

Abstract of “Aspects of High-Strain-Rate Microcavitation in Soft Solids”

by Anastasia Tzoumaka, Ph.D., Brown University, October 2022.

Inertial microcavitation rheometry (IMR) is an effective method for mechanical characterization of soft materials at high strain rates. In IMR, the dynamics of an isolated bubble is measured, and a theoretical modeling framework is used to extract information regarding the mechanical behavior of the soft material. IMR’s ability to provide accurate material characterization at high strain rates relies on the predictive capability of the modeling framework, and this thesis addresses several problems related to robust modeling of inertial microcavitation in soft solids.

Modeling approaches based on the Rayleigh-Plesset equation are commonly used but cannot address violent bubble collapse that involves appreciable compressible behavior and limit the nonlinear constitutive models that may be employed. We develop a simulation capability for inertial microcavitation of spherical bubbles, enabling appreciable compressibility to be accounted for and more complex viscoelastic constitutive laws to be incorporated. We apply the capability to understanding the role of compressibility during violent bubble collapse and consider more complex viscoelastic constitutive models for the surrounding material to determine high strain-rate material parameters for polyacrylamide gels.

We apply IMR to characterize the mechanical behavior of fixed porcine gray matter. Because brain tissue samples are not optically clear, microcavitation experiments are performed in thin layers, which are constrained between two walls. To account for the thin layer confinement, we develop a two-dimensional axisymmetric finite-element approach, which is a high-fidelity numerical tool that allows us to account for nonlinear interactions between the bubble and the walls. We then apply the framework to estimate the mechanical properties of fixed porcine brain tissue at high-strain rates.

We investigate the conditions under which non-spherical instabilities arise due to cavitation dynamics using three-dimensional post-buckling-type simulations. Comparisons are made against a theoretical linear stability analysis, and based on the simulated amplitude of the non-spherical bubble shape, a growth-rate based threshold for the onset of non-spherical instability is proposed.

Finally, we develop a three-dimensional finite-element framework to account for material anisotropy by implementing a quadratic standard reinforcing model and study how the bubble aspect ratio depends on anisotropic material properties during inertial microcavitation.

Aspects of High-Strain-Rate Microcavitation in Soft Solids

by

Anastasia Tzoumaka

Sc.M., Engineering, Brown University, RI, USA, 2018

Dipl. M.E., University of Thessaly, 2016

Submitted in partial fulfillment of the requirements for the
Degree of Doctor of Philosophy in the School of Engineering at Brown University

Providence, Rhode Island

October 2022

© Copyright 2022 by Anastasia Tzoumaka

This dissertation by Anastasia Tzoumaka is accepted in its present form
by the School of Engineering as satisfying the
dissertation requirement for the degree of Doctor of Philosophy.

Date _____

David L. Henann, Ph.D., Advisor

Recommended to the Graduate Council

Date _____

Pradeep Guduru, Ph.D., Reader

Date _____

Mauro Rodriguez Jr., Ph.D., Reader

Approved by the Graduate Council

Date _____

Thomas A. Lewis, Ph.D.
Dean of the Graduate School

Curriculum Vitæ

For undergraduate work, Anastasia Tzoumaka received a Diploma in Mechanical Engineering from the University of Thessaly in 2016. She was granted a Master of Science in Engineering from Brown University in 2018. She received an outstanding teaching assistant award in 2020 for ENGN0310.

Refereed Journal Publications

Anastasia Tzoumaka, Jin Yang, Selda Buyukozturk, Christian Franck, and David L. Henann. “Modeling high strain-rate microcavitation in soft materials: The role of material behavior in bubble dynamics.” *Soft Matter* submitted.

Jin Yang, **Anastasia Tzoumaka**, Kazuya Murakami, Eric Johnsen, David L. Henann, and Christian Franck. “Predicting complex nonspherical instability shapes of inertial cavitation bubbles in viscoelastic soft matter.” *Physical Review E* 104 (2021): 045108.

Conference Proceedings

“Modeling High-Strain-Rate Inertial Microcavitation in Soft Materials: The Role of Material Response in Bubble Dynamics.” US National Congress on Theoretical and Applied Mechanics, Austin, TX, June 23, 2022.

“Modeling high-strain-rate microcavitation in soft materials.” New England Workshop on the Mechanics of Materials and Structures (NEW.Mech), Cambridge, MA, May 6, 2022.

