

$$A_\mu \ \rightarrow \ \partial_\mu A_\nu \ - \ \partial_\nu A_\mu \ = \ F_{\mu\nu} \quad , \quad \mathcal{L}_\Phi \ = \ -\tfrac{1}{4}\, F^{\mu\nu}\, F_{\mu\nu} \tag{9.1.13}$$

and

$$\textit{Bianchi Identity} \quad \epsilon^{\mu\nu\rho\sigma}\partial_\mu F_{\nu\rho} \ = \ 0 \quad , \quad \textit{EoM}: \ \partial^\mu F_{\mu\nu} \ = \ 0 \quad . \tag{9.1.14}$$

Now we treat the *EoM* as a constraint to see if it has a solution. It is easily shown that

$$F_{\mu\nu} \ \equiv \ \epsilon_{\mu\nu}{}^{\rho\sigma}\partial_\rho b_\sigma \tag{9.1.15}$$

solves the second equation in (9.1.14) and upon substituting this into the *Bianchi Identity*, one finds $\partial^\mu[\,\partial_\mu b_\nu \ - \ \partial_\nu b_\mu\,] = 0$. So the exchange of the *EoM* and the *Bianchi Identity* occurs, but the dual field b_μ is also a 1-form.

Motivated by these arguments, the work in [157] contained a proposal that there should be a mapping operation acting on the $\mathbb{S}_4$-space of permutations that is the "shadow" of the behavior discussed in Eq. (9.1.9) - Eq. (9.1.15). It turns out that the realization of such a mapping operation (denoted by *) is straightforward. The ()-notation shown in Fig. 9.4 is most efficient in this realization.

The * map is defined by simply reversing the direction of reading the cycles in ()-notation for the permutations. Thus, we define

$$\begin{aligned}
{}^*\{\text{CM}\} &\equiv \{\,\mathbf{(234)}\quad,\ \mathbf{(213)}\quad,\ \mathbf{(143)}\quad,\ \mathbf{(124)}\,\} \\
{}^*\{\text{TM}\} &\equiv \{\,\mathbf{(243)}\quad,\ \mathbf{(142)}\quad,\ \mathbf{(123)}\quad,\ \mathbf{(134)}\,\} \\
{}^*\{\text{VM}\} &\equiv \{\,\mathbf{(1342)}\ ,\ \mathbf{(23)}\quad,\ \mathbf{(14)}\quad,\ \mathbf{(1243)}\,\} \\
{}^*\{\text{VM}_1\} &\equiv \{\,\mathbf{(1234)}\ ,\ \mathbf{(24)}\quad,\ \mathbf{(1432)}\,,\ \mathbf{(13)}\,\} \\
{}^*\{\text{VM}_2\} &\equiv \{\,\mathbf{(1423)}\ ,\ \mathbf{(1324)}\ ,\ \mathbf{(12)}\quad,\ \mathbf{(34)}\,\} \\
{}^*\{\text{VM}_3\} &\equiv \{\,\mathbf{(13)(24)},\ \mathbf{(14)(23)},\quad \mathbf{()}\quad,\ \mathbf{(12)(34)}\,\}
\end{aligned} \tag{9.1.16}$$

and if we consider only unordered sets these imply

$$ {}^*\{CM\} = \{TM\} \ , \quad {}^*\{TM\} = \{CM\} \ , \quad {}^*\{VM\} = \{VM\} \tag{9.1.17}$$

for the permutations associated with the (S01.), (S05.) and (S07.) supermultiplets. Furthermore

$$ {}^*\{VM_1\} = \{VM_1\} \ , \quad {}^*\{VM_2\} = \{VM_2\} \ , \quad {}^*\{VM_3\} = \{VM_3\} \tag{9.1.18}$$

for the permutations that were not constructed from projections. These results can be summarized in the Venn diagram shown in Fig. 9.5.

It is important to realize that the definition of the * operation acting on $\mathbf{L}_{\rm I}$ matrices is *not* unique. In principle, one can define the * operation by first performing a $\mathbb{O}(4)\times\mathbb{O}(4)$ transformation

$$ \mathbf{L}_{\rm I} \ \to \ \mathcal{X}\,\mathbf{L}_{\rm I}\,\mathcal{Y} \quad , \tag{9.1.19}$$

where

$$ \mathcal{X}\,(\mathcal{X})^t \ = \ (\mathcal{X})^t\,\mathcal{X} \ = \ \mathcal{Y}\,(\mathcal{Y})^t \ = \ (\mathcal{Y})^t\,\mathcal{Y} \ = \ \mathbb{I}_4 \quad , \tag{9.1.20}$$

and only afterward then read the ()-basis cycles in reverse order. The definition of the * operation we use is simply the minimal one, but more baroque ones are also consistent.

However, though the subsets may now be consistently interpreted as projections of

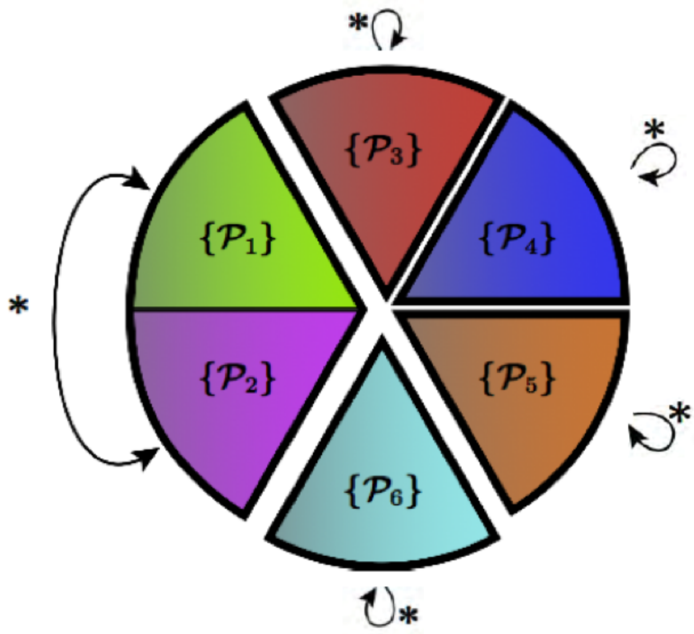


Figure 9.5: "Hodge Duality" Mapping of $\mathbb{BC}_4$

the supermultiplets, there still remains a mystery. If one does not invoke the argument about projection from supermultiplets, what structure in $\mathbb{S}_4$ is operative without an appeal to SUSY to determine these quartets?

In the remainder of this section, we will propose an answer to this question.

9.1.4 Bruhat Weak Ordering

The concept of the *Hamming distance* between two strings of bits [162] is a well known one for bitstrings of equal length.

One simply counts the number of bits that need to be "flipped" to go from one string to another. In this sense, one can regard the Hamming distance as a choice of a metric on the space of bitstrings.

Within the realm of the mathematics of permutation groups, there exist the concepts of *weak left Bruhat ordering* and *weak right Bruhat ordering*. The mathematician Adaya-Nand Verma (1933-2012) initiated the combinatorial study of Bruhat order and named it after earlier work completed by mathematician Francois Bruhat (1929-2017). Further details about weak left- and right- Bruhat ordering can be found in the work of [163].

The realization of either weak left- or right- Bruhat ordering can be used to define a metric on the space of elements of a permutation group. In this "spirit," at least for our purposes, the existence of these ordering prescriptions for the elements of the permutation group, is rather similar to the role of the Hamming distance on the space of bitstrings. We will begin by illustrating our use of weak ordering in Fig. 9.6 and then turn to an explanation of the image.

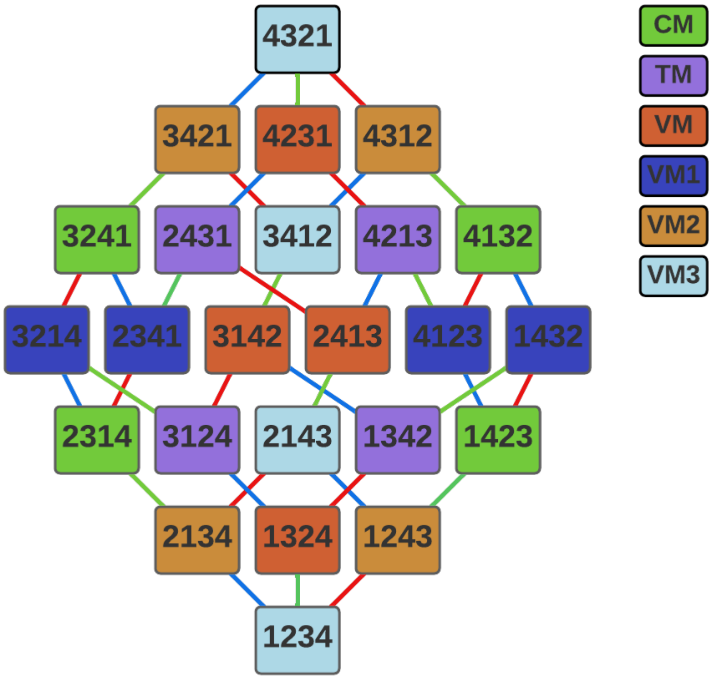


Figure 9.6: Weak Ordering in $\mathbb{BC}_4$

On the rightmost portion of the image, illustrated in a color-coded column, are the six subsets containing the quartets. Distributed throughout the main portion of the figure are the individual permutation elements listed by their "bra-ket" addresses and enclosed by the color of the quartet subset to which each is a member. Once this figure is obtained it immediately implies an intrinsic definition of a "minimal Bruhat distance" between any two elements by simply counting the number of links that connect the two elements along a minimum path.

There remains the task of explaining how this ordering was obtained.

To describe the procedure, let us start with the identity permutation that is illustrated as the bottom-most permutation in the figure.

The identity permutation element has the "bra-ket address" $\langle 1234 \rangle$. There are three pair-wise adjacent 2-cycle permutations (12), (23), and (34) that respectively provide instructions to change the first two numbers in the address, the second two numbers in the address, and finally the last two numbers in the address. The (12) permutation will send the identity address into the leftmost address at the first elevated level of Fig. 9.6. The (23) permutation will send the identity address into the middle address at the first elevated level of Fig. 9.6. Finally, the (34) permutation will send the identity address into the rightmost address at the first elevated level of Fig. 9.6. Next the three permutations (12), (23), and (34) are applied to the three addresses at the first elevated level. Sometimes, this will result in a restoration of the identity address. However, when this does not occur, the new addresses will be those that occur at the second elevated level. One simply repeats this procedure until the addresses of all the permutations have been reached. Thus, the links shown in the network of Fig. 9.6. represent the orbits of the addresses of the permutation elements under the action of the three pair-wise adjacent 2-cycle permutations (12), (23), and (34).

9.1.5 The permutohedron

In the mathematics literature, the concept of permutohedra seems first to have been a topic of a study carried out by Pieter Schoute (1846 -1923) in 1911 [164]. Further informative readings on this subject can be found in the references [165–168].

The appropriate permutohedron, in the context we require, is a polytope that can be constructed by using the network shown in Fig. 9.6 and conceiving of that image as a net. One can begin with a truncated octahedron[2] where each vertex of the truncated octahedron represents a permutation element.

The colors of the links are related to the three permutations links (12), (23), and (34) respective by blue, green, and red as shown by the epigram to the lower right side of Fig. 9.7.

[2] A simplified discussion of the truncated octahedron can be found at https://en.wikipedia.org/wiki/Truncated_octahedron

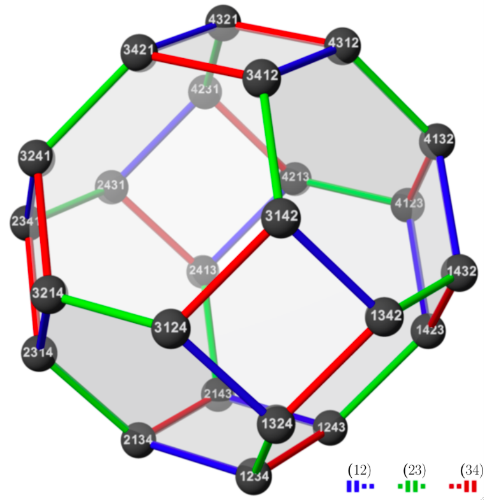


Figure 9.7: $\mathbb{S}_4$ Permutation Elements and the Truncated Octahedron

The network shown in Fig. 9.6 perfectly aligns with the permutohedron shown in Fig. 9.7.

A way to see this is to begin with the image in Fig. 9.7 and treat it as a frame that can be 'adorned' by the individual permutations listed in Fig. 9.2. By matching each "bra-ket address," one is led to a figurative "Christmas tree" seen in Fig. 9.8.

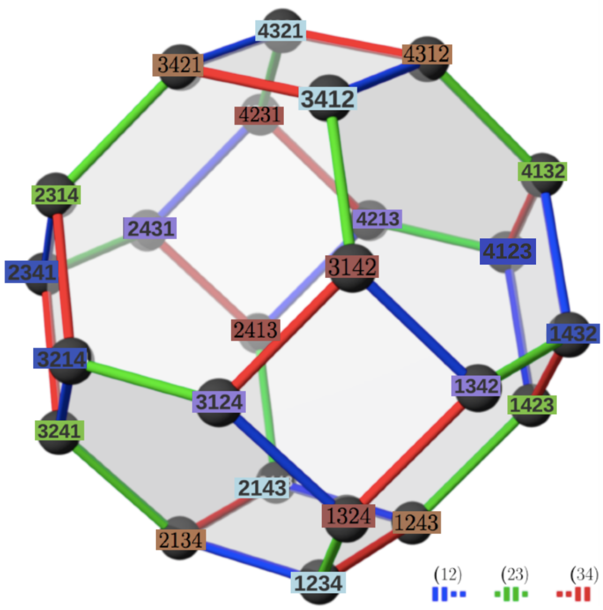


Figure 9.8: $\{\mathcal{P}_{[1]}\} \cdots \{\mathcal{P}_{[6]}\}$ addresses adorning the permutohedron

Having completely adorned the permutohedron vertices with the color-coded labels of the individual permutations from the subsets, one can check that the thirty-six edges connect the vertices in precisely the same manner as shown in the network of Fig. 9.6.

From a visual examination of Fig. 9.8, it is clear that the subset elements $\{\boldsymbol{\mathcal{P}}_{[1]}\} \cdots \{\boldsymbol{\mathcal{P}}_{[6]}\}$ occur in a highly symmetrical arrangement of the four colors about the adorned permutohedron. So the next obvious step is to mathematically quantify this symmetrical arrangement of the four colors.

Stated in a different fashion, the appearance of the 4D, $\mathcal{N} = 1$ supersymmetrical minimal quartets is equivalent to solving a coloring problem. Start with four colors and the truncated octahedron *with no labeling of nodes*, by using the colors, what is the maximally symmetrical choice for painting the nodes? The answer is the SUSY quartets.

9.1.6 The permutohedron, $\mathcal{X}$, and $\mathcal{Y}$ Transformations

The existence of the permutohedron also brings into clear focus the meaning of the transformations shown in Eq. (9.1.19) and Eq. (9.1.20). Let us illustrate this in the case of one example. Consider only the $\boldsymbol{\mathcal{P}}_{[6]}$ (illustrated in Fig. 9.14 below). If the (12) permutation is applied to it, the four light blue nodes shown in Fig. 9.14 "move" to the locations of the brown nodes in Fig. 9.13. If the (23) permutation is applied after the (12) permutation, then the four light blue nodes shown in Fig. 9.14 "move" to the locations of the green nodes in Fig. 9.9. The $\mathcal{X}$, and $\mathcal{Y}$ transformation correspond to left or right multiplication of the elements of the quartets by different orders and power of the permutation elements (12), (23), or (34).

The remainder of this section is devoted to the mathematical calculation that quantifies all these observations of this section.

9.1.7 The 300 "Correlators" of the $\mathbb{S}_4$ permutohedron

In the previous sections, the presentation contained arguments and visual images based only on the properties of the truncated octahedron to propose that the $\mathbf{L}_{\rm I}$ matrices that can be used to define SUSY systems in 1D are in fact solutions to a four-color problem on the truncated octahedron. From here on, we will present the calculational evidence that supports the discussions in the earlier sections.

For this purpose, there is introduced a function denoted by $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|A]}, \mathcal{P}_{[b|B]}]$ that we will call a "two-point correlator." This function assigns a number of the minimal Bruhat distance between the elements $\mathcal{P}_{[a|A]}$ and $\mathcal{P}_{[b|B]}$ contained in $\mathbb{S}_{\mathbf{4}}$. This means this is a symmetric 24 × 24 matrix with 300 possible entries. The symbol $\mathcal{P}_{[a|A]}$ is meant to denote the a-th element in the dissected subset A. If the two subsets are such that $A = B$, we will call these "intra-quartet correlators", and if the two subsets are such that $A \neq B$, we then call these "inter-quartet correlators."

We can begin our calculations by considering the case where $A = B = 1$ that corresponds to the first of the dissected subsets according to Fig. 9.2. The four permutation elements belonging to this subset have the corresponding vertices "blocked out" by green quadrilaterals in Fig. 9.9.

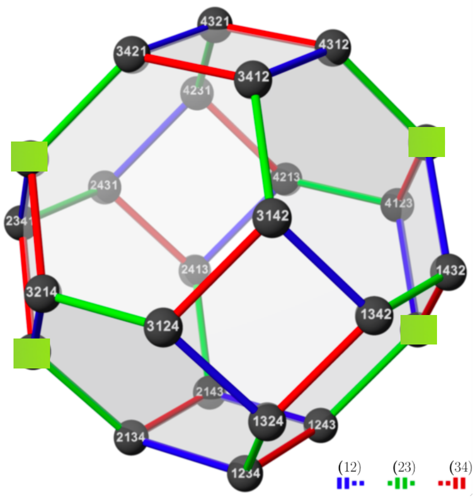


Figure 9.9: permutohedron & Colored $\mathcal{P}_{[1]}$ Subset Elements

From simply counting the number of links it takes to "travel" from a given specified element a in the set $\mathcal{P}_{[1]}$ to a second specified element b in $\mathcal{P}_{[1]}$ we are able to create a table. The rows of the table are listed according to the lexicographical ordering of the element in $\mathcal{P}_{[1]}$ and the columns of the table are listed according to the lexicographical ordering of the elements in $\mathcal{P}_{[1]}$ also.

$\boldsymbol{\mathcal{P}}_{[\mathbf{1}]}$ \ $\boldsymbol{\mathcal{P}}_{[\mathbf{1}]}$	$\langle 1423\rangle$	$\langle 2314\rangle$	$\langle 3241\rangle$	$\langle 4132\rangle$
$\langle 1423\rangle$	0	4	6	2
$\langle 2314\rangle$	4	0	2	6
$\langle 3241\rangle$	6	2	0	4
$\langle 4132\rangle$	2	6	4	0

Table 9.1: $\{CM\} - \{CM\}$ Two – Point Correlator Values

The information in the table above can be used to determine the values of $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|1]}, \mathcal{P}_{[b|1]}]$ in the form of a matrix.

$$\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|1]}, \mathcal{P}_{[b|1]}] = \begin{bmatrix} 0 & 4 & 6 & 2 \\ 4 & 0 & 2 & 6 \\ 6 & 2 & 0 & 4 \\ 2 & 6 & 4 & 0 \end{bmatrix} \tag{9.1.21}$$

It is clear that the trace of this matrix vanishes and a direct calculation of its eigenvalues yield $\{12,\, 0,\, -4,\, -8\}$ expressed in descending order.

There are five more cases ((9.1.22), (9.1.23), (9.1.24), (9.1.25), and (9.1.26)) where $A = B$. Direct calculations for each of these yield the same value for the traces as well for the sets of eigenvalues. This is true even though the six matrices involved are all distinct. However, the six matrices possess many similarities. For instance, each one contains six entries that are either 2's, 4's, or 6's. In no row or column do these numbers appear twice. There are exactly $3! = 6$ ways to satisfy this condition, hence six quartets.

Intuitively, something like this should have been an expectation by looking back at the image in Fig. 9.8. If we assign the standard right-handed x-y-z coordinate axes to the image in Fig. 9.8, where x-axis points out of the paper, it is apparent that each quartet has two of its members on the opposite faces of square faces that are perpendicular to the coordinate axes.

In the remainder of this subsection, we will illustrate the explicit analogous results for the remaining cases were $A = B$.

Intra-quartet $\mathcal{P}_{[2]}$ Correlators of the $\mathbb{S}_4$ permutohedron In this subsection, we illustrate the intra-quartet two-point correlators associated with $\boldsymbol{\mathcal{P}}_{[2]}$. The quartet members are indicated by purple quadrilaterals on some of the vertices of the permutohedron, the correlators are shown in tabular form, and finally, this same data is presented as a matrix.

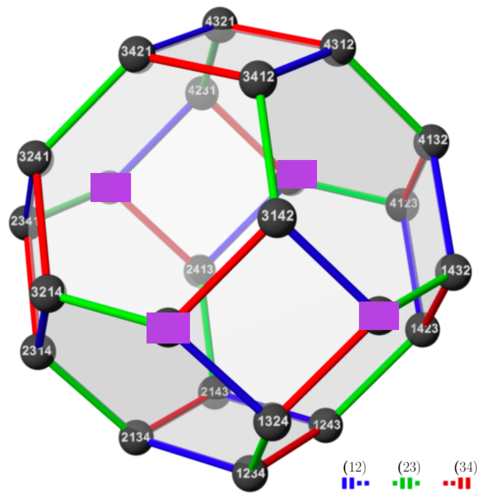


Figure 9.10: permutohedron & $\mathcal{P}_{[2]}$ Subset Elements

The data contained in $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|2]}, \mathcal{P}_{[b|2]}]$ is presented below first in tabular and in

$\boldsymbol{\mathcal{P}}_{[2]}$ \ $\boldsymbol{\mathcal{P}}_{[2]}$	$\langle 1342\rangle$	$\langle 2431\rangle$	$\langle 3124\rangle$	$\langle 4213\rangle$
$\langle 1342\rangle$	0	6	2	4
$\langle 2431\rangle$	6	0	4	2
$\langle 3124\rangle$	2	4	0	6
$\langle 4213\rangle$	4	2	6	0

Table 9.2: $\{TM\} - \{TM\}$ Two $-$ Point Correlator Values

matrix form.

$$\mathcal{A}_{\ell x}[\mathcal{P}_{[a|2]}, \mathcal{P}_{[b|2]}] = \begin{bmatrix} 0 & 6 & 2 & 4 \\ 6 & 0 & 4 & 2 \\ 2 & 4 & 0 & 6 \\ 4 & 2 & 6 & 0 \end{bmatrix} \tag{9.1.22}$$

Intra-quartet $\mathcal{P}_{[3]}$ Correlators of the $\mathbb{S}_4$ permutohedron In this subsection, we illustrate the intra-quartet two-point correlators associated with $\mathcal{P}_{[3]}$. The quartet members are indicated by rust color quadrilaterals on some of the vertices of the permutohedron, the correlators are shown in tabular form, and finally, this same data is presented as a matrix.

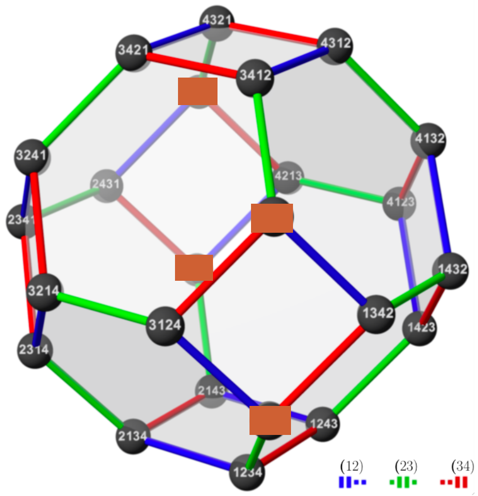


Figure 9.11: permutohedron & $\mathcal{P}_{[3]}$ Subset Elements

The data contained in $\mathcal{A}_{\ell x}[\mathcal{P}_{[a|3]}, \mathcal{P}_{[b|3]}]$ is presented below first in tabular and in

$\mathcal{P}_{[3]}$ \ $\mathcal{P}_{[3]}$	$\langle 1324 \rangle$	$\langle 2413 \rangle$	$\langle 3142 \rangle$	$\langle 4231 \rangle$
$\langle 1324 \rangle$	0	4	2	6
$\langle 2413 \rangle$	4	0	6	2
$\langle 3142 \rangle$	2	6	0	4
$\langle 4231 \rangle$	6	2	4	0

Table 9.3: $\{VM\} - \{VM\}$ Two – Point Correlator Values

matrix form.

$$\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|3]}, \mathcal{P}_{[b|3]}] = \begin{bmatrix} 0 & 4 & 2 & 6 \\ 4 & 0 & 6 & 2 \\ 2 & 6 & 0 & 4 \\ 6 & 2 & 4 & 0 \end{bmatrix} \tag{9.1.23}$$

Intra-quartet $\boldsymbol{\mathcal{P}}_{[4]}$ Correlators of the $\mathbb{S}_4$ permutohedron In this subsection, we illustrate the intra-quartet two-point correlators associated with $\boldsymbol{\mathcal{P}}_{[4]}$. The quartet members are indicated by dark blue quadrilaterals on some of the vertices of the permutohedron, the correlators are shown in tabular form, and finally, this same data is presented as a matrix.

The data contained in $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|4]}, \mathcal{P}_{[b|4]}]$ is presented below first in tabular and in

$\mathcal{P}_{[4]}$ \ $\mathcal{P}_{[4]}$	$\langle 1432 \rangle$	$\langle 2341 \rangle$	$\langle 3214 \rangle$	$\langle 4123 \rangle$
$\langle 1432 \rangle$	0	6	4	2
$\langle 2341 \rangle$	6	0	2	4
$\langle 3214 \rangle$	4	2	0	6
$\langle 4123 \rangle$	2	4	6	0

Table 9.4: $\{VM\}_1 - \{VM\}_1$ Two – Point Correlator Values

matrix form.

$$\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|4]}, \mathcal{P}_{[b|4]}] = \begin{bmatrix} 0 & 6 & 4 & 2 \\ 6 & 0 & 2 & 4 \\ 4 & 2 & 0 & 6 \\ 2 & 4 & 6 & 0 \end{bmatrix} \tag{9.1.24}$$

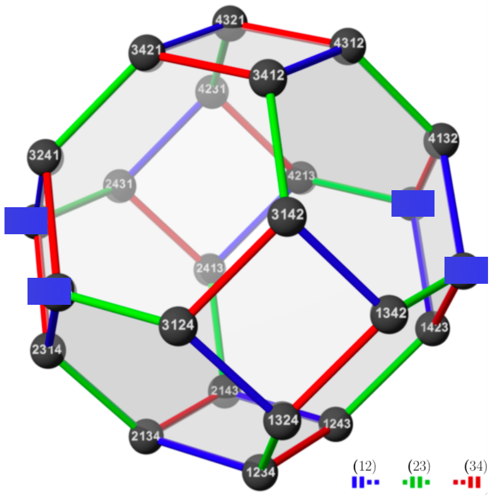


Figure 9.12: permutohedron & $\mathcal{P}_{[4]}$ Subset Elements

Intra-quartet $\boldsymbol{\mathcal{P}_{[5]}}$ Correlators of the $\mathbb{S}_4$ permutohedron In this subsection, we illustrate the intra-quartet two-point correlators associated with $\boldsymbol{\mathcal{P}_{[5]}}$. The quartet members are indicated by brown quadrilaterals on some of the vertices of the permutohedron, the correlators are shown in tabular form, and finally, this same data is presented as a matrix.

The data contained in $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|5]}, \mathcal{P}_{[b|5]}]$ is presented below first in tabular and in

$\mathcal{P}_{[5]}$ \ $\mathcal{P}_{[5]}$	$\langle 1243\rangle$	$\langle 2134\rangle$	$\langle 3421\rangle$	$\langle 4312\rangle$
$\langle 1243\rangle$	0	2	6	4
$\langle 2134\rangle$	2	0	4	6
$\langle 3421\rangle$	6	4	0	2
$\langle 4312\rangle$	4	6	2	0

Table 9.5: $\{VM\}_2 - \{VM\}_2$ Two $-$ Point Correlator Values

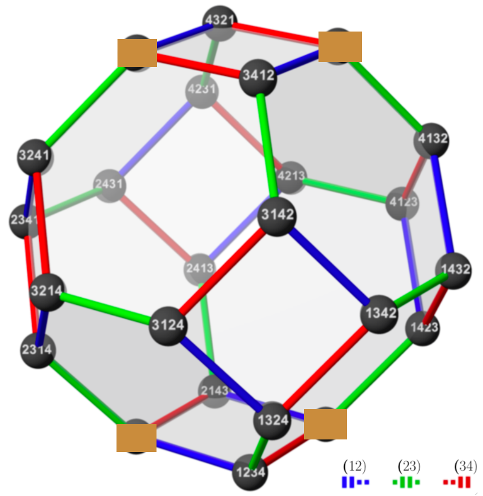


Figure 9.13: permutohedron & $\mathcal{P}_{[5]}$ Subset Elements

matrix form.

$$\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|5]}, \mathcal{P}_{[b|5]}] = \begin{bmatrix} 0 & 2 & 6 & 4 \\ 2 & 0 & 4 & 6 \\ 6 & 4 & 0 & 2 \\ 4 & 6 & 2 & 0 \end{bmatrix} \qquad (9.1.25)$$

Intra-quartet $\boldsymbol{\mathcal{P}}_{[6]}$ Correlators of the $\mathbb{S}_4$ permutohedron In this subsection, we illustrate the intra-quartet two-point correlators associated with $\boldsymbol{\mathcal{P}}_{[6]}$. The quartet members are indicated by light blue quadrilaterals on some of the vertices of the permutohedron, the correlators are shown in tabular form, and finally, this same data is presented as a matrix.

The data contained in $\boldsymbol{\mathcal{A}}_{\ell x}[\mathcal{P}_{[a|6]}, \mathcal{P}_{[b|6]}]$ is presented below first in tabular and in

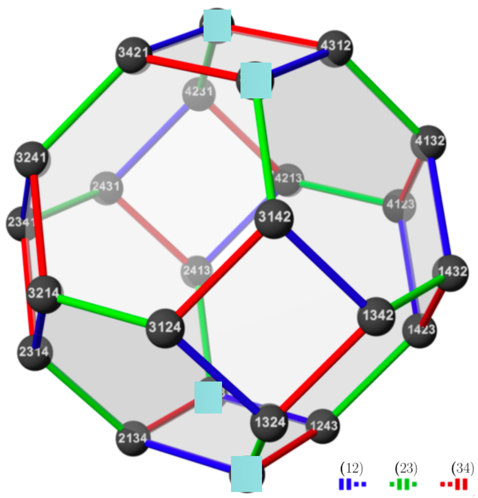


Figure 9.14: permutohedron & $\mathcal{P}_{[6]}$ Subset Elements

$\mathcal{P}_{[6]}$ \ $\mathcal{P}_{[6]}$	$\langle 1234 \rangle$	$\langle 2143 \rangle$	$\langle 3412 \rangle$	$\langle 4321 \rangle$
$\langle 1234 \rangle$	0	2	4	6
$\langle 2143 \rangle$	2	0	6	4
$\langle 3412 \rangle$	4	6	0	2
$\langle 4321 \rangle$	6	4	2	0

Table 9.6: $\{VM\}_3 - \{VM\}_3$ Two $-$ Point Correlator Values

matrix form.

$$\mathcal{A}_{\ell x}[\mathcal{P}_{[a|6]}, \mathcal{P}_{[b|6]}] = \begin{bmatrix} 0 & 2 & 4 & 6 \\ 2 & 0 & 6 & 4 \\ 4 & 6 & 0 & 2 \\ 6 & 4 & 2 & 0 \end{bmatrix} \tag{9.1.26}$$

9.1.8 *Inter-Quartet Results*

In this section, we will give the explicit numerical result for the inter-quartet calculations without reference to any diagrams.

Given the six $\boldsymbol{\mathcal{P}}_{[A]}$ subsets, the minimal number of links it takes to travel from a given specified element a in the set $\boldsymbol{\mathcal{P}}_{[A]}$ to a second specified element b in the set $\boldsymbol{\mathcal{P}}_{[B]}$ where $A \neq B$ can form a table/matrix. From simply counting $6 \times 5 = 30$, we can create in total 30 inter-quartet correlator matrices. Note that all these matrices satisfy the property that their transpose matrices have the same eigenvalues as themselves. Therefore we will only present half of them.

$\boldsymbol{\mathcal{P}}_{[6]}$ \ $\boldsymbol{\mathcal{P}}_{[1]}$	$\langle 1423 \rangle$	$\langle 2314 \rangle$	$\langle 3241 \rangle$	$\langle 4132 \rangle$
$\langle 1234 \rangle$	2	2	4	4
$\langle 2143 \rangle$	2	2	4	4
$\langle 3412 \rangle$	4	4	2	2
$\langle 4321 \rangle$	4	4	2	2

(9.1.27)

Matrix Eigenvalues: $\{12, -4, 0, 0\}$ Matrix Trace $= 8$

$\mathcal{P}_{[6]}$ \ $\mathcal{P}_{[2]}$	$\langle 1342 \rangle$	$\langle 2431 \rangle$	$\langle 3124 \rangle$	$\langle 4213 \rangle$
$\langle 1234 \rangle$	2	4	2	4
$\langle 2143 \rangle$	4	2	4	2
$\langle 3412 \rangle$	2	4	2	4
$\langle 4321 \rangle$	4	2	4	2

(9.1.28)

Matrix Eigenvalues: $\{12, -4, 0, 0\}$ Matrix Trace = 8

$\mathcal{P}_{[6]}$ \ $\mathcal{P}_{[3]}$	$\langle 1324 \rangle$	$\langle 2413 \rangle$	$\langle 3142 \rangle$	$\langle 4231 \rangle$
$\langle 1234 \rangle$	1	3	3	5
$\langle 2143 \rangle$	3	1	5	3
$\langle 3412 \rangle$	3	5	1	3
$\langle 4321 \rangle$	5	3	3	1

(9.1.29)

Matrix Eigenvalues: $\{12, -4, -4, 0\}$ Matrix Trace = 4

$\mathcal{P}_{[6]}$ \ $\mathcal{P}_{[4]}$	$\langle 1432 \rangle$	$\langle 2341 \rangle$	$\langle 3214 \rangle$	$\langle 4123 \rangle$
$\langle 1234 \rangle$	3	3	3	3
$\langle 2143 \rangle$	3	3	3	3
$\langle 3412 \rangle$	3	3	3	3
$\langle 4321 \rangle$	3	3	3	3

(9.1.30)

Matrix Eigenvalues: $\{12, 0, 0, 0\}$ Matrix Trace = 12

$\mathcal{P}_{[6]}$ \ $\mathcal{P}_{[5]}$	$\langle 1243\rangle$	$\langle 2134\rangle$	$\langle 3421\rangle$	$\langle 4312\rangle$
$\langle 1234\rangle$	1	1	5	5
$\langle 2143\rangle$	1	1	5	5
$\langle 3412\rangle$	5	5	1	1
$\langle 4321\rangle$	5	5	1	1

(9.1.31)

Matrix Eigenvalues: $\{12, -8, 0, 0\}$ Matrix Trace $= 4$

$\mathcal{P}_{[5]}$ \ $\mathcal{P}_{[1]}$	$\langle 1423\rangle$	$\langle 2314\rangle$	$\langle 3241\rangle$	$\langle 4132\rangle$
$\langle 1243\rangle$	1	3	5	3
$\langle 2134\rangle$	3	1	3	5
$\langle 3421\rangle$	5	3	1	3
$\langle 4312\rangle$	3	5	3	1

(9.1.32)

Matrix Eigenvalues: $\{12, -4, -4, 0\}$ Matrix Trace $= 4$

$\mathcal{P}_{[5]}$ \ $\mathcal{P}_{[2]}$	$\langle 1342\rangle$	$\langle 2431\rangle$	$\langle 3124\rangle$	$\langle 4213\rangle$
$\langle 1243\rangle$	3	3	3	3
$\langle 2134\rangle$	3	3	3	3
$\langle 3421\rangle$	3	3	3	3
$\langle 4312\rangle$	3	3	3	3

(9.1.33)

Matrix Eigenvalues: $\{12, 0, 0, 0\}$ Matrix Trace $= 12$

$\mathcal{P}_{[5]} \backslash \mathcal{P}_{[3]}$	$\langle 1324 \rangle$	$\langle 2413 \rangle$	$\langle 3142 \rangle$	$\langle 4231 \rangle$
$\langle 1243 \rangle$	2	2	4	4
$\langle 2134 \rangle$	2	2	4	4
$\langle 3421 \rangle$	4	4	2	2
$\langle 4312 \rangle$	4	4	2	2

(9.1.34)

Matrix Eigenvalues: $\{12, -4, 0, 0\}$ Matrix Trace = 8

$\mathcal{P}_{[5]} \backslash \mathcal{P}_{[4]}$	$\langle 1432 \rangle$	$\langle 2341 \rangle$	$\langle 3214 \rangle$	$\langle 4123 \rangle$
$\langle 1243 \rangle$	2	4	4	2
$\langle 2134 \rangle$	4	2	2	4
$\langle 3421 \rangle$	4	2	2	4
$\langle 4312 \rangle$	2	4	4	2

(9.1.35)

Matrix Eigenvalues: $\{12, -4, 0, 0\}$ Matrix Trace = 8

$\mathcal{P}_{[4]} \backslash \mathcal{P}_{[1]}$	$\langle 1423 \rangle$	$\langle 2314 \rangle$	$\langle 3241 \rangle$	$\langle 4132 \rangle$
$\langle 1432 \rangle$	1	5	5	1
$\langle 2341 \rangle$	5	1	1	5
$\langle 3214 \rangle$	5	1	1	5
$\langle 4123 \rangle$	1	5	5	1

(9.1.36)

Matrix Eigenvalues: $\{12, -8, 0, 0\}$ Matrix Trace = 4

$\mathcal{P}_{[4]} \backslash \mathcal{P}_{[2]}$	$\langle 1342 \rangle$	$\langle 2431 \rangle$	$\langle 3124 \rangle$	$\langle 4213 \rangle$
$\langle 1432 \rangle$	1	5	3	3
$\langle 2341 \rangle$	5	1	3	3
$\langle 3214 \rangle$	3	3	1	5
$\langle 4123 \rangle$	3	3	5	1

(9.1.37)

Matrix Eigenvalues: $\{12, -4, -4, 0\}$ Matrix Trace $= 4$

$\mathcal{P}_{[4]} \backslash \mathcal{P}_{[3]}$	$\langle 1324 \rangle$	$\langle 2413 \rangle$	$\langle 3142 \rangle$	$\langle 4231 \rangle$
$\langle 1432 \rangle$	2	4	2	4
$\langle 2341 \rangle$	4	2	4	2
$\langle 3214 \rangle$	2	4	2	4
$\langle 4123 \rangle$	4	2	4	2

(9.1.38)

Matrix Eigenvalues: $\{12, -4, 0, 0\}$ Matrix Trace $= 8$

$\mathcal{P}_{[3]} \backslash \mathcal{P}_{[1]}$	$\langle 1423 \rangle$	$\langle 2314 \rangle$	$\langle 3241 \rangle$	$\langle 4132 \rangle$
$\langle 1324 \rangle$	3	3	3	3
$\langle 2413 \rangle$	3	3	3	3
$\langle 3142 \rangle$	3	3	3	3
$\langle 4231 \rangle$	3	3	3	3

(9.1.39)

Matrix Eigenvalues: $\{12, 0, 0, 0\}$ Matrix Trace $= 12$

$\mathcal{P}_{[3]}$ \ $\mathcal{P}_{[2]}$	$\langle 1342\rangle$	$\langle 2431\rangle$	$\langle 3124\rangle$	$\langle 4213\rangle$
$\langle 1324\rangle$	1	5	1	5
$\langle 2413\rangle$	5	1	5	1
$\langle 3142\rangle$	1	5	1	5
$\langle 4231\rangle$	5	1	5	1

(9.1.40)

Matrix Eigenvalues: $\{12,\ -4,\ 0,\ 0\}$ Matrix Trace $= 4$

$\mathcal{P}_{[2]}$ \ $\mathcal{P}_{[1]}$	$\langle 1423\rangle$	$\langle 2314\rangle$	$\langle 3241\rangle$	$\langle 4132\rangle$
$\langle 1342\rangle$	2	4	4	2
$\langle 2431\rangle$	4	2	2	4
$\langle 3124\rangle$	4	2	2	4
$\langle 4213\rangle$	2	4	4	2

(9.1.41)

Matrix Eigenvalues: $\{12,\ -4,\ 0,\ 0\}$ Matrix Trace $= 8$

9.1.9 20,736 permutohedronic "Correlators" Via Eigenvalue Equivalence Classes

For an adinkra, if we remove all dashing considerations, the remaining graph is called the chromotopology of the adinkra. In the context of chromotopology, there's only one "seed" adinkra required to generate all four-color minimal adinkras: the Vierergruppe[3] [169]. The total number of four-color minimal adinkras carrying different chromotopology is $\mathrm{ord}(\mathbb{S}_3)\times\mathrm{ord}(\mathbb{S}_4) = 144$. More generally, as shown in [169], left cosets of the Vierergruppe via S_3 generate all elements of S_4 (which are our $\mathbf{L}_I$), and the Coxeter group $BC_4^{\pm a\mu A} = \pm H^a S_3^\mu \mathcal{V}^A$, where H^a just corresponds to a sign flip and $\mathcal{V}^A$ is the Vierergruppe.

[3] The Vierergruppe or Klein 4-group corresponds $\mathcal{P}_{[6]}$.

In order to generate these 144 adinkras, we can start from the six $\boldsymbol{\mathcal{P}}_{[i]}$ sets we presented before. Each of them belongs to a family of adinkras that are generated by performing color permutations. For example, look at $\mathcal{P}_{[1]} = \{\langle 1423\rangle, \langle 2314\rangle, \langle 3241\rangle, \langle 4132\rangle\}$. Although it looks like a set, the order of matrices does matter! We can assign **L**-matrices for this adinkra as

$$\mathbf{L}_1 \ = \ \langle 1423\rangle \ , \ \mathbf{L}_2 \ = \ \langle 2314\rangle \ , \ \mathbf{L}_3 \ = \ \langle 3241\rangle \ , \ \mathbf{L}_4 \ = \ \langle 4132\rangle \tag{9.1.42}$$

Note that the index I attached on L-matrices $\mathbf{L}_{\rm I}$ labels the color. By doing color permutations, there are in total $4! = 24$ descendant adinkras. One example is to switch color 1 and color 2, i.e.

$$\mathbf{L}_1 \ = \ \langle 2314\rangle \ , \ \mathbf{L}_2 \ = \ \langle 1423\rangle \ , \ \mathbf{L}_3 \ = \ \langle 3241\rangle \ , \ \mathbf{L}_4 \ = \ \langle 4132\rangle \tag{9.1.43}$$

We can label these six families as $\{\boldsymbol{\mathcal{P}}_{[i]}\}$, where each family has 4! adinkras. Clearly, there's no intersection between any two families.

Then, we can study all $144 \times 144 = 20,736$ distance matrices and classify them by eigenvalues. In order to give a clear presentation as well as maximize computational efficiency, these 20,736 matrices can be classified based on their origins as below.

(1.) $3,456 = 4! \times 4! \times 6$ matrices generated by two adinkras from the same family $\{\boldsymbol{\mathcal{P}}_{[i]}\}$.

(2.) $17280 = 4! \times 4! \times 2 \times 15$ matrices generated by two adinkras from two different families $\{\boldsymbol{\mathcal{P}}_{[i]}\}$ and $\{\boldsymbol{\mathcal{P}}_{[j]}\}$. Note that the results in section 9.1.8 imply that this class can be further divided into four subclasses:

(2.1) $(i, j) = (1, 2), (1, 6), (2, 6), (3, 4), (3, 5), (4, 5)$

(2.2) $(i, j) = (2, 3), (5, 6), (1, 4)$

(2.3) $(i, j) = (1, 3), (2, 5), (4, 6)$

(2.4) $(i, j) = (1, 5), (2, 4), (3, 6)$

It turns out that many of these 20,736 distance matrices have the same eigenvalues. Finally, there are only 15 unique sets of eigenvalues. Table 9.7 shows the summary of eigenvalue equivalent classes of 20,736 distance matrices. You can find intermediate tables obtained from class (1) to class (2.4) in Appendix H.

Eigenvalues	# of Matrices	Trace
$12, 8, 4, 0$	144	24
$12, 8, 0, 0$	864	20
$12, 4, 4, 0$	144	20
$12, 4, 0, 0$	2,016	16
$12, 8, -4, 0$	144	16
$12, 0, 0, 0$	12,672	12
$12, -4, 4, 0$	576	12
$12, 4i, -4i, 0$	288	12
$12, -4\sqrt{2}, 4\sqrt{2}, 0$	288	12
$12, -4i\sqrt{2}, 4i\sqrt{2}, 0$	288	12
$12, -4, 0, 0$	2,016	8
$12, -8, 4, 0$	144	8
$12, -8, 0, 0$	864	4
$12, -4, -4, 0$	144	4
$12, -8, -4, 0$	144	0

Table 9.7: Eigenvalue Equivalent Classes of 20,736 Distance Matrices

In Section 9.1.7 and 9.1.8, we discussed a small subset of this analysis. Let's discuss one more explicit example: consider an adinkra A belong in family $\{\boldsymbol{\mathcal{P}}_{[1]}\}$ with **L**-matrices

$$\mathbf{L}_{1,A} = \langle 2314 \rangle \ , \ \mathbf{L}_{2,A} = \langle 3241 \rangle \ , \ \mathbf{L}_{3,A} = \langle 1423 \rangle \ , \ \mathbf{L}_{4,A} = \langle 4132 \rangle \tag{9.1.44}$$

and adinkra B belong in family $\{\boldsymbol{\mathcal{P}}_{[1]}\}$ with **L**-matrices

$$\mathbf{L}_{1,B} = \langle 2314 \rangle \ , \ \mathbf{L}_{2,B} = \langle 1423 \rangle \ , \ \mathbf{L}_{3,B} = \langle 3241 \rangle \ , \ \mathbf{L}_{4,B} = \langle 4132 \rangle \tag{9.1.45}$$

Then the distance matrix corresponding to these two adinkras is

A \ B	$\langle 2314\rangle$	$\langle 1423\rangle$	$\langle 3241\rangle$	$\langle 4132\rangle$
$\langle 2314\rangle$	0	4	2	6
$\langle 3241\rangle$	2	6	0	4
$\langle 1423\rangle$	4	0	6	2
$\langle 4132\rangle$	6	2	4	0

(9.1.46)

whose eigenvalues are $\{12, -4\sqrt{2}, 4\sqrt{2}, 0\}$.

9.1.10 The View Beyond 4D, $\mathcal{N}$ = 1 SUSY

In this section, we wish to connect the current presentation relating the geometry of the orbits of elements described by the permutohedron to previous discussions [75,76,170] that go beyond the exemplary case of $\mathbb{BC}_4$ used in this work.

The faces that minimally connect all the elements contained in any single quartet are squares. Among the subsets $\boldsymbol{\mathcal{P}}_{[1]}, \cdots, \boldsymbol{\mathcal{P}}_{[6]}$, there is an element that has the minimum values with respect to lexicographical ordering. This is the identity element () and we note $() \in \boldsymbol{\mathcal{P}}_{[6]}$. In the following, we will go beyond the case related to 4D, $\mathcal{N} = 1$ SUSY to discuss similar structures that have been found for arbitrary values of $\mathcal{N}$. Let us recall that $\boldsymbol{\mathcal{P}}_{[6]}$ is "Klein's Vierergruppe" and we will concentrate on this generalization for arbitrary $\mathcal{N}$.

Already in the work of [75,76] the existence of a recursive procedure was demonstrated for how to construct the analog of the Vierergruppe that is needed for all values of $\mathcal{N}$. We will present this recursion in an appendix. In the work of [170], the case for $\mathcal{N} = 8$, the analog of the elements of the Vierergruppe, was presented as $\boldsymbol{\mathcal{P}}_{\widehat{I}}$ where explicitly we see

$$
\begin{aligned}
&\mathcal{P}_1 = () \quad , \quad \mathcal{P}_2 = (12)(34)(56)(78) \quad , \quad \mathcal{P}_3 = (13)(24)(57)(68) \quad , \quad \mathcal{P}_4 = (14)(23)(58)(67) \quad , \\
&\mathcal{P}_5 = (15)(26)(37)(48) \quad , \quad \mathcal{P}_6 = (16)(25)(38)(47) \quad , \quad \mathcal{P}_7 = (17)(28)(35)(46) \quad , \\
&\mathcal{P}_8 = (18)(27)(36)(45) \quad .
\end{aligned}
\tag{9.1.47}
$$

as the index $\widehat{\boldsymbol{I}}$ ranges from 1, . . . , 8.

Looking back at Fig. 9.9 the Vierergruppe consists of the permutation elements $\langle 1234\rangle$, $\langle 2134\rangle$, $\langle 1243\rangle$, and $\langle 2143\rangle$, all of which occur at the 'base" of the permutohedron related to 4D, $\mathcal{N} = 1$ minimal supersymmetry representations. By comparison, the permutation elements of $\mathbb{S}_8$ that appear in Eq. (9.1.47) must occur at the "base" of the permutohedron related to 4D, $\mathcal{N} = 2$ minimal supersymmetry representations. Application of the pair-wise adjacent permutations (23), (34), (45), (56), (67), and (78) to all the permutation elements that appear in Eq. (9.1.47) will lead to the complete permutohedron associated with 4D, $\mathcal{N} = 2$ minimal supersymmetry representations.

The size of the permutation matrices "jumps" by a factor of the $\mathcal{N}$ values belong to sequence of numbers given by

$$
1,\ 2,\ 4,\ 8,\ 9,\ 10,\ 12,\ 16,\ 17,\ 18,\ 20,\ 25, \dots \tag{9.1.48}
$$

We borrow language from nuclear physics [4] and call these "magical" values of $\mathcal{N}$. Once the matrix representation of the permutations is known, it is straightforward by tedious matter to convert the results into either $\langle\rangle$-notation of ()-notation.

Using the recursion formula of [170] we previously generated the explicit matrix formulations of the "bases" for any value of $\mathcal{N}$. For the magic number cases where $1 \le \mathcal{N} \le 16$ leads to

$$
\begin{aligned}
\boldsymbol{\mathcal{P}}_1 &= \mathbb{I}_2 \quad , \\
\boldsymbol{\mathcal{P}}_2 &= \boldsymbol{\sigma}^1 \quad ,
\end{aligned}
\tag{9.1.49}
$$

[4] See the webpage at https://en.wikipedia.org/wiki/Magic_number_(physics) on-line.

for $\mathcal{N} = 2$

$$\begin{array}{rcccccl} \boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_2 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_3 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_4 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \end{array} \tag{9.1.50}$$

for $\mathcal{N} = 4$,

$$\begin{array}{rcccccccl} \boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_2 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_3 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbf{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_4 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_5 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_6 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_7 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_8 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \end{array} \tag{9.1.51}$$

for $\mathcal{N} = 8$,

$$\begin{array}{rcccccccccl} \boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_2 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_3 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_4 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_5 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_6 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_7 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\ \boldsymbol{\mathcal{P}}_8 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\ \boldsymbol{\mathcal{P}}_9 & = & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \end{array} \tag{9.1.52}$$

for $\mathcal{N} = 9$,

$$\begin{array}{lcccccccccccl}
\boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_2 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_3 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_4 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_5 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_6 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_7 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_8 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_9 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_{10} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & ,
\end{array} \tag{9.1.53}$$

for $\mathcal{N} = 10$,

$$\begin{array}{lcccccccccccccl}
\boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_2 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_{2\times 2} & & , \\
\boldsymbol{\mathcal{P}}_3 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_4 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_5 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_{2\times 2} & & , \\
\boldsymbol{\mathcal{P}}_6 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_7 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^3 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_8 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_9 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_{10} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & & , \\
\boldsymbol{\mathcal{P}}_{11} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & & , \\
\boldsymbol{\mathcal{P}}_{12} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & & ,
\end{array} \tag{9.1.54}$$

for $\mathcal{N} = 12$, and

$$
\begin{array}{lclclclclclclclc}
\boldsymbol{\mathcal{P}}_1 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_2 & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_3 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_4 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & , \\
\boldsymbol{\mathcal{P}}_5 & = & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_6 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & , \\
\boldsymbol{\mathcal{P}}_7 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_8 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbf{I}_2 & , \\
\boldsymbol{\mathcal{P}}_9 & = & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_{10} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & , \\
\boldsymbol{\mathcal{P}}_{11} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_{12} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & , \\
\boldsymbol{\mathcal{P}}_{13} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & ; \\
\boldsymbol{\mathcal{P}}_{14} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & , \\
\boldsymbol{\mathcal{P}}_{15} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & , \\
\boldsymbol{\mathcal{P}}_{16} & = & \boldsymbol{\sigma}^1 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \mathbb{I}_2 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & \otimes & \boldsymbol{\sigma}^1 & ,
\end{array}
\tag{9.1.55}
$$

for $\mathcal{N} = 16$. In these equations, the quantities $\boldsymbol{\sigma}^1$,$\boldsymbol{\sigma}^2$,and $\boldsymbol{\sigma}^3$ refer to the usual 2×2 Pauli matrices and $\mathbf{I}_2$ refer to the identity matrix.

After these are converted into $\langle\rangle$-notation, it is possible to calculate the correlators for any of these cases. This is a topic for future study. Since there exists a recursion formula to derive the last eight of these from the first eight, it is conceivable that it should be possible to derive the intra-$\mathcal{N}$-tet correlators, along with their eigenvalues, and traces, in complete generality.

Collectively the sets of permutation objects that appear in Eq.(9.1.49) to Eq.(9.1.55) have informally been given the names of “diadems.”

9.1.11 Conclusion

In 9.1, we have used structures that are intrinsic to the Coxeter Group $\mathbb{BC}_4$ in order to discuss how their elements are organized to provide representations of four-dimensional $\mathcal{N} = 1$ SUSY.

It is of interest to note none of the eigenvalues for the inter-quartet correlators align with that of the intra-quartet correlators. It is also true that none of the traces of the inter-quartet correlators vanish as in the case of the intra-quartet correlators.

As seen from the results in (9.1.27) to (9.1.41) only three values of traces and four sets of eigenvalues occurred for the distributions of the various inter-quartet eigenvalues and traces. These are shown in the table below.

Eigenvalues \ Traces	4	8	12
$(12, 0, 0, 0)$	-	-	$(\mathcal{P}_{[1]}\vert\mathcal{P}_{[3]})(\mathcal{P}_{[2]}\vert\mathcal{P}_{[5]})$ $(\mathcal{P}_{[4]}\vert\mathcal{P}_{[6]})$
$(12, -4, 0, 0)$	$(\mathcal{P}_{[2]}\vert\mathcal{P}_{[3]})$	$(\mathcal{P}_{[1]}\vert\mathcal{P}_{[2]})(\mathcal{P}_{[1]}\vert\mathcal{P}_{[6]})$ $(\mathcal{P}_{[2]}\vert\mathcal{P}_{[6]})(\mathcal{P}_{[3]}\vert\mathcal{P}_{[4]})$ $(\mathcal{P}_{[3]}\vert\mathcal{P}_{[5]})(\mathcal{P}_{[4]}\vert\mathcal{P}_{[5]})$	-
$(12, -4, -4, 0)$	$(\mathcal{P}_{[1]}\vert\mathcal{P}_{[5]})(\mathcal{P}_{[2]}\vert\mathcal{P}_{[4]})$ $(\mathcal{P}_{[3]}\vert\mathcal{P}_{[6]})$	-	-
$(12, -8, 0, 0)$	$(\mathcal{P}_{[1]}\vert\mathcal{P}_{[4]})(\mathcal{P}_{[5]}\vert\mathcal{P}_{[6]})$	-	-

Table 9.8: Eigenvalue/Trace Distribution of Two-Point Correlator Values

These can be compared to the eigenvalues and traces of the intra-quartet correlators were given by $(12, 0, -4, -8)$ and 0 respectively.

If we calculate the "length" (by use of the usual Euclidean metric) of the eigenvectors associated with the intra-quartet correlators, it takes the value of $\sqrt{224}$. This may be compared to the "lengths" found for the inter-quartet correlator eigenvectors. Working from top to bottom in the rightmost column of Table 23, the respective lengths are $\sqrt{144}$, $\sqrt{160}$, $\sqrt{176}$, and $\sqrt{208}$, respectively.

Thus, we conclude that the SUSY quartets in $\mathbb{S}_4$ are precisely the ones that lead to the maximum possible value of the lengths of the eigenvectors for quartet correlators.

The use of the two-point quartet correlators reveals symmetries of how the SUSY quartets are embedded in the permutohedron in other ways also. One of the most amusing interpretations is that the SUSY quartets are the solutions to a set of Sudoku puzzles!

For example, looking at the results in Equations (9.1.21), (9.1.22), (9.1.23), (9.1.24), (9.1.25), and (9.1.26), it is apparent that every column and every row in these matrices sum to twelve. One is required to place zeros in every diagonal entry and then follow the remaining rules stated under the first full paragraph under (9.1.21).

This Sudoku-puzzle solution interpretation also follows for every table (and therefore every associated matrix) in the list of results shown in Equations (9.1.27) - (9.1.41) where the sums of the columns and rows are the same. The rules in these cases start with a "source set" of numbers 1, . . . , 5, 6, but become considerably more baroque to state.

We believe the most powerful implication of the observations in this work is that the representation theory for SUSY for all values of $\mathcal{N}$ can be interpreted as a Sudoku puzzle where the diadems set the start of rules. This means an essential part of a comprehensive understanding of the representation rules for SUSY can be studied solely and a mathematical problem of diadem embeddings into $\mathbb{S}_d$...with no reference at all to QFT.

9.2 Adinkra Height Yielding Matrix Numbers: Eigenvalue Analysis of Minimal Four-Color Adinkra Isomorphisms

This section is based on [85].

9.2.1 Introduction

The discovery of the role played by the $\mathcal{GR}$(d, N) "Garden Algebras" [75, 76] in the generation of representations of spacetime SUSY triggered the introduction of adinkras [77] in order to allow the technology of graph theory to include and expand prior existing results. Among these results was an algorithm for generating dimensions of the minimal linear representations for all possible supersymmetrical systems. This is encoded in a function denoted by $\mathrm{d}_{min}(N)$ [75, 76] where N is the number of one-dimensional supercharges. In an adinkra graph, links denote the orbits of the one-dimensional projections of space-time functions in a supermultiplet under the action of space-time supercharges. The nodes of an adinkra graph are projections of the fields of a supermultiplet when their functional dependence is restricted to solely depend on time. Since their introduction, adinkras have also opened gateways to insights in the physics and mathematics of SUSY.

On the physics side, one of the most striking implications uncovered is every off-shell linear supersymmetrical system that allows for the solution of the initial value problem involves adinkras containing error-correcting codes [78–80]. This apparently implies an information-theoretic foundation is present in all supersymmetrical particle, field, and string theories.

On the mathematics side, in a correspondingly striking implication, it is now known adinkras, due to the work of [91, 92], may be regarded as examples of Grothendieck's "dessin d' enfant" and thereby define a class of Riemann surfaces. From this vantage point, the heights of the nodes in an adinkra correspond to an integer-valued discrete Morse function defined over these Riemann surfaces.

In 1970, the mathematician Banchoff [37][5] introduced a form of discrete Morse functions for oriented triangular meshes. For the case of the Riemann surfaces associated with adinkras, it is the value of the heights of the nodes (made into a piecewise-linear function over the meshes constructed by linear extension across the edges and faces of the plaquettes) that are used to define the Morse function. So the height assigned to each node may be also be called the"Banchoff index" of the node.

Within 4D theories generally not all of the bosonic fields contained within a supermultiplet possess the same engineering dimensions. In a similar manner, in the general case, not all fermionic fields contained in a supermultiplet possess the same engineering dimensions. At the level of adinkra graphs, the engineering dimension of a particular projection of a space-time field determines the literal height at which the node representing that projection appears in the graphs. Fields with the lowest engineering dimension appear at the lowest level of an adinkra graph. Fields of higher engineering dimensions appear at higher levels in the graph.

[5] The publications [91, 92] pointed toward the relevance of this work.

A special class of adinkras is those in which all the bosonic nodes in the adinkra appear at the same height and all the fermionic nodes in the adinkra have the same height (but which is different from the bosonic node height). These are called "valise" adinkras. In terms of the field theories of which such adinkras are their projections, all bosonic fields in the supermultiplet possess the same engineering dimension. A similar statement can be made about the fermionic fields of the supermultiplet.

Even before the introduction of adinkras, the phenomenon of "node lifting" and "node lowering" had been noticed [84]. The effect of lifting nodes in adinkra graphs was investigated in [107, 108, 171–175]. In this paper, we study all 36,864 four-color valise adinkras associated with the Coxeter Group BC_4 and define an isomorphism algorithm of adinkras. To study the effect of lifting bosons in a four-color valise adinkra on isomorphism, we developed a symbolic program. Using the program, one can compute eigenvalues of "Banchoff-matrices" of all sixteen corresponding adinkras after lifting bosonic nodes given L-matrices of a valise adinkra. We are able to analyze calculation results for all 36,864 valise adinkras and find that dashing plays only a minor role and there is a mirror symmetry for lifting bosonic nodes in matters of isomorphism. A summary of the calculations can be found in a *Mathematica* notebook available at the HEPTHools Data Repository on GitHub.

9.2.2 Review of 36,864 Four-Color Valise Adinkras

First, we will briefly review L-matrices. Here we use the 4D, $\mathcal{N}=1$ chiral multiplet as an example. We have in our conventions for 4D quantities,

$$\begin{aligned}
&\mathrm{D}_a A = \psi_a \ , \ \mathrm{D}_a B = i(\gamma^5)_a{}^b\psi_b \ , \ \mathrm{D}_a F = (\gamma^\mu)_a{}^b\partial_\mu\psi_b \ , \ \mathrm{D}_a G = i(\gamma^5\gamma^\mu)_a{}^b\partial_\mu\psi_b \ , \\
&\mathrm{D}_a\psi_b = i(\gamma^\mu)_{ab}\partial_\mu A - (\gamma^5\gamma^\mu)_{ab}\partial_\mu B - iC_{ab}F + (\gamma^5)_{ab}G \ .
\end{aligned} \tag{9.2.1}$$

These equations are still valid if we restrict the functions only to be dependent on the t-coordinate. Under this restriction, we get the 4D, $\mathcal{N}=1$ chiral multiplet on the

0-Brane. Define

$$
\begin{aligned}
&\psi_1 = i\Psi_1 \quad , \quad \psi_2 = i\Psi_2 \quad , \quad \psi_3 = i\Psi_3 \quad , \quad \psi_4 = i\Psi_4 \quad , \\
&\Phi_1 = A \quad , \quad \Phi_2 = B \quad , \quad \partial_0\Phi_3 = F \quad , \quad \partial_0\Phi_4 = G \quad ,
\end{aligned}
\tag{9.2.2}
$$

and rewrite (9.2.1) on the 0-Brane in the form

$$
\mathrm{D}_I\Phi_i = i(L_I)_{i\hat{k}}\Psi_{\hat{k}} \quad , \tag{9.2.3a}
$$

$$
\mathrm{D}_I\Psi_{\hat{k}} = (R_I)_{\hat{k}i}\frac{d}{dt}\Phi_i \quad , \tag{9.2.3b}
$$

where $I = 1, 2, 3, 4$ is the color index.

Note that in (2.2) the projections of the 4D fields A and B are directly related to adinkra nodal functions according to $\Phi_1 = A$, $\Phi_2 = B$, while the projections of the 4D fields F and G are related to adinkra nodal functions according to $\partial_0\Phi_3 = F$, $\partial_0\Phi_4 = G$. That is the adinkra nodal functions Φ_3 and Φ_4 are the integrals of the projections of the 4D fields F and G. Whenever an integral is used to define an adinkra node starting from the projection of a 4D field, we refer to this process as "nodal lowering." This concept was first presented in the work of [84].

Consequently, we derive the explicit form of L-matrices:

$$
\begin{aligned}
(L_1)_{i\hat{k}} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{pmatrix} \quad , \quad (L_2)_{i\hat{k}} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \quad , \\
(L_3)_{i\hat{k}} &= \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \quad (L_4)_{i\hat{k}} = \begin{pmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{9.2.4}
$$

The R-matrices can be obtained by the relation $(R_I) = [(L_I)]^T$, where T superscript means transposition. The general algebra for d× d matrices describing N supersymmetries, known as the $\mathcal{GR}$(d, N) algebra (garden algebra) is

$$
\begin{aligned}
(L_I)_{i\hat{k}}(R_J)_{\hat{k}j} + (L_J)_{i\hat{k}}(R_I)_{\hat{k}j} &= 2\delta_{IJ}\delta_{ij} \quad , \\
(R_I)_{i\hat{k}}(L_J)_{\hat{k}j} + (R_J)_{i\hat{k}}(L_I)_{\hat{k}j} &= 2\delta_{IJ}\delta_{ij} \quad .
\end{aligned}
\tag{9.2.5}
$$

The $(L_I)_{i\hat{k}}$ and $(R_I)_{\hat{k}i}$ matrices described by Eqs. (9.2.4) satisfy the $\mathcal{GR}(4,4)$ algebra.

The adinkra diagram for the off-shell chiral multiplet where the associated L-matrices appear in Eq. (9.2.4) is shown as Figure 9.15, where D$_1$-green, D$_2$-violet, D$_3$-orange, D$_4$-red.

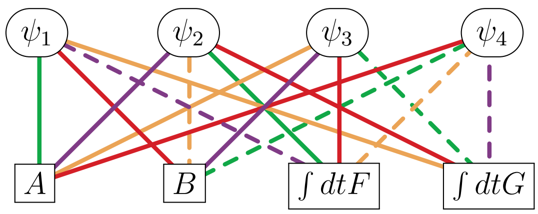


Figure 9.15: Valise Adinkra for Chiral Supermultiplet with F & G Nodes Lowered

Next, we will briefly introduce an algorithm to obtain all four-color valise adinkras. The recent work [144] studied how to generate all 36,864 four-color valise adinkras from only two basis valise adinkras given the titles of "quaternion adinkras."

By definition, BC_4 is the group of signed permutations of four elements. BC_4 can be expressed as plus or minus one times a sign flip element times a permutation element times an element of the Viererrgruppe.

$$
BC_4^{\pm a\mu A} = \pm H^a S_3^\mu \mathcal{V}^A, \tag{9.2.6}
$$

where H^a is a sign flip element and $a = 1, 2, ..., 8$

$$
H^a = \{(), (\overline{12}), (\overline{13}), (\overline{23}), (\overline{1}), (\overline{2}), (\overline{3}), (\overline{123})\}, \tag{9.2.7}
$$

S_3^μ is a permutation element and $\mu = 1, 2, ..., 6$

$$
S_3^\mu = \{(), (12), (13), (23), (123), (132)\}, \tag{9.2.8}
$$

$\mathcal{V}^A$ is an element of the Viererrgruppe and $A = 1, 2, 3, 4$

$$\mathcal{V}^A = \{(), (12)(34), (13)(24), (14)(23)\}. \tag{9.2.9}$$

The explicit matrix forms for these elements are given in [144]. As in [144], we refer to sign flips such as H^a as simply flips and permutations such as $S_3^\mu \mathcal{V}^A$ as *flops*.

All four-color valise adinkras can be generated by combinations of transformations.

$$\begin{aligned} (L_I^{\pm a\mu A b\nu})_i{}^{\hat{j}} &= (BC_4^{\pm a\mu A})_i{}^j (H^b S_3^\nu)_I{}^J (L_J^{(Q)})_j{}^{\hat{j}}, \\ (\widetilde{L}_I^{\pm a\mu A b\nu})_i{}^{\hat{j}} &= (BC_4^{\pm a\mu A})_i{}^j (H^b S_3^\nu)_I{}^J (\widetilde{L}_J^{(\widetilde{Q})})_j{}^{\hat{j}}. \end{aligned} \tag{9.2.10}$$

where $(L_J^{(Q)})_j{}^{\hat{j}}$ and $(\widetilde{L}_J^{(\widetilde{Q})})_j{}^{\hat{j}}$ denote the L-matrices associated with the quaternionic basis adinkras to be given shortly.

The counting is summarized in the following table. The total product is 36,864. Thus, there are totally 36,864 four-color valise adinkras.

index	$\pm$	a	μ	A	b	ν	$\sim$
count	2	8	6	4	8	6	2

We define BC_4 color transformations as $(BC_4^{\pm a\mu A})_I{}^J$, where I, J are the color indices; BC_4 boson transformations as $(BC_4^{\pm a\mu A})_i{}^j$, where i, j are the bosonic indices; BC_4 fermion transformations as

$(BC_4^{\pm a\mu A})_{\hat{i}}{}^{\hat{j}}$, where $\hat{i}, \hat{j}$ are the fermionic indices. The definitions of the L-matrices associated with the quaternion basis adinkras Q and $\widetilde{Q}$ take the explicit forms,

$$
\begin{aligned}
L_1^{(Q)} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad L_2^{(Q)} = \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \\
L_3^{(Q)} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \end{pmatrix} \quad , \quad L_4^{(Q)} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & -1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad , \\
L_1^{(\widetilde{Q})} &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad , \quad L_2^{(\widetilde{Q})} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix} \quad , \\
L_3^{(\widetilde{Q})} &= \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix} \quad , \quad L_4^{(\widetilde{Q})} = \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{9.2.11}
$$

Since the early 1980s [176] and in later work (see for example [177–180]) it has been understood that division algebras play a significant role in the representations of spacetime SUSY. The results in (2.11) situate these previous observations in the context of $\mathcal{GR}$(d, N) algebras. In fact, the embedding of the quaternions realized by the matrices in (2.11) also matches the initial presentation on $\mathcal{GR}$(d, N) algebras in [75, 76]. The recursion formulae given in these earliest works on Garden Algebras also solve a puzzle.

The division algebras only exist up to the octonions. This is equivalent to 4D, $\mathcal{N}$ = 2 spacetime SUSY. Since realizations of SUSY exist beyond 4D, $\mathcal{N}$ = 2, the puzzle was to ask, "What replaces the division algebras for the higher theories?" In the works of [75, 76] using a recursion formula that realizes Bott periodicity, such extensions were created for all values of $\mathcal{N}$.

9.2.3 Adinkra Height Related 4D, $\mathcal{N}$ = 1 Minimal Supermultiplets

The chiral supermultiplet is only one member of a group of ten such supermultiplets that lead to adinkras of the same general form as that shown in Fig. 9.15.

There are three versions of the 4D, $\mathcal{N} = 1$ chiral supermultiplet related to the standard version discussed above by Hodge duality transformation applied to the auxiliary fields. These take the forms shown in equations (9.2.12), (9.2.13), and (9.2.14), respectively.

$Hodge - Dual \ \#1 \ Chiral \ Supermultiplet: \ (A, \, B, \, \psi_a, \, \mathrm{f}_{\mu\,\nu\,\rho}, \, G)$

$$
\begin{aligned}
\mathrm{D}_a A \ &= \ \psi_a \qquad\qquad , \qquad \mathrm{D}_a B \ = \ i\,(\gamma^5)_a{}^b\,\psi_b \qquad , \\
\mathrm{D}_a \psi_b \ &= \ i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A \ - \ (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B \ - \ i\,\frac{1}{3!}\,C_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \mathrm{f}_{\mu\,\nu\,\rho}) \ + \ (\gamma^5)_{a\,b} G \ \ , \\
\mathrm{D}_a \mathrm{f}_{\mu\,\nu\,\rho} \ &= \ -\,(\gamma^\sigma)_a{}^b\,\epsilon_{\sigma\mu\nu\rho}\,\psi_b \quad , \quad \mathrm{D}_a G \ = \ i\,(\gamma^5\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b \quad ,
\end{aligned}
\tag{9.2.12}
$$

$Hodge - Dual \ \#2 \ Chiral \ Supermultiplet: \ (A, \, B, \, \psi_a, \, F, \, \mathrm{g}_{\mu\,\nu\,\rho})$

$$
\begin{aligned}
\mathrm{D}_a A \ &= \ \psi_a \qquad\qquad , \qquad \mathrm{D}_a B \ = \ i\,(\gamma^5)_a{}^b\,\psi_b \qquad , \\
\mathrm{D}_a \psi_b \ &= \ i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A \ - \ (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B \ - \ i\,C_{a\,b}\,F \ + \ \frac{1}{3!}\,(\gamma^5)_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \mathrm{g}_{\mu\,\nu\,\rho}) \ \ , \\
\mathrm{D}_a F \ &= \ (\gamma^\mu)_a{}^b\,\partial_\mu\,\psi_b \quad , \quad \mathrm{D}_a \mathrm{g}_{\mu\,\nu\,\rho} \ = \ -\,(\gamma^5\gamma^\sigma)_a{}^b\,\epsilon_{\sigma\mu\nu\rho}\,\psi_b \quad ,
\end{aligned}
\tag{9.2.13}
$$

$Hodge - Dual \ \#3 \ Chiral \ Supermultiplet: \ (A, \, B, \, \psi_a, \, \mathrm{f}_{\mu\,\nu\,\rho}, \, \mathrm{g}_{\mu\,\nu\,\rho})$

$$
\begin{aligned}
\mathrm{D}_a A \ &= \ \psi_a \qquad\qquad , \qquad \mathrm{D}_a B \ = \ i\,(\gamma^5)_a{}^b\,\psi_b \qquad , \\
\mathrm{D}_a \psi_b \ &= \ i\,(\gamma^\mu)_{a\,b}\,\partial_\mu A \ - \ (\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu B \\
&\quad \ - \ i\,\frac{1}{3!}\,C_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \mathrm{f}_{\mu\,\nu\,\rho}) \ + \ \frac{1}{3!}\,(\gamma^5)_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \mathrm{g}_{\mu\,\nu\,\rho}) \ \ , \\
\mathrm{D}_a \mathrm{f}_{\mu\,\nu\,\rho} \ &= \ -\,(\gamma^\sigma)_a{}^b\,\epsilon_{\sigma\mu\nu\rho}\,\psi_b \quad , \quad \mathrm{D}_a \mathrm{g}_{\mu\,\nu\,\rho} \ = \ -\,(\gamma^5\gamma^\sigma)_a{}^b\,\epsilon_{\sigma\mu\nu\rho}\,\psi_b \quad .
\end{aligned}
\tag{9.2.14}
$$

The engineering dimensions of the three-form gauge fields that appear above are the same as those of the A and B fields. So in an adinkra without node lowering, these three-form fields appear at the same height as the A and B fields. This has implications for how the component fields in these supermultiplets are related to the graph shown in Fig. 9.15.

For the $Hodge - Dual \ \#1 \ Chiral \ Supermultiplet$ one should perform the replacement in Fig. # 1. of the auxiliary fields according to

$$\int dt\, F \ \rightarrow \ \mathrm{f}_{123} \quad , \quad G \ \rightarrow \ G \quad , \tag{9.2.15}$$

where f_{123} is the purely spatial component of the Lorentz 3-form f_{123}.

For the $Hodge - Dual \ \#2 \ Chiral \ Supermultiplet$ one should perform the replacement in Fig. # 1. of the auxiliary fields according to

$$F \ \rightarrow \ F \quad , \quad \int dt\, G \ \rightarrow \ \mathrm{g}_{123} \quad , \tag{9.2.16}$$

where g_{123} is the purely spatial component of the Lorentz 3-form g_{123}.

For the $Hodge - Dual \ \#3 \ Chiral \ Supermultiplet$ one should perform the replacement in Fig. # 1. of the auxiliary fields according to

$$\int dt\, F \ \rightarrow \ \mathrm{f}_{123} \quad , \quad \int dt\, G \ \rightarrow \ \mathrm{g}_{123} \quad , \tag{9.2.17}$$

where f_{123} is the purely spatial component of the Lorentz 3-form f_{123} and g_{123} is the purely spatial component of the Lorentz 3-form g_{123}.

The parity transformations and Hodge dual transformations carried out on the standard chiral supermultiplet can be extended to vector supermultiplets as well. This leads to the supermultiplets described by the four equations seen in (9.2.18) - (9.2.21) according to:

$Vector \ Supermultiplet: \ (A_\mu, \, \lambda_b, \, \mathrm{d})$

$$\begin{aligned} \mathrm{D}_a\, A_\mu \ &= \ (\gamma_\mu)_a{}^b\, \lambda_b \quad , \\ \mathrm{D}_a \lambda_b \ &= \ -\, i\, \tfrac{1}{4}([\,\gamma^\mu\, , \, \gamma^\nu\,])_{ab}\, (\, \partial_\mu\, A_\nu \ - \ \partial_\nu\, A_\mu\,) \ + \ (\gamma^5)_{a\, b}\, \mathrm{d} \quad , \\ \mathrm{D}_a\, \mathrm{d} \ &= \ i\, (\gamma^5 \gamma^\mu)_a{}^b\, \partial_\mu \lambda_b \quad , \end{aligned} \tag{9.2.18}$$

$Axial - Vector\ Supermultiplet:\ (U_\mu,\, \widetilde{\lambda}_b,\, \widetilde{\mathrm{d}})$

$$
\begin{aligned}
\mathrm{D}_a\, U_\mu \;&=\; i\,(\gamma^5\gamma_\mu)_a{}^b\,\widetilde{\lambda}_b \quad , \\
\mathrm{D}_a\widetilde{\lambda}_b \;&=\; \tfrac{1}{4}(\gamma^5[\,\gamma^\mu\,,\,\gamma^\nu\,])_{ab}\,(\,\partial_\mu\, U_\nu \;-\; \partial_\nu\, U_\mu\,) \;+\; i\, C_{a\, b}\,\widetilde{\mathrm{d}} \quad , \\
\mathrm{D}_a\,\widetilde{\mathrm{d}} \;&=\; -\,(\gamma^\mu)_a{}^b\;\partial_\mu\widetilde{\lambda}_b \quad ,
\end{aligned}
\tag{9.2.19}
$$

$Hodge - Dual\ Vector\ Supermultiplet:\ (A_\mu,\, \lambda_b,\, \mathrm{d}_{\mu\,\nu\,\rho})$

$$
\begin{aligned}
\mathrm{D}_a\, A_\mu \;&=\; (\gamma_\mu)_a{}^b\,\lambda_b \quad , \\
\mathrm{D}_a\lambda_b \;&=\; -\,i\,\tfrac{1}{4}([\,\gamma^\mu\,,\,\gamma^\nu\,])_{ab}\,(\,\partial_\mu\, A_\nu \;-\; \partial_\nu\, A_\mu\,) \;+\; \frac{1}{3!}\,(\gamma^5)_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \mathrm{d}_{\mu\,\nu\,\rho}) \quad , \\
\mathrm{D}_a\,\mathrm{d} \;&=\; i\,(\gamma^5\gamma^\mu)_a{}^b\;\partial_\mu\lambda_b \quad ,
\end{aligned}
\tag{9.2.20}
$$

$Hodge - Dual\ Axial - Vector\ Supermultiplet:\ (A_\mu,\, \widetilde{\lambda}_b,\, \widetilde{\mathrm{d}}_{\mu\,\nu\,\rho})$

$$
\begin{aligned}
\mathrm{D}_a\, U_\mu \;&=\; i\,(\gamma^5\gamma_\mu)_a{}^b\,\widetilde{\lambda}_b \quad , \\
\mathrm{D}_a\widetilde{\lambda}_b \;&=\; \tfrac{1}{4}(\gamma^5[\,\gamma^\mu\,,\,\gamma^\nu\,])_{ab}\,(\,\partial_\mu\, U_\nu \;-\; \partial_\nu\, U_\mu\,) \;+\; i\,\frac{1}{3!}\, C_{a\,b}\,(\epsilon^{\sigma\mu\nu\rho}\,\partial_\sigma \widetilde{\mathrm{d}}_{\mu\,\nu\,\rho}) \quad , \\
\mathrm{D}_a\,\widetilde{\mathrm{d}} \;&=\; -\,(\gamma^\mu)_a{}^b\,\partial_\mu\widetilde{\lambda}_b \quad ,
\end{aligned}
\tag{9.2.21}
$$

One can continue by applying Hodge transformations to the propagating physical bosons of the chiral supermultiplet. There are theories associated with performing the duality on either the scalar or pseudoscalar in the supermultiplet. In the former case, this leads to the tensor supermultiplet shown in (9.2.22), while in the latter case the duality transformation leads to the axial-tensor supermultiplet shown in (9.2.23).

$Tensor\ Supermultiplet:\ (\varphi,\, B_{\mu\,\nu},\, \chi_a)$

$$
\begin{aligned}
\mathrm{D}_a\varphi \;&=\; \chi_a \quad , \qquad \mathrm{D}_a B_{\mu\,\nu} \;=\; -\,\tfrac{1}{4}([\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\chi_b \quad , \\
\mathrm{D}_a\chi_b \;&=\; i\,(\gamma^\mu)_{a\,b}\,\partial_\mu\varphi \;-\; (\gamma^5\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho B_{\sigma\,\tau} \quad ,
\end{aligned}
\tag{9.2.22}
$$

$Axial - Tensor\ Supermultiplet:\ (\widetilde{\varphi},\, C_{\mu\,\nu},\, \widetilde{\chi}_a)$

$$\begin{aligned} \mathrm{D}_a \widetilde{\varphi} &= -i\,(\gamma^\mu)_a{}^b\,\widetilde{\chi}_a \quad , \qquad \mathrm{D}_a C_{\mu\,\nu} = i\,\tfrac{1}{4}(\gamma^5[\,\gamma_\mu\,,\,\gamma_\nu\,])_a{}^b\,\widetilde{\chi}_b \quad , \\ \mathrm{D}_a \widetilde{\chi}_b &= -\,(\gamma^5\gamma^\mu)_{a\,b}\,\partial_\mu \widetilde{\varphi} \;-\; i\,(\gamma^\mu)_{a\,b}\,\epsilon_\mu{}^{\rho\,\sigma\,\tau}\partial_\rho C_{\sigma\,\tau} \quad , \end{aligned} \tag{9.2.23}$$

From these results, it is seen there are no explicit auxiliary fields in either case. Implicitly, such fields are present in the gauge two-forms contained in each supermultiplet.

One might be tempted to continue the process of performing a Hodge duality on the remaining scalar or pseudoscalar in either of the two versions of a tensor supermultiplet above. This was carried out in a little-known work by Freedman many years ago. However, the field theory that results from this "dualization" is distinguished from all the other supermultiplets described in this section. Namely, one can use the transformation laws of the ten supermultiplets given in (9.2.1), (9.2.12), . . . , (9.2.23) to show the condition

$$\{\,\mathrm{D}_a\ ,\ \mathrm{D}_b\,\}\ =\ i\,2(\gamma^\mu)_{a\,b}\partial_\mu \tag{9.2.24}$$

is satisfied (up to gauge transformations) on every component field...except in the case where Hodge dualization is carried on scalar and pseudoscalar in the two tensor supermultiplets above.

So all toll, there are ten minimal *off-shell* 4D, $\mathcal{N} = 1$ supermultiplets.

9.2.4 HYMNs: Height Yielding Matrix Numbers

Valise adinkras have the advantage of being the simplest type of adinkra. However, we know that when attempting to use adinkra-based arguments to re-construct higher dimensional field theory representations, the simplicity of valise adinkras interacts with the requirement of the realization of Lorentz symmetry in the higher dimensions [107,108, 173]. There can arise obstructions as pointedly noted in [173, 175] where the conditions for selecting which adinkras can be used as a basis for the re-construction of 2D world sheet supersymmetric theories. These works demonstrate that the shape of a non-valise adinkra is an important attribute to knowing what higher dimensional physics relates to which adinkra systems.

In the works of [171, 172, 174] two apparently distinct approaches were presented as a way to characterize the shape of non-valise adinkras. In particular, the second of these approaches in [174] makes explicit the appearance of a particular matrix derived from the L-matrices and R-matrices. These derived matrices can be called the "left B-matrix" and the "right B-matrix." Due to their relations to the work in [37] this is a highly appropriate name. The eigenvalues of the B-matrices relate to the shape of all adinkras including therefore, non-valise ones. In the following discussion, we are going to define the B-matrices in a manner that is slightly different from the work [174].

In the work of [174] a proposition was made for when two adinkras are isomorphic. If two adinkras are isomorphic, their eigenvalues for $B_L = L_N R_{N-1} L_{N-2} \cdots$ and $B_R = R_N L_{N-1} R_{N-2} \cdots$ and chromocharacters are the same. For non-valise adinkras, the chromocharacter is defined as the trace of the B_L matrix with setting all HYMN's to 1.