“Modeling high-strain-rate microcavitation in soft materials: The role of material response.” APS March Meeting, Boston, MA, March 4, 2019.

Teaching Experience at Brown University

ENGN 1750: Advanced Mechanics of Solids, STA (Fall 2021)

ENGN 0310: Mechanics of Solids and Structures, STA (Fall 2017, 2018, 2019)

STEM instructor: Introduction in Engineering for high school students. (Summer,2018)

Acknowledgements

First and foremost, I would like to express my deepest gratitude to my advisor, Professor David Henann, for being such a great mentor and for his support and guidance during my Ph.D. studies. His passion for research, dedication to the field and excellence in teaching has contributed immensely in my personal development as a researcher in the field. I feel very fortunate to have him as my advisor in my PhD studies. This was the best journey in my career so far.

I would like to thank my committee members Professor Pradeep Guduru for his valuable input and feedback to my research as well as being a great examiner during my program review and prelims and Professor Mauro Rodriguez for the valuable input and feedback to my research as well as all the great discussions and constructive criticism I received. Furthermore, I would like to thank Professor Alan Bower not only for giving valuable feedback on my research during my progress review but also for teaching classes in such a unique way that made me appreciate the beauty of mechanics.

I would like to thank Professor Nikolaos Aravas who was my first mentor and my undergraduate thesis advisor and helped me make my dream for graduate studies in the US come true.

I was very lucky to be part of a big collaborative, the BRC project. Thus, I would like to thank all my collaborators in the BRC project: Prof. Jin Young, Dr. Selda Buyukozturk, Elizabeth Bremer, Dr. Kazuya Murakami, Prof. Christian Franck, Prof. Jon Estrada and Prof. Eric Johnsen.

The Henann's lab is a great place to be. I feel very lucky to be sharing the lab all these years with some amazing people. I would like to thank my friends and colleagues Daren Liu, Shihong Li, Xiuqi Li and Harkirat Singh and especially Shihong and Harkirat for their valuable discussions and support during my job search period.

I would like to thank my parents for their support and encouragement all these years and last but not least, I would like to thank my husband Nick for his constant support and encouragement all these years despite the fact he was pursuing his PhD at Harvard. His companionship and understanding through the years (since undergrad) was a constant source of motivation and dedication to continue pursuing my dreams.

This work was supported by funds from the Office of Naval Research under grants N00014-18-1-2625, N00014-21-1-2109, and N00014-22-1-2094 (under Dr. Timothy Bentley). Part of this research was conducted using computational resources at the Center for Computation and Visualization (CCV) at Brown University.

TO MY FAMILY

Contents

List of Tables	xiii
List of Figures	xv
1 Introduction	1
1.1 Background on inertial microcavitation in soft solids	1
1.2 Thesis contributions	3
1.2.1 Modeling one-dimensional spherical bubble dynamics using finite elements	4
1.2.2 Material characterization of fixed porcine brain tissue	4
1.2.3 Predicting the onset of non-spherical instability	5
1.2.4 The role of material anisotropy on bubble dynamics	6
1.3 Thesis outline	6
2 Modeling inertial microcavitation in soft materials: The role of material behavior in bubble dynamics	9
2.1 Introduction	9
2.2 Continuum framework	12
2.2.1 Kinematics	13
2.2.2 Viscoelastic constitutive models for the surrounding material	14
2.2.3 Balance of linear momentum	19
2.3 Modeling of bubble dynamics	21
2.3.1 Keller-Miksis equation	21
2.3.2 Full-field solution using finite elements	23
2.3.3 Initial conditions in laser-induced and ultrasound-induced cavitation bubbles	25
2.4 Effect of material compressibility on bubble dynamics	27
2.5 Characterizing the high-strain-rate behavior of polyacrylamide gels	34
2.5.1 Stiff polyacrylamide gel	34
2.5.2 Soft polyacrylamide gel	37
2.6 Concluding remarks	40
3 Characterizing the mechanical behavior of fixed porcine brain tissue at high strain rates	43
3.1 Introduction	43
3.2 LIC experiments in porcine brain tissue	46
3.3 Axisymmetric finite-element modeling of LIC in a thin layer	49
3.4 High-strain rate characterization of fixed porcine gray matter	52

3.5	Discussion	54
3.5.1	Extended Keller-Miksis equation for thin layers	55
3.5.2	Comparison between the two methods	60
3.6	Concluding remarks	66
4	Predicting the onset of non-spherical instability during inertial microcavitation in soft solids	69
4.1	Introduction	69
4.2	Continuum framework	71
4.2.1	Three-dimensional governing equations	71
4.2.2	Linear perturbation analysis	73
4.2.3	Nonlinear dynamic post-buckling analysis	80
4.3	Results and discussion	83
4.3.1	Effect of radius ratio	83
4.3.2	Effect of viscosity and surface tension on the bubble stability	89
4.3.3	Strain and stress localization	94
4.4	Concluding remarks	96
5	The role of material anisotropy in inertial microcavitation	99
5.1	Introduction	99
5.2	Methods	100
5.2.1	Transversely isotropic hyperelastic material model	101
5.2.2	Finite-element modeling	103
5.3	Results and discussion	105
5.3.1	Ultrasound-induced cavitation	105
5.3.2	Laser-induced cavitation	108
5.4	Concluding remarks	109
6	Conclusions and outlook	111
6.1	Concluding remarks	111
6.2	Future directions	113
A	Physics of the bubble contents	117
B	Driving conditions for laser-induced and ultrasound-induced cavitation in Abaqus	123
B.1	Laser-induced cavitation	123
B.2	Ultrasound-induced cavitation	126
C	Effect of experimental sampling frequency on fitted parameters	127

List of Tables

2.1	Parameters used in the studies of Sections 2.4 and 2.5	27
4.1	Baseline material properties for the qKV model used in the nonlinear dynamic post-buckling analyses	82
5.1	Material properties used for the study of the effect of anisotropy on bubble dynamics	103
A.1	Parameters describing the physics of the bubble contents	118
A.2	Dimensionless quantities involved in the physics of the bubble contents . .	120

List of Figures

1.1	Illustration of material characterization techniques at different strain rates for different characteristic stress levels	2
2.1	Schematic representation of the reference and deformed configurations . . .	14
2.2	Rheological schematics of the nonlinear, finite-deformation viscoelastic modeling approaches for KV-type and SNS-type models	15
2.3	Effect of the radius ratio $R_{\max}/R_0$ and the viscosity of the surrounding material μ on normalized radius versus time curves for LIC	28
2.4	Effect of the peak tensile pressure $ p_A /p_\infty$ and the viscosity of the surrounding material μ on normalized radius versus time curves for UIC	29
2.5	Plots of ΔR_{RMS} versus the maximum Mach number for LIC and UIC	31
2.6	Plots of $\Delta_{\max}$ versus the maximum Mach number for LIC and UIC	32
2.7	Comparison of experimental data, numerical KM-IMR results, and numerical FE-IMR results for the normalized radius versus time history for stiff PA gel	35
2.8	Comparison of experimental data, numerical FE-IMR results using the Newtonian qKV model, and numerical FE-IMR results using the non-Newtonian model for normalized radius versus time histories for soft PA gel	38
3.1	Illustration of the experimental protocol followed to prepare optically clear samples for LIC experiments in porcine brain tissue	47
3.2	Illustration of the experimental setup for LIC in a $350\mu\text{m}$ thick slice of fixed porcine brain tissue, images from the lifespan of a single LIC bubble within a thin layer of a fixed porcine brain tissue, and experimentally obtained radius versus time curves for three different LIC experiments	48
3.3	Illustration of the two-dimensional axisymmetric finite-element configuration. We simulate the one fourth of the original geometry. The top surface is the glass plate boundary where, we account for a no-slip boundary condition.	50
3.4	Application of axisymmetric, finite-element-based IMR to characterization of fixed porcine gray matter	53
3.5	Illustration of two interacting bubbles with different radii $R_1(t)$ and $R_2(t)$ separated by a distance h	55
3.6	Illustration of an infinite sequence of image bubbles in the Keller-Miksis framework for a thin layer.	59
3.7	Contour plots of the ΔR_{RMS}^* difference between the axisymmetric finite-element approach and the modified Keller-Miksis approach with two image bubbles in a $500\mu\text{m}$ thick sample	62