We have recently developed a new algorithm to determine whether any two adinkras are isomorphic. In this section, we will briefly describe the isomorphism algorithm using the simplest non-trivial adinkras. We begin by revisiting the valise adinkra for the chiral multiplet, where we notice the mass dimensions of the fields are:

$$[\Phi_i] = 1 \quad , \quad [\Psi_{\hat{i}}] = 3/2 \quad . \tag{9.2.25}$$

For any valise adinkra, the bosons will all have the same mass dimension and the fermions will all have the same mass dimension that is one-half higher than the bosons. We can raise the nodes associated with the auxiliary bosons F and G via the raising operator M, defined below along with its inverse, the lowering operator M^{-1}

$$M = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \partial_0 & 0 \\ 0 & 0 & 0 & \partial_0 \end{pmatrix} \quad , \quad M^{-1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \int dt & 0 \\ 0 & 0 & 0 & \int dt \end{pmatrix} \quad . \tag{9.2.26}$$

The M operator raises nodes by performing the field redefinition:

$$\tilde{\Phi} = M\Phi = \begin{pmatrix} A \\ B \\ F \\ G \end{pmatrix} \quad . \tag{9.2.27}$$

This results in the following mass dimensions for the tilded fields $\tilde{\Phi}_i$:

$$[\tilde{\Phi}_1] = [\tilde{\Phi}_2] = 1 \quad , \quad [\tilde{\Phi}_3] = [\tilde{\Phi}_4] = 2 \quad . \tag{9.2.28}$$

Operating on Eq. (9.2.3a) from the left with M results in the following:

$$\begin{aligned} \mathrm{D}_I M\Phi &= iML_I\Psi \quad , \\ \mathrm{D}_I\tilde{\Phi} &= i\tilde{L}_I\tilde{\Psi} \quad , \end{aligned} \tag{9.2.29}$$

where we have defined $\tilde{\Psi} = \Psi$ and $\tilde{L}_I$ as follows

$$\tilde{L}_I = ML_I \quad . \tag{9.2.30}$$

Inserting $M^{-1}M = I$ into Eq. (9.2.3b) results in

$$\begin{aligned} \mathrm{D}_I\Psi &= R_I M^{-1}M\Phi \quad , \\ \mathrm{D}_I\tilde{\Psi} &= \tilde{R}_I\tilde{\Phi} \quad . \end{aligned} \tag{9.2.31}$$

We write the results in Eqs. (9.2.29) and (9.2.31) succinctly as the transformation laws for $\tilde{\Phi}$ and $\tilde{\Psi}$:

$$\mathrm{D}_I\tilde{\Phi} = i\tilde{L}_I\tilde{\Psi} \quad , \tag{9.2.32a}$$

$$\mathrm{D}_I\tilde{\Psi} = \tilde{R}_I\tilde{\Phi} \quad . \tag{9.2.32b}$$

It is straightforward to show that $\tilde{L}_I$ and $\tilde{R}_I$ satisfy the garden algebra, Eq. (9.2.5), written instead in terms of these tilded matrices. The adinkra matrices $\tilde{L}_I$ and $\tilde{R}_I$ now have height information, in the derivatives and integrals, and can be expressed as the three-level adinkra in Fig. 9.16

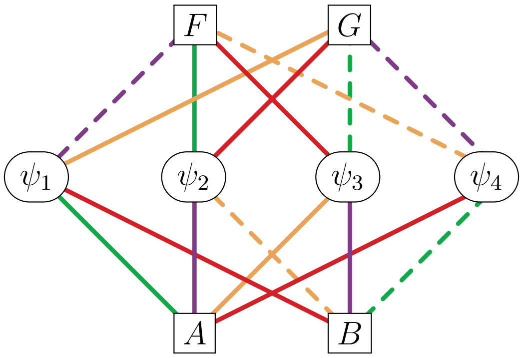


Figure 9.16: Adinkra for Chiral Supermultiplet without F & G Nodes Lowered

To investigate such node liftings of adinkras, it will be advantageous to forget the integral/derivative nature of lowering/raising nodes and instead simply use mass parameters to keep track of the lowered/raised nodes. We make the substitution

$$\partial_0 F \to mF \quad , \quad \partial_0 G \to mG \quad . \tag{9.2.33}$$

For instance, for the chiral multiplet, M and its inverse could have been defined instead as

$$M(m) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & m \end{pmatrix} \quad , \tag{9.2.34}$$

and Eqs. (9.2.29) through (9.2.32) would have followed the same as before. We refer to mass parameters such as m as HYMNs: height-yielding matrix numbers.

Next, we generalize HYMN's to arbitrarily sized $d \times d$ adinkras. The $\mathcal{GR}(d, N)$ SUSY transformation laws are

$$\mathrm{D}_I \Phi \ = iL_I \Psi \quad , \tag{9.2.35}$$

$$\mathrm{D}_I\Psi = R_I\dot{\Phi} \quad , \tag{9.2.36}$$

where a dot above a field indicates a time derivative: $\dot{\Phi} = d\Phi/dt$. The L_I and R_I matrices realize the closed $N = 4$ SUSY algebra for d bosons and d fermions as the $\mathcal{GR}(d, N)$ algebra

$$L_I R_J + L_J R_I = 2\delta_{IJ} I_{d\times d} \quad , \tag{9.2.37}$$

$$R_I L_J + R_J L_I = 2\delta_{IJ} I_{d\times d} \quad , \tag{9.2.38}$$

with $I_{d\times d}$ the $d \times d$ identity matrix.

We define the node raising operator $M(m, w)$ that acts on an arbitrary number of d bosons as

$$M(m, w) \equiv \begin{pmatrix} m^{p_1} & 0 & 0 & \dots & 0 \\ 0 & m^{p_1} & 0 & \dots & 0 \\ 0 & 0 & m^{p_3} & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & 0 \\ 0 & 0 & 0 & \dots & m^{p_d} \end{pmatrix} \quad , \tag{9.2.39}$$

$$w \equiv p_1 2^0 + p_2 2^1 + p_3 2^2 + \cdots + p_d 2^{d-1} \quad , \quad \text{with } p_i = 0, 1 \tag{9.2.40}$$

where the word parameter w is as in [157]. The node raising operator $M(m, w)$ has the following combination properties

$$M(1, w) = I_{d\times d} \quad , \tag{9.2.41}$$

$$M(m, w)M(\mu, w) = M(m\mu, w) \quad , \tag{9.2.42}$$

$$M(m, w_1)M(m, w_2) = M(m, w_1 + w_2) \quad , \tag{9.2.43}$$

$$[M(m, w)]^{-1} = M(m^{-1}, w) \quad , \tag{9.2.44}$$

$$M(m, w)[M(\mu, w)]^{-1} = M(m/\mu, w) \quad , \tag{9.2.45}$$

We denote a raised node boson collection as $\Phi(m,w)$

$$\Phi(m,w) \ = M(m,w)\Phi \quad . \tag{9.2.46}$$

Multipling Eq. (9.2.35) from the left by $M(m,w)$ and inserting $I_{d\times d} = M(\mu,w)[M(\mu,w)]^{-1}$ into the right hand side of Eq. (9.2.36) results in

$$\begin{aligned} \mathrm{D}_I M(m,w)\Phi \ &= iM(m,w)L_I\Psi \quad , \\ \mathrm{D}_I\Psi \ &= R_I[M(\mu,w)]^{-1}M(\mu,w)\dot{\Phi} = \ R_I M(\mu^{-1},w)M(\mu,w)\dot{\Phi} \quad , \end{aligned} \tag{9.2.47}$$

We now make the matrix redefinitions

$$L_I(m,w) = M(m,w)L_I \quad , \quad R_I(\mu,w) = R_I M(\mu^{-1},w) \tag{9.2.48}$$

which along with the field redefinitions in Eq. (9.2.46) reduces Eqs. (9.2.47) to

$$\mathrm{D}_I\Phi(m,w) = iL_I(m,w)\Psi \quad , \quad \mathrm{D}_I\Psi = R_I(\mu,w)\dot{\Phi}(\mu,w) \tag{9.2.49}$$

It is important to make clear the meaning of the matrices $L_I(m,w)$ and $R_I(\mu,w)$ as these are the "deformed" version of the L_I and R_I for adinkras where some number of bosonic and fermionic nodes have been lifted from the corresponding level in a valise adinkra within the context of BC_4 related adinkras. It will be necessary to modify our formalism in the future to handle the cases where nodes can be lifted to a height that is greater than one about that which the node appears in a corresponding valise adinkra.

The redefined matrices $L_I(m,w)$ and $R_I(\mu,w)$ satisfy the $GR(d,N)$ algebra in the $\mu \to m$ limit:

$$\begin{aligned} L_I(m,w)R_J(\mu,w) + L_J(m,w)R_I(\mu,w) &= M(m,w)(L_IR_J + L_JR_I)M(\mu^{-1},w) \\ &= 2\delta_{IJ}M(m,w)M(\mu^{-1},w) \\ &= 2\delta_{IJ}M(m/\mu,w) \\ &\to 2\delta_{IJ}I_{d\times d} \quad , \quad \text{for } \mu\to m \end{aligned} \tag{9.2.50}$$

where going from the second to third line we have made use of the property in Eq. (9.2.41) and in going from the third to last line have made use of the property in Eq. (9.2.45). The same results holds in the $\mu \to m$ limit for the $R_I L_J + R_J L_I$ algebra in Eq. (9.2.38), though the details are different:

$$
\begin{aligned}
&R_I(\mu, w)L_J(m, w) + R_J(\mu, w)L_I(m, w) \\
&= R_I M(\mu^{-1}, w)M(m, w)L_J + R_J M(\mu^{-1}, w)M(m, w)L_I \\
&= R_I M(m/\mu, w)L_J + R_J M(m/\mu, w)L_I \\
&\to R_I L_J + R_J L_I = 2\delta_{IJ} I_{d\times d} \quad , \text{for } \mu \to m \quad .
\end{aligned}
\tag{9.2.51}
$$

In order to describe adinkras uniquely, the color-dependent block matrix C_I associated with the I-th color is defined as

$$
C_I = \begin{pmatrix} 0 & L_I \\ R_I & 0 \end{pmatrix} \quad . \tag{9.2.52}
$$

Since every path can be covered by looking at the N distinct color paths from a boson or fermion, we can define the B matrix by multiplying all of the color matrices

$$
B = C_N C_{N-1} C_{N-2} \cdots C_1 \quad . \tag{9.2.53}
$$

If N is odd,

$$
B = \begin{pmatrix} 0 & L_N R_{N-1} \cdots L_1 \\ R_N L_{N-1} \cdots R_1 & 0 \end{pmatrix} \equiv \begin{pmatrix} 0 & B_L \\ B_R & 0 \end{pmatrix} \quad , \tag{9.2.54}
$$

and if N is even,

$$
B = \begin{pmatrix} L_N R_{N-1} \cdots R_1 & 0 \\ 0 & R_N L_{N-1} \cdots L_1 \end{pmatrix} \equiv \begin{pmatrix} B_L & 0 \\ 0 & B_R \end{pmatrix} \quad . \tag{9.2.55}
$$

Any reader familiar with our work in [174] will recognize these definitions from that work. We here make the additional identification that the non-vanishing upper left-hand entry in the last expression of (9.2.55) is what we mean by the left Banchoff matrix and accordingly the non-vanishing upper right-hand entry in the last expression of (9.2.55) is the right Banchoff matrix. It also is of note that in the case of odd values of N, the square of the matrix in (9.2.54) can be used to define the left Banchoff and right Banchoff matrices.

Permuting fermionic or bosonic nodes does not influence the eigenvalues of the B matrix. We prove this in the following. We can assign that if we relabel fermionic nodes, the permutation matrix is P_F; if we relabel bosonic nodes, the permutation matrix is P_B. Thus, after relabeling,

$$L_I \to P_B L_I P_F, \; R_I \to P_F^T R_I P_B^T \quad , \tag{9.2.56}$$

Consequently, when N is odd,

$$\begin{aligned} L_N R_{N-1} \cdots L_1 &\to P_B L_N P_F P_F^T R_{N-1} P_B^T ... P_B L_1 P_F = P_B (L_N R_{N-1} \cdots L_1) P_F \quad , \\ R_N L_{N-1} \cdots R_1 &\to P_F^T R_N P_B^T P_B L_{N-1} P_F \cdots P_F^T R_1 P_B^T = P_F^T (R_N L_{N-1} \cdots R_1) P_B^T \quad . \end{aligned} \tag{9.2.57}$$

When N is even,

$$\begin{aligned} L_N R_{N-1} \cdots R_1 &\to P_B L_N P_F P_F^T R_{N-1} P_B^T \cdots P_F^T R_1 P_B^T = P_B (L_N R_{N-1} \cdots R_1) P_B^T \quad , \\ R_N L_{N-1} \cdots L_1 &\to P_F^T R_N P_B^T P_B L_{N-1} P_F \cdots P_B L_1 P_F = P_F^T (R_N L_{N-1} \cdots L_1) P_F \quad . \end{aligned} \tag{9.2.58}$$

For all N, we can obtain

$$B \to \begin{pmatrix} P_B & 0 \\ 0 & P_F^T \end{pmatrix} B \begin{pmatrix} P_B^T & 0 \\ 0 & P_F \end{pmatrix} \quad . \tag{9.2.59}$$

Thus, eigenvalues of B will not be changed after relabeling nodes, although eigenvalues of $L_N ... L_1$ and $R_N ... R_1$ may be changed when N is odd.

We thus make the conjecture the eigenvalues of B matrix carry all information about shape isomorphism. Two adinkras are isomorphic if and only if eigenvalues of B_L and

B_R and chromocharacters are the same.

The parameter χ_o defines the two equivalence classes for $\mathcal{GR}(4,4)$ valise adinkras with respect to signed permutations of bosonic and fermionic nodes [144]. The general definition of χ_o for adinkras with an arbitrary number of colors N is [171, 172]

$$\chi_o \;=\; \frac{1}{d_{min}(N)}\frac{1}{N!}\epsilon^{I_1 I_2 \cdots I_N}\,Tr(L_{I_1}R_{I_2}\cdots L/R_{I_N}) \quad , \tag{9.2.60}$$

where the last matrix in the trace will be L_{I_N} if N is odd and R_{I_N} if N is even. The quantity $d_{min}(N)$ is a function of N first identified in the original works on Garden Algebras [75, 76]

$$d_{min}(N) \;=\; \begin{cases} 2^{\frac{N-1}{2}},\ N\equiv 1,7 & \mod (8) \\ 2^{\frac{N}{2}},\ N\equiv 2,4,6 & \mod (8) \\ 2^{\frac{N+1}{2}},\ N\equiv 3,5 & \mod (8) \\ 2^{\frac{N-2}{2}},\ N\equiv 8 & \mod (8) \end{cases} \quad . \tag{9.2.61}$$

So far, the effect of lifting nodes on adinkra isomorphism is unknown generally. In order to study this effect, we are developing a symbolic program described through the examples given in the following subsections.

9.2.5 HYMNs for $\mathcal{GR}(2,2)$

Figure 9.17 shows one of the simplest two-color adinkras. The adinkra matrices for this adinkra are the following where L_1-green and L_2-purple:

$$\begin{aligned} L_1 &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \ , \ R_1 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \quad , \\ L_2 &= \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} , \ R_2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \quad . \end{aligned} \tag{9.2.62}$$

In Fig. 9.17 for instance, we can raise the first bosonic node with $M(m,1)$, the second with $M(m,2)$ and both with $M(m,3) = M(m,1)M(m,2)$. These matrices and

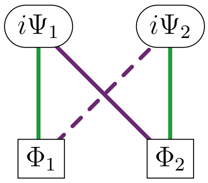


Figure 9.17: An example of a $\mathcal{GR}(2,2)$ valise adinkra.

the corresponding raised boson collections $\Phi(m,w)$ are

$$\Phi(m,1) = \begin{pmatrix} m\Phi_1 \\ \Phi_2 \end{pmatrix} \quad , \quad M(m,1) = \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \quad , \tag{9.2.63a}$$

$$\Phi(m,2) = \begin{pmatrix} \Phi_1 \\ m\Phi_2 \end{pmatrix} \quad , \quad M(m,2) = \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \quad , \tag{9.2.63b}$$

$$\Phi(m,3) = \begin{pmatrix} m\Phi_1 \\ m\Phi_2 \end{pmatrix} \quad , \quad M(m,3) = \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad . \tag{9.2.63c}$$

The resulting adinkras for each of these three cases are given in Fig. 9.18.

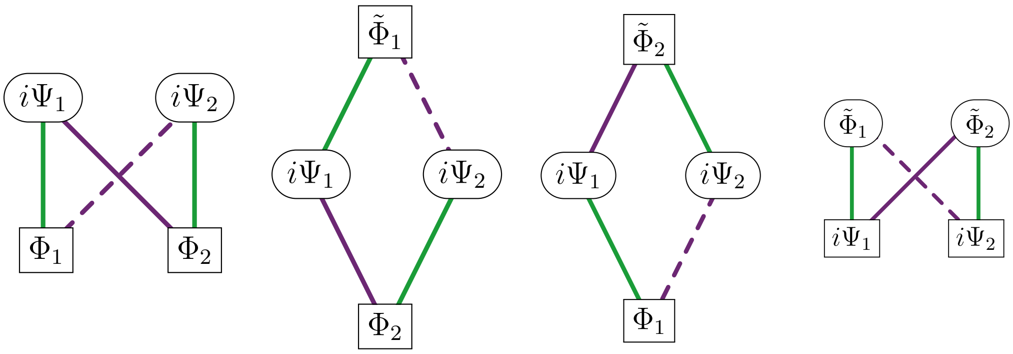


Figure 9.18: Valise and raised versions of the $\mathcal{GR}(2,2)$ adinkra of Fig. 9.17. The tilded bosons are identified as $\tilde{\Phi}_i = m\Phi_i$.

According to Eq. (9.2.48), the adinkra matrices associated with Fig. 9.18 and nodal redefinitions in Eq. (9.2.63a) are as follows. The adinkra matrices corresponding to the leftmost adinkra in Fig. 9.18 with boson one raised are

$$
\begin{aligned}
L_1(m,1) &= \begin{pmatrix} m & 0 \\ 0 & 1 \end{pmatrix} \quad , \; R_1(\mu,1) = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & 1 \end{pmatrix} \quad , \\
L_2(m,1) &= \begin{pmatrix} 0 & -m \\ 1 & 0 \end{pmatrix} , \; R_2(\mu,1) = \begin{pmatrix} 0 & 1 \\ -\mu^{-1} & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{9.2.64}
$$

The adinkra matrices corresponding to the middle adinkra in Fig. 9.18 with boson two raised are

$$
\begin{aligned}
L_1(m,2) &= \begin{pmatrix} 1 & 0 \\ 0 & m \end{pmatrix} \quad , \; R_1(\mu,2) = \begin{pmatrix} 1 & 0 \\ 0 & \mu^{-1} \end{pmatrix} \quad , \\
L_2(m,2) &= \begin{pmatrix} 0 & -1 \\ m & 0 \end{pmatrix} \; , \; R_2(\mu,2) = \begin{pmatrix} 0 & \mu^{-1} \\ -1 & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{9.2.65}
$$

The adinkra matrices corresponding to the rightmost adinkra in Fig. 9.18 with both bosons raised are

$$
\begin{aligned}
L_1(m,3) &= \begin{pmatrix} m & 0 \\ 0 & m \end{pmatrix} \quad , \; R_1(\mu,3) = \begin{pmatrix} \mu^{-1} & 0 \\ 0 & \mu^{-1} \end{pmatrix} \quad , \\
L_2(m,3) &= \begin{pmatrix} 0 & -m \\ m & 0 \end{pmatrix} \quad , \; R_2(\mu,3) = \begin{pmatrix} 0 & \mu^{-1} \\ -\mu^{-1} & 0 \end{pmatrix} \quad .
\end{aligned}
\tag{9.2.66}
$$

The two color C_I matrix and B matrix given by Eqs. (9.2.52) and (9.2.89) are

$$
C_I = \begin{pmatrix} 0 & L_I \\ R_I & 0 \end{pmatrix} \quad , \quad B = C_2 C_1 = \begin{pmatrix} L_2 R_1 & 0 \\ 0 & R_2 L_1 \end{pmatrix} \quad . \tag{9.2.67}
$$

The eigenvalues of B are invariant with respect to fermionic and bosonic nodal transformations as described previously for the arbitrary N color case. With respect to color transformations, the eigenvalues of B are invariant with respect to even numbers of flips and flops, but odd numbers of flips and flops negate all eigenvalues. This is because of the $\mathcal{GR}(2,2)$ algebra relation $L_2 R_1 = -L_1 R_2$. The consequences of raising nodes are unknown as to the eigenvalues, so we next turn to the tabulation of all $\mathcal{GR}(2,2)$ adinkras, valise and raised.

In total, there are 16 distinct $\mathcal{GR}(2,2)$ valise adinkras as shown in Fig. 9.19. These are tabulated in terms of sign flips H_1^a and permuations S_2^μ where $a = 1,2$ and $\mu = 1,2$

$$H_1^a = \{(), (\bar{1})\} \quad , \quad S_2^\mu = \{(), (12)\} \quad . \tag{9.2.68}$$

Analogous to the BC_4 notation of [144] that was summarized in Sec˙ 9.2.2, all 16 adinkras in Fig. 9.19 are succinctly labeled as

$$L_I^{\pm a\mu b} = \pm\, H_1^a S_2^\mu L_I^{(0)} H_1^b \quad , \quad R_I^{\pm a\mu b} = (L_I^{\pm a\mu b})^T \quad , \tag{9.2.69}$$

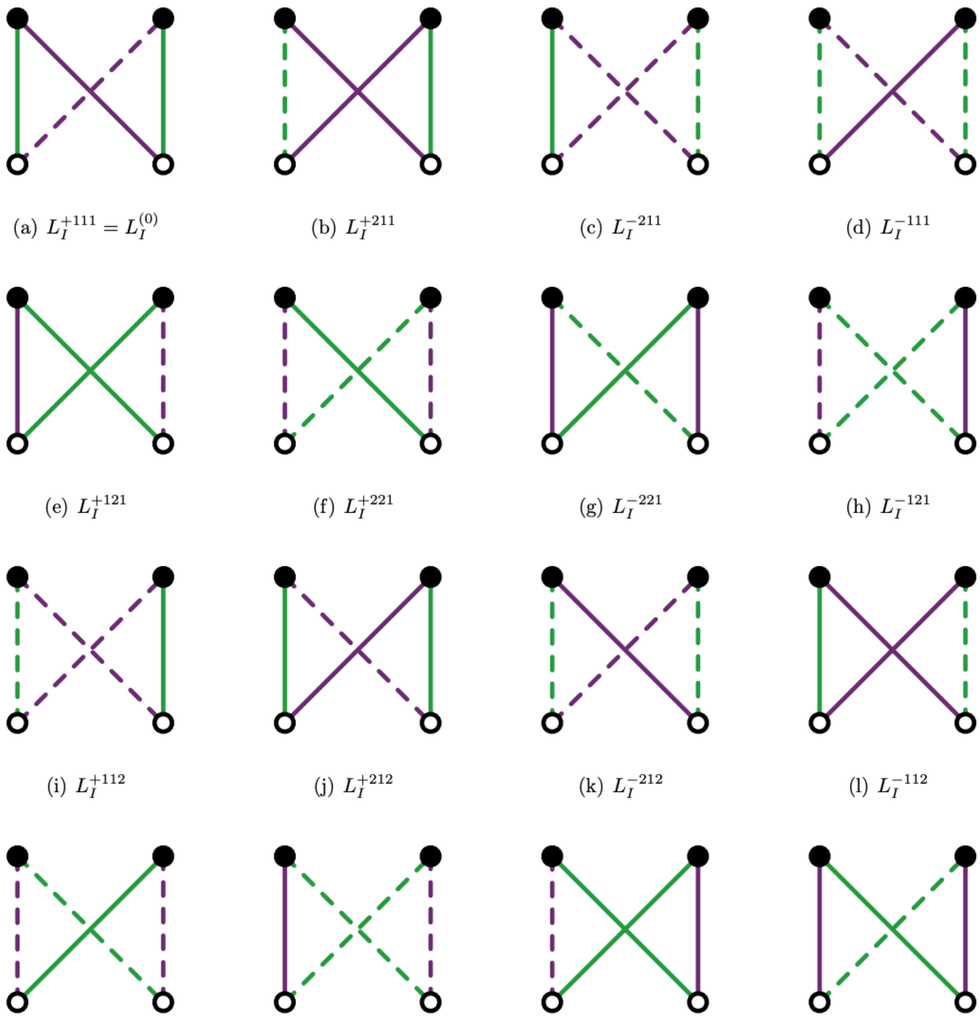
where $L_I^{(0)}$ corresponds to the adinkra in Fig. 9.17 with adinkra matrices as in Eq. (9.2.62) and the T superscript means transpose. This is analogoous to the BC_4 notation of [144] that was summarized in Sec˙ 9.2.2.

Similar to the BC_4 case, there are isometries of $\mathcal{GR}(2,2)$ adinkras that reduce the total number of distinct adinkras from the 64 total possible BC_2 boson $\times BC_2$ fermion transformations to the 16 distinct adinkras in Fig. 9.19. These isometries are:

$$L_I = \mathcal{X} L_I \mathcal{Y} \tag{9.2.70a}$$

$$(\mathcal{X}, \mathcal{Y}) \in \begin{cases} ((), ()) \\ ((\bar{1})(12), (\bar{1})(12)) \\ ((\bar{2})(12), (\bar{2})(12)) \\ (\overline{12}, (\overline{12})) \end{cases} \tag{9.2.70b}$$

These isometries are the reason that Eq. (9.2.69) can be written as it is, in terms of only flips on boson one, flops of bosons one and two, flips of fermion one, and an overall factor of plus or minus one. Altogether, these are the 16 possibilities that take a given starting $\mathcal{GR}(2,2)$ valise adinkra to all other possible $\mathcal{GR}(2,2)$ valise adinkras. There is nothing special about our choice for $L_I^{(0)} = L_I^{+111}$, it was chosen at random.



(a) $L_I^{+111} = L_I^{(0)}$ (b) L_I^{+211} (c) L_I^{-211} (d) L_I^{-111}

(e) L_I^{+121} (f) L_I^{+221} (g) L_I^{-221} (h) L_I^{-121}

(i) L_I^{+112} (j) L_I^{+212} (k) L_I^{-212} (l) L_I^{-112}

(m) L_I^{+122} (n) L_I^{+222} (o) L_I^{-222} (p) L_I^{-122}

Figure 9.19: All 16 $\mathcal{GR}(2,2)$ valise adinkras. The white nodes are bosons, the black fermions, in numerical left to right order as in Fig. 9.17.

Next, we construct the 64 possible raisings of each of the 16 $\mathcal{GR}(2,2)$ adinkras as in Eq. (9.2.48):

$$L_I^{\pm a\mu b}(m,w) = M(m,w)L_I^{\pm a\mu b} \quad , \quad R_I^{\pm a\mu b}(\mu,w) = R_I^{\pm a\mu b}M(\mu^{-1},w) \quad . \tag{9.2.71}$$

As in Eq. (9.2.67), we construct $B^{\pm a\mu b(i)}$

$$B^{\pm a\mu b}(m,\mu,w) = \begin{pmatrix} L_2^{\pm a\mu b}(m,w)R_1^{\pm a\mu b}(\mu,w) & 0 \\ 0 & R_2^{\pm a\mu b}(\mu,w)L_1^{\pm a\mu b}(m,w) \end{pmatrix}$$
$$\text{no } a,\mu,b \text{ sum.} \tag{9.2.72}$$

This inspires us to calculate the eigenvalues of the following six matrices (no a,μ,b sum):

$$L_2^{\pm a\mu b}(m,w)R_1^{\pm a\mu b}(\mu,w) = -L_1^{\pm a\mu b}(m,w)R_2^{\pm a\mu b}(\mu,w) \quad , \tag{9.2.73}$$
$$M(m/\mu,w) = L_1^{\pm a\mu b}(m,w)R_1^{\pm a\mu b}(\mu,w) = L_2^{\pm a\mu b}(m,w)R_2^{\pm a\mu b}(\mu,w) \quad , \tag{9.2.74}$$
$$R_2^{\pm a\mu b}(\mu,w)L_1^{\pm a\mu b}(m,w) \quad , \quad R_1^{\pm a\mu b}(\mu,w)L_2^{\pm a\mu b}(m,w) \quad , \tag{9.2.75}$$
$$R_1^{\pm a\mu b}(\mu,w)L_1^{\pm a\mu b}(m,w) \quad , \quad R_2^{\pm a\mu b}(\mu,w)L_2^{\pm a\mu b}(m,w) \quad . \tag{9.2.76}$$

The calculation of these eigenvalues amounts to solving the following characteristic polynomials (no a,μ,b sum):

$$0 = \det\left(L_2^{\pm a\mu b}(m,w)R_1^{\pm a\mu b}(\mu,w) \pm \lambda I_{2\times 2}\right) \quad , \tag{9.2.77}$$
$$0 = \det\left(M(m/\mu,w) - \lambda I_{2\times 2}\right) \quad , \tag{9.2.78}$$
$$0 = \det\left(R_2^{\pm a\mu b}(\mu,w)L_1^{\pm a\mu b}(m,w) - \lambda I_{2\times 2}\right) \quad , \tag{9.2.79}$$
$$0 = \det\left(R_1^{\pm a\mu b}(\mu,w)L_2^{\pm a\mu b}(m,w) - \lambda I_{2\times 2}\right) \quad , \tag{9.2.80}$$
$$0 = \det\left(R_1^{\pm a\mu b}(\mu,w)L_1^{\pm a\mu b}(m,w) - \lambda I_{2\times 2}\right) \quad , \tag{9.2.81}$$
$$0 = \det\left(R_2^{\pm a\mu b}(\mu,w)L_2^{\pm a\mu b}(m,w) - \lambda I_{2\times 2}\right) \quad . \tag{9.2.82}$$

Owing to Eq. (9.2.73), the plus sign in Eq. (9.2.77) corresponds to the eigenvalues of $L_1^{\pm a\mu b}(m,w)R_2^{\pm a\mu b}(\mu,w)$.

9.2.6 HYMNS for $\mathcal{GR}(4,4)$

Consider lifting one, two, three, and four bosonic nodes in a four-color valise adinkra with four bosons and four fermions. From Sec. 9.2.2, we have known there are 36,864 $\mathcal{GR}(4,4)$ valise adinkras in total. Similar to the analysis in $\mathcal{GR}(2,2)$ case, we define node raising operators:

$$M(m,1) = \begin{pmatrix} m & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; M(\mu^{-1},1) = \begin{pmatrix} \mu_1^{-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{9.2.83}$$

$$M(m,2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & m & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; M(\mu^{-1},2) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & \mu_2^{-1} & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{9.2.84}$$

$$M(m,4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & m & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \; M(\mu^{-1},4) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & \mu^{-1} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} , \tag{9.2.85}$$

$$M(m,8) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & m \end{pmatrix} , \; M(\mu^{-1},8) = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & \mu^{-1} \end{pmatrix} , \tag{9.2.86}$$

For example, if we lift the i-th bosonic node, then the corresponding L_I and R_I for the lifted adinkra are

$$L_I(m,w) = M(m,w)L_I \ , \ R_I(\mu,w) = R_I M(\mu^{-1},w) \quad , \tag{9.2.87}$$

where $w = 2^{i-1}$. Lifting more than one bosonic nodes can be described in the similar way. For example, the $L_I$and R_I matrices after lifting the $i-th$ and $j-th$ bosonic nodes are:

$$\begin{aligned} L_I(m,w) &= M(m,2^{i-1})M(m,2^{j-1})L_I \quad , \\ R_I(\mu,w) &= R_I M(\mu^{-1},2^{i-1})M(\mu^{-1},2^{j-1}) \ . \end{aligned} \tag{9.2.88}$$

where node raising operators commute to each other, which means that lifting the $i-th$ node first then the $j-th$ node describes the same thing as lifting the $j-th$ node first then the $i-th$ node. In order to study isomorphic properties of non-valise adinkras, we have to study the B matrix

$$\begin{aligned} B &= \begin{pmatrix} 0 & L_4 \\ R_4 & 0 \end{pmatrix} \begin{pmatrix} 0 & L_3 \\ R_3 & 0 \end{pmatrix} \begin{pmatrix} 0 & L_2 \\ R_2 & 0 \end{pmatrix} \begin{pmatrix} 0 & L_1 \\ R_1 & 0 \end{pmatrix} \quad , \\ &= \begin{pmatrix} L_4R_3L_2R_1 & 0 \\ 0 & R_4L_3R_2L_1 \end{pmatrix} \quad . \end{aligned} \tag{9.2.89}$$

Next, we construct the 589,824 possible raisings of each of the 36,864 $\mathcal{GR}(4,4)$ valise adinkras. Then we have

$$B_L(m,\mu,w) = L_4(m,w)R_3(\mu,w)L_2(m,w)R_1(\mu,w) \quad , \tag{9.2.90}$$

$$B_R(m,\mu,w) = R_4(\mu,w)L_3(m,w)R_2(\mu,w)L_1(m,w) \quad , \tag{9.2.91}$$

where the word parameter w can be $0,1,2,3,\ldots 15$. A summary using a *Mathematica* notebook can be found at the HEPTHools Data Repository on GitHub.

Let us define the HYMN ratio parameter ρ to connect m and μ,

$$\rho \equiv \frac{m}{\mu} \quad . \tag{9.2.92}$$

We shall find that the eigenvalues of the matrices described previously will depend only on this ratio parameter.

9.2.7 Eigenvalues for $\mathcal{GR}(2,2)$

As shown by Eq. (9.2.71), there are 64 possible $\mathcal{GR}(2,2)$ valise and raised adinkras. The eigenvalues of $L_1(m,w)R_1(\mu,w)$, $L_2(m,w)R_1(\mu,w)$, $R_1(\mu,w)L_1(m,w)$, $R_1(\mu,w)L_2(m,w)$, $R_2(\mu,w)L_1(m,w)$, and

$R_2(\mu,w)L_2(m,w)$ define equivalence classes of $\mathcal{GR}(2,2)$ adinkras. We find

- all sixteen valise $\mathcal{GR}(2,2)$ adinkras give the same eigenvalues:
  $L_1(m,w)R_1(\mu,w) : \{1\ ,\ 1\}$; $L_2(m,w)R_1(\mu,w) : \{i\ ,\ -i\}$;
  $R_1(\mu,w)L_1(m,w) : \{1\ ,\ 1\}$;
  $R_1(\mu,w)L_2(m,w) : \{i\ ,\ -i\}$; $R_2(\mu,w)L_1(m,w) : \{i\ ,\ -i\}$;
  $R_2(\mu,w)L_2(m,w) : \{1\ ,\ 1\}$
- all $\mathcal{GR}(2,2)$ adinkras with one boson at the top give the same eigenvalues:
  $L_1(m,w)R_1(\mu,w) : \{1\ ,\ \rho\}$; $L_2(m,w)R_1(\mu,w) : \{i\sqrt{\rho}\ ,\ -i\sqrt{\rho}\}$;
  $R_1(\mu,w)L_1(m,w) : \{1\ ,\ \rho\}$;
  $R_1(\mu,w)L_2(m,w) : \{i\sqrt{\rho}\ ,\ -i\sqrt{\rho}\}$; $R_2(\mu,w)L_1(m,w) : \{i\sqrt{\rho}\ ,\ -i\sqrt{\rho}\}$;
  $R_2(\mu,w)L_2(m,w) : \{1\ ,\ \rho\}$
- all $\mathcal{GR}(2,2)$ adinkras with two bosons at the top give the same eigenvalues:
  $L_1(m,w)R_1(\mu,w) : \{\rho\ ,\ \rho\}$; $L_2(m,w)R_1(\mu,w) : \{i\rho\ ,\ -i\rho\}$;
  $R_1(\mu,w)L_1(m,w) : \{\rho\ ,\ \rho\}$;
  $R_1(\mu,w)L_2(m,w) : \{i\rho\ ,\ -i\rho\}$; $R_2(\mu,w)L_1(m,w) : \{i\rho\ ,\ -i\rho\}$;
  $R_2(\mu,w)L_2(m,w) : \{\rho\ ,\ \rho\}$

These eigenvalue equivalence classes are summarized in Table 9.9

Table 9.9: Eigenvalue equivalence classes of $\mathcal{GR}(2,2)$

Type	w	Total Number	Eigenvalue equivalence class	# of adinkras in each class
Valise Adinkras	0	16	all	16
Adinkras with one boson at the top	1,2	32	all	32
Adinkras with two bosons at the top	3	16	all	16

9.2.8 Eigenvalues for $\mathcal{GR}(4,4)$

As $B_L(m,\mu,w)$ and $B_R(m,\mu,w)$ are found to depend only on ρ, we refer to these here as $B_L(\rho,w)$ and $B_R(\rho,w)$. We have calculated the eigenvalues of $B_L(\rho,w)$ and $B_R(\rho,w)$ for all 589,824 possible valise and raised adinkras. These eigenvalue results are as follows:

[1] 36,864 $\mathcal{GR}(4,4)$ valise adinkras split into two classes: $\chi_0 = \pm 1$, in each class adinkras give the same eigenvalues
$B_L(\rho,w):\ \chi_0\,\{1,1,1,1\}$; $\quad B_R(\rho,w):\ \chi_0\,\{-1,-1,-1,-1\}$

[2] 147,456 $\mathcal{GR}(4,4)$ adinkras with one boson raised split into two classes: $\chi_0 = \pm 1$, in each class adinkras give the same eigenvalues
$B_L(\rho,w):\ \chi_0\,\{1,1,\rho,\rho\}$; $\quad B_R(\rho,w):\ \chi_0\,\{-1,-1,-\rho,-\rho\}$

[3] 227,184 $\mathcal{GR}(4,4)$ adinkras with two bosons raised split into six eigenvalue equivalence classes $EB_1^{(\chi_0)}$, $EB_2^{(\chi_0)}$, and $EB_3^{(\chi_0)}$. In each class, the matrices $B_L(\rho,w)$ and $B_R(\rho,w)$ have the same eigenvalues
$EB_1^{(\chi_0)}$: $B_L(\rho,w):\chi_0\,\{\rho,\rho,\rho,\rho\}$; $B_R(\rho,w):\ \chi_0\,\{-1,-1,-\rho^2,-\rho^2\}$
$EB_2^{(\chi_0)}$: $B_L(\rho,w):\ \chi_0\,\{1,1,\rho^2,\rho^2\}$; $B_R(\rho,w):\ \chi_0\,\{-\rho,-\rho,-\rho,-\rho\}$
$EB_3^{(\chi_0)}$: $B_L(\rho,w):\ \chi_0\,\{\rho,\rho,\rho,\rho\}$; $B_R(\rho,w):\ \chi_0\,\{-\rho,-\rho,-\rho,-\rho\}$

[4] 147,456 $\mathcal{GR}(4,4)$ adinkras with three bosons raised split into two classes: $\chi_0 = \pm 1$, in each class adinkras give the same eigenvalues
$B_L(\rho,w):\ \chi_0\,\{\rho,\rho,\rho^2,\rho^2\}$; $B_R(\rho,w):\ \chi_0\,\{-\rho,-\rho,-\rho^2,-\rho^2\}$

[5] 36,864 $\mathcal{GR}(4,4)$ adinkras with all four bosons raised split into two classes: $\chi_0 = \pm 1$, in each class adinkras give the same eigenvalues
$B_L(\rho,w):\ \chi_0\,\{\rho^2,\rho^2,\rho^2,\rho^2\}$; $B_R(\rho,w):\ \chi_0\,\{-\rho^2,-\rho^2,-\rho^2,-\rho^2\}$

These eigenvalue equivalence classes are summarized in Table 9.10.

Table 9.10: Number of Adinkras in each Eigenvalue equivalence classes of $\mathcal{GR}(4,4)$

Type	w	Total Number	Eigenvalue Equivalence Classes	# of adinkras in each class
Valise Adinkras	0	36,864	$\chi_0 = \pm 1$	18,432
One Boson Raised Adinkras	1,2,4,8	147,456	$\chi_0 = \pm 1$	73,728
Two Bosons Raised Adinkras	3,5,6,9,10,12	221,184	$EB_1^{(\pm)}$, $EB_2^{(\pm)}$, $EB_3^{(\pm)}$	36,864
Three Bosons Raised Adinkras	7,11,13,14	147,456	$\chi_0 = \pm 1$	73,728
Four Bosons Raised Adinkras	15	36,864	$\chi_0 = \pm 1$	18,432

There is an interesting relationship between adinkras within the $EB_i^{(\pm)}$ equivalence classes for two raised bosons. For a given valise adinkra, raising bosons one and two are in the same equivalence class as raising bosons three and four. The same holds for raising one and three or two and four as well as for one and four or two and three. In terms of words, for a given valise adinkra raising two bosons with either the word code w or $15 - w$ will lead to the same equivalence class. Specifically, these word pairs are $(3, 12)$, $(5, 10)$ and $(6, 9)$. It is important to note that this does not mean that all adinkras raised with word $w = 3$ have the same B_L and B_R eigenvalues as all adinkras raised with word $w = 12$, but merely that given a particular valise adinkra, raising with word $w = 3$ will lead to the same B_L and B_R eigenvalues as instead raising the same valise adinkra with word $w = 12$. Possibly, some adinkras raised with the word $w = 3$ may be in the same equivalence class as other adinkras raised with the word $w = 5$ for instance, it depends on how the valise adinkras are related by signed bosons and fermion permutations.

In order to relate this discussion back to the supermultiplets discussed in section 9.2.3, let us make a series of observations about the structure of the superfields that describe them

When *no* node lowering is applied to the 4D, $\mathcal{N} = 1$ tensor supermultiplet or Hodge Dual # 3 chiral supermultiplet, under the action of projection to 1D, $N = 4$ supermultiplets, these are within the class of adinkras with $w = 0$. When *no* node lowering is applied to the 4D, $\mathcal{N} = 1$ vector supermultiplet, the Hodge Dual # 1, or Hodge Dual # 2 chiral 4D, $\mathcal{N} = 1$ chiral supermultiplets, under the action of projection to 1D, N

= 4 supermultiplets, these are within the class of adinkras with w = 1,2,4, or 8. When *no* node lowering is applied to the 4D, $\mathcal{N}$ = 1 chiral supermultiplet, under the action of projection to a 1D, N = 4 supermultiplet, it is within the class of adinkras with w = 3, 5, 6, 9, 10, and 12. When *no* node lowering is applied to the 4D, $\mathcal{N}$ = 1 field strengths of the tensor supermultiplet, Hodge Dual # 1, or Hodge Dual # 2 chiral supermultiplet, under the action of projection to 1D, N = 4 supermultiplets, these are within the class of adinkras with w = 7, 11, 13, 14. The spinor field strengths for all of these supermultiplets correspond to w = 15.

9.2.9 Conclusion

From Figure 9.18, it was clear starting from the reference valise adinkra a number of other adinkras with lifted nodes that can be constructed from it occur as follows. There was one adinkra with no nodes lifted, two adinkras with one node lifted, and finally, one adinkra with two nodes lifted. Looking vertically in the column that labels "Total Number" in Table 9.9, we see the ratios of 1:2:1, i. e. the binomial coefficient for two choose an integer. In turn, this implies that the actions of lifting nodes versus using flips and flops to generate all sixty-four adinkras commute one with the other.

Looking vertically in the column that labels "Total Number" in Table 9.10, we see the ratios 1:4:6:4:1, i. e. the binomial coefficient for four choose an integer. As in the two-color case, this also implies that the actions of lifting nodes versus using flips and flops to generate all adinkras commute one with the other.

This analysis also provides a simple way to see the number[6] of adinkras associated with BC_4 and with all possibility of raised nodes is equal to 4! × 36,864 = 884,736.

The observations in this work reveal that the HYMN's, i. e. the eigenvalues of the Banchoff matrices for the adinkras investigated, can be used to cleanly partition these sets of adinkras into distinct classes. Apparently, the actions of the flipping and flopping operation preserve these classes. This offers an explicit route whereby considering instead of individual adinkras as the basis for higher-dimensional supermultiplets, it is the emergent class structure of adinkras that provides such a basis.

[6] This number counts both $\chi_o = \pm$ sectors in the work of [24].

Adinkras are interesting and useful tools to study supersymmetry for their abilities to encode information about higher dimensional supersymmetry. In this paper, we applied the HYMN's to study non-valise adinkras. The B matrix is defined by HYMN's and its eigenvalues carry all information about isomorphisms in shape. A program to calculate eigenvalues of B-matrices has been developed in the Python language.

The introduction of the matrix $M(m, w)$ as the nodal raising operator has implications for future directions of research. Previous mathematical "devices" introduced for the analysis of adinkras include holoraumy [181] and the Gadget [182]. But all such previous discussions have been restricted to valise adinkras. Clearly, these can now be modified by appropriate introductions of $M(m, w)$ into their definitions. As well, future study of the holoraumy, Gadget, HYMN's, and $\chi_{\rm o}$ all seems indicated to ascertain their dependence on adinkra dashing as well as the impacts of $M(m, w)$. Finally with the introduction of $M(m, w)$ a re-examination of the work in [107] and [108] is also possible to be extended to the entire 36,864 BC_4 related adinkras.

Appendix A

Superspace Conventions

A.1 Notation in 4D, $\mathcal{N} = 1$ Superspace

A.1.1 Two-component Notation

In this appendix, we review the two-component notation in 4D, $\mathcal{N} = 1$ superspace. First, the algebra that super-covariant derivatives satisfy is

$$\{\mathrm{D}_\alpha, \mathrm{D}_\beta\} = \{\overline{\mathrm{D}}_{\dot\alpha}, \overline{\mathrm{D}}_{\dot\beta}\} = 0 \tag{A.1.1}$$

$$\{\mathrm{D}_\alpha, \overline{\mathrm{D}}_{\dot\beta}\} = i\partial_{\alpha\dot\beta} \tag{A.1.2}$$

where $\alpha, \ \dot\alpha = 1, 2$.

$$\mathrm{D}^2 = \tfrac{1}{2}\mathrm{D}^\alpha \, \mathrm{D}_\alpha \tag{A.1.3}$$

$$\overline{\mathrm{D}}^2 = \tfrac{1}{2}\overline{\mathrm{D}}^{\dot\alpha} \, \overline{\mathrm{D}}_{\dot\alpha} \tag{A.1.4}$$

$$\Box = \tfrac{1}{2}\partial^{\alpha\dot\alpha} \, \partial_{\alpha\dot\alpha} \tag{A.1.5}$$

$$\int d^2\theta d^2\bar\theta = \mathrm{D}^2 \, \overline{\mathrm{D}}^2 \, |_{\theta\to 0, \bar\theta\to 0} \tag{A.1.6}$$

$$(\mathrm{D}_\alpha)^* = -\overline{\mathrm{D}}_{\dot\alpha} \tag{A.1.7}$$

$$(\mathrm{D}^{\alpha})^* \ = \overline{\mathrm{D}}^{\dot{\alpha}} \tag{A.1.8}$$

$$\to (\mathrm{D}^2)^* \ = \overline{\mathrm{D}}^2 \tag{A.1.9}$$

where $*$ means hermitian conjugation.

Useful identities:

$$\mathrm{D}_{\alpha}\,\mathrm{D}^2 \ = \ \overline{\mathrm{D}}_{\dot{\alpha}}\,\overline{\mathrm{D}}^2 \ = \ 0 \tag{A.1.10}$$

$$\mathrm{D}^{\alpha}\,\overline{\mathrm{D}}^2 \ = \ \overline{\mathrm{D}}^2\,\mathrm{D}^{\alpha} \ + \ i\partial^{\alpha\dot{\alpha}}\,\overline{\mathrm{D}}_{\dot{\alpha}} \tag{A.1.11}$$

$$\overline{\mathrm{D}}^{\dot{\alpha}}\,\mathrm{D}^2 \ = \ \mathrm{D}^2\,\overline{\mathrm{D}}^{\dot{\alpha}} \ + \ i\partial^{\alpha\dot{\alpha}}\,\mathrm{D}_{\alpha} \tag{A.1.12}$$

$$\mathrm{D}_{\alpha}\,\mathrm{D}_{\beta} \ = \ -C_{\alpha\beta}\,\mathrm{D}^2 \tag{A.1.13}$$

$$\mathrm{D}^{\gamma}\,\overline{\mathrm{D}}^2\,\mathrm{D}_{\gamma} \ = \ \overline{\mathrm{D}}^{\dot{\gamma}}\,\mathrm{D}^2\,\overline{\mathrm{D}}_{\dot{\gamma}} \ = \ 2\,\mathrm{D}^2\,\overline{\mathrm{D}}^2 \ + \ i\partial^{\alpha\dot{\alpha}}\,\mathrm{D}_{\alpha}\,\overline{\mathrm{D}}_{\dot{\alpha}} \tag{A.1.14}$$

$$\overline{\mathrm{D}}^2\,\mathrm{D}^2 \ = \ \mathrm{D}^2\,\overline{\mathrm{D}}^2 \ + \ \Box \ + \ i\partial^{\alpha\dot{\alpha}}\,\mathrm{D}_{\alpha}\,\overline{\mathrm{D}}_{\dot{\alpha}} \tag{A.1.15}$$

$$\mathrm{D}^2\,\overline{\mathrm{D}}^2 \ = \ \overline{\mathrm{D}}^2\,\mathrm{D}^2 \ + \ \Box \ + \ i\partial^{\alpha\dot{\alpha}}\,\overline{\mathrm{D}}_{\dot{\alpha}}\,\mathrm{D}_{\alpha} \tag{A.1.16}$$

We can relate a vector label $\underset{\sim}{a}$ in a Cartesian coordinate basis to the $\alpha\dot{\alpha}$ basis by a set of Clebsch-Gordan coefficients, the Pauli matrices:

$$\text{For fields}: V^{\alpha\dot{\alpha}} \ = \frac{1}{\sqrt{2}}\,(\sigma_{\underset{\sim}{b}})^{\alpha\dot{\alpha}}\,V^{\underset{\sim}{b}} \ \ , \ \ V^{\underset{\sim}{b}} \ = \ \frac{1}{\sqrt{2}}\,(\sigma^{\underset{\sim}{b}})_{\alpha\dot{\alpha}}\,V^{\alpha\dot{\alpha}} \tag{A.1.17}$$

$$\text{For derivatives}: \partial_{\alpha\dot{\alpha}} \ = (\sigma^{\underset{\sim}{b}})_{\alpha\dot{\alpha}}\,\partial_{\underset{\sim}{b}} \ \ , \ \ \partial_{\underset{\sim}{b}} \ = \ \frac{1}{2}\,(\sigma_{\underset{\sim}{b}})^{\alpha\dot{\alpha}}\,\partial_{\alpha\dot{\alpha}} \tag{A.1.18}$$

$$\text{For coordinates}: x^{\alpha\dot{\alpha}} \ = \frac{1}{2}\,(\sigma_{\underset{\sim}{b}})^{\alpha\dot{\alpha}}\,x^{\underset{\sim}{b}} \ \ , \ \ x^{\underset{\sim}{b}} \ = \ (\sigma^{\underset{\sim}{b}})_{\alpha\dot{\alpha}}\,x^{\alpha\dot{\alpha}} \tag{A.1.19}$$

Pauli matrices satisfy:

$$(\sigma_{\underset{\sim}{a}})^{\alpha\dot{\alpha}}\,(\sigma^{\underset{\sim}{b}})_{\alpha\dot{\alpha}} \ = 2\,\delta_{\underset{\sim}{a}}{}^{\underset{\sim}{b}} \tag{A.1.20}$$

$$(\sigma^{\underset{\sim}{b}})^{\alpha\dot{\alpha}}\,(\sigma_{\underset{\sim}{b}})_{\beta\dot{\beta}} \ = 2\,\delta_{\alpha}{}^{\beta}\,\delta_{\dot{\alpha}}{}^{\dot{\beta}} \tag{A.1.21}$$

A.1.2 Four-component Notation

Using the Kronecker product, we can create the covariant gamma matrices using the Pauli matrices:

$$[\gamma^0]_a{}^b = i(\sigma^3 \otimes \sigma^2) \qquad [\gamma^1]_a{}^b = (\mathbb{I}_2 \otimes \sigma^1) \tag{A.1.22}$$

$$[\gamma^2]_a{}^b = (\sigma^2 \otimes \sigma^2) \qquad [\gamma^3]_a{}^b = (\mathbb{I}_2 \otimes \sigma^3) \tag{A.1.23}$$

These can all be seen to be real. When talking about the entry in the a^{th} row and b^{th} column of the μ^{th} gamma matrix, the symbol would be $(\gamma^\mu)_a{}^b$.

The metric:

$$\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.1.24}$$

Then, through calculating, we can see that

$$\gamma^0 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{bmatrix} \qquad \gamma^1 = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$$
$$\gamma^2 = \begin{bmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ -1 & 0 & 0 & 0 \end{bmatrix} \qquad \gamma^3 = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{bmatrix} \tag{A.1.25}$$

We define the fifth gamma matrix as follows

$$[\gamma^5] = -(\sigma^1 \otimes \sigma^2) \qquad \text{and so} \qquad \gamma^5 = \begin{bmatrix} 0 & 0 & 0 & i \\ 0 & 0 & -i & 0 \\ 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \end{bmatrix} \tag{A.1.26}$$

With gamma matrices, the algebra reads:

$$[D_\alpha, D_\beta\} = i(\gamma^{\underline{a}})_{\alpha\beta}\partial_{\underline{a}} \quad , \tag{A.1.27}$$

$$[\mathcal{M}_{\underline{ab}}, D_\alpha\} = \frac{1}{2}(\gamma_{\underline{ab}})_\alpha{}^\beta D_\beta \quad , \tag{A.1.28}$$

$$[\mathcal{M}_{\underline{ab}}, \partial_{\underline{c}}] = \eta_{\underline{c}[\underline{a}}\partial_{\underline{b}]} \quad , \tag{A.1.29}$$

$$[\mathcal{M}_{\underline{ab}}, \mathcal{M}_{\underline{cd}}] = \eta_{\underline{c}[\underline{a}}\mathcal{M}_{\underline{b}]\underline{d}} - \eta_{\underline{d}[\underline{a}}\mathcal{M}_{\underline{b}]\underline{c}} \quad . \tag{A.1.30}$$

Multiplication Table:

$$\gamma^{\underline{a}}\gamma^{\underline{b}} = \gamma^{\underline{ab}} + \eta^{\underline{ab}}\mathbb{I} \tag{A.1.31}$$

$$\gamma^{\underline{a}}\gamma^{\underline{bc}} = \eta^{\underline{a}[\underline{b}}\gamma^{\underline{c}]} + i\epsilon^{\underline{abcd}}\gamma^5\gamma_{\underline{d}} \tag{A.1.32}$$

$$\gamma^{\underline{a}}i\gamma^5 = -i\gamma^5\gamma^{\underline{a}} \tag{A.1.33}$$

$$\gamma^{\underline{a}}i\gamma^5\gamma^{\underline{b}} = -\frac{1}{2}\epsilon^{\underline{ab}[2]}\gamma_{[2]} - i\eta^{\underline{ab}}\gamma^5 \tag{A.1.34}$$

$$\gamma^{\underline{ab}}\gamma^{\underline{c}} = -\eta^{\underline{c}[\underline{a}}\gamma^{\underline{b}]} + i\epsilon^{\underline{abcd}}\gamma^5\gamma_{\underline{d}} \tag{A.1.35}$$

$$\gamma_{\underline{ab}}\gamma^{\underline{cd}} = \delta_{[\underline{a}}{}^{[\underline{c}}\gamma^{\underline{d}]}{}_{\underline{b}]} - \delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}} + i\epsilon_{\underline{ab}}{}^{\underline{cd}}\gamma^5 \tag{A.1.36}$$

$$\gamma^{\underline{ab}}i\gamma^5 = \frac{1}{2}\epsilon^{\underline{ab}[2]}\gamma_{[2]} \tag{A.1.37}$$

$$\gamma^{\underline{ab}}i\gamma^5\gamma^{\underline{c}} = -\epsilon^{\underline{abcd}}\gamma_{\underline{d}} - i\eta^{\underline{c}[\underline{a}}\gamma^5\gamma^{\underline{b}]} \tag{A.1.38}$$

$$i\gamma^5\gamma^{\underline{a}} = i\gamma^5\gamma^{\underline{a}} \tag{A.1.39}$$

$$i\gamma^5\gamma^{\underline{ab}} = \frac{1}{2}\epsilon^{\underline{ab}[2]}\gamma_{[2]} \tag{A.1.40}$$

$$i\gamma^5 i\gamma^5 = -\mathbb{I} \tag{A.1.41}$$

$$i\gamma^5 i\gamma^5\gamma^{\underline{a}} = -\gamma^{\underline{a}} \tag{A.1.42}$$

$$i\gamma^5\gamma^{\underline{a}}\gamma^{\underline{b}} = \frac{1}{2}\epsilon^{\underline{ab}[2]}\gamma_{[2]} + i\eta^{\underline{ab}}\gamma^5 \tag{A.1.43}$$

$$i\gamma^5\gamma^{\underline{a}}\gamma^{\underline{bc}} = i\eta^{\underline{a}[\underline{b}}\gamma^5\gamma^{\underline{c}]} - \epsilon^{\underline{abcd}}\gamma_{\underline{d}} \tag{A.1.44}$$

$$i\gamma^5\gamma^{\underline{a}}i\gamma^5 = \gamma^{\underline{a}} \tag{A.1.45}$$

$$i\gamma^5\gamma^{\underline{a}}i\gamma^5\gamma^{\underline{b}} = \gamma^{\underline{ab}} + \eta^{\underline{ab}}\mathbb{I} \tag{A.1.46}$$

Fierz Identities:

$$\delta_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = -\frac{1}{2}(\gamma^{\underline{a}})_{\alpha\beta}(\gamma_{\underline{a}})^{\gamma\delta} + \frac{1}{4}(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\gamma\delta} \tag{A.1.47}$$

$$\begin{aligned}(\gamma^{\underline{a}})_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = &-\frac{1}{2}(\gamma^{[1]})_{\alpha\beta}(\gamma_{[1]}{}^{\underline{a}})^{\gamma\delta} + \frac{1}{2}(\gamma^{\underline{a}})_{\alpha\beta}C^{\gamma\delta} \\ &-\frac{1}{2}(\gamma^{\underline{a}[1]})_{\alpha\beta}(\gamma_{[1]})^{\gamma\delta} - i\frac{1}{4}\epsilon^{\underline{a}[1][2]}(\gamma_{[2]})_{\alpha\beta}(\gamma^5\gamma_{[1]})^{\gamma\delta}\end{aligned} \tag{A.1.48}$$

$$\begin{aligned}(\gamma^{\underline{ab}})_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = &-\frac{1}{2}(\gamma^{[\underline{a}})_{\alpha\beta}(\gamma^{\underline{b}]})^{\gamma\delta} + i\frac{1}{2}\epsilon^{\underline{abcd}}(\gamma_{\underline{c}})_{\alpha\beta}(\gamma^5\gamma_{\underline{d}})^{\gamma\delta} \\ &-\frac{1}{2}(\gamma^{[1][\underline{a}})_{\alpha\beta}(\gamma^{\underline{b}]}{}_{[1]})^{\gamma\delta} + \frac{1}{2}(\gamma^{\underline{ab}})_{\alpha\beta}C^{\gamma\delta} \\ &-i\frac{1}{4}\epsilon^{\underline{ab}[2]}(\gamma_{[2]})_{\alpha\beta}(\gamma^5)^{\gamma\delta}\end{aligned} \tag{A.1.49}$$

$$i(\gamma^5)_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = -i\frac{1}{2}(\gamma^{[1]})_{\alpha\beta}(\gamma^5\gamma_{[1]})^{\gamma\delta} + \frac{1}{8}\epsilon^{[2][\overline{2}]}(\gamma_{[2]})_{\alpha\beta}(\gamma_{[\overline{2}]})^{\gamma\delta} \tag{A.1.50}$$

$$\begin{aligned}i(\gamma^5\gamma^{\underline{a}})_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = &-\frac{1}{4}\epsilon^{\underline{a}[1][2]}(\gamma_{[1]})_{\alpha\beta}(\gamma_{[2]})^{\gamma\delta} - i\frac{1}{2}(\gamma_{\underline{a}})_{\alpha\beta}(\gamma^5)^{\gamma\delta} \\ &-\frac{1}{4}\epsilon^{\underline{a}[1][2]}(\gamma_{[2]})_{\alpha\beta}(\gamma_{[1]})^{\gamma\delta} + i\frac{1}{2}(\gamma^{\underline{a}[1]})_{\alpha\beta}(\gamma^5\gamma_{[1]})^{\gamma\delta}\end{aligned} \tag{A.1.51}$$

A.2 Notations & Conventions in 10D

In this section, we briefly summarize the convention that we adopted for 10D sigma matrices. The Clifford algebra is

$$(\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{\underline{b}})^{\beta\gamma} + (\sigma^{\underline{b}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} = 2\eta^{\underline{ab}}\delta_{\alpha}{}^{\gamma} \quad , \tag{A.2.1}$$

where the inverse metric $\eta^{\underline{ab}}$ is:

$$\eta^{\underline{ab}} = \mathrm{diag}(-1,+1,+1,+1,+1,+1,+1,+1,+1,+1) \quad . \tag{A.2.2}$$

In 10D, the Dirac spinor has $2^{D/2} = 32$ components. We use undotted Greek index to denote 16 component left-handed Majorana spinor, and dotted index to denote right-handed ones,

$$(\psi^{\alpha})^{*} = \psi^{\alpha} \quad , \qquad (\psi^{\dot{\alpha}})^{*} = \psi^{\dot{\alpha}} \tag{A.2.3}$$

where $\alpha = 1, \dots, 16$ and $\dot{\alpha} = 1, \dots, 16$. We raise and lower the spinor indices by spinor metric $C_{\alpha\dot{\beta}}$ as follows:

$$\begin{aligned} &\psi_{\dot{\beta}} = \psi^{\alpha} C_{\alpha\dot{\beta}} \quad , \qquad \psi_{\alpha} = \psi^{\dot{\beta}} C_{\alpha\dot{\beta}} \quad , \\ &C_{\alpha\dot{\gamma}} C^{\alpha\dot{\beta}} = \delta_{\dot{\gamma}}{}^{\dot{\beta}} \quad , \quad C_{\gamma\dot{\beta}} C^{\alpha\dot{\beta}} = \delta_{\gamma}{}^{\alpha} \quad . \end{aligned} \tag{A.2.4}$$

The sigma matrices are bispinors. There are three types of them: purely left-handed:

$$(\sigma^{\underline{a}})_{\alpha\beta} \quad , \qquad (\sigma^{\underline{abc}})_{\alpha\beta} \quad , \qquad (\sigma^{\underline{abcde}})_{\alpha\beta} \quad ; \tag{A.2.5}$$

purely right-handed (related to purely left-handed by the following):

$$\begin{aligned} (\sigma^{\underline{a}})^{\alpha\beta} = C^{\alpha\dot{\alpha}} C^{\beta\dot{\beta}} (\sigma^{\underline{a}})_{\dot{\alpha}\dot{\beta}} \quad , \qquad (\sigma^{\underline{abc}})^{\alpha\beta} = C^{\alpha\dot{\alpha}} C^{\beta\dot{\beta}} (\sigma^{\underline{abc}})_{\dot{\alpha}\dot{\beta}} \quad , \\ (\sigma^{\underline{abcde}})^{\alpha\beta} = C^{\alpha\dot{\alpha}} C^{\beta\dot{\beta}} (\sigma^{\underline{abcde}})_{\dot{\alpha}\dot{\beta}} \quad ; \end{aligned} \tag{A.2.6}$$

and mixed bispinors:

$$C_{\alpha\dot{\beta}} \quad , \qquad (\sigma^{\underline{ab}})_{\alpha\dot{\beta}} \quad , \qquad (\sigma^{\underline{abcd}})_{\alpha\dot{\beta}} \quad , \tag{A.2.7}$$

which have relations

$$\delta_\alpha{}^\beta = C^{\beta\dot\beta}C_{\alpha\dot\beta} \quad , \qquad (\sigma^{\underline{ab}})_\alpha{}^\beta = C^{\beta\dot\beta}(\sigma^{\underline{ab}})_{\alpha\dot\beta} \quad , \qquad (\sigma^{\underline{abcd}})_\alpha{}^\beta = C^{\beta\dot\beta}(\sigma^{\underline{abcd}})_{\alpha\dot\beta} \quad . \tag{A.2.8}$$

Definition of σ-matrices with more Lorentz indices:

$$(\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{\underline{b}})^{\beta\gamma} = (\sigma^{\underline{ab}})_\alpha{}^\gamma + \eta^{\underline{ab}}\delta_\alpha{}^\gamma \tag{A.2.9}$$

$$(\sigma^{\underline{b}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} = -(\sigma^{\underline{ab}})_\alpha{}^\gamma + \eta^{\underline{ab}}\delta_\alpha{}^\gamma \tag{A.2.10}$$

$$(\sigma^{\underline{a}})^{\alpha\beta}(\sigma^{\underline{bc}})_\beta{}^\gamma = (\sigma^{\underline{abc}})^{\alpha\gamma} + \eta^{\underline{a}[\underline{b}}(\sigma^{\underline{c}]})^{\alpha\gamma} \tag{A.2.11}$$

$$(\sigma^{\underline{bc}})_\alpha{}^\beta(\sigma^{\underline{a}})_{\beta\gamma} = (\sigma^{\underline{abc}})_{\alpha\gamma} - \eta^{\underline{a}[\underline{b}}(\sigma^{\underline{c}]})_{\alpha\gamma} \tag{A.2.12}$$

$$(\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{\underline{bcd}})^{\beta\gamma} = (\sigma^{\underline{abcd}})_\alpha{}^\gamma + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cd}]})_\alpha{}^\gamma \tag{A.2.13}$$

$$(\sigma^{\underline{bcd}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} = -(\sigma^{\underline{abcd}})_\alpha{}^\gamma + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cd}]})_\alpha{}^\gamma \tag{A.2.14}$$

$$(\sigma^{\underline{a}})^{\alpha\beta}(\sigma^{\underline{bcde}})_\beta{}^\gamma = (\sigma^{\underline{abcde}})^{\alpha\gamma} + \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cde}]})^{\alpha\gamma} \tag{A.2.15}$$

$$(\sigma^{\underline{bcde}})_\alpha{}^\beta(\sigma^{\underline{a}})_{\beta\gamma} = (\sigma^{\underline{abcde}})_{\alpha\gamma} - \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cde}]})_{\alpha\gamma} \tag{A.2.16}$$

$$(\sigma^{\underline{a}})_{\alpha\beta}(\sigma^{\underline{bcdef}})^{\beta\gamma} = \tfrac{1}{4!}\epsilon^{\underline{abcdef}[4]}(\sigma_{[4]})_\alpha{}^\gamma + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cdef}]})_\alpha{}^\gamma \tag{A.2.17}$$

$$(\sigma^{\underline{bcdef}})_{\alpha\beta}(\sigma^{\underline{a}})^{\beta\gamma} = -\tfrac{1}{4!}\epsilon^{\underline{abcdef}[4]}(\sigma_{[4]})_\alpha{}^\gamma + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cdef}]})_\alpha{}^\gamma \tag{A.2.18}$$

and

$$(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}(\sigma^{\underline{b}})^{\dot\beta\dot\gamma} = (\sigma^{\underline{ab}})_{\dot\alpha}{}^{\dot\gamma} + \eta^{\underline{ab}}\delta_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.19}$$

$$(\sigma^{\underline{b}})_{\dot\alpha\dot\beta}(\sigma^{\underline{a}})^{\dot\beta\dot\gamma} = -(\sigma^{\underline{ab}})_{\dot\alpha}{}^{\dot\gamma} + \eta^{\underline{ab}}\delta_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.20}$$

$$(\sigma^{\underline{a}})^{\dot\alpha\dot\beta}(\sigma^{\underline{bc}})_{\dot\beta}{}^{\dot\gamma} = (\sigma^{\underline{abc}})^{\dot\alpha\dot\gamma} + \eta^{\underline{a}[\underline{b}}(\sigma^{\underline{c}]})^{\dot\alpha\dot\gamma} \tag{A.2.21}$$

$$(\sigma^{\underline{bc}})_{\dot\alpha}{}^{\dot\beta}(\sigma^{\underline{a}})_{\dot\beta\dot\gamma} = (\sigma^{\underline{abc}})_{\dot\alpha\dot\gamma} - \eta^{\underline{a}[\underline{b}}(\sigma^{\underline{c}]})_{\dot\alpha\dot\gamma} \tag{A.2.22}$$

$$(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}(\sigma^{\underline{bcd}})^{\dot\beta\dot\gamma} = (\sigma^{\underline{abcd}})_{\dot\alpha}{}^{\dot\gamma} + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cd}]})_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.23}$$

$$(\sigma^{\underline{bcd}})_{\dot\alpha\dot\beta}(\sigma^{\underline{a}})^{\dot\beta\dot\gamma} = -(\sigma^{\underline{abcd}})_{\dot\alpha}{}^{\dot\gamma} + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cd}]})_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.24}$$

$$(\sigma^{\underline{a}})^{\dot\alpha\dot\beta}(\sigma^{\underline{bcde}})_{\dot\beta}{}^{\dot\gamma} = (\sigma^{\underline{abcde}})^{\dot\alpha\dot\gamma} + \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cde}]})^{\dot\alpha\dot\gamma} \tag{A.2.25}$$

$$(\sigma^{\underline{bcde}})_{\dot\alpha}{}^{\dot\beta}(\sigma^{\underline{a}})_{\dot\beta\dot\gamma} = (\sigma^{\underline{abcde}})_{\dot\alpha\dot\gamma} - \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cde}]})_{\dot\alpha\dot\gamma} \tag{A.2.26}$$

$$(\sigma^{\underline{a}})_{\dot\alpha\dot\beta}(\sigma^{\underline{bcdef}})^{\dot\beta\dot\gamma} = -\tfrac{1}{4!}\epsilon^{\underline{abcdef}[4]}(\sigma_{[4]})_{\dot\alpha}{}^{\dot\gamma} + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cdef}]})_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.27}$$

$$(\sigma^{\underline{bcdef}})_{\dot\alpha\dot\beta}(\sigma^{\underline{a}})^{\dot\beta\dot\gamma} = \tfrac{1}{4!}\epsilon^{\underline{abcdef}[4]}(\sigma_{[4]})_{\dot\alpha}{}^{\dot\gamma} + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}(\sigma^{\underline{cdef}]})_{\dot\alpha}{}^{\dot\gamma} \tag{A.2.28}$$

The sigma matrices with five vector indices satisfy the self-dual / anti-self-dual identities:

$$(\sigma_{[5]})_{\alpha\beta} = \frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})_{\alpha\beta} \tag{A.2.29}$$

$$(\sigma_{[5]})^{\alpha\beta} = -\frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})^{\alpha\beta} \tag{A.2.30}$$

$$(\sigma_{[5]})_{\dot{\alpha}\dot{\beta}} = -\frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})_{\dot{\alpha}\dot{\beta}} \tag{A.2.31}$$

$$(\sigma_{[5]})^{\dot{\alpha}\dot{\beta}} = \frac{1}{5!}\epsilon_{[5]}{}^{[\bar{5}]}(\sigma_{[\bar{5}]})^{\dot{\alpha}\dot{\beta}} \tag{A.2.32}$$

The symmetric relations of the gamma matrices are given by:

$$(\sigma^{\underline{a}})_{\alpha\beta} = (\sigma^{\underline{a}})_{\beta\alpha} \tag{A.2.33}$$

$$(\sigma^{\underline{abc}})_{\alpha\beta} = -(\sigma^{\underline{abc}})_{\beta\alpha} \tag{A.2.34}$$

$$(\sigma^{\underline{abcde}})_{\alpha\beta} = (\sigma^{\underline{abcde}})_{\beta\alpha} \tag{A.2.35}$$

From the definition, we can easily work out the trace identities:

$$(\sigma_{\underline{a}})_{\alpha\beta}(\sigma^{\underline{b}})^{\alpha\beta} = 16\delta_{\underline{a}}{}^{\underline{b}}, \tag{A.2.36}$$

$$(\sigma_{\underline{ab}})_{\alpha}{}^{\beta}(\sigma^{\underline{cd}})_{\beta}{}^{\alpha} = -16\delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}}, \tag{A.2.37}$$

$$(\sigma_{\underline{abc}})_{\alpha\beta}(\sigma^{\underline{def}})^{\alpha\beta} = 16\delta_{[\underline{a}}{}^{\underline{d}}\delta_{\underline{b}}{}^{\underline{e}}\delta_{\underline{c}]}{}^{\underline{f}}, \tag{A.2.38}$$

$$(\sigma_{\underline{abcd}})_{\alpha}{}^{\beta}(\sigma^{\underline{efgh}})_{\beta}{}^{\alpha} = 16\delta_{[\underline{a}}{}^{\underline{e}}\delta_{\underline{b}}{}^{\underline{f}}\delta_{\underline{c}}{}^{\underline{g}}\delta_{\underline{d}]}{}^{\underline{h}}, \tag{A.2.39}$$

$$(\sigma_{\underline{abcde}})_{\alpha\beta}(\sigma^{\underline{fghij}})^{\alpha\beta} = 16[\delta_{[\underline{a}}{}^{\underline{f}}\delta_{\underline{b}}{}^{\underline{g}}\delta_{\underline{c}}{}^{\underline{h}}\delta_{\underline{d}}{}^{\underline{i}}\delta_{\underline{e}]}{}^{\underline{j}} + \epsilon_{\underline{abcde}}{}^{\underline{fghij}}], \tag{A.2.40}$$

and

$$(\sigma_{\underline{a}})_{\dot{\alpha}\dot{\beta}}(\sigma^{\underline{b}})^{\dot{\alpha}\dot{\beta}} = 16\delta_{\underline{a}}{}^{\underline{b}}, \tag{A.2.41}$$

$$(\sigma_{\underline{ab}})_{\dot{\alpha}}{}^{\dot{\beta}}(\sigma^{\underline{cd}})_{\dot{\beta}}{}^{\dot{\alpha}} = -16\delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}}, \tag{A.2.42}$$

$$(\sigma_{\underline{abc}})_{\dot{\alpha}\dot{\beta}}(\sigma^{\underline{def}})^{\dot{\alpha}\dot{\beta}} = 16\delta_{[\underline{a}}{}^{\underline{d}}\delta_{\underline{b}}{}^{\underline{e}}\delta_{\underline{c}]}{}^{\underline{f}}, \tag{A.2.43}$$

$$(\sigma_{\underline{abcd}})_{\dot{\alpha}}{}^{\dot{\beta}}(\sigma^{\underline{efgh}})_{\dot{\beta}}{}^{\dot{\alpha}} = 16\delta_{[\underline{a}}{}^{\underline{e}}\delta_{\underline{b}}{}^{\underline{f}}\delta_{\underline{c}}{}^{\underline{g}}\delta_{\underline{d}]}{}^{\underline{h}}, \tag{A.2.44}$$

$$(\sigma_{\underline{abcde}})_{\dot{\alpha}\dot{\beta}}(\sigma^{\underline{fghij}})^{\dot{\alpha}\dot{\beta}} = 16[\delta_{[\underline{a}}{}^{\underline{f}}\delta_{\underline{b}}{}^{\underline{g}}\delta_{\underline{c}}{}^{\underline{h}}\delta_{\underline{d}}{}^{\underline{i}}\delta_{\underline{e}]}{}^{\underline{j}} - \epsilon_{\underline{abcde}}{}^{\underline{fghij}}]. \tag{A.2.45}$$

From the definition, we can also derive the following 10D sigma matrices identities:

$$(\sigma_{\underline{a}})^{\delta\gamma}(\sigma^{[2]})_{\gamma}{}^{\alpha}\sigma_{[2]})_{\delta}{}^{\beta} = 54(\sigma_{\underline{a}})^{\alpha\beta} \tag{A.2.46}$$

$$(\sigma_{\underline{a}})^{\delta\gamma}(\sigma_{\underline{c}}{}^{\underline{d}})_{\delta}{}^{\alpha}(\sigma^{\underline{ce}})_{\gamma}{}^{\beta} = 6(\sigma_{\underline{a}}{}^{\underline{de}})^{\alpha\beta} + 7\eta^{\underline{de}}(\sigma_{\underline{a}})^{\alpha\beta} - 8\delta_{\underline{a}}{}^{(\underline{d}}(\sigma^{\underline{e})})^{\alpha\beta} \tag{A.2.47}$$

$$(\sigma_{[5]})^{\alpha\beta}(\sigma^{[2]})_{\alpha}{}^{\delta}(\sigma_{[2]})_{\beta}{}^{\gamma} = -10(\sigma_{[5]})^{\delta\gamma} \tag{A.2.48}$$

$$(\sigma_{[5]})^{\dot{\alpha}\dot{\beta}}(\sigma^{[2]})_{\dot{\alpha}}{}^{\dot{\delta}}(\sigma_{[2]})_{\dot{\beta}}{}^{\dot{\gamma}} = -10(\sigma_{[5]})^{\dot{\delta}\dot{\gamma}} \tag{A.2.49}$$

$$(\sigma_{[5]})_{\alpha\beta}(\sigma^{\underline{c}})^{\alpha\delta}(\sigma_{\underline{c}})^{\beta\gamma} = 0 \tag{A.2.50}$$

as well as the following Fierz identities:

$$\delta_{\alpha}{}^{\delta}\delta_{\beta}{}^{\gamma} = \frac{1}{16}\Big\{(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\gamma} + \frac{1}{3!}(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\delta\gamma} + \frac{1}{2\times 5!}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\delta\gamma}\Big\} \tag{A.2.51}$$

$$\delta_{(\alpha}{}^{\delta}\delta_{\beta)}{}^{\gamma} = \frac{1}{16}\Big\{2(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\gamma} + \frac{1}{5!}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\delta\gamma}\Big\} \tag{A.2.52}$$

$$\delta_{(\dot{\alpha}}{}^{\dot{\delta}}\delta_{\dot{\beta})}{}^{\dot{\gamma}} = \frac{1}{16}\Big\{2(\sigma^{\underline{c}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{c}})^{\dot{\delta}\dot{\gamma}} + \frac{1}{5!}(\sigma^{[5]})_{\dot{\alpha}\dot{\beta}}(\sigma_{[5]})^{\dot{\delta}\dot{\gamma}}\Big\} \tag{A.2.53}$$

$$(\sigma^{[2]})_{\alpha}{}^{\delta}(\sigma_{[2]})_{\beta}{}^{\gamma} = \frac{27}{8}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\gamma} + \frac{1}{16}(\sigma^{[3]})_{\alpha\beta}(\sigma_{[3]})^{\delta\gamma} - \frac{1}{384}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\delta\gamma} \tag{A.2.54}$$

$$(\sigma^{[2]})_{(\alpha}{}^{\delta}(\sigma_{[2]})_{\beta)}{}^{\gamma} = \frac{27}{4}(\sigma^{\underline{c}})_{\alpha\beta}(\sigma_{\underline{c}})^{\delta\gamma} - \frac{1}{192}(\sigma^{[5]})_{\alpha\beta}(\sigma_{[5]})^{\delta\gamma} \tag{A.2.55}$$

$$(\sigma^{[2]})_{(\dot{\alpha}}{}^{\dot{\delta}}(\sigma_{[2]})_{\dot{\beta})}{}^{\dot{\gamma}} = \frac{27}{4}(\sigma^{\underline{c}})_{\dot{\alpha}\dot{\beta}}(\sigma_{\underline{c}})^{\dot{\delta}\dot{\gamma}} - \frac{1}{192}(\sigma^{[5]})_{\dot{\alpha}\dot{\beta}}(\sigma_{[5]})^{\dot{\delta}\dot{\gamma}} \tag{A.2.56}$$

$$\begin{aligned}\delta_{\alpha}{}^{\beta}(\sigma_{\underline{b}})^{\gamma\delta} = \frac{1}{16}\Big\{&\delta_{\alpha}{}^{\gamma}(\sigma_{\underline{b}})^{\beta\delta} + \frac{1}{2}(\sigma^{[2]})_{\alpha}{}^{\gamma}(\sigma_{\underline{b}[2]})^{\beta\delta} - (\sigma_{\underline{b}[1]})_{\alpha}{}^{\gamma}(\sigma^{[1]})^{\beta\delta}\\ &+ \frac{1}{4!}(\sigma^{[4]})_{\alpha}{}^{\gamma}(\sigma_{\underline{b}[4]})^{\beta\delta} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_{\alpha}{}^{\gamma}(\sigma^{[3]})^{\beta\delta}\Big\}\end{aligned} \tag{A.2.57}$$

$$\begin{aligned}\delta_{\dot{\alpha}}{}^{\dot{\beta}}(\sigma_{\underline{b}})^{\dot{\gamma}\dot{\delta}} = \frac{1}{16}\Big\{&\delta_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma_{\underline{b}})^{\dot{\beta}\dot{\delta}} + \frac{1}{2}(\sigma^{[2]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma_{\underline{b}[2]})^{\dot{\beta}\dot{\delta}} - (\sigma_{\underline{b}[1]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma^{[1]})^{\dot{\beta}\dot{\delta}}\\ &+ \frac{1}{4!}(\sigma^{[4]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma_{\underline{b}[4]})^{\dot{\beta}\dot{\delta}} - \frac{1}{3!}(\sigma_{\underline{b}[3]})_{\dot{\alpha}}{}^{\dot{\gamma}}(\sigma^{[3]})^{\dot{\beta}\dot{\delta}}\Big\}\end{aligned} \tag{A.2.58}$$

$$\begin{aligned}(\sigma^{\underline{de}})_{\alpha}{}^{\gamma}\delta_{\beta}{}^{\delta} = \frac{1}{16}\Big\{&-(\sigma_{[1]})_{\alpha\beta}(\sigma^{[1]\underline{de}})^{\gamma\delta} + (\sigma^{[\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}]})^{\gamma\delta}\\ &- \frac{1}{3!}(\sigma_{[3]})_{\alpha\beta}(\sigma^{[3]\underline{de}})^{\gamma\delta} + \frac{1}{2}(\sigma^{[2][\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}]}{}_{[2]})^{\gamma\delta} + (\sigma^{\underline{de}[1]})_{\alpha\beta}(\sigma_{[1]})^{\gamma\delta}\\ &+ \frac{1}{2\times 4!}(\sigma_{[4]}{}^{[\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}][4]})^{\gamma\delta} + \frac{1}{3!}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\Big\}\end{aligned} \tag{A.2.59}$$

$$\begin{aligned}(\sigma^{\underline{de}})_{(\alpha}{}^{\gamma}\delta_{\beta)}{}^{\delta} = \frac{1}{16}\Big\{&-2(\sigma_{[1]})_{\alpha\beta}(\sigma^{[1]\underline{de}})^{\gamma\delta} + 2(\sigma^{[\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}]})^{\gamma\delta}\\ &+ \frac{1}{4!}(\sigma_{[4]}{}^{[\underline{d}})_{\alpha\beta}(\sigma^{\underline{e}][4]})^{\gamma\delta} + \frac{1}{3}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\Big\}\end{aligned} \tag{A.2.60}$$

$$
\begin{aligned}
(\sigma_{\underline{ab}})_\alpha{}^\delta(\sigma_{\underline{c}})^{\beta\gamma} =& \frac{1}{16}\Big\{\delta_\alpha{}^\beta(\sigma_{\underline{abc}})^{\gamma\delta} + \delta_\alpha{}^\beta\eta_{\underline{c}[\underline{a}}(\sigma_{\underline{b}]})^{\gamma\delta} \\
&- \frac{1}{2}(\sigma^{[2]})_\alpha{}^\beta(\sigma_{\underline{abc}[2]})^{\gamma\delta} - \frac{1}{2}(\sigma^{[2]})_\alpha{}^\beta\eta_{\underline{c}[\underline{a}}(\sigma_{\underline{b}][2]})^{\gamma\delta} \\
&+ (\sigma^{[1]}{}_{\underline{c}})_\alpha{}^\beta(\sigma_{\underline{ab}[1]})^{\gamma\delta} - (\sigma^{[1]}{}_{[\underline{a}})_\alpha{}^\beta(\sigma_{\underline{b}]\underline{c}[1]})^{\gamma\delta} \\
&+ (\sigma_{\underline{ab}})_\alpha{}^\beta(\sigma_{\underline{c}})^{\gamma\delta} - (\sigma_{\underline{c}[\underline{a}})_\alpha{}^\beta(\sigma_{\underline{b}]})^{\gamma\delta} + \eta_{\underline{c}[\underline{a}}(\sigma_{\underline{b}][1]})_\alpha{}^\beta(\sigma^{[1]})^{\gamma\delta} \\
&+ \frac{1}{4!}(\sigma^{[4]})_\alpha{}^\beta\eta_{\underline{c}[\underline{a}}(\sigma_{\underline{b}][4]})^{\gamma\delta} - \frac{1}{3!}(\sigma^{[3]}{}_{\underline{c}})_\alpha{}^\beta(\sigma_{\underline{ab}[3]})^{\gamma\delta} \\
&+ \frac{1}{3!}(\sigma^{[3]}{}_{[\underline{a}})_\alpha{}^\beta(\sigma_{\underline{b}]\underline{c}[3]})^{\gamma\delta} - \frac{1}{3!}\eta_{\underline{c}[\underline{a}}(\sigma_{\underline{b}][3]})_\alpha{}^\beta(\sigma^{[3]})^{\gamma\delta} \\
&- \frac{1}{2}(\sigma^{[2]}{}_{\underline{ab}})_\alpha{}^\beta(\sigma_{\underline{c}[2]})^{\gamma\delta} + \frac{1}{2}(\sigma^{[2]}{}_{\underline{c}[\underline{a}})_\alpha{}^\beta(\sigma_{\underline{b}][2]})^{\gamma\delta} \\
&+ \frac{1}{4!3!}\epsilon_{\underline{abc}[4][3]}(\sigma^{[4]})_\alpha{}^\beta(\sigma^{[3]})^{\gamma\delta} - (\sigma_{\underline{abc}[1]})_\alpha{}^\beta(\sigma^{[1]})^{\gamma\delta}\Big\}
\end{aligned} \tag{A.2.61}
$$

$$
\begin{aligned}
(\sigma_{\underline{b}}{}^{\underline{c}})_\alpha{}^\beta(\sigma_{\underline{c}})^{\delta\gamma} =& \frac{1}{16}\Big\{ - 9\delta_\alpha{}^\gamma(\sigma_{\underline{b}})^{\beta\delta} - \frac{5}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} - 7(\sigma_{\underline{b}[1]})_\alpha{}^\gamma(\sigma^{[1]})^{\beta\delta} \\
&- \frac{1}{4!}(\sigma^{[4]})_\alpha{}^\gamma(\sigma_{\underline{b}[4]})^{\beta\delta} - \frac{1}{2}(\sigma_{\underline{b}[3]})_\alpha{}^\gamma(\sigma^{[3]})^{\beta\delta}\Big\}
\end{aligned} \tag{A.2.62}
$$

$$
\begin{aligned}
(\sigma_{[\underline{a}})^{\alpha\gamma}(\sigma_{\underline{b}]}{}^{\underline{de}})^{\beta\delta} =& \frac{1}{16}\Big\{ - 2(\sigma_{[1]})^{\alpha\beta}(\sigma_{\underline{ab}}{}^{\underline{de}[1]})^{\gamma\delta} \\
&+ (\sigma_{[1]})^{\alpha\beta}\delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}]}{}^{\underline{e}][1]})^{\gamma\delta} - 2(\sigma^{[\underline{d}})^{\alpha\beta}(\sigma^{\underline{e}]}{}_{\underline{ab}})^{\gamma\delta} \\
&+ \delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}]})^{\alpha\beta}(\sigma^{\underline{e}]})^{\gamma\delta} - \delta_{[\underline{a}}{}^{[\underline{d}}(\sigma^{\underline{e}]})^{\alpha\beta}(\sigma_{\underline{b}]})^{\gamma\delta} \\
&+ \frac{1}{3!}(\sigma_{[3]})^{\alpha\beta}\delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}]}{}^{\underline{e}][3]})^{\gamma\delta} \\
&+ (\sigma_{[2][\underline{a}})^{\alpha\beta}(\sigma_{\underline{b}]}{}^{\underline{de}[2]})^{\gamma\delta} - (\sigma^{[2][\underline{d}})^{\alpha\beta}(\sigma^{\underline{e}]}{}_{\underline{ab}[2]})^{\gamma\delta} \\
&- \frac{1}{18}\epsilon_{\underline{ab}}{}^{\underline{de}[3][\bar{3}]}(\sigma_{[3]})^{\alpha\beta}(\sigma_{[\bar{3}]})^{\gamma\delta} \\
&- 2(\sigma_{[1]\underline{ab}})^{\alpha\beta}(\sigma^{\underline{de}[1]})^{\gamma\delta} + 2(\sigma^{[1]\underline{de}})^{\alpha\beta}(\sigma_{\underline{ab}[1]})^{\gamma\delta} \\
&+ \frac{1}{2}\delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}][2]})^{\alpha\beta}(\sigma^{\underline{e}][2]})^{\gamma\delta} - \frac{1}{2}\delta_{[\underline{a}}{}^{[\underline{d}}(\sigma^{\underline{e}][2]})^{\alpha\beta}(\sigma_{\underline{b}][2]})^{\gamma\delta} \\
&- \delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}]}{}^{\underline{e}][1]})^{\alpha\beta}(\sigma_{[1]})^{\gamma\delta} - 2(\sigma_{\underline{ab}}{}^{[\underline{d}})^{\alpha\beta}(\sigma^{\underline{e}]})^{\gamma\delta} \\
&- \frac{1}{3}(\sigma_{[3]\underline{ab}})^{\alpha\beta}(\sigma^{\underline{de}[3]})^{\gamma\delta} + \frac{1}{3!}(\sigma_{[3][\underline{a}}{}^{[\underline{d}})^{\alpha\beta}(\sigma_{\underline{b}]}{}^{\underline{e}][3]})^{\gamma\delta} \\
&- \frac{1}{3!}\delta_{[\underline{a}}{}^{[\underline{d}}(\sigma_{\underline{b}]}{}^{\underline{e}][3]})^{\alpha\beta}(\sigma_{[3]})^{\gamma\delta} - (\sigma_{[2]\underline{ab}}{}^{[\underline{d}})^{\alpha\beta}(\sigma^{\underline{e}][2]})^{\gamma\delta} \\
&+ 2(\sigma_{\underline{ab}}{}^{\underline{de}[1]})^{\alpha\beta}(\sigma_{[1]})^{\gamma\delta}\Big\}
\end{aligned} \tag{A.2.63}
$$

$$
\begin{aligned}
(\sigma^{[2]})_\alpha{}^\beta(\sigma_{\underline{b}[2]})^{\delta\gamma} =& - \frac{9}{2}\delta_\alpha{}^\gamma(\sigma_{\underline{b}})^{\beta\delta} - \frac{1}{2}(\sigma^{[2]})_\alpha{}^\gamma(\sigma_{\underline{b}[2]})^{\beta\delta} + \frac{5}{2}(\sigma_{\underline{b}[1]})_\alpha{}^\gamma(\sigma^{[1]})^{\beta\delta} \\
&+ \frac{1}{48}(\sigma^{[4]})_\alpha{}^\gamma(\sigma_{\underline{b}[4]})^{\beta\delta}
\end{aligned} \tag{A.2.64}
$$

$$
\begin{aligned}
(\sigma^{\underline{a}[\underline{d}})_\alpha{}^\gamma(\sigma_{\underline{a}}{}^{\underline{e}]})_\beta{}^\delta =& \frac{1}{16}\Big\{12(\sigma_{[1]})_{\alpha\beta}(\sigma^{\underline{de}[1]})^{\gamma\delta} + 12(\sigma^{\underline{de}[1]})_{\alpha\beta}(\sigma_{[1]})^{\gamma\delta} \\
&+ \frac{2}{3}(\sigma_{[3]})_{\alpha\beta}(\sigma^{\underline{de}[3]})^{\gamma\delta} + \frac{2}{3}(\sigma^{\underline{de}[3]})_{\alpha\beta}(\sigma_{[3]})^{\gamma\delta}\Big\}
\end{aligned} \tag{A.2.65}
$$

$$\delta_\alpha{}^\beta(\sigma_{\underline{a}})^{\dot\gamma\dot\delta} = \frac{1}{16}\Big\{(\sigma_{\underline{a}})_\alpha{}^{\dot\gamma}C^{\beta\dot\delta} - (\sigma^{[1]})_\alpha{}^{\dot\gamma}(\sigma_{\underline{a}[1]})^{\beta\dot\delta} - \frac{1}{3!}(\sigma^{[3]})_\alpha{}^{\dot\gamma}(\sigma_{\underline{a}[3]})^{\beta\ dot\delta} + \frac{1}{2}(\sigma_{\underline{a}[2]})_\alpha{}^{\dot\gamma}(\sigma^{[2]})^{\beta\dot\delta} + \frac{1}{4!}(\sigma_{\underline{a}[4]})_\alpha{}^{\dot\gamma}(\sigma^{[4]})^{\beta\dot\delta}\Big\} \tag{A.2.66}$$

$$\delta_{\dot\alpha}{}^{\dot\beta}(\sigma_{\underline{a}})^{\gamma\delta} = \frac{1}{16}\Big\{(\sigma_{\underline{a}})_{\dot\alpha}{}^{\gamma}C^{\delta\dot\beta} + (\sigma^{[1]})_{\dot\alpha}{}^{\gamma}(\sigma_{\underline{a}[1]})^{\delta\dot\beta} - \frac{1}{3!}(\sigma^{[3]})_{\dot\alpha}{}^{\gamma}(\sigma_{\underline{a}[3]})^{\delta\dot\beta} - \frac{1}{2}(\sigma_{\underline{a}[2]})_{\dot\alpha}{}^{\gamma}(\sigma^{[2]})^{\delta\dot\beta} + \frac{1}{4!}(\sigma_{\underline{a}[4]})_{\dot\alpha}{}^{\gamma}(\sigma^{[4]})^{\delta\dot\beta}\Big\} \tag{A.2.67}$$

Finally, we list the explicit (real) representations of the sigma matrices in terms of tensor products of Pauli matrices:

$$(\sigma^0)_{\alpha\beta} = \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.68}$$

$$(\sigma^1)_{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \tag{A.2.69}$$

$$(\sigma^2)_{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \tag{A.2.70}$$

$$(\sigma^3)_{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \tag{A.2.71}$$

$$(\sigma^4)_{\alpha\beta} = \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.2.72}$$

$$(\sigma^5)_{\alpha\beta} = \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.2.73}$$

$$(\sigma^6)_{\alpha\beta} = \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^2 \tag{A.2.74}$$

$$(\sigma^7)_{\alpha\beta} = \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \otimes \sigma^2 \tag{A.2.75}$$

$$(\sigma^8)_{\alpha\beta} = \sigma^1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.76}$$

$$(\sigma^9)_{\alpha\beta} = \sigma^3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.77}$$

and

$$(\sigma^0)^{\alpha\beta} = -\mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.78}$$

$$(\sigma^1)^{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \tag{A.2.79}$$

$$(\sigma^2)^{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \tag{A.2.80}$$

$$(\sigma^3)^{\alpha\beta} = \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \tag{A.2.81}$$

$$(\sigma^4)^{\alpha\beta} = \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.2.82}$$

$$(\sigma^5)^{\alpha\beta} = \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.2.83}$$

$$(\sigma^6)^{\alpha\beta} = \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^2 \tag{A.2.84}$$

$$(\sigma^7)^{\alpha\beta} = \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \otimes \sigma^2 \tag{A.2.85}$$

$$(\sigma^8)^{\alpha\beta} = \sigma^1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.86}$$

$$(\sigma^9)^{\alpha\beta} = \sigma^3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.2.87}$$

Basis: A 16×16 matrix has 256 independent components. And we have six kinds of 16×16 matrices: $M_{\alpha\beta}$, $M^{\alpha\beta}$, $M_{\dot{\alpha}\dot{\beta}}$, $M^{\dot{\alpha}\dot{\beta}}$, $M_{\alpha}{}^{\beta}$, $M_{\dot{\alpha}}{}^{\dot{\beta}}$. The basis is:

$$\begin{aligned} (\sigma^{\underline{a}})_{\alpha\beta} &= (\sigma^{\underline{a}})_{\beta\alpha} \to \{10\} \\ (\sigma^{[5]})_{\alpha\beta} &= (\sigma^{[5]})_{\beta\alpha} \to \{126\} \\ (\sigma^{[3]})_{\alpha\beta} &= (\sigma^{[3]})_{\beta\alpha} \to \{120\} \\ \delta_{\alpha}{}^{\beta} &\to \{1\} \\ (\sigma^{[2]})_{\alpha}{}^{\beta} &\to \{45\} \\ (\sigma^{[4]})_{\alpha}{}^{\beta} &\to \{210\} \end{aligned} \tag{A.2.88}$$

For first four kinds of matrices we can also consider the sym and antisym basis: Sym part: $\{136\} = \{10\} + \{126\}$ and the antisym part:$\{120\}$. The other two kinds of matrices: $\{256\} = \{1\} + \{45\} + \{210\}$.

A.3 Notations & Conventions in 11D

In this section we briefly summerize the convention that we adopted for 11D gamma matrices. Our 32×32 gamma matrices are defined by the Clifford algebra:

$$\{\gamma^{\underline{a}}, \gamma^{\underline{b}}\} = 2\eta^{\underline{ab}}\,\mathbb{I} \quad , \tag{A.3.1}$$

where $\mathbb{I}$ denotes the 32×32 identity matrix and the inverse metric $\eta^{\underline{ab}}$ follows the "most plus" signature:

$$\eta^{\underline{ab}} = \mathrm{diag}(-1,+1,+1,+1,+1,+1,+1,+1,+1,+1,+1) \quad . \tag{A.3.2}$$

It is known that D-dimensional space-time Dirac spinor has $2^{\frac{D-1}{2}}$ components when D is odd, and $2^{D/2}$ components when D is even. Hence in 11D, the spinor indices of the gamma matrices, denoted by α, β, and so forth, run from 1 to 32.

One can raise and lower the spinor indices via the "spinor metric", $C_{\alpha\beta}$, which satisfies:

$$C_{\alpha\beta} = -C_{\beta\alpha} \quad , \quad C_{\alpha\beta}C^{\gamma\beta} = \delta_\alpha{}^\gamma \quad . \tag{A.3.3}$$

The gamma matrices with multiple vector indices are defined through the equations:

$$\gamma^{\underline{a}}\gamma^{\underline{b}} = \gamma^{\underline{ab}} + \eta^{\underline{ab}} \tag{A.3.4}$$

$$\gamma^{\underline{b}}\gamma^{\underline{a}} = -\gamma^{\underline{ab}} + \eta^{\underline{ab}} \tag{A.3.5}$$

$$\gamma^{\underline{a}}\gamma^{\underline{bc}} = \gamma^{\underline{abc}} + \eta^{\underline{a}[\underline{b}}\gamma^{\underline{c}]} \tag{A.3.6}$$

$$\gamma^{\underline{bc}}\gamma^{\underline{a}} = \gamma^{\underline{abc}} - \eta^{\underline{a}[\underline{b}}\gamma^{\underline{c}]} \tag{A.3.7}$$

$$\gamma^{\underline{a}}\gamma^{\underline{bcd}} = \gamma^{\underline{abcd}} + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cd}]} \tag{A.3.8}$$

$$\gamma^{\underline{bcd}}\gamma^{\underline{a}} = -\gamma^{\underline{abcd}} + \tfrac{1}{2}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cd}]} \tag{A.3.9}$$

$$\gamma^{\underline{a}}\gamma^{\underline{bcde}} = \gamma^{\underline{abcde}} + \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cde}]} \tag{A.3.10}$$

$$\gamma^{\underline{bcde}}\gamma^{\underline{a}} = \gamma^{\underline{abcde}} - \tfrac{1}{3!}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cde}]} \tag{A.3.11}$$

$$\gamma^{\underline{a}}\gamma^{\underline{bcdef}} = \tfrac{1}{5!}\epsilon^{\underline{abcdef}[5]}\gamma_{[5]} + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cdef}]} \tag{A.3.12}$$

$$\gamma^{\underline{bcdef}}\gamma^{\underline{a}} = -\tfrac{1}{5!}\epsilon^{\underline{abcdef}[5]}\gamma_{[5]} + \tfrac{1}{4!}\eta^{\underline{a}[\underline{b}}\gamma^{\underline{cdef}]} \tag{A.3.13}$$

The symmetric relations of the gamma matrices are given by:

$$(\gamma^{\underline{a}})_{\alpha\beta} = (\gamma^{\underline{a}})_{\beta\alpha} \tag{A.3.14}$$

$$(\gamma^{\underline{ab}})_{\alpha\beta} = (\gamma^{\underline{ab}})_{\beta\alpha} \tag{A.3.15}$$

$$(\gamma^{\underline{abc}})_{\alpha\beta} = -(\gamma^{\underline{abc}})_{\beta\alpha} \tag{A.3.16}$$

$$(\gamma^{\underline{abcd}})_{\alpha\beta} = -(\gamma^{\underline{abcd}})_{\beta\alpha} \tag{A.3.17}$$

$$(\gamma^{\underline{abcde}})_{\alpha\beta} = (\gamma^{\underline{abcde}})_{\beta\alpha} \tag{A.3.18}$$

From the definitions, one can easily work out the following trace identities:

$$(\gamma_{\underline{a}})_\alpha{}^\beta(\gamma^{\underline{b}})_\beta{}^\alpha = 32\delta_{\underline{a}}{}^{\underline{b}} \tag{A.3.19}$$

$$(\gamma_{\underline{ab}})_\alpha{}^\beta(\gamma^{\underline{cd}})_\beta{}^\alpha = -32\delta_{[\underline{a}}{}^{\underline{c}}\delta_{\underline{b}]}{}^{\underline{d}} \tag{A.3.20}$$

$$(\gamma_{\underline{abc}})_\alpha{}^\beta(\gamma^{\underline{def}})_\beta{}^\alpha = -32\delta_{[\underline{a}}{}^{\underline{d}}\delta_{\underline{b}}{}^{\underline{e}}\delta_{\underline{c}]}{}^{\underline{f}} \tag{A.3.21}$$

$$(\gamma_{\underline{abcd}})_\alpha{}^\beta(\gamma^{\underline{efgh}})_\beta{}^\alpha = 32\delta_{[\underline{a}}{}^{\underline{e}}\delta_{\underline{b}}{}^{\underline{f}}\delta_{\underline{c}}{}^{\underline{g}}\delta_{\underline{d}]}{}^{\underline{h}} \tag{A.3.22}$$

$$(\gamma_{\underline{abcde}})_\alpha{}^\beta(\gamma^{\underline{fghij}})_\beta{}^\alpha = 32\delta_{[\underline{a}}{}^{\underline{f}}\delta_{\underline{b}}{}^{\underline{g}}\delta_{\underline{c}}{}^{\underline{h}}\delta_{\underline{d}}{}^{\underline{i}}\delta_{\underline{e}]}{}^{\underline{j}} \tag{A.3.23}$$

as well as the following Fierz identities:

$$\delta_{(\alpha}{}^\delta\delta_{\beta)}{}^\gamma = \frac{1}{16}\Big\{ -(\gamma^{\underline{c}})_{\alpha\beta}(\gamma_{\underline{c}})^{\delta\gamma} + \frac{1}{2}(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\delta\gamma} - \frac{1}{5!}(\gamma^{[5]})_{\alpha\beta}(\gamma_{[5]})^{\delta\gamma}\Big\} \tag{A.3.24}$$

$$(\gamma^{[2]})_{(\alpha}{}^\delta(\gamma_{[2]})_{\beta)}{}^\gamma = \frac{1}{16}\Big\{ -70(\gamma^{\underline{c}})_{\alpha\beta}(\gamma_{\underline{c}})^{\delta\gamma} + 19(\gamma^{[2]})_{\alpha\beta}(\gamma_{[2]})^{\delta\gamma} + \frac{1}{12}(\gamma^{[5]})_{\alpha\beta}(\gamma_{[5]})^{\delta\gamma}\Big\} \tag{A.3.25}$$

Finally, we list the explicit representations of 11D gamma matrices in terms of tensor products of Pauli matrices:

Spinor metric:

$$C_{\alpha\beta} = -i\sigma^2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.3.26}$$

Gamma matrices:

$$(\gamma^{\underline{0}})_\alpha{}^\beta = i\sigma^2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.3.27}$$

$$(\gamma^{\underline{1}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \otimes \sigma^2 \tag{A.3.28}$$

$$(\gamma^{\underline{2}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \tag{A.3.29}$$

$$(\gamma^{\underline{3}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \tag{A.3.30}$$

$$(\gamma^{\underline{4}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.3.31}$$

$$(\gamma^{\underline{5}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \sigma^3 \otimes \sigma^2 \otimes \mathbb{I}_2 \tag{A.3.32}$$

$$(\gamma^{\underline{6}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^1 \otimes \sigma^2 \tag{A.3.33}$$

$$(\gamma^{\underline{7}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^2 \otimes \mathbb{I}_2 \otimes \sigma^3 \otimes \sigma^2 \tag{A.3.34}$$

$$(\gamma^{\underline{8}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^1 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.3.35}$$

$$(\gamma^{\underline{9}})_\alpha{}^\beta = \sigma^1 \otimes \sigma^3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.3.36}$$

$$(\gamma^{\underline{10}})_\alpha{}^\beta = \sigma^3 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \otimes \mathbb{I}_2 \tag{A.3.37}$$

Basis: A 32×32 matrix has 1024 independent components. The basis is:

$$
\begin{aligned}
(\gamma^{\underline{a}})_{\alpha\beta} &= (\gamma^{\underline{a}})_{\beta\alpha} \to \{11\} \\
(\gamma^{[2]})_{\alpha\beta} &= (\gamma^{[2]})_{\beta\alpha} \to \{55\} \\
(\gamma^{[5]})_{\alpha\beta} &= (\gamma^{[5]})_{\beta\alpha} \to \{462\} \\
C_{\alpha\beta} &= -\, C_{\beta\alpha} \to \{1\} \\
(\gamma^{[3]})_{\alpha\beta} &= -\, (\gamma^{[3]})_{\beta\alpha} \to \{165\} \\
(\gamma^{[4]})_{\alpha\beta} &= -\, (\gamma^{[4]})_{\beta\alpha} \to \{330\}
\end{aligned}
\tag{A.3.38}
$$

Sym part: $\{528\} = \{11\} + \{55\} + \{462\}$ and the antisym part:$\{496\} = \{1\} + \{165\} + \{330\}$.

In the derivation of the cubic irreducible monomials, we encountered a number of Fierz identities. In the following, we list the ones relevant to derive our results.

A.3.1 For {32} θ-monomials

$$
C_{[\delta\epsilon}\, C_{\beta]\alpha} \;=\; \frac{1}{32}\Big\{ C_{[\epsilon\beta}\, C_{\delta]\alpha} + \frac{1}{3!}(\gamma^{[3]})_{[\epsilon\beta}\,(\gamma_{[3]})_{\delta]\alpha} - \frac{1}{4!}(\gamma^{[4]})_{[\epsilon\beta}\,(\gamma_{[4]})_{\delta]\alpha} \Big\} \quad , \tag{A.3.39}
$$

$$
(\gamma^{[3]})_{[\delta\epsilon}\,(\gamma_{[3]})_{\beta]\alpha} \;=\; \frac{1}{32}\Big\{ 990 C_{[\epsilon\beta}\, C_{\delta]\alpha} - 5(\gamma^{[3]})_{[\epsilon\beta}\,(\gamma_{[3]})_{\delta]\alpha} - \frac{11}{4}(\gamma^{[4]})_{[\epsilon\beta}\,(\gamma_{[4]})_{\delta]\alpha} \Big\} \quad , \tag{A.3.40}
$$

$$
(\gamma^{[4]})_{[\delta\epsilon}\,(\gamma_{[4]})_{\beta]\alpha} \;=\; -\frac{495}{2} C_{[\epsilon\beta}\, C_{\delta]\alpha} - \frac{11}{4}(\gamma^{[3]})_{[\epsilon\beta}\,(\gamma_{[3]})_{\delta]\alpha} + \frac{3}{16}(\gamma^{[4]})_{[\epsilon\beta}\,(\gamma_{[4]})_{\delta]\alpha} \quad . \tag{A.3.41}
$$

A.3.2 For {320} θ-monomials

$$
\begin{aligned}
C_{[\delta\epsilon}(\gamma^{\underline{a}})_{\beta]\alpha} \;=\; \frac{1}{32}\Big\{ &- C_{[\epsilon\beta}(\gamma^{\underline{a}})_{\delta]\alpha} - \frac{1}{3!}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[3]})_{\delta]\alpha} + \frac{1}{2}(\gamma^{\underline{a}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
&- \frac{1}{4!}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[4]})_{\delta]\alpha} + \frac{1}{3!}(\gamma^{\underline{a}[3]})_{[\epsilon\beta}(\gamma_{[3]})_{\delta]\alpha} \Big\} \quad ,
\end{aligned}
\tag{A.3.42}
$$

$$
\begin{aligned}
(\gamma^{\underline{a}[2]})_{[\delta\epsilon}\,(\gamma_{[2]})_{\beta]\alpha} \;=\; \frac{1}{32}\Big\{ &90 C_{[\epsilon\beta}(\gamma^{\underline{a}})_{\delta]\alpha} - (\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[3]})_{\delta]\alpha} - 13(\gamma^{\underline{a}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
&- \frac{1}{4}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[4]})_{\delta]\alpha} + (\gamma^{\underline{a}[3]})_{[\epsilon\beta}(\gamma_{[3]})_{\delta]\alpha} \Big\} \quad ,
\end{aligned}
\tag{A.3.43}
$$

$$
\begin{aligned}
(\gamma^{[3]})_{[\delta\epsilon}\,(\gamma^{\underline{a}}{}_{[3]})_{\beta]\alpha} \;=\; & -\frac{45}{2}C_{[\epsilon\beta}(\gamma^{\underline{a}})_{\delta]\alpha} - \frac{1}{4}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[3]})_{\delta]\alpha} - \frac{3}{4}(\gamma^{\underline{a}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& + \frac{1}{16}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[4]})_{\delta]\alpha} - \frac{1}{4}(\gamma^{\underline{a}[3]})_{[\epsilon\beta}(\gamma_{[3]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.44}
$$

$$
\begin{aligned}
(\gamma^{\underline{a}[3]})_{[\delta\epsilon}\,(\gamma_{[3]})_{\beta]\alpha} \;=\; & \frac{45}{2}C_{[\epsilon\beta}(\gamma^{\underline{a}})_{\delta]\alpha} - \frac{1}{4}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[3]})_{\delta]\alpha} + \frac{3}{4}(\gamma^{\underline{a}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& - \frac{1}{16}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[4]})_{\delta]\alpha} - \frac{1}{4}(\gamma^{\underline{a}[3]})_{[\epsilon\beta}(\gamma_{[3]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.45}
$$

$$
\begin{aligned}
(\gamma^{[4]})_{[\delta\epsilon}\,(\gamma^{\underline{a}}{}_{[4]})_{\beta]\alpha} \;=\; & -\frac{315}{2}C_{[\epsilon\beta}(\gamma^{\underline{a}})_{\delta]\alpha} + \frac{7}{4}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[3]})_{\delta]\alpha} - \frac{21}{4}(\gamma^{\underline{a}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& - \frac{1}{16}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{\underline{a}}{}_{[4]})_{\delta]\alpha} - \frac{7}{4}(\gamma^{\underline{a}[3]})_{[\epsilon\beta}(\gamma_{[3]})_{\delta]\alpha} \quad .
\end{aligned} \tag{A.3.46}
$$

A.3.3 For $\{1, 408\}$ θ-monomials

$$
\begin{aligned}
(\gamma^{\underline{ab}[2]})_{[\delta\epsilon}\,(\gamma_{[2]})_{\beta]\alpha} \;=\; & \frac{9}{4}C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} + \frac{1}{4}(\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[2]})_{\delta]\alpha} + \frac{5}{4}(\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \\
& - \frac{1}{4\times 4!5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha} - \frac{1}{4}(\gamma^{\underline{ab}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.47}
$$

$$
\begin{aligned}
(\gamma^{[3][\underline{a}})_{[\delta\epsilon}\,(\gamma^{\underline{b}]}{}_{[3]})_{\beta]\alpha} \;=\; & \frac{63}{2}C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} + \frac{1}{2}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{ab}}{}_{[3]})_{\delta]\alpha} - \frac{21}{2}(\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \\
& - \frac{1}{16\times 5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.48}
$$

$$
\begin{aligned}
(\gamma^{\underline{ab}[1]})_{[\delta\epsilon}\,(\gamma_{[1]})_{\beta]\alpha} \;=\; & -\frac{9}{32}C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} - \frac{1}{64}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{ab}}{}_{[3]})_{\delta]\alpha} + \frac{5}{64}(\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[2]})_{\delta]\alpha} \\
& + \frac{7}{32}(\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} - \frac{1}{64}(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[3]})_{\delta]\alpha} + \frac{5}{64}(\gamma^{\underline{ab}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& - \frac{1}{32\times 4!5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.49}
$$

$$
\begin{aligned}
(\gamma^{[2][\underline{a}})_{[\delta\epsilon}\,(\gamma^{\underline{b}]}{}_{[2]})_{\beta]\alpha} \;=\; & \frac{9}{2}C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} + \frac{5}{2}(\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} + \frac{1}{2}(\gamma^{\underline{ab}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& - \frac{1}{2\times 4!5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.50}
$$

$$
\begin{aligned}
(\gamma^{[3]})_{[\delta\epsilon}\,(\gamma^{\underline{ab}}{}_{[3]})_{\beta]\alpha} \;=\; & \frac{63}{4}C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} - \frac{1}{4}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{ab}}{}_{[3]})_{\delta]\alpha} - \frac{21}{4}(\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \\
& + \frac{1}{4}(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[3]})_{\delta]\alpha} - \frac{1}{32\times 5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha} \quad ,
\end{aligned} \tag{A.3.51}
$$

$$
\begin{aligned}
C_{[\delta\epsilon}(\gamma^{\underline{ab}})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ - C_{[\epsilon\beta}(\gamma^{\underline{ab}})_{\delta]\alpha} + \frac{1}{3!}(\gamma^{[3]})_{[\epsilon\beta}(\gamma^{\underline{ab}}{}_{[3]})_{\delta]\alpha} + \frac{1}{2}(\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[2]})_{\delta]\alpha} \\
& - (\gamma^{\underline{ab}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} + \frac{1}{3!}(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{b}]}{}_{[3]})_{\delta]\alpha} + \frac{1}{2}(\gamma^{\underline{ab}[2]})_{[\epsilon\beta}(\gamma_{[2]})_{\delta]\alpha} \\
& - \frac{1}{4!5!}\epsilon^{\underline{ab}}{}_{[4][5]}(\gamma^{[4]})_{[\epsilon\beta}(\gamma^{[5]})_{\delta]\alpha}\Big\} \quad .
\end{aligned} \tag{A.3.52}
$$

A.3.4 For {3, 520} θ-monomials

$$
\begin{aligned}
(\gamma^{\underline{abc}})_{[\delta\epsilon}C_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; - \; C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \; + \; \frac{1}{5!3!}\,\epsilon^{\underline{abc}[3][5]}\,(\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& + \; \frac{1}{4}\,(\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \; - \; \frac{1}{2}\,(\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \; - \; (\gamma^{\underline{abc}})_{[\epsilon\beta}C_{\delta]\alpha} \\
& - \; \frac{1}{4!4!}\,\epsilon^{\underline{abc}[4][\bar{4}]}\,(\gamma_{[4]})_{[\epsilon\beta}(\gamma_{[\bar{4}]})_{\delta]\alpha} \; + \; \frac{1}{12}\,(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \; + \; \frac{1}{4}\,(\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \\
& + \; (\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha}\Big\} \quad ,
\end{aligned} \tag{A.3.53}
$$

$$
\begin{aligned}
C_{[\delta\epsilon}(\gamma^{\underline{abc}})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; - \; C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \; - \; \frac{1}{5!3!}\,\epsilon^{\underline{abc}[3][5]}\,(\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& + \; \frac{1}{4}\,(\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \; + \; \frac{1}{2}\,(\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \; - \; (\gamma^{\underline{abc}})_{[\epsilon\beta}C_{\delta]\alpha} \\
& - \; \frac{1}{4!4!}\,\epsilon^{\underline{abc}[4][\bar{4}]}\,(\gamma_{[4]})_{[\epsilon\beta}(\gamma_{[\bar{4}]})_{\delta]\alpha} \; - \; \frac{1}{12}\,(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \; + \; \frac{1}{4}\,(\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \\
& - \; (\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha}\Big\} \quad ,
\end{aligned} \tag{A.3.54}
$$

$$
\begin{aligned}
(\gamma^{\underline{abc}[1]})_{[\delta\epsilon}(\gamma_{[1]})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; - \; 8C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \; - \; \frac{2}{5!3!}\,\epsilon^{\underline{abc}[3][5]}\,(\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& - \; (\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \; + \; 3\,(\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \; + \; 8\,(\gamma^{\underline{abc}})_{[\epsilon\beta}C_{\delta]\alpha} \\
& + \; \frac{1}{6}\,(\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \; + \; (\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \; + \; 6\,(\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha}\Big\} \quad ,
\end{aligned} \tag{A.3.55}
$$

$$
\begin{aligned}
(\gamma^{[1][\underline{ab}})_{[\delta\epsilon}(\gamma^{\underline{c}]}{}_{[1]})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; 48\, C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \;-\; \frac{12}{5!3!}\, \epsilon^{\underline{abc}[3][5]}\, (\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& -\; 2\, (\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \;-\; 6\, (\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \;-\; 48\, (\gamma^{\underline{abc}})_{[\epsilon\beta} C_{\delta]\alpha} \\
& -\; \frac{1}{3}\, (\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \;+\; 2\, (\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \;+\; 36\, (\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \Big\} \quad ,
\end{aligned}
\tag{A.3.56}
$$

$$
\begin{aligned}
(\gamma^{[2][\underline{ab}})_{[\delta\epsilon}(\gamma^{\underline{c}]}{}_{[2]})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; 336\, C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \;-\; \frac{4!}{5!3!}\, \epsilon^{\underline{abc}[3][5]}\, (\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& +\; 4\, (\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \;+\; 28\, (\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \;+\; 336\, (\gamma^{\underline{abc}})_{[\epsilon\beta} C_{\delta]\alpha} \\
& -\; \frac{48}{4!4!}\, \epsilon^{\underline{abc}[4][\bar{4}]}\, (\gamma_{[4]})_{[\epsilon\beta}(\gamma_{[\bar{4}]})_{\delta]\alpha} \;+\; \frac{2}{3}\, (\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \;+\; 4\, (\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \\
& +\; 168\, (\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \Big\} \quad ,
\end{aligned}
\tag{A.3.57}
$$

$$
\begin{aligned}
(\gamma^{[2][\underline{a}})_{[\delta\epsilon}(\gamma^{\underline{bc}]}{}_{[2]})_{\beta]\alpha} \;=\; & \frac{1}{32}\Big\{ \; 336\, C_{[\epsilon\beta}(\gamma^{\underline{abc}})_{\delta]\alpha} \;+\; \frac{4!}{5!3!}\, \epsilon^{\underline{abc}[3][5]}\, (\gamma_{[3]})_{[\epsilon\beta}(\gamma_{[5]})_{\delta]\alpha} \\
& +\; 4\, (\gamma^{[2][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[2]})_{\delta]\alpha} \;-\; 28\, (\gamma^{[1][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[1]})_{\delta]\alpha} \;+\; 336\, (\gamma^{\underline{abc}})_{[\epsilon\beta} C_{\delta]\alpha} \\
& -\; \frac{48}{4!4!}\, \epsilon^{\underline{abc}[4][\bar{4}]}\, (\gamma_{[4]})_{[\epsilon\beta}(\gamma_{[\bar{4}]})_{\delta]\alpha} \;-\; \frac{2}{3}\, (\gamma^{[3][\underline{a}})_{[\epsilon\beta}(\gamma^{\underline{bc}]}{}_{[3]})_{\delta]\alpha} \;+\; 4\, (\gamma^{[2][\underline{ab}})_{[\epsilon\beta}(\gamma^{\underline{c}]}{}_{[2]})_{\delta]\alpha} \\
& -\; 168\, (\gamma^{\underline{abc}[1]})_{[\epsilon\beta}(\gamma_{[1]})_{\delta]\alpha} \Big\} \quad .
\end{aligned}
\tag{A.3.58}
$$

A.3.5 11D Gamma Matrix Multiplication Table

We have given previously the definitions we use for the 11D γ-matrices. For the convenience of our readers, we review a number of these below and as well present some new results.

Identities with unique expressions

$$\gamma^{\underline{a}}\gamma_{\underline{b}} \;=\; \gamma^{\underline{a}}{}_{\underline{b}} + \delta_{\underline{b}}{}^{\underline{a}}\mathbb{I} \quad , \tag{A.3.59}$$

$$\gamma^{\underline{a}}\gamma_{\underline{bc}} \;=\; \gamma^{\underline{a}}{}_{\underline{bc}} + \delta_{[\underline{b}}{}^{\underline{a}}\gamma_{\underline{c}]} \quad , \tag{A.3.60}$$

$$\gamma^{\underline{a}}\gamma_{\underline{bcd}} \;=\; \gamma^{\underline{a}}{}_{\underline{bcd}} + \tfrac{1}{2}\delta_{[\underline{b}}{}^{\underline{a}}\gamma_{\underline{cd}]} \quad , \tag{A.3.61}$$

$$\gamma^{\underline{a}}\gamma_{\underline{bcde}} \;=\; \gamma^{\underline{a}}{}_{\underline{bcde}} + \tfrac{1}{3!}\delta_{[\underline{b}}{}^{\underline{a}}\gamma_{\underline{cde}]} \quad , \tag{A.3.62}$$

$$\gamma^{\underline{a}}\gamma_{\underline{bcdef}} \;=\; \tfrac{1}{5!}\epsilon^{\underline{a}}{}_{\underline{bcdef}}{}^{[5]}\gamma_{[5]} + \tfrac{1}{4!}\delta_{[\underline{b}}{}^{\underline{a}}\gamma_{\underline{cdef}]} \quad . \tag{A.3.63}$$

$$\gamma^{\underline{ab}}\gamma_{\underline{c}} = \gamma^{\underline{ab}}{}_{\underline{c}} - \delta_{\underline{c}}{}^{[\underline{a}}\gamma^{\underline{b}]} \quad , \tag{A.3.64}$$

$$\gamma^{\underline{ab}}\gamma_{\underline{cd}} = \gamma^{\underline{ab}}{}_{\underline{cd}} + \delta_{[\underline{c}}{}^{[\underline{a}}\gamma_{\underline{d}]}{}^{\underline{b}]} - \delta_{\underline{c}}{}^{[\underline{a}}\delta_{\underline{d}}{}^{\underline{b}]} \quad , \tag{A.3.65}$$

$$\gamma^{\underline{ab}}\gamma_{\underline{cde}} = \gamma^{\underline{ab}}{}_{\underline{cde}} - \tfrac{1}{2}\delta_{[\underline{c}}{}^{[\underline{a}}\gamma_{\underline{de}]}{}^{\underline{b}]} - \delta_{[\underline{c}}{}^{\underline{a}}\delta_{\underline{d}}{}^{\underline{b}}\gamma_{\underline{e}]} \quad , \tag{A.3.66}$$

$$\gamma^{\underline{ab}}\gamma_{\underline{cdef}} = \tfrac{1}{5!}\epsilon^{\underline{ab}}{}_{\underline{cdef}}{}^{[5]}\gamma_{[5]} + \tfrac{1}{3!}\delta_{[\underline{c}}{}^{[\underline{a}}\gamma_{\underline{def}]}{}^{\underline{b}]} - \tfrac{1}{2}\delta_{[\underline{c}}{}^{\underline{a}}\delta_{\underline{d}}{}^{\underline{b}}\gamma_{\underline{ef}]} \quad , \tag{A.3.67}$$

$$\gamma^{\underline{ab}}\gamma_{\underline{cdefg}} = \tfrac{1}{4!}\epsilon^{\underline{ab}}{}_{\underline{cdefg}}{}^{[4]}\gamma_{[4]} - \tfrac{1}{4!}\delta_{[\underline{c}}{}^{[\underline{a}}\gamma_{\underline{defg}]}{}^{\underline{b}]} - \tfrac{1}{3!}\delta_{[\underline{c}}{}^{\underline{a}}\delta_{\underline{d}}{}^{\underline{b}}\gamma_{\underline{efg}]} \quad . \tag{A.3.68}$$

$$\gamma^{\underline{abc}}\gamma_{\underline{d}} = \gamma^{\underline{abc}}{}_{\underline{d}} + \tfrac{1}{2}\delta_{\underline{d}}{}^{[\underline{a}}\gamma^{\underline{bc}]} \quad , \tag{A.3.69}$$

$$\gamma^{\underline{abc}}\gamma_{\underline{de}} = \gamma^{\underline{abc}}{}_{\underline{de}} + \tfrac{1}{2}\delta_{[\underline{d}}{}^{[\underline{a}}\gamma_{\underline{e}]}{}^{\underline{bc}]} - \delta_{\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma^{\underline{c}]} \quad , \tag{A.3.70}$$

$$\gamma^{\underline{abc}}\gamma_{\underline{def}} = \tfrac{1}{5!}\epsilon^{\underline{abc}}{}_{\underline{def}}{}^{[5]}\gamma_{[5]} + \tfrac{1}{4}\delta_{[\underline{d}}{}^{[\underline{a}}\gamma_{\underline{ef}]}{}^{\underline{bc}]} + \tfrac{1}{2}\delta_{[\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma_{\underline{f}]}{}^{\underline{c}]} - \delta_{\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\delta_{\underline{f}}{}^{\underline{c}]} \quad , \tag{A.3.71}$$

$$\gamma^{\underline{abc}}\gamma_{\underline{defg}} = \tfrac{1}{4!}\epsilon^{\underline{abc}}{}_{\underline{defg}}{}^{[4]}\gamma_{[4]} + \tfrac{1}{12}\delta_{[\underline{d}}{}^{[\underline{a}}\gamma_{\underline{efg}]}{}^{\underline{bc}]} - \tfrac{1}{4}\delta_{[\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma_{\underline{fg}]}{}^{\underline{c}]} - \delta_{[\underline{d}}{}^{\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\delta_{\underline{f}}{}^{\underline{c}}\gamma_{\underline{g}]} \quad . \tag{A.3.72}$$

$$\gamma^{\underline{abcd}}\gamma_{\underline{e}} = \gamma^{\underline{abcd}}{}_{\underline{e}} - \tfrac{1}{3!}\delta_{\underline{e}}{}^{[\underline{a}}\gamma^{\underline{bcd}]} \quad , \tag{A.3.73}$$

$$\gamma^{\underline{abcd}}\gamma_{\underline{ef}} = \tfrac{1}{5!}\epsilon^{\underline{abcd}}{}_{\underline{ef}}{}^{[5]}\gamma_{[5]} + \tfrac{1}{3!}\delta_{[\underline{e}}{}^{[\underline{a}}\gamma_{\underline{f}]}{}^{\underline{bcd}]} - \tfrac{1}{2}\delta_{\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma^{\underline{cd}]} \quad , \tag{A.3.74}$$

$$\gamma^{\underline{abcd}}\gamma_{\underline{efg}} = \tfrac{1}{4!}\epsilon^{\underline{abcd}}{}_{\underline{efg}}{}^{[4]}\gamma_{[4]} - \tfrac{1}{12}\delta_{[\underline{e}}{}^{[\underline{a}}\gamma_{\underline{fg}]}{}^{\underline{bcd}]} - \tfrac{1}{4}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{g}]}{}^{\underline{cd}]} + \delta_{\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma^{\underline{d}]} \quad . \tag{A.3.75}$$

$$\gamma^{\underline{abcde}}\gamma_{\underline{f}} = \tfrac{1}{5!}\epsilon^{\underline{abcde}}{}_{\underline{f}}{}^{[5]}\gamma_{[5]} + \tfrac{1}{4!}\delta_{\underline{f}}{}^{[\underline{a}}\gamma^{\underline{bcde}]} \quad , \tag{A.3.76}$$

$$\gamma^{\underline{abcde}}\gamma_{\underline{fg}} = \tfrac{1}{4!}\epsilon^{\underline{abcde}}{}_{\underline{fg}}{}^{[4]}\gamma_{[4]} + \tfrac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\gamma_{\underline{g}]}{}^{\underline{bcde}]} - \tfrac{1}{3!}\delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma^{\underline{cde}]} \quad . \tag{A.3.77}$$

Identities with multiple expressions Sometimes, multiplication of our general 11D γ matrices can yield multiple equivalent expressions. The following cases of this phenomenon are relevant to know about in the discussion of irreducible monomials.

For $\gamma^{[3]}\gamma_{[5]}$, the $\gamma_{[5]}$-term has multiple expressions.

$$
\begin{aligned}
&\gamma^{\underline{abc}}\gamma_{\underline{defgh}} \\
&= \frac{1}{5!4!2!}\delta_{[\underline{d}}{}^{[\underline{a}}\epsilon_{\underline{efgh}]}{}^{\underline{bc}][5]}\gamma_{[5]} - \frac{1}{3!}\epsilon^{\underline{abc}}{}_{\underline{defgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma_{\underline{fgh}]}{}^{\underline{c}]} - \frac{1}{2}\delta_{[\underline{d}}{}^{\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\delta_{\underline{f}}{}^{\underline{c}}\gamma_{\underline{gh}]} \\
&= \frac{1}{4!2!}\epsilon^{[4]}{}_{\underline{defgh}}{}^{[\underline{ab}}\gamma^{\underline{c}]}{}_{[4]} - \frac{1}{3!}\epsilon^{\underline{abc}}{}_{\underline{defgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma_{\underline{fgh}]}{}^{\underline{c}]} - \frac{1}{2}\delta_{[\underline{d}}{}^{\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\delta_{\underline{f}}{}^{\underline{c}}\gamma_{\underline{gh}]} \\
&= \frac{1}{4!4!}\epsilon^{[4]\underline{abc}}{}_{[\underline{defg}}\gamma_{\underline{h}][4]} - \frac{1}{3!}\epsilon^{\underline{abc}}{}_{\underline{defgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{d}}{}^{[\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\gamma_{\underline{fgh}]}{}^{\underline{c}]} - \frac{1}{2}\delta_{[\underline{d}}{}^{\underline{a}}\delta_{\underline{e}}{}^{\underline{b}}\delta_{\underline{f}}{}^{\underline{c}}\gamma_{\underline{gh}]} \quad .
\end{aligned}
\tag{A.3.78}
$$

For $\gamma^{[4]}\gamma_{[\bar{4}]}$, the $\gamma_{[5]}$-term has multiple expressions.

$$
\begin{aligned}
&\gamma^{\underline{abcd}}\gamma_{\underline{efgh}} \\
&= \frac{1}{5!3!3!}\delta_{[\underline{e}}{}^{[\underline{a}}\epsilon_{\underline{fgh}]}{}^{\underline{bcd}][5]}\gamma_{[5]} - \frac{1}{3!}\epsilon^{\underline{abcd}}{}_{\underline{efgh}}{}^{[3]}\gamma_{[3]} - \frac{1}{8}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{gh}]}{}^{\underline{cd}]} - \frac{1}{3!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{h}]}{}^{\underline{d}]} + \delta_{\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}]} \\
&= \frac{1}{4!3!}\epsilon^{[4]\underline{abcd}}{}_{[\underline{efg}}\gamma_{\underline{h}][4]} - \frac{1}{3!}\epsilon^{\underline{abcd}}{}_{\underline{efgh}}{}^{[3]}\gamma_{[3]} - \frac{1}{8}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{gh}]}{}^{\underline{cd}]} - \frac{1}{3!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{h}]}{}^{\underline{d}]} + \delta_{\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}]} \\
&= -\frac{1}{4!3!}\epsilon^{[4]}{}_{\underline{efgh}}{}^{[\underline{abc}}\gamma^{\underline{d}]}{}_{[4]} - \frac{1}{3!}\epsilon^{\underline{abcd}}{}_{\underline{efgh}}{}^{[3]}\gamma_{[3]} - \frac{1}{8}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{gh}]}{}^{\underline{cd}]} - \frac{1}{3!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{h}]}{}^{\underline{d}]} + \delta_{\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}]} \quad .
\end{aligned}
\tag{A.3.79}
$$

For $\gamma^{[4]}\gamma_{[5]}$, the $\gamma_{[4]}$-term has multiple expressions.

$$
\begin{aligned}
&\gamma^{\underline{abcd}}\gamma_{\underline{efghi}} \\
&= -\frac{1}{4!4!3!}\delta_{[\underline{e}}{}^{[\underline{a}}\epsilon_{\underline{fghi}]}{}^{\underline{bcd}][4]}\gamma_{[4]} - \frac{1}{2}\epsilon^{\underline{abcd}}{}_{\underline{efghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{ghi}]}{}^{\underline{cd}]} \\
&\quad + \frac{1}{12}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{hi}]}{}^{\underline{d}]} + \delta_{[\underline{e}}{}^{\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}}\gamma_{\underline{i}]} \\
&= -\frac{1}{3!3!}\epsilon^{[3]}{}_{\underline{efghi}}{}^{[\underline{abc}}\gamma^{\underline{d}]}{}_{[3]} - \frac{1}{2}\epsilon^{\underline{abcd}}{}_{\underline{efghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{ghi}]}{}^{\underline{cd}]} \\
&\quad + \frac{1}{12}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{hi}]}{}^{\underline{d}]} + \delta_{[\underline{e}}{}^{\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}}\gamma_{\underline{i}]} \\
&= \frac{1}{4!3!}\epsilon^{[3]\underline{abcd}}{}_{[\underline{efgh}}\gamma_{\underline{i}][3]} - \frac{1}{2}\epsilon^{\underline{abcd}}{}_{\underline{efghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\gamma_{\underline{ghi}]}{}^{\underline{cd}]} \\
&\quad + \frac{1}{12}\delta_{[\underline{e}}{}^{[\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\gamma_{\underline{hi}]}{}^{\underline{d}]} + \delta_{[\underline{e}}{}^{\underline{a}}\delta_{\underline{f}}{}^{\underline{b}}\delta_{\underline{g}}{}^{\underline{c}}\delta_{\underline{h}}{}^{\underline{d}}\gamma_{\underline{i}]} \quad .
\end{aligned}
\tag{A.3.80}
$$

For $\gamma^{[5]}\gamma_{[3]}$, the $\gamma_{[5]}$-term has multiple expressions.

$$\begin{aligned}\gamma^{\underline{abcde}}\gamma_{\underline{fgh}} =& \frac{1}{5!4!2!}\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{gh}]}{}^{\underline{bcde}][5]}\gamma_{[5]} - \frac{1}{3!}\epsilon^{\underline{abcde}}{}_{\underline{fgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{h}]}{}^{\underline{cde}]} - \frac{1}{2}\delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma^{\underline{de}]} \\ =& \frac{1}{4!2!}\epsilon^{[4]\underline{abcde}}{}_{[\underline{fg}}\gamma_{\underline{h}][4]} - \frac{1}{3!}\epsilon^{\underline{abcde}}{}_{\underline{fgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{h}]}{}^{\underline{cde}]} - \frac{1}{2}\delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma^{\underline{de}]} \\ =& \frac{1}{4!4!}\epsilon^{[4]}{}_{\underline{fgh}}{}^{[\underline{abcd}}\gamma^{\underline{e}]}{}_{[4]} - \frac{1}{3!}\epsilon^{\underline{abcde}}{}_{\underline{fgh}}{}^{[3]}\gamma_{[3]} + \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{h}]}{}^{\underline{cde}]} - \frac{1}{2}\delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma^{\underline{de}]} \quad .\end{aligned} \tag{A.3.81}$$

For $\gamma^{[5]}\gamma_{[4]}$, the $\gamma_{[4]}$-term has multiple expressions.

$$\begin{aligned}&\gamma^{\underline{abcde}}\gamma_{\underline{fghi}} \\ =& \frac{1}{4!4!3!}\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{ghi}]}{}^{\underline{bcde}][4]}\gamma_{[4]} - \frac{1}{2}\epsilon^{\underline{abcde}}{}_{\underline{fghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{hi}]}{}^{\underline{cde}]} \\ &- \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma_{\underline{i}]}{}^{\underline{de}]} + \delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\delta_{\underline{i}}{}^{\underline{d}}\gamma^{\underline{e}]} \\ =& \frac{1}{3!3!}\epsilon^{[3]\underline{abcde}}{}_{[\underline{fgh}}\gamma_{\underline{i}][3]} - \frac{1}{2}\epsilon^{\underline{abcde}}{}_{\underline{fghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{hi}]}{}^{\underline{cde}]} \\ &- \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma_{\underline{i}]}{}^{\underline{de}]} + \delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\delta_{\underline{i}}{}^{\underline{d}}\gamma^{\underline{e}]} \\ =& -\frac{1}{4!3!}\epsilon^{[3]}{}_{\underline{fghi}}{}^{[\underline{abcd}}\gamma^{\underline{e}]}{}_{[3]} - \frac{1}{2}\epsilon^{\underline{abcde}}{}_{\underline{fghi}}{}^{[2]}\gamma_{[2]} - \frac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{hi}]}{}^{\underline{cde}]} \\ &- \frac{1}{12}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma_{\underline{i}]}{}^{\underline{de}]} + \delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\delta_{\underline{i}}{}^{\underline{d}}\gamma^{\underline{e}]} \quad .\end{aligned} \tag{A.3.82}$$

For $\gamma^{[5]}\gamma_{[\bar{5}]}$, both $\gamma_{[5]}$-term and $\gamma_{[3]}$-term have multiple expressions. The different $\gamma_{[5]}$-term expressions are:

$\gamma_{[5]}$-term	coefficient
$\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\epsilon_{\underline{hij}]}{}^{\underline{cde}][5]}\gamma_{[5]}$	$\frac{1}{5!3!3!2!}$
$\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{ghi}}{}^{\underline{bcde}][4]}\gamma_{\underline{j}][4]}$	$\frac{1}{4!4!3!2!}$
$\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{ghij}]}{}^{\underline{bcd}}{}_{[4]}\gamma^{\underline{e}][4]}$	$-\frac{1}{4!4!3!2!}$
$\epsilon^{[3]\underline{abcde}}{}_{[\underline{fgh}}\gamma_{\underline{ij}][3]}$	$-\frac{1}{3!3!2!}$
$\epsilon^{[3]}{}_{\underline{fghij}}{}^{[\underline{abc}}\gamma^{\underline{de}]}{}_{[3]}$	$\frac{1}{3!3!2!}$

The different $\gamma_{[3]}$-term expressions are:

$\gamma_{[3]}$-term	coefficient
$\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{ghij}]}{}^{\underline{bcde}][3]}\gamma_{[3]}$	$-\frac{1}{4!4!3!}$
$\epsilon^{[2]\underline{abcde}}{}_{[\underline{fghi}}\gamma_{\underline{j}][2]}$	$-\frac{1}{4!2!}$
$\epsilon^{[2]}{}_{\underline{fghij}}{}^{[\underline{abcd}}\gamma^{\underline{e}]}{}_{[2]}$	$-\frac{1}{4!2!}$

Therefore, we have $5 \times 3 = 15$ different expressions of $\gamma^{\underline{abcde}}\gamma_{\underline{fghij}}$ with right symmetries and good coefficients. One example would be

$$\begin{aligned}
\gamma^{\underline{abcde}}\gamma_{\underline{fghij}} \;=\;& \tfrac{1}{5!3!3!2!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\epsilon_{\underline{hij}]}{}^{\underline{cde}][5]}\gamma_{[5]} - \tfrac{1}{4!4!3!}\delta_{[\underline{f}}{}^{[\underline{a}}\epsilon_{\underline{ghij}]}{}^{\underline{bcde}][3]}\gamma_{[3]} \\
&+ \epsilon^{\underline{abcde}}{}_{\underline{fghij}}{}^{[1]}\gamma_{[1]} - \tfrac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\gamma_{\underline{ij}]}{}^{\underline{de}]} - \tfrac{1}{4!}\delta_{[\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\delta_{\underline{i}}{}^{\underline{d}}\gamma_{\underline{j}]}{}^{\underline{e}]} \\
&+ \delta_{\underline{f}}{}^{[\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\delta_{\underline{h}}{}^{\underline{c}}\delta_{\underline{i}}{}^{\underline{d}}\delta_{\underline{j}}{}^{\underline{e}]} \quad .
\end{aligned} \tag{A.3.83}$$

A.3.6 Additional Useful Identities for 11D Gamma Matrices

Over and above previous results, the list of identities below are useful particularly with regards to the discussion on deriving irreducible θ monomials.

$$\gamma_{[1]}\gamma^{[1]} = 11 \quad , \tag{A.3.84}$$

$$\gamma_{[2]}\gamma^{[2]} = -110 \quad , \tag{A.3.85}$$

$$\gamma_{[3]}\gamma^{[3]} = -990 \quad , \tag{A.3.86}$$

$$\gamma_{[4]}\gamma^{[4]} = 7920 \quad , \tag{A.3.87}$$

$$\gamma_{[5]}\gamma^{[5]} = 55440 \quad , \tag{A.3.88}$$

$$\gamma^{\underline{a}[2]}\gamma_{[2]} = -90\gamma^{\underline{a}} \quad , \tag{A.3.89}$$

$$\gamma^{\underline{a}[2]}\gamma_{\underline{def}}\gamma_{[2]} = -6\gamma^{\underline{a}}{}_{\underline{def}} - 13\delta_{[\underline{d}}{}^{\underline{a}}\gamma_{\underline{ef}]} \quad , \tag{A.3.90}$$

$$\gamma^{\underline{a}[2]}\gamma_{\underline{defg}}\gamma_{[2]} = 6\gamma^{\underline{a}}{}_{\underline{defg}} - \delta_{[\underline{d}}{}^{\underline{a}}\gamma_{\underline{efg}]} \quad , \tag{A.3.91}$$

$$\gamma_{[3]}\gamma^{[3]\underline{a}} = -720\gamma^{\underline{a}} \quad , \tag{A.3.92}$$

$$\gamma^{\underline{a}[3]}\gamma_{[3]} = -720\gamma^{\underline{a}} \quad , \tag{A.3.93}$$

$$\gamma_{[3]}\gamma_{\underline{efg}}\gamma^{[3]\underline{a}} = 48\gamma^{\underline{a}}{}_{\underline{efg}} + 24\delta_{[\underline{e}}{}^{\underline{a}}\gamma_{\underline{fg}]} \quad , \tag{A.3.94}$$

$$\gamma^{\underline{a}[3]}\gamma_{\underline{efg}}\gamma_{[3]} = -48\gamma^{\underline{a}}{}_{\underline{efg}} + 24\delta_{[\underline{e}}{}^{\underline{a}}\gamma_{\underline{fg}]} \quad , \tag{A.3.95}$$

$$\gamma_{[3]}\gamma_{\underline{efgh}}\gamma^{[3]\underline{a}} = 48\gamma^{\underline{a}}{}_{\underline{efgh}} - 8\delta_{[\underline{e}}{}^{\underline{a}}\gamma_{\underline{fgh}]} \quad , \tag{A.3.96}$$

$$\gamma^{\underline{a}[3]}\gamma_{\underline{efgh}}\gamma_{[3]} = 48\gamma^{\underline{a}}{}_{\underline{efgh}} + 8\delta_{[\underline{e}}{}^{\underline{a}}\gamma_{\underline{fgh}]} \quad , \tag{A.3.97}$$

$$\gamma_{[4]}\gamma^{[4]\underline{a}} = 5040\gamma^{\underline{a}} \quad , \tag{A.3.98}$$

$$\gamma_{[4]}\gamma_{\underline{fgh}}\gamma^{[4]\underline{a}} = 336\gamma^{\underline{a}}{}_{\underline{fgh}} - 168\delta_{[\underline{f}}{}^{\underline{a}}\gamma_{\underline{gh}]} \quad , \tag{A.3.99}$$

$$\gamma_{[4]}\gamma_{\underline{fghi}}\gamma^{[4]\underline{a}} = 48\gamma^{\underline{a}}{}_{\underline{fghi}} + 56\delta_{[\underline{f}}{}^{\underline{a}}\gamma_{\underline{ghi}]} \quad , \tag{A.3.100}$$

$$\gamma^{[\underline{a}}\gamma_{\underline{hij}}\gamma^{\underline{b}]} = -2\gamma^{\underline{ab}}{}_{\underline{hij}} - 2\delta_{[\underline{h}}{}^{\underline{a}}\delta_{\underline{i}}{}^{\underline{b}}\gamma_{\underline{j}]} \quad , \tag{A.3.101}$$

$$\gamma^{\underline{ab}[2]}\gamma_{\underline{hij}}\gamma_{[2]} = 8\delta_{[\underline{h}}{}^{[\underline{a}}\gamma^{\underline{b}]}{}_{\underline{ij}]} + 40\delta_{[\underline{h}}{}^{\underline{a}}\delta_{\underline{i}}{}^{\underline{b}}\gamma_{\underline{j}]} \quad , \tag{A.3.102}$$

$$\gamma^{[\underline{a}}\gamma_{\underline{ijlm}}\gamma^{\underline{b}]} = \frac{2}{5!}\epsilon^{\underline{ab}}{}_{\underline{ijlm}[5]}\gamma^{[5]} + \delta_{[\underline{i}}{}^{\underline{a}}\delta_{\underline{j}}{}^{\underline{b}}\gamma_{\underline{lm}]} \quad , \tag{A.3.103}$$

$$\gamma^{\underline{ab}[2]}\gamma_{\underline{ijlm}}\gamma_{[2]} = \frac{1}{15}\epsilon^{\underline{ab}}{}_{\underline{ijlm}[5]}\gamma^{[5]} + 8\delta_{\underline{i}}{}^{[\underline{a}}\delta_{\underline{j}}{}^{\underline{b}}\gamma_{\underline{lm}]} \quad , \tag{A.3.104}$$

$$\gamma^{[3][\underline{a}}\gamma^{\underline{b}]}{}_{[3]} = -1008\gamma^{\underline{ab}} \quad , \tag{A.3.105}$$

$$\gamma^{[3][\underline{a}}\gamma_{\underline{fgh}}\gamma^{\underline{b}]}{}_{[3]} = 96\gamma^{\underline{ab}}{}_{\underline{fgh}} - 336\delta_{[\underline{f}}{}^{\underline{a}}\delta_{\underline{g}}{}^{\underline{b}}\gamma_{\underline{h}]} \quad , \tag{A.3.106}$$

$$\gamma^{[3][\underline{a}}\gamma_{\underline{fghi}}\gamma^{\underline{b}]}{}_{[3]} = \frac{2}{5}\epsilon^{\underline{ab}}{}_{\underline{fghi}[5]}\gamma^{[5]} \quad , \tag{A.3.107}$$

$$\gamma^{\underline{ab}[1]}\gamma_{\underline{hij}}\gamma_{[1]} = -3\gamma^{\underline{ab}}{}_{\underline{hij}} + \frac{5}{2}\delta_{[\underline{h}}{}^{[\underline{a}}\gamma^{\underline{b}]}{}_{\underline{ij}]} + 7\delta_{[\underline{h}}{}^{\underline{a}}\delta_{\underline{i}}{}^{\underline{b}}\gamma_{\underline{j}]} \quad , \tag{A.3.108}$$

$$\gamma^{\underline{ab}[1]}\gamma_{\underline{ijlm}}\gamma_{[1]} = \frac{1}{5!}\epsilon^{\underline{ab}}{}_{\underline{ijlm}[5]}\gamma^{[5]} + \frac{1}{2}\delta_{[\underline{i}}{}^{[\underline{a}}\gamma_{\underline{jlm}]}{}^{\underline{b}]} - \frac{5}{2}\delta_{[\underline{i}}{}^{\underline{a}}\delta_{\underline{j}}{}^{\underline{b}}\gamma_{\underline{lm}]} \quad , \tag{A.3.109}$$

$$\gamma^{[2][\underline{a}}\gamma^{\underline{b}]}{}_{[2]} = -144\gamma^{\underline{ab}} \quad , \tag{A.3.110}$$

$$\gamma^{[2][\underline{a}}\gamma_{\underline{hij}}\gamma^{\underline{b}]}{}_{[2]} = 80\delta_{[\underline{h}}{}^{\underline{a}}\delta_{\underline{i}}{}^{\underline{b}}\gamma_{\underline{j}]} \quad , \tag{A.3.111}$$

$$\gamma^{[2][\underline{a}}\gamma_{\underline{ijlm}}\gamma^{\underline{b}]}{}_{[2]} = \frac{2}{15}\epsilon^{\underline{ab}}{}_{\underline{ijlm}[5]}\gamma^{[5]} - 16\delta_{[\underline{i}}{}^{\underline{a}}\delta_{\underline{j}}{}^{\underline{b}}\gamma_{\underline{lm}]} \quad , \tag{A.3.112}$$

$$\gamma^{[3]}\gamma^{\underline{ab}}{}_{[3]} = -504\gamma^{\underline{ab}} \quad , \tag{A.3.113}$$

$$\gamma^{[3]}\gamma_{\underline{ijk}}\gamma^{\underline{ab}}{}_{[3]} = -48\gamma^{\underline{ab}}{}_{\underline{ijk}} - 168\delta_{[\underline{i}}{}^{\underline{a}}\delta_{\underline{j}}{}^{\underline{b}}\gamma_{\underline{k}]} \quad , \tag{A.3.114}$$

$$\gamma^{[3]}\gamma_{\underline{jklm}}\gamma^{\underline{ab}}{}_{[3]} = \frac{1}{5}\epsilon^{\underline{ab}}{}_{\underline{jklm}[5]}\gamma^{[5]} + 8\delta_{[\underline{j}}{}^{[\underline{a}}\gamma^{\underline{b}]}{}_{\underline{klm}]} \quad . \tag{A.3.115}$$

Appendix B

Chiral and Vector Supermultiplets from the 4D, $\mathcal{N} = 1$ Unconstrained Scalar Superfield

In this appendix, we present an expanded discussion of the mathematical structures that support the validity of the "splitting" of the adinkras presented in Figure 3.3. The vector supermultiplet shown in the figure is well familiar, but this is not so for the chiral supermultiplet shown. We will begin with the vector supermultiplet. The notational conventions used in the following discussion are those of *Superspace* [100].

The vector supermultiplet contains a 1-form gauge potential $A_{\underline{a}}$ that is part of a super 1-form with different sectors given by

$$A_\alpha \ = \ i\mathrm{D}_\alpha V \quad , \quad A_{\dot\alpha} \ = \ -i\overline{\mathrm{D}}_{\dot\alpha} V \quad , \quad A_{\underline{a}} \ = \ [\overline{\mathrm{D}}_{\dot\alpha}, \mathrm{D}_\alpha] V \quad . \tag{B.0.1}$$

The vector supermultiplet field strength superfield W_α is determined in terms of an unconstrained real scalar superfield V by

$$W_\alpha \ = \ i\overline{\mathrm{D}}^2 \mathrm{D}_\alpha V \quad , \quad \overline{W}_{\dot\alpha} \ = \ -i\mathrm{D}^2 \overline{\mathrm{D}}_{\dot\alpha} V \quad , \quad V \ = \ \overline{V} \quad . \tag{B.0.2}$$

The definition of W^α is invariant under gauge transformations with a chiral parameter Λ that takes the form

$$V' = V + i(\overline{\Lambda} - \Lambda) \quad , \quad \overline{\mathrm{D}}_{\dot{\alpha}}\Lambda = \mathrm{D}_\alpha\overline{\Lambda} = 0 \quad . \tag{B.0.3}$$

The components of the prepotential superfield V can be defined by the projection method

$$\begin{aligned} C &= V| \quad , \quad \chi_\alpha = i\mathrm{D}_\alpha V| \quad , \quad \overline{\chi}_{\dot{\alpha}} = -i\overline{\mathrm{D}}_{\dot{\alpha}}V| \quad , \\ M &= \mathrm{D}^2V| \quad , \quad \overline{M} = \overline{\mathrm{D}}^2V| \quad , \quad A_{\alpha\dot{\alpha}} = \frac{1}{2}[\overline{\mathrm{D}}_{\dot{\alpha}} \, , \, \mathrm{D}_\alpha]V| \quad , \\ \lambda_\alpha &= i\overline{\mathrm{D}}^2\mathrm{D}_\alpha V| \quad , \quad \overline{\lambda}_{\dot{\alpha}} = -i\mathrm{D}^2\overline{\mathrm{D}}_{\dot{\alpha}}V| \quad , \quad D' = \frac{1}{2}\mathrm{D}^\alpha\overline{\mathrm{D}}^2\mathrm{D}_\alpha V| \quad . \end{aligned} \tag{B.0.4}$$

All the components of V can be gauged away by nonderivative gauge transformations except for $A_{\alpha\dot{\alpha}}$, λ_α, $\overline{\lambda}_{\dot{\alpha}}$, and D', where the vector and spinor are the physical components and D' is an auxiliary component. We find the action to be

$$S = \int d^4xd^2\theta\, W^2 = \frac{1}{2}\int d^4xd^4\theta\; V\,\mathrm{D}^\alpha\overline{\mathrm{D}}^2\mathrm{D}_\alpha V \quad . \tag{B.0.5}$$

However, a superfield with the same structure as V can also be used to describe an entirely different supermultiplet. To distinguish this second supermultiplet from the first, we will use the symbol $\mathcal{V}$ to denote its prepotential. This second supermultiplet contains a 3-form gauge potential $A_{\underline{abc}}$ that is part of a super 3-form with different sectors given by

$$\begin{aligned} A_{\alpha\beta\gamma} &= A_{\alpha\beta\dot{\gamma}} = A_{\alpha\beta\underline{c}} = 0 \quad , \quad A_{\alpha\dot{\beta}\underline{c}} = iC_{\alpha\gamma}C_{\dot{\beta}\dot{\gamma}}\mathcal{V} \quad , \\ A_{\alpha\underline{bc}} &= -C_{\dot{\beta}\dot{\gamma}}C_{\alpha(\beta}\mathrm{D}_{\gamma)}\mathcal{V} \quad , \quad A_{\underline{abc}} = \epsilon_{\underline{dabc}}[\overline{\mathrm{D}}^{\dot{\delta}}, \mathrm{D}^\delta]\mathcal{V} \quad . \end{aligned} \tag{B.0.6}$$

This super 3-form theory has been discussed previously in the works of [100, 183] and more recently in [184] in terms of component fields.

The chiral supermultiplet that is contained in a real unconstrained scalar superfield has a gauge-invariant superfield field strength of the form

$$\overline{\Pi} = \mathrm{D}^2\mathcal{V} \quad , \quad \Pi = \overline{\mathrm{D}}^2\mathcal{V} \quad , \quad \mathcal{V} = \overline{\mathcal{V}} \quad . \tag{B.0.7}$$

where the prepotential $\mathcal{V}$ has gauge transformations

$$\mathcal{V}' \;=\; \mathcal{V} - \frac{1}{2}\big(\mathrm{D}^{\alpha}\omega_{\alpha} + \overline{\mathrm{D}}^{\dot{\alpha}}\overline{\omega}_{\dot{\alpha}}\big) \quad , \quad \mathrm{D}_{\alpha}\overline{\omega}_{\dot{\alpha}} \;=\; 0 \quad . \tag{B.0.8}$$

The physical component fields of this gauge 3-form multiplet are

$$\begin{aligned} \overline{\phi} \;&=\; \overline{\Pi}| \;=\; \mathrm{D}^2\mathcal{V}| \quad , \quad \phi \;=\; \Pi| \;=\; \overline{\mathrm{D}}^2\mathcal{V}| \quad , \\ \psi_{\alpha} \;&=\; \mathrm{D}_{\alpha}\Pi| \;=\; \mathrm{D}_{\alpha}\overline{\mathrm{D}}^2\mathcal{V}| \quad , \quad \overline{\psi}_{\dot{\alpha}} \;=\; \overline{\mathrm{D}}_{\dot{\alpha}}\overline{\Pi}| \;=\; \overline{\mathrm{D}}_{\dot{\alpha}}\mathrm{D}^2\mathcal{V}| \quad , \\ h \;&=\; \big(\mathrm{D}^2\Pi + \overline{\mathrm{D}}^2\overline{\Pi}\big)| \;=\; \{\mathrm{D}^2, \overline{\mathrm{D}}^2\}\mathcal{V}| \quad , \\ f \;&=\; -i\big(\mathrm{D}^2\Pi - \overline{\mathrm{D}}^2\overline{\Pi}\big)| \;=\; \frac{1}{2\cdot 3!}\epsilon_{\underline{abcd}}\partial^{\underline{a}}A^{\underline{bcd}}| \;=\; \frac{1}{2}\partial^{\alpha\dot{\alpha}}[\overline{\mathrm{D}}_{\dot{\alpha}}, \mathrm{D}_{\alpha}]\mathcal{V}| \quad . \end{aligned} \tag{B.0.9}$$

The quantity f is the field strength of the component gauge 3-form $A_{\underline{abc}}$, so that the field strength f is invariant. Interpreted the other way, the chiral supermultiplet is that with component content $(\overline{\phi}, \phi, \psi_{\alpha}, \overline{\psi}_{\dot{\alpha}}, h, f)$, and the supermultiplet corresponding to the bottom right diagram of Figure 3.3 is one of the Hodge-dual variants[1] of the chiral supermultiplet with component content $(\overline{\phi}, \phi, \psi_{\alpha}, \overline{\psi}_{\dot{\alpha}}, h, A_{\underline{abc}})$. This is done by taking one of the auxiliary fields f to its Hodge-dual gauge 3-form $A_{\underline{abc}}$ as shown in the last line of (B.0.9).

The action for this supermultiplet is given by

$$S \;=\; \int d^4x d^2\theta\, \overline{\Pi}\,\Pi \;=\; \frac{1}{2}\int d^4x\, d^4\theta\, d^4\overline{\theta}\; \mathcal{V}\,\{\mathrm{D}^2\overline{\mathrm{D}}^2 \;+\; \overline{\mathrm{D}}^2\mathrm{D}^2\}\mathcal{V} \quad . \tag{B.0.10}$$

and when this is evaluated at the level of component fields, this action contains the d'Alembertian for the complex component scalar field ϕ, the Dirac operator for the component spinor ψ_{α}, the square of f - the kinetic term for the component 3-form and the square of h - real spin-0 auxiliary field.

The first result in (B.0.9) shows that the propagating spin-0 states associate with ϕ occur at the quadratic θ level of $\mathcal{V}$. However, the final equation in (B.0.6) shows that the component 3-form is at the same level.

[1] There are three of them. Two of them can correspond to Figure 3.3 with one of the auxiliary fields replaced by its Hodge-dual - see equations (3.6) and (3.7) in [184]. The third one is obtained by replacing both of the auxiliary fields to their Hodge-duals, as shown in (3.8) in the reference.

Appendix C

SO(10) Irreducible Representations

Here we list the SO(10) irreducible representations by Dynkin labels and dimensions [68].

Dynkin label	Dimension
(00000)	$\mathbf{1}$
(10000)	$\mathbf{10}$
(00001)	$\mathbf{16}$
(00010)	$\overline{\mathbf{16}}$
(01000)	$\mathbf{45}$
(20000)	$\mathbf{54}$
(00100)	$\mathbf{120}$
(00020)	$\mathbf{126}$
(00002)	$\overline{\mathbf{126}}$
(10010)	$\mathbf{144}$
(10001)	$\overline{\mathbf{144}}$
(00011)	$\mathbf{210}$
(30000)	$\mathbf{210'}$
(11000)	$\mathbf{320}$
(01001)	$\mathbf{560}$
(01010)	$\overline{\mathbf{560}}$
(40000)	$\mathbf{660}$
(00030)	$\mathbf{672}$
(00003)	$\overline{\mathbf{672}}$
(20001)	$\mathbf{720}$
(20010)	$\overline{\mathbf{720}}$
(02000)	$\mathbf{770}$
(10100)	$\mathbf{945}$
(10020)	$\mathbf{1,050}$
(10002)	$\overline{\mathbf{1,050}}$
(00110)	$\mathbf{1,200}$
(00101)	$\overline{\mathbf{1,200}}$

Dynkin label	Dimension
(21000)	$1,386$
(00012)	$1,440$
(00021)	$\overline{1,440}$
(10011)	$1,728$
(50000)	$1,782$
(30010)	$2,640$
(30001)	$\overline{2,640}$
(00040)	$2,772$
(00004)	$\overline{2,772}$
(01100)	$2,970$
(11010)	$3,696$
(11001)	$\overline{3,696}$
(01020)	$3,696'$
(01002)	$\overline{3,696'}$
(00200)	$4,125$
(60000)	$4,290$
(20100)	$4,312$
(12000)	$4,410$
(31000)	$4,608$
(20020)	$4,950$
(20002)	$\overline{4,950}$
(10003)	$5,280$
(10030)	$\overline{5,280}$
(01011)	$5,940$
(00120)	$6,930$
(00102)	$\overline{6,930}$
(00031)	$6,930'$
(00013)	$\overline{6,930'}$
(03000)	$7,644$
(40001)	$7,920$
(40010)	$\overline{7,920}$
(02001)	$8,064$
(02010)	$\overline{8,064}$
(20011)	$8,085$
(10101)	$8,800$
(10110)	$\overline{8,800}$
(00022)	$8,910$
(70000)	$9,438$
(00005)	$9,504$
(00050)	$\overline{9,504}$
(00111)	$10,560$

Table C.1: SO(10) irreducible representations

Appendix D

More About Two-Forms

Let the projection operators be

$$[\mathcal{P}^{(\pm)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \;=\; \tfrac{1}{4}\left[\delta_{[\underline{a}}{}^{\underline{c}}\,\delta_{\underline{b}]}{}^{\underline{d}} \pm i\,\epsilon_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}}\right] \quad . \tag{D.0.1}$$

Note that it satisfies the following properties of projection operators,

$$[\mathcal{P}^{(+)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} + [\mathcal{P}^{(-)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \;=\; \tfrac{1}{2}\,\delta_{[\underline{a}}{}^{\underline{c}}\,\delta_{\underline{b}]}{}^{\underline{d}} \quad , \tag{D.0.2}$$

$$[\mathcal{P}^{(\pm)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}}\,[\mathcal{P}^{(\pm)}]_{\underline{c}\,\underline{d}}{}^{\underline{e}\,\underline{f}} \;=\; [\mathcal{P}^{(\pm)}]_{\underline{a}\,\underline{b}}{}^{\underline{e}\,\underline{f}} \quad , \tag{D.0.3}$$

$$[\mathcal{P}^{(\pm)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}}\,[\mathcal{P}^{(\mp)}]_{\underline{c}\,\underline{d}}{}^{\underline{e}\,\underline{f}} \;=\; 0 \quad . \tag{D.0.4}$$

Recall Equation (6.4.30), that a generic 2-form in 4D can be decomposed as follows,

$$A_{\underline{a}\,\underline{b}} \;=\; A^{(+)}_{\underline{a}\,\underline{b}} \;+\; A^{(-)}_{\underline{a}\,\underline{b}} \quad . \tag{D.0.5}$$

In terms of projection operators defined in (D.0.1), we have

$$A^{(\pm)}_{\underline{a}\,\underline{b}} \;=\; [\mathcal{P}^{(\pm)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}}\,A_{\underline{c}\,\underline{d}} \quad . \tag{D.0.6}$$

From Equations (D.0.3) and (D.0.4), we know that contractions among self-dual 2-forms or contractions among anti-self-dual 2-forms would retain, while the contraction between a self-dual 2-form and an anti-self-dual 2-form would vanish.

Moreover, recall the identity

$$\gamma^5 \, \gamma^{\underline{a}\underline{b}} \;=\; -i \, \tfrac{1}{2} \, \epsilon^{\underline{a}\underline{b}\underline{c}\underline{d}} \, \gamma_{\underline{c}\underline{d}} \tag{D.0.7}$$

and the definition of the spinor projection operators

$$(\mathrm{P}^{(\pm)})_{\alpha}{}^{\beta} \;=\; \tfrac{1}{2} \left[\, \delta_{\alpha}{}^{\beta} \;\pm\; (\gamma^5)_{\alpha}{}^{\beta} \, \right] \quad . \tag{D.0.8}$$

The direct calculation tells us

$$\Big(\mathrm{P}^{(\pm)} \, \gamma^{\underline{a}\underline{b}} \Big)_{\alpha}{}^{\beta} \;=\; [\mathcal{P}^{(\mp)}]_{\underline{a}\,\underline{b}}{}^{\underline{c}\,\underline{d}} \, (\gamma_{\underline{c}\underline{d}})_{\alpha}{}^{\beta} \tag{D.0.9}$$

Appendix E

10D Weyl Scaling Properties of Weight $\geq \frac{3}{2}$ Super Tensors

In this appendix, we record the Weyl scaling transformation laws for the super torsion and super curvature tensors of scale weight equal or greater than three-halves in the respective cases of the $\mathcal{N} = 2$A, $\mathcal{N} = 2$B and $\mathcal{N} = 1$ theories.

E.1 10D, $\mathcal{N} = 2$A

$$\begin{aligned}
\delta_S T_{\underline{a}\underline{b}}{}^{\gamma} =\ & \tfrac{3}{2} L T_{\underline{a}\underline{b}}{}^{\gamma} + i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\nabla_{\alpha} L) - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha} L) T_{\beta|\underline{b}]}{}^{\gamma} \\
& - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\beta}}(\nabla_{\dot{\alpha}} L) T_{\dot{\beta}|\underline{b}]}{}^{\gamma} + i\tfrac{1}{5}(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_{\delta} L) T_{\underline{a}\underline{b}}{}^{\underline{c}} \quad ,
\end{aligned} \tag{E.1.1}$$

$$\begin{aligned}
\delta_S T_{\underline{a}\underline{b}}{}^{\dot{\gamma}} =\ & \tfrac{3}{2} L T_{\underline{a}\underline{b}}{}^{\dot{\gamma}} + i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\gamma}}(\nabla_{\underline{b}]}\nabla_{\dot{\alpha}} L) - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha} L) T_{\beta|\underline{b}]}{}^{\dot{\gamma}} \\
& - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\beta}}(\nabla_{\dot{\alpha}} L) T_{\dot{\beta}|\underline{b}]}{}^{\dot{\gamma}} + i\tfrac{1}{5}(\sigma_{\underline{c}})^{\dot{\gamma}\dot{\delta}}(\nabla_{\dot{\delta}} L) T_{\underline{a}\underline{b}}{}^{\underline{c}} \quad ,
\end{aligned} \tag{E.1.2}$$

$$\begin{aligned}
\delta_S R_{\alpha\underline{b}}{}^{\underline{d}\underline{e}} =\ & \tfrac{3}{2} L R_{\alpha\underline{b}}{}^{\underline{d}\underline{e}} - T_{\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \tfrac{1}{5} T_{\alpha\underline{b}}{}^{\gamma}(\sigma^{\underline{d}\underline{e}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \tfrac{1}{5} T_{\alpha\underline{b}}{}^{\dot{\gamma}}(\sigma^{\underline{d}\underline{e}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L) \\
& + \tfrac{1}{5}(\sigma^{\underline{d}\underline{e}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\nabla_{\beta} L) + i\tfrac{1}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta} L) R_{\alpha\beta}{}^{\underline{d}\underline{e}} + i\tfrac{1}{5}(\sigma_{\underline{b}})^{\dot{\delta}\dot{\beta}}(\nabla_{\dot{\delta}} L) R_{\alpha\dot{\beta}}{}^{\underline{d}\underline{e}} \\
& - (\nabla_{\alpha}\nabla^{[\underline{d}} L)\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned} \tag{E.1.3}$$

$$\delta_S R_{\dot{\alpha}\underline{b}}{}^{\underline{d}\underline{e}} = \tfrac{3}{2} L R_{\dot{\alpha}\underline{b}}{}^{\underline{d}\underline{e}} - T_{\dot{\alpha}\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]} L) + \tfrac{1}{5} T_{\dot{\alpha}\underline{b}}{}^{\gamma}(\sigma^{\underline{d}\underline{e}})_{\gamma}{}^{\delta}(\nabla_{\delta} L) + \tfrac{1}{5} T_{\dot{\alpha}\underline{b}}{}^{\dot{\gamma}}(\sigma^{\underline{d}\underline{e}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}} L)$$

$$
\begin{aligned}
&+ \tfrac{1}{5}(\sigma^{\underline{de}})_{\dot{\alpha}}{}^{\dot{\beta}}(\nabla_{\underline{b}}\nabla_{\dot{\beta}}L) + i\tfrac{1}{5}(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta}L)R_{\dot{\alpha}\beta}{}^{\underline{de}} + i\tfrac{1}{5}(\sigma_{\underline{b}})^{\dot{\delta}\dot{\beta}}(\nabla_{\dot{\delta}}L)R_{\dot{\alpha}\dot{\beta}}{}^{\underline{de}} \\
&- (\nabla_{\dot{\alpha}}\nabla^{[\underline{d}}L)\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned} \tag{E.1.4}
$$

$$
\begin{aligned}
\delta_S R_{\underline{ab}}{}^{\underline{de}} =\ & 2LR_{\underline{ab}}{}^{\underline{de}} - T_{\underline{ab}}{}^{[\underline{d}}(\nabla^{\underline{e}]}L) + \tfrac{1}{5}T_{\underline{ab}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta}L) + \tfrac{1}{5}T_{\underline{ab}}{}^{\dot{\gamma}}(\sigma^{\underline{de}})_{\dot{\gamma}}{}^{\dot{\delta}}(\nabla_{\dot{\delta}}L) \\
& - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}L)R_{\beta|\underline{b}]}{}^{\underline{de}} - i\tfrac{1}{5}(\sigma_{[\underline{a}})^{\dot{\alpha}\dot{\beta}}(\nabla_{\dot{\alpha}}L)R_{\dot{\beta}|\underline{b}]}{}^{\underline{de}} - (\nabla_{[\underline{a}}\nabla^{[\underline{d}}L)\delta_{\underline{b}]}{}^{\underline{e}]} \quad .
\end{aligned} \tag{E.1.5}
$$

E.2 10D, $\mathcal{N} = 2$B

$$
\begin{aligned}
\delta_S T_{\underline{ab}}{}^{\gamma} =\ & (\tfrac{1}{2}L + \overline{L})T_{\underline{ab}}{}^{\gamma} + i(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\overline{\nabla}_{\alpha}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L})) - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_{\alpha}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\beta|\underline{b}]}{}^{\gamma} \\
& - i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\beta}|\underline{b}]}{}^{\gamma} + i(\sigma_{\underline{c}})^{\gamma\delta}(\overline{\nabla}_{\delta}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\underline{ab}}{}^{\underline{c}} \quad ,
\end{aligned} \tag{E.2.1}
$$

$$
\begin{aligned}
\delta_S T_{\underline{ab}}{}^{\overline{\gamma}} =\ & (\tfrac{1}{2}\overline{L} + L)T_{\underline{ab}}{}^{\overline{\gamma}} + i(\sigma_{[\underline{a}})^{\alpha\gamma}(\nabla_{\underline{b}]}\nabla_{\alpha}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L)) - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_{\alpha}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))T_{\beta|\underline{b}]}{}^{\overline{\gamma}} \\
& - i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\overline{\beta}|\underline{b}]}{}^{\overline{\gamma}} + i(\sigma_{\underline{c}})^{\gamma\delta}(\nabla_{\delta}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))T_{\underline{ab}}{}^{\underline{c}} \quad .
\end{aligned} \tag{E.2.2}
$$

$$
\begin{aligned}
\delta_S R_{\alpha\underline{b}}{}^{\underline{de}} =\ & (\tfrac{1}{2}\overline{L} + L)R_{\alpha\underline{b}}{}^{\underline{de}} - \tfrac{1}{2}T_{\alpha\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\alpha\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta}L) + \tfrac{1}{5}T_{\alpha\underline{b}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\overline{\nabla}_{\delta}\overline{L}) \\
& + \tfrac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\nabla_{\beta}L) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_{\delta}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))R_{\alpha\beta}{}^{\underline{de}} \\
& + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))R_{\alpha\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}(\nabla_{\alpha}\nabla^{[\underline{d}}(L + \overline{L}))\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned} \tag{E.2.3}
$$

$$
\begin{aligned}
\delta_S R_{\overline{\alpha}\underline{b}}{}^{\underline{de}} =\ & (\tfrac{1}{2}L + \overline{L})R_{\overline{\alpha}\underline{b}}{}^{\underline{de}} - \tfrac{1}{2}T_{\overline{\alpha}\underline{b}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\overline{\alpha}\underline{b}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta}L) + \tfrac{1}{5}T_{\overline{\alpha}\underline{b}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\overline{\nabla}_{\delta}\overline{L}) \\
& + \tfrac{1}{5}(\sigma^{\underline{de}})_{\alpha}{}^{\beta}(\nabla_{\underline{b}}\overline{\nabla}_{\beta}\overline{L}) + i(\sigma_{\underline{b}})^{\delta\beta}(\overline{\nabla}_{\delta}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))R_{\overline{\alpha}\beta}{}^{\underline{de}} \\
& + i(\sigma_{\underline{b}})^{\delta\beta}(\nabla_{\delta}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))R_{\overline{\alpha}\overline{\beta}}{}^{\underline{de}} - \tfrac{1}{2}(\overline{\nabla}_{\alpha}\nabla^{[\underline{d}}(L + \overline{L}))\delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned} \tag{E.2.4}
$$

$$
\begin{aligned}
\delta_S R_{\underline{ab}}{}^{\underline{de}} =\ & (L + \overline{L})R_{\underline{ab}}{}^{\underline{de}} - \tfrac{1}{2}T_{\underline{ab}}{}^{[\underline{d}}(\nabla^{\underline{e}]}(L + \overline{L})) + \tfrac{1}{5}T_{\underline{ab}}{}^{\gamma}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\nabla_{\delta}L) + \tfrac{1}{5}T_{\underline{ab}}{}^{\overline{\gamma}}(\sigma^{\underline{de}})_{\gamma}{}^{\delta}(\overline{\nabla}_{\delta}\overline{L}) \\
& - i(\sigma_{[\underline{a}})^{\alpha\beta}(\overline{\nabla}_{\alpha}(\tfrac{1}{32}L + \tfrac{27}{160}\overline{L}))R_{\beta|\underline{b}]}{}^{\underline{de}} - i(\sigma_{[\underline{a}})^{\alpha\beta}(\nabla_{\alpha}(\tfrac{1}{32}\overline{L} + \tfrac{27}{160}L))R_{\overline{\beta}|\underline{b}]}{}^{\underline{de}}
\end{aligned}
$$

$$- \tfrac{1}{2}(\nabla_{[\underline{a}}\nabla^{[\underline{d}}(L+\overline{L}))\delta_{\underline{b}]}{}^{\underline{e}]} \quad . \tag{E.2.5}$$

E.3 10D, $\mathcal{N} = 1$

$$\begin{aligned}
\delta_S T_{\underline{ab}}{}^{\gamma} \;=&\; \tfrac{3}{2} L\, T_{\underline{ab}}{}^{\gamma} \;+\; i\, \tfrac{2}{5} (\sigma_{[\underline{a}})^{\alpha\gamma} (\nabla_{\underline{b}]} \nabla_{\alpha} L) \;-\; i\, \tfrac{2}{5}\, (\sigma_{[\underline{a}})^{\alpha\beta} (\nabla_{\alpha} L)\, T_{\beta|\underline{b}]}{}^{\gamma} \\
&+\; i\, \tfrac{2}{5}\, (\sigma_{\underline{c}})^{\gamma\delta} (\nabla_{\delta} L)\, T_{\underline{ab}}{}^{\underline{c}} \quad ,
\end{aligned} \tag{E.3.1}$$

$$\begin{aligned}
\delta_S R_{\alpha\underline{b}}{}^{\underline{de}} \;=&\; \tfrac{3}{2} L\, R_{\alpha\underline{b}}{}^{\underline{de}} \;-\; T_{\alpha\underline{b}}{}^{[\underline{d}} (\nabla^{\underline{e}]} L) \;+\; \tfrac{1}{5}\, T_{\alpha\underline{b}}{}^{\gamma}\, (\sigma^{\underline{de}})_{\gamma}{}^{\delta}\, (\nabla_{\delta} L) \\
&+\; \tfrac{1}{5}\, (\sigma^{\underline{de}})_{\alpha}{}^{\beta} (\nabla_{\underline{b}} \nabla_{\beta} L) \;+\; i\, \tfrac{2}{5} (\sigma_{\underline{b}})^{\delta\beta} (\nabla_{\delta} L) R_{\alpha\beta}{}^{\underline{de}} \;-\; (\nabla_{\alpha} \nabla^{[\underline{d}} L) \delta_{\underline{b}}{}^{\underline{e}]} \quad ,
\end{aligned} \tag{E.3.2}$$

$$\begin{aligned}
\delta_S R_{\underline{ab}}{}^{\underline{de}} \;=&\; 2L\, R_{\underline{ab}}{}^{\underline{de}} \;-\; T_{\underline{ab}}{}^{[\underline{d}} (\nabla^{\underline{e}]} L) \;+\; \tfrac{1}{5} T_{\underline{ab}}{}^{\gamma} (\sigma^{\underline{de}})_{\gamma}{}^{\delta} (\nabla_{\delta} L) \\
&-\; i\, \tfrac{2}{5} (\sigma_{[\underline{a}})^{\alpha\beta} (\nabla_{\alpha} L) R_{\beta|\underline{b}]}{}^{\underline{de}} \;-\; (\nabla_{[\underline{a}} \nabla^{[\underline{d}} L) \delta_{\underline{b}]}{}^{\underline{e}]} \quad .
\end{aligned} \tag{E.3.3}$$

Appendix F

Adynkra Libraries in 10D

In this appendix, we explicitly present the adynkra libraries of all superfields that appear in equation (8.0.4) in the text.

F.1 Dynkin Label Library of $\mathcal{V}_{[0,0,1,0,0]}$

- Level-0: $[0,0,1,0,0]$
- Level-1: $[0,0,0,0,1] \oplus [1,0,0,1,0] \oplus [0,1,0,0,1] \oplus [0,0,1,1,0]$
- Level-2: $[0,0,0,0,0] \oplus [0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (2)[0,0,0,1,1] \oplus [0,2,0,0,0] \oplus [1,0,1,0,0] \oplus [1,0,0,2,0] \oplus [1,0,0,0,2] \oplus [0,0,2,0,0] \oplus [0,1,0,1,1]$
- Level-3: $[0,0,0,1,0] \oplus (2)[1,0,0,0,1] \oplus (2)[0,1,0,1,0] \oplus [0,0,0,0,3] \oplus [2,0,0,1,0] \oplus (2)[0,0,1,0,1] \oplus [0,0,0,2,1] \oplus (2)[1,1,0,0,1] \oplus [0,2,0,1,0] \oplus [1,0,1,1,0] \oplus [1,0,0,1,2] \oplus [0,1,1,0,1]$
- Level-4: $(2)[0,0,1,0,0] \oplus [0,0,0,0,2] \oplus (2)[1,1,0,0,0] \oplus (3)[1,0,0,1,1] \oplus (2)[0,1,1,0,0] \oplus [0,1,0,2,0] \oplus (3)[0,1,0,0,2] \oplus (2)[2,0,1,0,0] \oplus [1,2,0,0,0] \oplus [2,0,0,0,2] \oplus [0,0,0,1,3] \oplus [0,0,1,1,1] \oplus [0,2,1,0,0] \oplus (2)[1,1,0,1,1] \oplus [1,0,1,0,2]$
- Level-5: $[1,0,0,1,0] \oplus (3)[0,1,0,0,1] \oplus (2)[2,0,0,0,1] \oplus [0,0,1,1,0] \oplus (2)[0,0,0,1,2] \oplus [3,0,0,1,0] \oplus (3)[1,1,0,1,0] \oplus (2)[1,0,0,0,3] \oplus (2)[0,2,0,0,1] \oplus (4)[1,0,1,0,1] \oplus [1,0,0,2,1] \oplus (2)[2,1,0,0,1] \oplus [0,1,1,1,0] \oplus [0,0,1,0,3] \oplus (2)[0,1,0,1,2] \oplus [2,0,1,1,0] \oplus [1,2,0,1,0] \oplus [2,0,0,1,2] \oplus [1,1,1,0,1]$

- Level-6: $[2,0,0,0,0] \oplus [0,0,0,1,1] \oplus [4,0,0,0,0] \oplus (2)[0,2,0,0,0] \oplus (2)[1,0,1,0,0] \oplus [1,0,0,2,0] \oplus (3)[1,0,0,0,2] \oplus (2)[2,1,0,0,0] \oplus [0,0,0,0,4] \oplus (2)[0,0,2,0,0] \oplus (3)[0,1,0,1,1] \oplus (2)[0,0,1,0,2] \oplus (4)[2,0,0,1,1] \oplus [0,0,0,2,2] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus (3)[1,1,1,0,0] \oplus [1,1,0,2,0] \oplus (3)[1,1,0,0,2] \oplus [1,0,0,1,3] \oplus (2)[1,0,1,1,1] \oplus [0,2,0,1,1] \oplus [2,0,2,0,0] \oplus [0,1,1,0,2] \oplus [2,1,0,1,1]$

- Level-7: $[1,0,0,0,1] \oplus [0,1,0,1,0] \oplus [0,0,0,0,3] \oplus (2)[2,0,0,1,0] \oplus (2)[0,0,1,0,1] \oplus [0,0,0,2,1] \oplus (3)[3,0,0,0,1] \oplus (4)[1,1,0,0,1] \oplus [4,0,0,1,0] \oplus [0,2,0,1,0] \oplus (4)[1,0,1,1,0] \oplus (3)[1,0,0,1,2] \oplus (3)[2,1,0,1,0] \oplus [0,1,0,0,3] \oplus [2,0,0,0,3] \oplus (3)[0,1,1,0,1] \oplus [0,0,2,1,0] \oplus [0,1,0,2,1] \oplus (3)[2,0,1,0,1] \oplus [1,2,0,0,1] \oplus [3,1,0,0,1] \oplus (2)[2,0,0,2,1] \oplus [0,0,1,1,2] \oplus [3,0,1,1,0] \oplus [1,1,1,1,0] \oplus [1,0,2,0,1] \oplus [1,1,0,1,2]$

- Level-8: $[0,0,1,0,0] \oplus [3,0,0,0,0] \oplus [1,1,0,0,0] \oplus (3)[1,0,0,1,1] \oplus (2)[0,1,1,0,0] \oplus (2)[0,1,0,2,0] \oplus (2)[0,1,0,0,2] \oplus (5)[2,0,1,0,0] \oplus [1,2,0,0,0] \oplus (2)[3,1,0,0,0] \oplus (2)[2,0,0,2,0] \oplus (2)[2,0,0,0,2] \oplus (3)[0,0,1,1,1] \oplus (2)[1,0,2,0,0] \oplus (3)[3,0,0,1,1] \oplus [0,2,1,0,0] \oplus (4)[1,1,0,1,1] \oplus [4,0,1,0,0] \oplus (2)[1,0,1,2,0] \oplus (2)[1,0,1,0,2] \oplus [1,0,0,2,2] \oplus [2,1,1,0,0] \oplus [0,0,3,0,0] \oplus [2,1,0,2,0] \oplus [2,1,0,0,2] \oplus [0,1,1,1,1] \oplus [2,0,1,1,1]$

- Level-9: $[1,0,0,1,0] \oplus [0,1,0,0,1] \oplus [0,0,0,3,0] \oplus (2)[2,0,0,0,1] \oplus (2)[0,0,1,1,0] \oplus [0,0,0,1,2] \oplus (3)[3,0,0,1,0] \oplus (4)[1,1,0,1,0] \oplus [4,0,0,0,1] \oplus [0,2,0,0,1] \oplus (4)[1,0,1,0,1] \oplus (3)[1,0,0,2,1] \oplus (3)[2,1,0,0,1] \oplus [0,1,0,3,0] \oplus [2,0,0,3,0] \oplus (3)[0,1,1,1,0] \oplus [0,0,2,0,1] \oplus [0,1,0,1,2] \oplus (3)[2,0,1,1,0] \oplus [1,2,0,1,0] \oplus [3,1,0,1,0] \oplus (2)[2,0,0,1,2] \oplus [0,0,1,2,1] \oplus [3,0,1,0,1] \oplus [1,1,1,0,1] \oplus [1,0,2,1,0] \oplus [1,1,0,2,1]$

- Level-10: $[2,0,0,0,0] \oplus [0,0,0,1,1] \oplus [4,0,0,0,0] \oplus (2)[0,2,0,0,0] \oplus (2)[1,0,1,0,0] \oplus (3)[1,0,0,2,0] \oplus [1,0,0,0,2] \oplus (2)[2,1,0,0,0] \oplus [0,0,0,4,0] \oplus (2)[0,0,2,0,0] \oplus (3)[0,1,0,1,1] \oplus (2)[0,0,1,2,0] \oplus (4)[2,0,0,1,1] \oplus [0,0,0,2,2] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus (3)[1,1,1,0,0] \oplus (3)[1,1,0,2,0] \oplus$

[1, 1, 0, 0, 2] ⊕ [1, 0, 0, 3, 1] ⊕ (2)[1, 0, 1, 1, 1] ⊕ [0, 2, 0, 1, 1] ⊕ [2, 0, 2, 0, 0] ⊕ [0, 1, 1, 2, 0] ⊕ [2, 1, 0, 1, 1]

- Level-11: [1, 0, 0, 0, 1] ⊕ (3)[0, 1, 0, 1, 0] ⊕ (2)[2, 0, 0, 1, 0] ⊕ [0, 0, 1, 0, 1] ⊕ (2)[0, 0, 0, 2, 1] ⊕ [3, 0, 0, 0, 1] ⊕ (3)[1, 1, 0, 0, 1] ⊕ (2)[1, 0, 0, 3, 0] ⊕ (2)[0, 2, 0, 1, 0] ⊕ (4)[1, 0, 1, 1, 0] ⊕ [1, 0, 0, 1, 2] ⊕ (2)[2, 1, 0, 1, 0] ⊕ [0, 1, 1, 0, 1] ⊕ [0, 0, 1, 3, 0] ⊕ (2)[0, 1, 0, 2, 1] ⊕ [2, 0, 1, 0, 1] ⊕ [1, 2, 0, 0, 1] ⊕ [2, 0, 0, 2, 1] ⊕ [1, 1, 1, 1, 0]

- Level-12: (2)[0, 0, 1, 0, 0] ⊕ [0, 0, 0, 2, 0] ⊕ (2)[1, 1, 0, 0, 0] ⊕ (3)[1, 0, 0, 1, 1] ⊕ (2)[0, 1, 1, 0, 0] ⊕ (3)[0, 1, 0, 2, 0] ⊕ [0, 1, 0, 0, 2] ⊕ (2)[2, 0, 1, 0, 0] ⊕ [1, 2, 0, 0, 0] ⊕ [2, 0, 0, 2, 0] ⊕ [0, 0, 0, 3, 1] ⊕ [0, 0, 1, 1, 1] ⊕ [0, 2, 1, 0, 0] ⊕ (2)[1, 1, 0, 1, 1] ⊕ [1, 0, 1, 2, 0]

- Level-13: [0, 0, 0, 0, 1] ⊕ (2)[1, 0, 0, 1, 0] ⊕ (2)[0, 1, 0, 0, 1] ⊕ [0, 0, 0, 3, 0] ⊕ [2, 0, 0, 0, 1] ⊕ (2)[0, 0, 1, 1, 0] ⊕ [0, 0, 0, 1, 2] ⊕ (2)[1, 1, 0, 1, 0] ⊕ [0, 2, 0, 0, 1] ⊕ [1, 0, 1, 0, 1] ⊕ [1, 0, 0, 2, 1] ⊕ [0, 1, 1, 1, 0]

- Level-14: [0, 0, 0, 0, 0] ⊕ [0, 1, 0, 0, 0] ⊕ [2, 0, 0, 0, 0] ⊕ (2)[0, 0, 0, 1, 1] ⊕ [0, 2, 0, 0, 0] ⊕ [1, 0, 1, 0, 0] ⊕ [1, 0, 0, 2, 0] ⊕ [1, 0, 0, 0, 2] ⊕ [0, 0, 2, 0, 0] ⊕ [0, 1, 0, 1, 1]

- Level-15: [0, 0, 0, 1, 0] ⊕ [1, 0, 0, 0, 1] ⊕ [0, 1, 0, 1, 0] ⊕ [0, 0, 1, 0, 1]

- Level-16: [0, 0, 1, 0, 0]

F.2 Dynkin Label Library of $\mathcal{V}_{[1,0,1,0,1]}$

- Level-0: [1, 0, 1, 0, 1]

- Level-1: [1, 0, 1, 0, 0] ⊕ [1, 0, 0, 0, 2] ⊕ [0, 0, 2, 0, 0] ⊕ [0, 1, 0, 1, 1] ⊕ [0, 0, 1, 0, 2] ⊕ [2, 0, 0, 1, 1] ⊕ [1, 1, 1, 0, 0] ⊕ [1, 1, 0, 0, 2] ⊕ [1, 0, 1, 1, 1]

- Level-2: [1, 0, 0, 0, 1] ⊕ [0, 1, 0, 1, 0] ⊕ [0, 0, 0, 0, 3] ⊕ [2, 0, 0, 1, 0] ⊕ (2)[0, 0, 1, 0, 1] ⊕ [0, 0, 0, 2, 1] ⊕ [3, 0, 0, 0, 1] ⊕ (3)[1, 1, 0, 0, 1] ⊕ [0, 2, 0, 1, 0] ⊕ (3)[1, 0, 1, 1, 0] ⊕ (3)[1, 0, 0, 1, 2] ⊕ [2, 1, 0, 1, 0] ⊕ [0, 1, 0, 0, 3] ⊕ [2, 0, 0, 0, 3] ⊕ (3)[0, 1, 1, 0, 1] ⊕ [0, 0, 2, 1, 0] ⊕ [0, 1, 0, 2, 1] ⊕ (2)[2, 0, 1, 0, 1] ⊕ [1, 2, 0, 0, 1] ⊕ [2, 0, 0, 2, 1] ⊕ [0, 0, 1, 1, 2] ⊕ [1, 1, 1, 1, 0] ⊕ [1, 0, 2, 0, 1] ⊕ [1, 1, 0, 1, 2]

- Level-3: $[0,0,1,0,0] \oplus [0,0,0,2,0] \oplus [0,0,0,0,2] \oplus [3,0,0,0,0] \oplus (2)[1,1,0,0,0] \oplus (5)[1,0,0,1,1] \oplus (4)[0,1,1,0,0] \oplus (3)[0,1,0,2,0] \oplus (4)[0,1,0,0,2] \oplus (4)[2,0,1,0,0] \oplus (2)[1,2,0,0,0] \oplus [3,1,0,0,0] \oplus (2)[2,0,0,2,0] \oplus (4)[2,0,0,0,2] \oplus [0,0,0,3,1] \oplus (2)[0,0,0,1,3] \oplus (5)[0,0,1,1,1] \oplus [1,0,0,0,4] \oplus (4)[1,0,2,0,0] \oplus (2)[3,0,0,1,1] \oplus (2)[0,2,1,0,0] \oplus (7)[1,1,0,1,1] \oplus [0,2,0,2,0] \oplus (2)[0,2,0,0,2] \oplus (2)[1,0,1,2,0] \oplus (5)[1,0,1,0,2] \oplus (2)[1,0,0,2,2] \oplus (2)[2,1,1,0,0] \oplus [0,0,3,0,0] \oplus [2,1,0,2,0] \oplus (2)[2,1,0,0,2] \oplus [0,0,2,0,2] \oplus [0,1,0,1,3] \oplus (3)[0,1,1,1,1] \oplus [2,0,0,1,3] \oplus (2)[2,0,1,1,1] \oplus [1,1,2,0,0] \oplus [1,2,0,1,1] \oplus [1,1,1,0,2]$

- Level-4: $(2)[1,0,0,1,0] \oplus (4)[0,1,0,0,1] \oplus (2)[0,0,0,3,0] \oplus (4)[2,0,0,0,1] \oplus (5)[0,0,1,1,0] \oplus (4)[0,0,0,1,2] \oplus (3)[3,0,0,1,0] \oplus (8)[1,1,0,1,0] \oplus (4)[1,0,0,0,3] \oplus [4,0,0,0,1] \oplus (5)[0,2,0,0,1] \oplus (11)[1,0,1,0,1] \oplus (7)[1,0,0,2,1] \oplus (7)[2,1,0,0,1] \oplus (2)[0,1,0,3,0] \oplus [2,0,0,3,0] \oplus (8)[0,1,1,1,0] \oplus (3)[0,0,1,0,3] \oplus (5)[0,0,2,0,1] \oplus (8)[0,1,0,1,2] \oplus (7)[2,0,1,1,0] \oplus [0,0,0,2,3] \oplus (4)[1,2,0,1,0] \oplus (2)[3,1,0,1,0] \oplus (7)[2,0,0,1,2] \oplus (3)[0,0,1,2,1] \oplus [0,3,0,0,1] \oplus [3,0,0,0,3] \oplus (4)[1,1,0,0,3] \oplus (3)[3,0,1,0,1] \oplus (9)[1,1,1,0,1] \oplus [2,2,0,0,1] \oplus [3,0,0,2,1] \oplus [1,0,0,1,4] \oplus (3)[1,0,2,1,0] \oplus (4)[1,1,0,2,1] \oplus (2)[0,2,1,1,0] \oplus (4)[1,0,1,1,2] \oplus (2)[0,2,0,1,2] \oplus (2)[0,1,2,0,1] \oplus [0,1,1,0,3] \oplus (2)[2,1,1,1,0] \oplus [2,0,2,0,1] \oplus (2)[2,1,0,1,2] \oplus [2,0,1,0,3] \oplus [1,2,1,0,1]$

- Level-5: $[0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (3)[0,0,0,1,1] \oplus [4,0,0,0,0] \oplus (4)[0,2,0,0,0] \oplus (7)[1,0,1,0,0] \oplus (6)[1,0,0,2,0] \oplus (5)[1,0,0,0,2] \oplus (5)[2,1,0,0,0] \oplus [0,0,0,4,0] \oplus [0,0,0,0,4] \oplus (6)[0,0,2,0,0] \oplus (13)[0,1,0,1,1] \oplus (6)[0,0,1,2,0] \oplus (7)[0,0,1,0,2] \oplus (2)[0,3,0,0,0] \oplus (12)[2,0,0,1,1] \oplus (5)[0,0,0,2,2] \oplus [4,1,0,0,0] \oplus (6)[3,0,1,0,0] \oplus (3)[2,2,0,0,0] \oplus (3)[3,0,0,2,0] \oplus (5)[3,0,0,0,2] \oplus (13)[1,1,1,0,0] \oplus (9)[1,1,0,2,0] \oplus (12)[1,1,0,0,2] \oplus (3)[1,0,0,3,1] \oplus (7)[1,0,0,1,3] \oplus (2)[0,1,0,0,4] \oplus (6)[0,1,2,0,0] \oplus (16)[1,0,1,1,1] \oplus (9)[0,2,0,1,1] \oplus (2)[4,0,0,1,1] \oplus (2)[2,0,0,0,4] \oplus (6)[2,0,2,0,0] \oplus (4)[0,1,1,2,0] \oplus (8)[0,1,1,0,2] \oplus (11)[2,1,0,1,1] \oplus (4)[1,2,1,0,0] \oplus (4)[0,1,0,2,2] \oplus (3)[2,0,1,2,0] \oplus (8)[2,0,1,0,2] \oplus (2)[3,1,1,0,0] \oplus (3)[0,0,2,1,1] \oplus (2)[0,0,1,1,3] \oplus (2)[1,2,0,2,0] \oplus (4)[1,2,0,0,2] \oplus (3)[2,0,0,2,2] \oplus [3,1,0,2,0] \oplus (2)[3,1,0,0,2] \oplus [1,0,3,0,0] \oplus [0,3,0,1,1] \oplus [0,2,2,0,0] \oplus [1,0,1,0,4] \oplus [3,0,0,1,3] \oplus (3)[1,1,0,1,3] \oplus$

$(2)[1,0,2,0,2] \oplus (2)[3,0,1,1,1] \oplus (6)[1,1,1,1,1] \oplus [0,2,1,0,2] \oplus [2,2,0,1,1] \oplus [2,1,2,0,0] \oplus [2,1,1,0,2]$

- Level-6: $[0,0,0,1,0] \oplus (3)[1,0,0,0,1] \oplus (7)[0,1,0,1,0] \oplus (2)[0,0,0,0,3] \oplus (6)[2,0,0,1,0] \oplus (7)[0,0,1,0,1] \oplus (7)[0,0,0,2,1] \oplus (6)[3,0,0,0,1] \oplus (13)[1,1,0,0,1] \oplus (5)[1,0,0,3,0] \oplus (3)[4,0,0,1,0] \oplus (10)[0,2,0,1,0] \oplus (17)[1,0,1,1,0] \oplus (14)[1,0,0,1,2] \oplus (12)[2,1,0,1,0] \oplus (6)[0,1,0,0,3] \oplus [5,0,0,0,1] \oplus (7)[2,0,0,0,3] \oplus (16)[0,1,1,0,1] \oplus (2)[0,0,0,1,4] \oplus (2)[0,0,1,3,0] \oplus (7)[0,0,2,1,0] \oplus (13)[0,1,0,2,1] \oplus (17)[2,0,1,0,1] \oplus (2)[0,0,0,3,2] \oplus (11)[1,2,0,0,1] \oplus (7)[3,1,0,0,1] \oplus (11)[2,0,0,2,1] \oplus (9)[0,0,1,1,2] \oplus [1,0,0,0,5] \oplus (3)[0,3,0,1,0] \oplus [3,0,0,3,0] \oplus (3)[1,1,0,3,0] \oplus (7)[3,0,1,1,0] \oplus [4,1,0,1,0] \oplus (16)[1,1,1,1,0] \oplus (4)[2,2,0,1,0] \oplus (7)[3,0,0,1,2] \oplus (10)[1,0,2,0,1] \oplus (6)[1,0,1,0,3] \oplus (3)[0,2,0,0,3] \oplus (16)[1,1,0,1,2] \oplus [4,0,0,0,3] \oplus (6)[0,2,1,0,1] \oplus (3)[1,0,0,2,3] \oplus [1,3,0,0,1] \oplus (2)[4,0,1,0,1] \oplus (6)[1,0,1,2,1] \oplus (4)[0,2,0,2,1] \oplus (4)[2,1,0,0,3] \oplus [0,0,3,0,1] \oplus [3,2,0,0,1] \oplus [4,0,0,2,1] \oplus (3)[0,1,2,1,0] \oplus [0,1,0,1,4] \oplus [0,0,2,0,3] \oplus (9)[2,1,1,0,1] \oplus (3)[2,0,2,1,0] \oplus [2,0,0,1,4] \oplus (4)[2,1,0,2,1] \oplus (4)[0,1,1,1,2] \oplus (2)[1,2,1,1,0] \oplus (4)[2,0,1,1,2] \oplus [3,1,1,1,0] \oplus (2)[1,2,0,1,2] \oplus [3,1,0,1,2] \oplus [3,0,2,0,1] \oplus (2)[1,1,2,0,1] \oplus [1,1,1,0,3]$

- Level-7: $[1,0,0,0,0] \oplus (4)[0,0,1,0,0] \oplus (3)[0,0,0,2,0] \oplus (2)[0,0,0,0,2] \oplus (2)[3,0,0,0,0] \oplus (5)[1,1,0,0,0] \oplus (13)[1,0,0,1,1] \oplus [5,0,0,0,0] \oplus (12)[0,1,1,0,0] \oplus (11)[0,1,0,2,0] \oplus (9)[0,1,0,0,2] \oplus (12)[2,0,1,0,0] \oplus (8)[1,2,0,0,0] \oplus (5)[3,1,0,0,0] \oplus (9)[2,0,0,2,0] \oplus (9)[2,0,0,0,2] \oplus (5)[0,0,0,3,1] \oplus (5)[0,0,0,1,3] \oplus (15)[0,0,1,1,1] \oplus [1,0,0,4,0] \oplus (3)[1,0,0,0,4] \oplus (13)[1,0,2,0,0] \oplus (12)[3,0,0,1,1] \oplus (10)[0,2,1,0,0] \oplus (28)[1,1,0,1,1] \oplus (3)[1,3,0,0,0] \oplus (4)[4,0,1,0,0] \oplus (8)[0,2,0,2,0] \oplus (9)[0,2,0,0,2] \oplus (13)[1,0,1,2,0] \oplus (15)[1,0,1,0,2] \oplus (2)[3,2,0,0,0] \oplus (2)[4,0,0,2,0] \oplus (4)[4,0,0,0,2] \oplus (12)[1,0,0,2,2] \oplus (14)[2,1,1,0,0] \oplus (3)[0,0,3,0,0] \oplus (9)[2,1,0,2,0] \oplus (13)[2,1,0,0,2] \oplus [0,0,1,0,4] \oplus [0,0,0,2,4] \oplus (2)[0,0,2,2,0] \oplus (4)[0,0,2,0,2] \oplus (4)[0,1,0,3,1] \oplus (7)[0,1,0,1,3] \oplus (16)[0,1,1,1,1] \oplus [5,0,0,1,1] \oplus (3)[2,0,0,3,1] \oplus (7)[2,0,0,1,3] \oplus [0,3,1,0,0] \oplus (3)[0,0,1,2,2] \oplus [3,0,0,0,4] \oplus (17)[2,0,1,1,1] \oplus (4)[3,0,2,0,0] \oplus (2)[1,1,0,0,4] \oplus [0,3,0,2,0] \oplus [0,3,0,0,2] \oplus (7)[1,1,2,0,0] \oplus (11)[1,2,0,1,1] \oplus (7)[3,1,0,1,1] \oplus [4,1,1,0,0] \oplus (2)[2,2,1,0,0] \oplus (2)[3,0,1,2,0] \oplus$

(5)[3, 0, 1, 0, 2] ⊕ (5)[1, 1, 1, 2, 0] ⊕ (9)[1, 1, 1, 0, 2] ⊕ [4, 1, 0, 0, 2] ⊕ [2, 2, 0, 2, 0] ⊕ (2)[2, 2, 0, 0, 2] ⊕ (2)[3, 0, 0, 2, 2] ⊕ [0, 1, 3, 0, 0] ⊕ (5)[1, 1, 0, 2, 2] ⊕ (4)[1, 0, 2, 1, 1] ⊕ (2)[1, 0, 1, 1, 3] ⊕ [0, 2, 0, 1, 3] ⊕ [2, 0, 3, 0, 0] ⊕ (2)[0, 2, 1, 1, 1] ⊕ [0, 1, 2, 0, 2] ⊕ [4, 0, 1, 1, 1] ⊕ [2, 1, 0, 1, 3] ⊕ [2, 0, 2, 0, 2] ⊕ (3)[2, 1, 1, 1, 1]

- Level-8: (2)[0, 0, 0, 0, 1] ⊕ (5)[1, 0, 0, 1, 0] ⊕ (8)[0, 1, 0, 0, 1] ⊕ (3)[0, 0, 0, 3, 0] ⊕ (6)[2, 0, 0, 0, 1]⊕(10)[0, 0, 1, 1, 0]⊕(7)[0, 0, 0, 1, 2]⊕(6)[3, 0, 0, 1, 0]⊕(16)[1, 1, 0, 1, 0]⊕ (5)[1, 0, 0, 0, 3] ⊕ (4)[4, 0, 0, 0, 1] ⊕ (12)[0, 2, 0, 0, 1] ⊕ (19)[1, 0, 1, 0, 1] ⊕ (17)[1, 0, 0, 2, 1] ⊕ (15)[2, 1, 0, 0, 1] ⊕ (6)[0, 1, 0, 3, 0] ⊕ [5, 0, 0, 1, 0] ⊕ (5)[2, 0, 0, 3, 0] ⊕ (19)[0, 1, 1, 1, 0] ⊕ [0, 0, 0, 4, 1] ⊕ (4)[0, 0, 1, 0, 3] ⊕ (8)[0, 0, 2, 0, 1] ⊕ (17)[0, 1, 0, 1, 2] ⊕ (19)[2, 0, 1, 1, 0] ⊕ (4)[0, 0, 0, 2, 3] ⊕ (14)[1, 2, 0, 1, 0] ⊕ (8)[3, 1, 0, 1, 0] ⊕ (16)[2, 0, 0, 1, 2]⊕(10)[0, 0, 1, 2, 1]⊕(5)[0, 3, 0, 0, 1]⊕(4)[3, 0, 0, 0, 3]⊕(8)[1, 1, 0, 0, 3]⊕ (12)[3, 0, 1, 0, 1]⊕(3)[4, 1, 0, 0, 1]⊕(22)[1, 1, 1, 0, 1]⊕(7)[2, 2, 0, 0, 1]⊕(7)[3, 0, 0, 2, 1]⊕ (2)[1, 0, 0, 1, 4]⊕(10)[1, 0, 2, 1, 0]⊕(3)[1, 0, 1, 3, 0]⊕(2)[0, 2, 0, 3, 0]⊕(18)[1, 1, 0, 2, 1]⊕ (7)[0, 2, 1, 1, 0]⊕(3)[1, 0, 0, 3, 2]⊕(2)[1, 3, 0, 1, 0]⊕(3)[4, 0, 1, 1, 0]⊕(12)[1, 0, 1, 1, 2]⊕ (7)[0, 2, 0, 1, 2] ⊕ (2)[2, 1, 0, 3, 0] ⊕ [0, 0, 3, 1, 0] ⊕ [3, 2, 0, 1, 0] ⊕ (3)[4, 0, 0, 1, 2] ⊕ (5)[0, 1, 2, 0, 1]⊕(2)[0, 1, 1, 0, 3]⊕(10)[2, 1, 1, 1, 0]⊕(2)[0, 1, 0, 2, 3]⊕(6)[2, 0, 2, 0, 1]⊕ (10)[2, 1, 0, 1, 2] ⊕ (3)[2, 0, 1, 0, 3] ⊕ [5, 0, 1, 0, 1] ⊕ (4)[0, 1, 1, 2, 1] ⊕ [0, 0, 2, 1, 2] ⊕ [1, 2, 0, 0, 3] ⊕ [2, 0, 0, 2, 3] ⊕ [3, 1, 0, 0, 3] ⊕ (3)[1, 2, 1, 0, 1] ⊕ (4)[2, 0, 1, 2, 1] ⊕ (3)[3, 1, 1, 0, 1] ⊕ (2)[1, 2, 0, 2, 1] ⊕ [3, 1, 0, 2, 1] ⊕ [3, 0, 2, 1, 0] ⊕ (2)[1, 1, 2, 1, 0] ⊕ [1, 0, 3, 0, 1] ⊕ [3, 0, 1, 1, 2] ⊕ (2)[1, 1, 1, 1, 2]

- Level-9: [0, 0, 0, 0, 0]⊕(3)[0, 1, 0, 0, 0]⊕(2)[2, 0, 0, 0, 0]⊕(6)[0, 0, 0, 1, 1]⊕[4, 0, 0, 0, 0]⊕ (6)[0, 2, 0, 0, 0]⊕(10)[1, 0, 1, 0, 0]⊕(8)[1, 0, 0, 2, 0]⊕(7)[1, 0, 0, 0, 2]⊕(6)[2, 1, 0, 0, 0]⊕ [0, 0, 0, 4, 0] ⊕ [0, 0, 0, 0, 4] ⊕ (8)[0, 0, 2, 0, 0] ⊕ (19)[0, 1, 0, 1, 1] ⊕ (9)[0, 0, 1, 2, 0] ⊕ (8)[0, 0, 1, 0, 2]⊕(5)[0, 3, 0, 0, 0]⊕(16)[2, 0, 0, 1, 1]⊕(8)[0, 0, 0, 2, 2]⊕(2)[4, 1, 0, 0, 0]⊕ (8)[3, 0, 1, 0, 0]⊕(6)[2, 2, 0, 0, 0]⊕(6)[3, 0, 0, 2, 0]⊕(6)[3, 0, 0, 0, 2]⊕(19)[1, 1, 1, 0, 0]⊕ (16)[1, 1, 0, 2, 0] ⊕ (15)[1, 1, 0, 0, 2] ⊕ (8)[1, 0, 0, 3, 1] ⊕ (8)[1, 0, 0, 1, 3] ⊕ [0, 4, 0, 0, 0] ⊕ [0, 1, 0, 4, 0] ⊕ [0, 1, 0, 0, 4] ⊕ (8)[0, 1, 2, 0, 0] ⊕ (24)[1, 0, 1, 1, 1] ⊕ (16)[0, 2, 0, 1, 1] ⊕ (5)[4, 0, 0, 1, 1] ⊕ [2, 0, 0, 4, 0] ⊕ [2, 0, 0, 0, 4] ⊕ [5, 0, 1, 0, 0] ⊕ (9)[2, 0, 2, 0, 0] ⊕ [2, 3, 0, 0, 0] ⊕ [5, 0, 0, 0, 2] ⊕ (9)[0, 1, 1, 2, 0] ⊕ (9)[0, 1, 1, 0, 2] ⊕ (20)[2, 1, 0, 1, 1] ⊕ [0, 0, 0, 3, 3] ⊕ (8)[1, 2, 1, 0, 0] ⊕ (9)[0, 1, 0, 2, 2] ⊕ (9)[2, 0, 1, 2, 0] ⊕ (10)[2, 0, 1, 0, 2] ⊕

$(5)[3,1,1,0,0] \oplus (5)[0,0,2,1,1] \oplus (2)[0,0,1,3,1] \oplus (2)[0,0,1,1,3] \oplus (6)[1,2,0,2,0] \oplus (7)[1,2,0,0,2] \oplus (8)[2,0,0,2,2] \oplus (3)[3,1,0,2,0] \oplus (5)[3,1,0,0,2] \oplus (2)[1,0,3,0,0] \oplus (2)[0,3,0,1,1] \oplus [0,2,2,0,0] \oplus [3,0,0,3,1] \oplus (2)[3,0,0,1,3] \oplus (3)[1,1,0,3,1] \oplus (4)[1,1,0,1,3] \oplus (2)[1,0,2,2,0] \oplus (2)[1,0,2,0,2] \oplus [4,0,2,0,0] \oplus (6)[3,0,1,1,1] \oplus (12)[1,1,1,1,1] \oplus [4,1,0,1,1] \oplus [0,2,1,2,0] \oplus [0,2,1,0,2] \oplus (3)[2,2,0,1,1] \oplus (2)[2,1,2,0,0] \oplus [4,0,1,0,2] \oplus (2)[1,0,1,2,2] \oplus [0,2,0,2,2] \oplus [2,1,1,2,0] \oplus (2)[2,1,1,0,2] \oplus [0,1,2,1,1] \oplus [2,1,0,2,2] \oplus [2,0,2,1,1]$

- Level-10: $(2)[0,0,0,1,0] \oplus (5)[1,0,0,0,1] \oplus (8)[0,1,0,1,0] \oplus (3)[0,0,0,0,3] \oplus (6)[2,0,0,1,0] \oplus (9)[0,0,1,0,1] \oplus (7)[0,0,0,2,1] \oplus (4)[3,0,0,0,1] \oplus (14)[1,1,0,0,1] \oplus (5)[1,0,0,3,0] \oplus (2)[4,0,0,1,0] \oplus (11)[0,2,0,1,0] \oplus (17)[1,0,1,1,0] \oplus (14)[1,0,0,1,2] \oplus (12)[2,1,0,1,0] \oplus (5)[0,1,0,0,3] \oplus [5,0,0,0,1] \oplus (4)[2,0,0,0,3] \oplus (16)[0,1,1,0,1] \oplus [0,0,0,1,4] \oplus (3)[0,0,1,3,0] \oplus (7)[0,0,2,1,0] \oplus (14)[0,1,0,2,1] \oplus (14)[2,0,1,0,1] \oplus (3)[0,0,0,3,2] \oplus (12)[1,2,0,0,1] \oplus (6)[3,1,0,0,1] \oplus (13)[2,0,0,2,1] \oplus (8)[0,0,1,1,2] \oplus (4)[0,3,0,1,0] \oplus (2)[3,0,0,3,0] \oplus (5)[1,1,0,3,0] \oplus (7)[3,0,1,1,0] \oplus [4,1,0,1,0] \oplus (17)[1,1,1,1,0] \oplus (5)[2,2,0,1,0] \oplus (6)[3,0,0,1,2] \oplus [1,0,0,4,1] \oplus (7)[1,0,2,0,1] \oplus (2)[1,0,1,0,3] \oplus (2)[0,2,0,0,3] \oplus (15)[1,1,0,1,2] \oplus [4,0,0,0,3] \oplus (6)[0,2,1,0,1] \oplus (3)[1,0,0,2,3] \oplus (2)[1,3,0,0,1] \oplus (2)[4,0,1,0,1] \oplus (9)[1,0,1,2,1] \oplus (5)[0,2,0,2,1] \oplus (2)[2,1,0,0,3] \oplus [3,2,0,0,1] \oplus [4,0,0,2,1] \oplus (3)[0,1,2,1,0] \oplus [0,1,1,3,0] \oplus (8)[2,1,1,0,1] \oplus [0,1,0,3,2] \oplus (3)[2,0,2,1,0] \oplus (6)[2,1,0,2,1] \oplus [2,0,1,3,0] \oplus (3)[0,1,1,1,2] \oplus [0,0,2,2,1] \oplus [2,0,0,3,2] \oplus (2)[1,2,1,1,0] \oplus (3)[2,0,1,1,2] \oplus [3,1,1,1,0] \oplus (2)[1,2,0,1,2] \oplus [3,1,0,1,2] \oplus [3,0,2,0,1] \oplus [1,1,2,0,1] \oplus [1,1,1,2,1]$

- Level-11: $[1,0,0,0,0] \oplus (4)[0,0,1,0,0] \oplus (2)[0,0,0,2,0] \oplus (3)[0,0,0,0,2] \oplus [3,0,0,0,0] \oplus (5)[1,1,0,0,0] \oplus (11)[1,0,0,1,1] \oplus (10)[0,1,1,0,0] \oplus (7)[0,1,0,2,0] \oplus (9)[0,1,0,0,2] \oplus (8)[2,0,1,0,0] \oplus (6)[1,2,0,0,0] \oplus (2)[3,1,0,0,0] \oplus (6)[2,0,0,2,0] \oplus (6)[2,0,0,0,2] \oplus (3)[0,0,0,3,1] \oplus (4)[0,0,0,1,3] \oplus (11)[0,0,1,1,1] \oplus [1,0,0,4,0] \oplus [1,0,0,0,4] \oplus (8)[1,0,2,0,0] \oplus (6)[3,0,0,1,1] \oplus (7)[0,2,1,0,0] \oplus (19)[1,1,0,1,1] \oplus (2)[1,3,0,0,0] \oplus [4,0,1,0,0] \oplus (5)[0,2,0,2,0] \oplus (6)[0,2,0,0,2] \oplus (9)[1,0,1,2,0] \oplus (9)[1,0,1,0,2] \oplus [3,2,0,0,0] \oplus [4,0,0,2,0] \oplus [4,0,0,0,2] \oplus (8)[1,0,0,2,2] \oplus (8)[2,1,1,0,0] \oplus [0,0,3,0,0] \oplus (6)[2,1,0,2,0] \oplus (6)[2,1,0,0,2] \oplus (2)[0,0,2,2,0] \oplus$

$[0,0,2,0,2] \oplus (3)[0,1,0,3,1] \oplus (3)[0,1,0,1,3] \oplus (10)[0,1,1,1,1] \oplus (3)[2,0,0,3,1] \oplus (2)[2,0,0,1,3] \oplus [0,3,1,0,0] \oplus (2)[0,0,1,2,2] \oplus (9)[2,0,1,1,1] \oplus [3,0,2,0,0] \oplus [0,3,0,0,2] \oplus (3)[1,1,2,0,0] \oplus (7)[1,2,0,1,1] \oplus (3)[3,1,0,1,1] \oplus [2,2,1,0,0] \oplus [3,0,1,2,0] \oplus [3,0,1,0,2] \oplus (3)[1,1,1,2,0] \oplus (3)[1,1,1,0,2] \oplus [2,2,0,0,2] \oplus [3,0,0,2,2] \oplus (3)[1,1,0,2,2] \oplus [1,0,2,1,1] \oplus [1,0,1,3,1] \oplus [0,2,1,1,1] \oplus [2,1,1,1,1]$

- Level-12: $[0,0,0,0,1] \oplus (3)[1,0,0,1,0] \oplus (6)[0,1,0,0,1] \oplus [0,0,0,3,0] \oplus (4)[2,0,0,0,1] \oplus (5)[0,0,1,1,0] \oplus (5)[0,0,0,1,2] \oplus (2)[3,0,0,1,0] \oplus (8)[1,1,0,1,0] \oplus (4)[1,0,0,0,3] \oplus (6)[0,2,0,0,1] \oplus (11)[1,0,1,0,1] \oplus (7)[1,0,0,2,1] \oplus (6)[2,1,0,0,1] \oplus (2)[0,1,0,3,0] \oplus (2)[2,0,0,3,0] \oplus (8)[0,1,1,1,0] \oplus (2)[0,0,1,0,3] \oplus (4)[0,0,2,0,1] \oplus (8)[0,1,0,1,2] \oplus (7)[2,0,1,1,0] \oplus [0,0,0,2,3] \oplus (5)[1,2,0,1,0] \oplus (2)[3,1,0,1,0] \oplus (6)[2,0,0,1,2] \oplus (4)[0,0,1,2,1] \oplus (2)[0,3,0,0,1] \oplus (2)[1,1,0,0,3] \oplus (2)[3,0,1,0,1] \oplus (8)[1,1,1,0,1] \oplus (2)[2,2,0,0,1] \oplus (2)[3,0,0,2,1] \oplus (3)[1,0,2,1,0] \oplus [1,0,1,3,0] \oplus (6)[1,1,0,2,1] \oplus (2)[0,2,1,1,0] \oplus [1,0,0,3,2] \oplus (3)[1,0,1,1,2] \oplus (2)[0,2,0,1,2] \oplus [0,1,2,0,1] \oplus (2)[2,1,1,1,0] \oplus (2)[2,1,0,1,2] \oplus [0,1,1,2,1] \oplus [1,2,1,0,1] \oplus [2,0,1,2,1]$

- Level-13: $[0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (2)[0,0,0,1,1] \oplus (2)[0,2,0,0,0] \oplus (4)[1,0,1,0,0] \oplus (2)[1,0,0,2,0] \oplus (4)[1,0,0,0,2] \oplus (2)[2,1,0,0,0] \oplus [0,0,0,0,4] \oplus (3)[0,0,2,0,0] \oplus (6)[0,1,0,1,1] \oplus (2)[0,0,1,2,0] \oplus (4)[0,0,1,0,2] \oplus [0,3,0,0,0] \oplus (5)[2,0,0,1,1] \oplus (2)[0,0,0,2,2] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus (5)[1,1,1,0,0] \oplus (3)[1,1,0,2,0] \oplus (5)[1,1,0,0,2] \oplus [1,0,0,3,1] \oplus (2)[1,0,0,1,3] \oplus (2)[0,1,2,0,0] \oplus (6)[1,0,1,1,1] \oplus (3)[0,2,0,1,1] \oplus [2,0,2,0,0] \oplus [0,1,1,2,0] \oplus (2)[0,1,1,0,2] \oplus (3)[2,1,0,1,1] \oplus [1,2,1,0,0] \oplus [0,1,0,2,2] \oplus [2,0,1,2,0] \oplus [2,0,1,0,2] \oplus [0,0,2,1,1] \oplus [1,2,0,0,2] \oplus [2,0,0,2,2] \oplus [1,1,1,1,1]$

- Level-14: $[1,0,0,0,1] \oplus [0,1,0,1,0] \oplus [0,0,0,0,3] \oplus [2,0,0,1,0] \oplus (2)[0,0,1,0,1] \oplus [0,0,0,2,1] \oplus [3,0,0,0,1] \oplus (3)[1,1,0,0,1] \oplus [0,2,0,1,0] \oplus (3)[1,0,1,1,0] \oplus (3)[1,0,0,1,2] \oplus [2,1,0,1,0] \oplus [0,1,0,0,3] \oplus [2,0,0,0,3] \oplus (3)[0,1,1,0,1] \oplus [0,0,2,1,0] \oplus [0,1,0,2,1] \oplus (2)[2,0,1,0,1] \oplus [1,2,0,0,1] \oplus [2,0,0,2,1] \oplus [0,0,1,1,2] \oplus [1,1,1,1,0] \oplus [1,0,2,0,1] \oplus [1,1,0,1,2]$

- Level-15: $[1,0,0,1,1] \oplus [0,1,1,0,0] \oplus [0,1,0,0,2] \oplus [2,0,1,0,0] \oplus [2,0,0,0,2] \oplus [0,0,1,1,1] \oplus [1,0,2,0,0] \oplus [1,1,0,1,1] \oplus [1,0,1,0,2]$

- Level-16: $[1,0,1,0,1]$

F.3 Dynkin Label Library of $\mathcal{V}_{[3,0,0,0,1]}$

- Level-0: $[3,0,0,0,1]$

- Level-1: $[3,0,0,0,0] \oplus [2,0,1,0,0] \oplus [3,1,0,0,0] \oplus [2,0,0,0,2] \oplus [3,0,0,1,1]$

- Level-2: $[2,0,0,0,1] \oplus (2)[3,0,0,1,0] \oplus [1,1,0,1,0] \oplus [4,0,0,0,1] \oplus [1,0,1,0,1] \oplus (2)[2,1,0,0,1] \oplus [2,0,1,1,0] \oplus [3,1,0,1,0] \oplus [2,0,0,1,2] \oplus [3,0,1,0,1]$

- Level-3: $[4,0,0,0,0] \oplus [0,2,0,0,0] \oplus [1,0,1,0,0] \oplus [1,0,0,2,0] \oplus (2)[2,1,0,0,0] \oplus [0,1,0,1,1] \oplus (3)[2,0,0,1,1] \oplus [4,1,0,0,0] \oplus (3)[3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus (2)[3,0,0,0,2] \oplus (2)[1,1,1,0,0] \oplus [1,1,0,2,0] \oplus [1,1,0,0,2] \oplus [1,0,1,1,1] \oplus [4,0,0,1,1] \oplus [2,0,2,0,0] \oplus (2)[2,1,0,1,1] \oplus [2,0,1,0,2] \oplus [3,1,1,0,0] \oplus [3,1,0,0,2]$

- Level-4: $(2)[0,1,0,1,0] \oplus (2)[2,0,0,1,0] \oplus [0,0,0,2,1] \oplus (3)[3,0,0,0,1] \oplus (3)[1,1,0,0,1] \oplus [1,0,0,3,0] \oplus (2)[4,0,0,1,0] \oplus (2)[0,2,0,1,0] \oplus (3)[1,0,1,1,0] \oplus [1,0,0,1,2] \oplus (4)[2,1,0,1,0] \oplus [5,0,0,0,1] \oplus [2,0,0,0,3] \oplus [0,1,1,0,1] \oplus [0,1,0,2,1] \oplus (4)[2,0,1,0,1] \oplus (2)[1,2,0,0,1] \oplus (4)[3,1,0,0,1] \oplus (2)[2,0,0,2,1] \oplus (2)[3,0,1,1,0] \oplus [4,1,0,1,0] \oplus (2)[1,1,1,1,0] \oplus [2,2,0,1,0] \oplus (2)[3,0,0,1,2] \oplus [1,0,2,0,1] \oplus [1,1,0,1,2] \oplus [4,0,0,0,3] \oplus [4,0,1,0,1] \oplus [2,1,0,0,3] \oplus [3,2,0,0,1] \oplus (2)[2,1,1,0,1]$

- Level-5: $[0,0,1,0,0] \oplus [0,0,0,2,0] \oplus [3,0,0,0,0] \oplus (2)[1,1,0,0,0] \oplus (3)[1,0,0,1,1] \oplus [5,0,0,0,0] \oplus (3)[0,1,1,0,0] \oplus (3)[0,1,0,2,0] \oplus [0,1,0,0,2] \oplus (4)[2,0,1,0,0] \oplus (3)[1,2,0,0,0] \oplus (3)[3,1,0,0,0] \oplus (3)[2,0,0,2,0] \oplus (2)[2,0,0,0,2] \oplus [0,0,0,3,1] \oplus [0,0,1,1,1] \oplus (2)[1,0,2,0,0] \oplus (5)[3,0,0,1,1] \oplus [5,1,0,0,0] \oplus (2)[0,2,1,0,0] \oplus (6)[1,1,0,1,1] \oplus [1,3,0,0,0] \oplus (3)[4,0,1,0,0] \oplus [0,2,0,2,0] \oplus [0,2,0,0,2] \oplus (2)[1,0,1,2,0] \oplus [1,0,1,0,2] \oplus (2)[3,2,0,0,0] \oplus [4,0,0,2,0] \oplus (3)[4,0,0,0,2] \oplus [1,0,0,2,2] \oplus (5)[2,1,1,0,0] \oplus (2)[2,1,0,2,0] \oplus (4)[2,1,0,0,2] \oplus [0,1,1,1,1] \oplus [5,0,0,1,1] \oplus [2,0,0,1,3] \oplus [3,0,0,0,4] \oplus (3)[2,0,1,1,1] \oplus [3,0,2,0,0] \oplus [1,1,2,0,0] \oplus (2)[1,2,0,1,1] \oplus (3)[3,1,0,1,1] \oplus [4,1,1,0,0] \oplus [2,2,1,0,0] \oplus (2)[3,0,1,0,2] \oplus [1,1,1,0,2] \oplus [4,1,0,0,2] \oplus [2,2,0,0,2]$

- Level-6: $[0,0,0,0,1] \oplus (2)[1,0,0,1,0] \oplus (3)[0,1,0,0,1] \oplus [0,0,0,3,0] \oplus (2)[2,0,0,0,1] \oplus (3)[0,0,1,1,0] \oplus [0,0,0,1,2] \oplus (3)[3,0,0,1,0] \oplus (6)[1,1,0,1,0] \oplus (3)[4,0,0,0,1] \oplus (4)[0,2,0,0,1] \oplus (4)[1,0,1,0,1] \oplus (4)[1,0,0,2,1] \oplus (6)[2,1,0,0,1] \oplus [0,1,0,3,0] \oplus (2)[5,0,0,1,0] \oplus [2,0,0,3,0] \oplus (4)[0,1,1,1,0] \oplus (2)[0,1,0,1,2] \oplus (6)[2,0,1,1,0] \oplus (5)[1,2,0,1,0] \oplus (5)[3,1,0,1,0] \oplus [6,0,0,0,1] \oplus (4)[2,0,0,1,2] \oplus [0,0,1,2,1] \oplus (2)[0,3,0,0,1] \oplus (2)[3,0,0,0,3] \oplus [1,1,0,0,3] \oplus (6)[3,0,1,0,1] \oplus (4)[4,1,0,0,1] \oplus (5)[1,1,1,0,1] \oplus (4)[2,2,0,0,1] \oplus (2)[3,0,0,2,1] \oplus [1,0,2,1,0] \oplus (3)[1,1,0,2,1] \oplus [0,2,1,1,0] \oplus [5,1,0,1,0] \oplus [1,3,0,1,0] \oplus (2)[4,0,1,1,0] \oplus [1,0,1,1,2] \oplus [0,2,0,1,2] \oplus [3,2,0,1,0] \oplus (2)[4,0,0,1,2] \oplus (3)[2,1,1,1,0] \oplus [2,0,2,0,1] \oplus (3)[2,1,0,1,2] \oplus [2,0,1,0,3] \oplus [5,0,1,0,1] \oplus [3,1,0,0,3] \oplus [1,2,1,0,1] \oplus (2)[3,1,1,0,1]$

- Level-7: $[0,0,0,0,0] \oplus (2)[0,1,0,0,0] \oplus [2,0,0,0,0] \oplus (3)[0,0,0,1,1] \oplus [4,0,0,0,0] \oplus (3)[0,2,0,0,0] \oplus (4)[1,0,1,0,0] \oplus (3)[1,0,0,2,0] \oplus (2)[1,0,0,0,2] \oplus (3)[2,1,0,0,0] \oplus (2)[0,0,2,0,0] \oplus [6,0,0,0,0] \oplus (6)[0,1,0,1,1] \oplus (2)[0,0,1,2,0] \oplus [0,0,1,0,2] \oplus (3)[0,3,0,0,0] \oplus (6)[2,0,0,1,1] \oplus [0,0,0,2,2] \oplus (3)[4,1,0,0,0] \oplus (5)[3,0,1,0,0] \oplus (4)[2,2,0,0,0] \oplus (3)[3,0,0,2,0] \oplus (3)[3,0,0,0,2] \oplus (7)[1,1,1,0,0] \oplus (5)[1,1,0,2,0] \oplus (4)[1,1,0,0,2] \oplus [1,0,0,3,1] \oplus [1,0,0,1,3] \oplus [0,4,0,0,0] \oplus [0,1,2,0,0] \oplus [6,1,0,0,0] \oplus (5)[1,0,1,1,1] \oplus (5)[0,2,0,1,1] \oplus (5)[4,0,0,1,1] \oplus (3)[5,0,1,0,0] \oplus (3)[2,0,2,0,0] \oplus [4,2,0,0,0] \oplus [2,3,0,0,0] \oplus [5,0,0,2,0] \oplus (2)[5,0,0,0,2] \oplus [0,1,1,2,0] \oplus [0,1,1,0,2] \oplus (9)[2,1,0,1,1] \oplus (4)[1,2,1,0,0] \oplus [0,1,0,2,2] \oplus (2)[2,0,1,2,0] \oplus (3)[2,0,1,0,2] \oplus (5)[3,1,1,0,0] \oplus (2)[1,2,0,2,0] \oplus (3)[1,2,0,0,2] \oplus (2)[2,0,0,2,2] \oplus (2)[3,1,0,2,0] \oplus (4)[3,1,0,0,2] \oplus [6,0,0,1,1] \oplus [0,3,0,1,1] \oplus [3,0,0,1,3] \oplus [1,1,0,1,3] \oplus [4,0,2,0,0] \oplus (3)[3,0,1,1,1] \oplus (2)[1,1,1,1,1] \oplus (2)[4,1,0,1,1] \oplus (2)[2,2,0,1,1] \oplus [2,1,2,0,0] \oplus [4,0,1,0,2] \oplus [2,1,1,0,2]$

- Level-8: $(2)[0,0,0,1,0] \oplus (3)[1,0,0,0,1] \oplus (4)[0,1,0,1,0] \oplus [0,0,0,0,3] \oplus (3)[2,0,0,1,0] \oplus (4)[0,0,1,0,1] \oplus (2)[0,0,0,2,1] \oplus (3)[3,0,0,0,1] \oplus (6)[1,1,0,0,1] \oplus [1,0,0,3,0] \oplus (3)[4,0,0,1,0] \oplus (5)[0,2,0,1,0] \oplus (6)[1,0,1,1,0] \oplus (4)[1,0,0,1,2] \oplus (7)[2,1,0,1,0] \oplus [0,1,0,0,3] \oplus (3)[5,0,0,0,1] \oplus [2,0,0,0,3] \oplus (5)[0,1,1,0,1] \oplus [0,0,2,1,0] \oplus (3)[0,1,0,2,1] \oplus (6)[2,0,1,0,1] \oplus (7)[1,2,0,0,1] \oplus (6)[3,1,0,0,1] \oplus (2)[6,0,0,1,0] \oplus (5)[2,0,0,2,1] \oplus [0,0,1,1,2] \oplus (3)[0,3,0,1,0] \oplus [3,0,0,3,0] \oplus [1,1,0,3,0] \oplus [7,0,0,0,1] \oplus (6)[3,0,1,1,0] \oplus (4)[4,1,0,1,0] \oplus (6)[1,1,1,1,0] \oplus$

(5)[2, 2, 0, 1, 0] ⊕ (4)[3, 0, 0, 1, 2] ⊕ [1, 0, 2, 0, 1] ⊕ [0, 2, 0, 0, 3] ⊕ (5)[1, 1, 0, 1, 2] ⊕ [4, 0, 0, 0, 3] ⊕ (2)[0, 2, 1, 0, 1] ⊕ [1, 0, 0, 2, 3] ⊕ (2)[5, 1, 0, 0, 1] ⊕ (2)[1, 3, 0, 0, 1] ⊕ (4)[4, 0, 1, 0, 1] ⊕ [1, 0, 1, 2, 1] ⊕ [0, 2, 0, 2, 1] ⊕ [2, 1, 0, 0, 3] ⊕ (2)[3, 2, 0, 0, 1] ⊕ (2)[4, 0, 0, 2, 1] ⊕ (5)[2, 1, 1, 0, 1] ⊕ [2, 0, 2, 1, 0] ⊕ (3)[2, 1, 0, 2, 1] ⊕ [5, 0, 1, 1, 0] ⊕ [5, 0, 0, 1, 2] ⊕ [1, 2, 1, 1, 0] ⊕ [2, 0, 1, 1, 2] ⊕ (2)[3, 1, 1, 1, 0] ⊕ [1, 2, 0, 1, 2] ⊕ [3, 1, 0, 1, 2] ⊕ [3, 0, 2, 0, 1]

- Level-9: [1, 0, 0, 0, 0] ⊕ (3)[0, 0, 1, 0, 0] ⊕ [0, 0, 0, 2, 0] ⊕ (2)[0, 0, 0, 0, 2] ⊕ [3, 0, 0, 0, 0] ⊕ (3)[1, 1, 0, 0, 0] ⊕ (5)[1, 0, 0, 1, 1] ⊕ [5, 0, 0, 0, 0] ⊕ (5)[0, 1, 1, 0, 0] ⊕ (2)[0, 1, 0, 2, 0] ⊕ (4)[0, 1, 0, 0, 2] ⊕ (5)[2, 0, 1, 0, 0] ⊕ (4)[1, 2, 0, 0, 0] ⊕ (3)[3, 1, 0, 0, 0] ⊕ (3)[2, 0, 0, 2, 0] ⊕ (3)[2, 0, 0, 0, 2] ⊕ [0, 0, 0, 1, 3] ⊕ [7, 0, 0, 0, 0] ⊕ (3)[0, 0, 1, 1, 1] ⊕ (3)[1, 0, 2, 0, 0] ⊕ (6)[3, 0, 0, 1, 1] ⊕ (2)[5, 1, 0, 0, 0] ⊕ (4)[0, 2, 1, 0, 0] ⊕ (9)[1, 1, 0, 1, 1] ⊕ (3)[1, 3, 0, 0, 0] ⊕ (4)[4, 0, 1, 0, 0] ⊕ (2)[0, 2, 0, 2, 0] ⊕ (3)[0, 2, 0, 0, 2] ⊕ (2)[1, 0, 1, 2, 0] ⊕ (3)[1, 0, 1, 0, 2] ⊕ (3)[3, 2, 0, 0, 0] ⊕ (3)[4, 0, 0, 2, 0] ⊕ (2)[4, 0, 0, 0, 2] ⊕ (2)[1, 0, 0, 2, 2] ⊕ (7)[2, 1, 1, 0, 0] ⊕ (5)[2, 1, 0, 2, 0] ⊕ (4)[2, 1, 0, 0, 2] ⊕ [0, 1, 0, 1, 3] ⊕ (2)[0, 1, 1, 1, 1] ⊕ (3)[5, 0, 0, 1, 1] ⊕ [2, 0, 0, 3, 1] ⊕ [2, 0, 0, 1, 3] ⊕ [6, 0, 1, 0, 0] ⊕ [0, 3, 1, 0, 0] ⊕ [6, 0, 0, 0, 2] ⊕ (5)[2, 0, 1, 1, 1] ⊕ (2)[3, 0, 2, 0, 0] ⊕ [0, 3, 0, 0, 2] ⊕ [1, 1, 2, 0, 0] ⊕ (5)[1, 2, 0, 1, 1] ⊕ (6)[3, 1, 0, 1, 1] ⊕ (2)[4, 1, 1, 0, 0] ⊕ (2)[2, 2, 1, 0, 0] ⊕ (2)[3, 0, 1, 2, 0] ⊕ [3, 0, 1, 0, 2] ⊕ [1, 1, 1, 2, 0] ⊕ [1, 1, 1, 0, 2] ⊕ [4, 1, 0, 2, 0] ⊕ [4, 1, 0, 0, 2] ⊕ [2, 2, 0, 2, 0] ⊕ [2, 2, 0, 0, 2] ⊕ [3, 0, 0, 2, 2] ⊕ [1, 1, 0, 2, 2] ⊕ [4, 0, 1, 1, 1] ⊕ [2, 1, 1, 1, 1]

- Level-10: [0, 0, 0, 0, 1] ⊕ (2)[1, 0, 0, 1, 0] ⊕ (4)[0, 1, 0, 0, 1] ⊕ (3)[2, 0, 0, 0, 1] ⊕ (2)[0, 0, 1, 1, 0] ⊕ (2)[0, 0, 0, 1, 2] ⊕ (3)[3, 0, 0, 1, 0] ⊕ (5)[1, 1, 0, 1, 0] ⊕ (2)[1, 0, 0, 0, 3] ⊕ (2)[4, 0, 0, 0, 1] ⊕ (4)[0, 2, 0, 0, 1] ⊕ (6)[1, 0, 1, 0, 1] ⊕ (2)[1, 0, 0, 2, 1] ⊕ (6)[2, 1, 0, 0, 1] ⊕ (2)[5, 0, 0, 1, 0] ⊕ [2, 0, 0, 3, 0] ⊕ (3)[0, 1, 1, 1, 0] ⊕ [0, 0, 1, 0, 3] ⊕ [0, 0, 2, 0, 1] ⊕ (3)[0, 1, 0, 1, 2] ⊕ (6)[2, 0, 1, 1, 0] ⊕ (5)[1, 2, 0, 1, 0] ⊕ (6)[3, 1, 0, 1, 0] ⊕ [6, 0, 0, 0, 1] ⊕ (4)[2, 0, 0, 1, 2] ⊕ (2)[0, 3, 0, 0, 1] ⊕ [1, 1, 0, 0, 3] ⊕ (4)[3, 0, 1, 0, 1] ⊕ (3)[4, 1, 0, 0, 1] ⊕ (5)[1, 1, 1, 0, 1] ⊕ (4)[2, 2, 0, 0, 1] ⊕ (4)[3, 0, 0, 2, 1] ⊕ [1, 0, 2, 1, 0] ⊕ (3)[1, 1, 0, 2, 1] ⊕ [4, 0, 0, 3, 0] ⊕ [0, 2, 1, 1, 0] ⊕ [5, 1, 0, 1, 0] ⊕ [1, 3, 0, 1, 0] ⊕ (3)[4, 0, 1, 1, 0] ⊕ [1, 0, 1, 1, 2] ⊕ [0, 2, 0, 1, 2] ⊕ [2, 1, 0, 3, 0] ⊕ (2)[3, 2, 0, 1, 0] ⊕ [4, 0, 0, 1, 2] ⊕ (4)[2, 1, 1, 1, 0] ⊕ (2)[2, 1, 0, 1, 2] ⊕ [5, 0, 1, 0, 1] ⊕ [1, 2, 1, 0, 1] ⊕ [2, 0, 1, 2, 1] ⊕ [3, 1, 1, 0, 1] ⊕ [3, 1, 0, 2, 1]

- Level-11: $[0, 1, 0, 0, 0] \oplus [2, 0, 0, 0, 0] \oplus [0, 0, 0, 1, 1] \oplus [4, 0, 0, 0, 0] \oplus (2)[0, 2, 0, 0, 0] \oplus (3)[1, 0, 1, 0, 0] \oplus [1, 0, 0, 2, 0] \oplus (3)[1, 0, 0, 0, 2] \oplus (3)[2, 1, 0, 0, 0] \oplus [0, 0, 0, 0, 4] \oplus [0, 0, 2, 0, 0] \oplus (3)[0, 1, 0, 1, 1] \oplus (2)[0, 0, 1, 0, 2] \oplus [0, 3, 0, 0, 0] \oplus (5)[2, 0, 0, 1, 1] \oplus (2)[4, 1, 0, 0, 0] \oplus (4)[3, 0, 1, 0, 0] \oplus (3)[2, 2, 0, 0, 0] \oplus (3)[3, 0, 0, 2, 0] \oplus (2)[3, 0, 0, 0, 2] \oplus (5)[1, 1, 1, 0, 0] \oplus (2)[1, 1, 0, 2, 0] \oplus (4)[1, 1, 0, 0, 2] \oplus [1, 0, 0, 1, 3] \oplus [0, 1, 2, 0, 0] \oplus (3)[1, 0, 1, 1, 1] \oplus (2)[0, 2, 0, 1, 1] \oplus (3)[4, 0, 0, 1, 1] \oplus [5, 0, 1, 0, 0] \oplus (2)[2, 0, 2, 0, 0] \oplus [4, 2, 0, 0, 0] \oplus [2, 3, 0, 0, 0] \oplus [5, 0, 0, 2, 0] \oplus [0, 1, 1, 0, 2] \oplus (6)[2, 1, 0, 1, 1] \oplus (2)[1, 2, 1, 0, 0] \oplus (2)[2, 0, 1, 2, 0] \oplus [2, 0, 1, 0, 2] \oplus (3)[3, 1, 1, 0, 0] \oplus [1, 2, 0, 2, 0] \oplus [1, 2, 0, 0, 2] \oplus [2, 0, 0, 2, 2] \oplus (3)[3, 1, 0, 2, 0] \oplus [3, 1, 0, 0, 2] \oplus [3, 0, 0, 3, 1] \oplus [3, 0, 1, 1, 1] \oplus [1, 1, 1, 1, 1] \oplus [4, 1, 0, 1, 1] \oplus [2, 2, 0, 1, 1]$

- Level-12: $[1, 0, 0, 0, 1] \oplus [0, 1, 0, 1, 0] \oplus [0, 0, 0, 0, 3] \oplus (2)[2, 0, 0, 1, 0] \oplus [0, 0, 1, 0, 1] \oplus (3)[3, 0, 0, 0, 1] \oplus (4)[1, 1, 0, 0, 1] \oplus (2)[4, 0, 0, 1, 0] \oplus [0, 2, 0, 1, 0] \oplus (2)[1, 0, 1, 1, 0] \oplus (2)[1, 0, 0, 1, 2] \oplus (4)[2, 1, 0, 1, 0] \oplus [0, 1, 0, 0, 3] \oplus [2, 0, 0, 0, 3] \oplus (2)[0, 1, 1, 0, 1] \oplus (4)[2, 0, 1, 0, 1] \oplus (2)[1, 2, 0, 0, 1] \oplus (3)[3, 1, 0, 0, 1] \oplus (2)[2, 0, 0, 2, 1] \oplus [3, 0, 0, 3, 0] \oplus (3)[3, 0, 1, 1, 0] \oplus (2)[4, 1, 0, 1, 0] \oplus (2)[1, 1, 1, 1, 0] \oplus (2)[2, 2, 0, 1, 0] \oplus [3, 0, 0, 1, 2] \oplus [1, 0, 2, 0, 1] \oplus [1, 1, 0, 1, 2] \oplus [3, 2, 0, 0, 1] \oplus [4, 0, 0, 2, 1] \oplus [2, 1, 1, 0, 1] \oplus [2, 1, 0, 2, 1]$

- Level-13: $[3, 0, 0, 0, 0] \oplus [1, 1, 0, 0, 0] \oplus [1, 0, 0, 1, 1] \oplus [0, 1, 1, 0, 0] \oplus [0, 1, 0, 0, 2] \oplus (3)[2, 0, 1, 0, 0] \oplus [1, 2, 0, 0, 0] \oplus (2)[3, 1, 0, 0, 0] \oplus [2, 0, 0, 2, 0] \oplus (2)[2, 0, 0, 0, 2] \oplus [1, 0, 2, 0, 0] \oplus (3)[3, 0, 0, 1, 1] \oplus (2)[1, 1, 0, 1, 1] \oplus [4, 0, 1, 0, 0] \oplus [1, 0, 1, 0, 2] \oplus [3, 2, 0, 0, 0] \oplus [4, 0, 0, 2, 0] \oplus (2)[2, 1, 1, 0, 0] \oplus [2, 1, 0, 2, 0] \oplus [2, 1, 0, 0, 2] \oplus [2, 0, 1, 1, 1] \oplus [3, 1, 0, 1, 1]$

- Level-14: $[2, 0, 0, 0, 1] \oplus (2)[3, 0, 0, 1, 0] \oplus [1, 1, 0, 1, 0] \oplus [4, 0, 0, 0, 1] \oplus [1, 0, 1, 0, 1] \oplus (2)[2, 1, 0, 0, 1] \oplus [2, 0, 1, 1, 0] \oplus [3, 1, 0, 1, 0] \oplus [2, 0, 0, 1, 2] \oplus [3, 0, 1, 0, 1]$

- Level-15: $[4, 0, 0, 0, 0] \oplus [2, 1, 0, 0, 0] \oplus [2, 0, 0, 1, 1] \oplus [3, 0, 1, 0, 0] \oplus [3, 0, 0, 0, 2]$

- Level-16: $[3, 0, 0, 0, 1]$

F.4 Dynkin Label Library of $\mathcal{V}_{[4,0,0,0,0]}$

- Level-0: $[4, 0, 0, 0, 0]$

- Level-1: $[3,0,0,0,1] \oplus [4,0,0,1,0]$
- Level-2: $[2,0,1,0,0] \oplus [3,1,0,0,0] \oplus [3,0,0,1,1] \oplus [4,0,1,0,0]$
- Level-3: $[3,0,0,1,0] \oplus [1,1,0,1,0] \oplus [4,0,0,0,1] \oplus [2,1,0,0,1] \oplus [2,0,1,1,0] \oplus [3,1,0,1,0] \oplus [3,0,1,0,1] \oplus [4,1,0,0,1]$
- Level-4: $[4,0,0,0,0] \oplus [0,2,0,0,0] \oplus [1,0,0,2,0] \oplus [2,1,0,0,0] \oplus [2,0,0,1,1] \oplus [4,1,0,0,0] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus [1,1,1,0,0] \oplus [1,1,0,2,0] \oplus [4,0,0,1,1] \oplus [2,0,2,0,0] \oplus [4,2,0,0,0] \oplus [5,0,0,0,2] \oplus [2,1,0,1,1] \oplus [3,1,1,0,0] \oplus [3,1,0,0,2]$
- Level-5: $[0,1,0,1,0] \oplus [2,0,0,1,0] \oplus [3,0,0,0,1] \oplus [1,1,0,0,1] \oplus [1,0,0,3,0] \oplus [4,0,0,1,0] \oplus [0,2,0,1,0] \oplus [1,0,1,1,0] \oplus (2)[2,1,0,1,0] \oplus [5,0,0,0,1] \oplus [2,0,1,0,1] \oplus [1,2,0,0,1] \oplus (2)[3,1,0,0,1] \oplus [2,0,0,2,1] \oplus [3,0,1,1,0] \oplus [4,1,0,1,0] \oplus [1,1,1,1,0] \oplus [2,2,0,1,0] \oplus [3,0,0,1,2] \oplus [4,0,0,0,3] \oplus [5,1,0,0,1] \oplus [4,0,1,0,1] \oplus [3,2,0,0,1] \oplus [2,1,1,0,1]$
- Level-6: $[0,0,1,0,0] \oplus [1,1,0,0,0] \oplus [1,0,0,1,1] \oplus [0,1,1,0,0] \oplus [0,1,0,2,0] \oplus (2)[2,0,1,0,0] \oplus [1,2,0,0,0] \oplus [3,1,0,0,0] \oplus [2,0,0,2,0] \oplus (2)[3,0,0,1,1] \oplus [5,1,0,0,0] \oplus [0,2,1,0,0] \oplus (2)[1,1,0,1,1] \oplus [1,3,0,0,0] \oplus (2)[4,0,1,0,0] \oplus [1,0,1,2,0] \oplus [3,2,0,0,0] \oplus [4,0,0,0,2] \oplus (2)[2,1,1,0,0] \oplus [2,1,0,2,0] \oplus [2,1,0,0,2] \oplus [5,0,0,1,1] \oplus [6,0,1,0,0] \oplus [2,0,1,1,1] \oplus [1,2,0,1,1] \oplus (2)[3,1,0,1,1] \oplus [4,1,1,0,0] \oplus [2,2,1,0,0] \oplus [3,0,1,0,2] \oplus [4,1,0,0,2]$
- Level-7: $[0,0,0,0,1] \oplus [1,0,0,1,0] \oplus [0,1,0,0,1] \oplus [2,0,0,0,1] \oplus [0,0,1,1,0] \oplus [3,0,0,1,0] \oplus (2)[1,1,0,1,0] \oplus [4,0,0,0,1] \oplus [0,2,0,0,1] \oplus [1,0,1,0,1] \oplus [1,0,0,2,1] \oplus (2)[2,1,0,0,1] \oplus [5,0,0,1,0] \oplus [0,1,1,1,0] \oplus (2)[2,0,1,1,0] \oplus (2)[1,2,0,1,0] \oplus (2)[3,1,0,1,0] \oplus [6,0,0,0,1] \oplus [2,0,0,1,2] \oplus [0,3,0,0,1] \oplus [7,0,0,1,0] \oplus (2)[3,0,1,0,1] \oplus (2)[4,1,0,0,1] \oplus [1,1,1,0,1] \oplus (2)[2,2,0,0,1] \oplus [3,0,0,2,1] \oplus [1,1,0,2,1] \oplus [5,1,0,1,0] \oplus [1,3,0,1,0] \oplus [4,0,1,1,0] \oplus [3,2,0,1,0] \oplus [4,0,0,1,2] \oplus [2,1,1,1,0] \oplus [2,1,0,1,2] \oplus [5,0,1,0,1] \oplus [3,1,1,0,1]$
- Level-8: $[0,0,0,0,0] \oplus [0,1,0,0,0] \oplus [2,0,0,0,0] \oplus [0,0,0,1,1] \oplus [4,0,0,0,0] \oplus [0,2,0,0,0] \oplus [1,0,1,0,0] \oplus [1,0,0,2,0] \oplus [1,0,0,0,2] \oplus [2,1,0,0,0] \oplus [0,0,2,0,0] \oplus$

$[6,0,0,0,0] \oplus [0,1,0,1,1] \oplus [0,3,0,0,0] \oplus (2)[2,0,0,1,1] \oplus [4,1,0,0,0] \oplus [3,0,1,0,0] \oplus (2)[2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus (2)[1,1,1,0,0] \oplus [8,0,0,0,0] \oplus [1,1,0,2,0] \oplus [1,1,0,0,2] \oplus [0,4,0,0,0] \oplus [6,1,0,0,0] \oplus [1,0,1,1,1] \oplus [0,2,0,1,1] \oplus (2)[4,0,0,1,1] \oplus [5,0,1,0,0] \oplus [2,0,2,0,0] \oplus [4,2,0,0,0] \oplus [2,3,0,0,0] \oplus [5,0,0,2,0] \oplus [5,0,0,0,2] \oplus (3)[2,1,0,1,1] \oplus [1,2,1,0,0] \oplus (2)[3,1,1,0,0] \oplus [1,2,0,2,0] \oplus [1,2,0,0,2] \oplus [2,0,0,2,2] \oplus [3,1,0,2,0] \oplus [3,1,0,0,2] \oplus [6,0,0,1,1] \oplus [4,0,2,0,0] \oplus [3,0,1,1,1] \oplus [4,1,0,1,1] \oplus [2,2,0,1,1]$

- Level-9: $[0,0,0,1,0] \oplus [1,0,0,0,1] \oplus [0,1,0,1,0] \oplus [2,0,0,1,0] \oplus [0,0,1,0,1] \oplus [3,0,0,0,1] \oplus (2)[1,1,0,0,1] \oplus [4,0,0,1,0] \oplus [0,2,0,1,0] \oplus [1,0,1,1,0] \oplus [1,0,0,1,2] \oplus (2)[2,1,0,1,0] \oplus [5,0,0,0,1] \oplus [0,1,1,0,1] \oplus (2)[2,0,1,0,1] \oplus (2)[1,2,0,0,1] \oplus (2)[3,1,0,0,1] \oplus [6,0,0,1,0] \oplus [2,0,0,2,1] \oplus [0,3,0,1,0] \oplus [7,0,0,0,1] \oplus (2)[3,0,1,1,0] \oplus (2)[4,1,0,1,0] \oplus [1,1,1,1,0] \oplus (2)[2,2,0,1,0] \oplus [3,0,0,1,2] \oplus [1,1,0,1,2] \oplus [5,1,0,0,1] \oplus [1,3,0,0,1] \oplus [4,0,1,0,1] \oplus [3,2,0,0,1] \oplus [4,0,0,2,1] \oplus [2,1,1,0,1] \oplus [2,1,0,2,1] \oplus [5,0,1,1,0] \oplus [3,1,1,1,0]$

- Level-10: $[0,0,1,0,0] \oplus [1,1,0,0,0] \oplus [1,0,0,1,1] \oplus [0,1,1,0,0] \oplus [0,1,0,0,2] \oplus (2)[2,0,1,0,0] \oplus [1,2,0,0,0] \oplus [3,1,0,0,0] \oplus [2,0,0,0,2] \oplus (2)[3,0,0,1,1] \oplus [5,1,0,0,0] \oplus [0,2,1,0,0] \oplus (2)[1,1,0,1,1] \oplus [1,3,0,0,0] \oplus (2)[4,0,1,0,0] \oplus [1,0,1,0,2] \oplus [3,2,0,0,0] \oplus [4,0,0,2,0] \oplus (2)[2,1,1,0,0] \oplus [2,1,0,2,0] \oplus [2,1,0,0,2] \oplus [5,0,0,1,1] \oplus [6,0,1,0,0] \oplus [2,0,1,1,1] \oplus [1,2,0,1,1] \oplus (2)[3,1,0,1,1] \oplus [4,1,1,0,0] \oplus [2,2,1,0,0] \oplus [3,0,1,2,0] \oplus [4,1,0,2,0]$

- Level-11: $[0,1,0,0,1] \oplus [2,0,0,0,1] \oplus [3,0,0,1,0] \oplus [1,1,0,1,0] \oplus [1,0,0,0,3] \oplus [4,0,0,0,1] \oplus [0,2,0,0,1] \oplus [1,0,1,0,1] \oplus (2)[2,1,0,0,1] \oplus [5,0,0,1,0] \oplus [2,0,1,1,0] \oplus [1,2,0,1,0] \oplus (2)[3,1,0,1,0] \oplus [2,0,0,1,2] \oplus [3,0,1,0,1] \oplus [4,1,0,0,1] \oplus [1,1,1,0,1] \oplus [2,2,0,0,1] \oplus [3,0,0,2,1] \oplus [4,0,0,3,0] \oplus [5,1,0,1,0] \oplus [4,0,1,1,0] \oplus [3,2,0,1,0] \oplus [2,1,1,1,0]$

- Level-12: $[4,0,0,0,0] \oplus [0,2,0,0,0] \oplus [1,0,0,0,2] \oplus [2,1,0,0,0] \oplus [2,0,0,1,1] \oplus [4,1,0,0,0] \oplus [3,0,1,0,0] \oplus [2,2,0,0,0] \oplus [3,0,0,2,0] \oplus [3,0,0,0,2] \oplus [1,1,1,0,0] \oplus [1,1,0,0,2] \oplus [4,0,0,1,1] \oplus [2,0,2,0,0] \oplus [4,2,0,0,0] \oplus [5,0,0,2,0] \oplus [2,1,0,1,1] \oplus [3,1,1,0,0] \oplus [3,1,0,2,0]$

- Level-13: $[3,0,0,0,1] \oplus [1,1,0,0,1] \oplus [4,0,0,1,0] \oplus [2,1,0,1,0] \oplus [2,0,1,0,1] \oplus [3,1,0,0,1] \oplus [3,0,1,1,0] \oplus [4,1,0,1,0]$
- Level-14: $[2,0,1,0,0] \oplus [3,1,0,0,0] \oplus [3,0,0,1,1] \oplus [4,0,1,0,0]$
- Level-15: $[3,0,0,1,0] \oplus [4,0,0,0,1]$
- Level-16: $[4,0,0,0,0]$

Appendix G

Permutations Expressed As Matrices

In this appendix, we simply express the permutations associated with $\mathbb{S}_4$ as 4×4 matrices.

$$() = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (12) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (13) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix},$$

$$(14) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (23) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (24) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$(34) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (123) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (124) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$(132) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}, \quad (134) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (142) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix},$$

$$(143) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (234) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (243) = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$(1234) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \end{bmatrix}, \quad (1243) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (1324) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix},$$

$$(1342) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (1423) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (1432) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{bmatrix},$$

$$(12)(34) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}, \quad (13)(24) = \begin{bmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{bmatrix}, \quad (14)(32) = \begin{bmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$

Appendix H

Eigenvalue Equivalence Classes

Eigenvalues	# of Matrices	Trace
12, 8, 4, 0	144	24
12, 8, 0, 0	288	20
12, 4, 0, 0	288	16
12, 8, -4, 0	144	16
12, 0, 0, 0	1,152	12
$12, -4\sqrt{2}, 4\sqrt{2}, 0$	288	12
$12, -4i\sqrt{2}, 4i\sqrt{2}, 0$	288	12
12, -4, 0, 0	288	8
12, -8, 4, 0	144	8
12, -8, 0, 0	288	4
12, -8, -4, 0	144	0

Table H.1: Eigenvalue Equivalent Classes of 3,456 Distance Matrices in Class (1)

Eigenvalues	# of Matrices	Trace
12, 4, 0, 0	1,152	16
12, 0, 0, 0	4,608	12
12, -4, 0, 0	1,152	8

Table H.2: Eigenvalue Equivalent Classes of 6,912 Distance Matrices in Class (2.1)

Eigenvalues	# of Matrices	Trace
12, 8, 0, 0	576	20
12, 0, 0, 0	2,304	12
12, -8, 0, 0	576	4

Table H.3: Eigenvalue Equivalent Classes of 3,456 Distance Matrices in Class (2.2)

Eigenvalues	# of Matrices	Trace
12, 0, 0, 0	3,456	12

Table H.4: Eigenvalue Equivalent Classes of 3,456 Distance Matrices in Class (2.3)

Eigenvalues	# of Matrices	Trace
12, 4, 4,0	144	20
12, 4, 0, 0	576	16
12, 0, 0, 0	1,152	12
12, -4, 4, 0	576	12
$12, 4i, -4i, 0$	288	12
12, -4, 0, 0	576	8
12, -4, -4, 0	144	4

Table H.5: Eigenvalue Equivalent Classes of 3,456 Distance Matrices in Class (2.4)

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