3.8	Contour plots of the ΔR_{RMS}^* difference between the axisymmetric finite-element approach and the modified Keller-Miksis approach with two image bubbles in a $350\mu\text{m}$ thick sample	63
3.9	Contour plots of the best-fitted G and μ for a $500\mu\text{m}$ layer thickness using the modified Keller-Miksis approach	64
3.10	Contour plots of the best-fitted G and μ for a $350\mu\text{m}$ layer thickness using the modified Keller-Miksis approach	65
4.1	Experimental snapshot of a non-spherical bubble shape and corresponding simulation domain with a perturbed geometry	81
4.2	Results of the nonlinear dynamic post-buckling analysis: Computed mean radius and perturbation magnitude versus time curves for three different radius ratios	84
4.3	Results of the nonlinear dynamic post-buckling analysis: Computed mean radius, η , and ξ versus time curves for two different radius ratios	85
4.4	Results of the nonlinear dynamic post-buckling analysis: Computed growth rate, mean radius, and perturbation amplitude versus time curves for three different radius ratios	87
4.5	Non-dimensionalized growth rate for different non-spherical mode shape numbers	89
4.6	Results of the nonlinear dynamic post-buckling analysis: Computed mean radius, η , and ξ versus time curves for two different viscosities	91
4.7	Results of the nonlinear dynamic post-buckling analysis: Computed growth rate, mean radius, and perturbation amplitude versus time curves for two different viscosities	92
4.8	Results of the nonlinear dynamic post-buckling analysis: mean radius, and perturbation amplitude versus time curves for three different elastocapillary numbers	94
4.9	Contours of the radial and circumferential components of the logarithmic strain tensor for spherical and perturbed bubbles	95
4.10	Contours of the radial and circumferential components of the Cauchy stress tensor for spherical and perturbed bubbles	96
5.1	Schematic of the finite-element configuration for one eighth of the spherical bubble geometry	104
5.2	Spatiotemporal evolution of a single bubble within a soft anisotropic solid during ultrasound-induced cavitation	105
5.3	Effect of anisotropy on bubble dynamics during UIC	106
5.4	Effect of the anisotropy parameter θ/G on the maximum bubble radii in UIC	107
5.5	Spatiotemporal evolution of a single bubble within a soft anisotropic solid during laser-induced cavitation	108
5.6	Effect of anisotropy on bubble dynamics during LIC	109
C.1	Regions of the radius history used for the purpose of fitting in the sampling frequency study	128
C.2	Contour plots showing the error in the fitting of the parameter α in the case of LIC for three different sampling frequencies	130
C.3	Contour plots showing the error in the fitting of the viscosity μ in the case of LIC for three different sampling frequencies	132

C.4	Contour plots showing the error in the fitting of the parameter α in the case of UIC (Region I) for three different sampling frequencies	133
C.5	Contour plots showing the error in the fitting of the viscosity μ in the case of UIC (Region I) for three different sampling frequencies	134
C.6	Contour plots showing the error in the fitting of the parameter α in the case of UIC (Region II) for three different sampling frequencies	135
C.7	Contour plots showing the error in the fitting of the viscosity μ in the case of UIC (Region II) for three different sampling frequencies	136

Chapter 1

Introduction

1.1 Background on inertial microcavitation in soft solids

A number of important engineering applications, such as modeling head injuries due to blunt impact and blast [1–4], estimating tissue damage during histotripsy procedures [5–7], and understanding the physics involved in laser-based surgical procedures [8, 9], necessitate mechanical characterization of soft materials at high-strain rates. Although existing techniques shown in Fig. 1.1 have contributed much to our current understanding of the mechanical behavior of soft materials, many of them—such as tension/compression testing [10–12], indentation [13], needle-induced cavitation (NIC) [14–16], and volume-controlled cavity expansion (VCCE) [17–19]—have focused on the low to moderate strain-rate regime. Moreover, high strain-rate methodologies, such as Kolsky bar or Taylor plate-impact testing, are inadequate to characterize soft solids due to the low impedance and slow wave speed of the latter. In recent years, significant effort has gone towards developing new characterization techniques capable of extracting information on the high-strain rate regime based on dynamic cavitation.

Dynamic cavitation, is the process whereby vapor cavities are produced in a fluid or solid due to pressure changes, leading to the dynamic growth, collapse, and rebound of

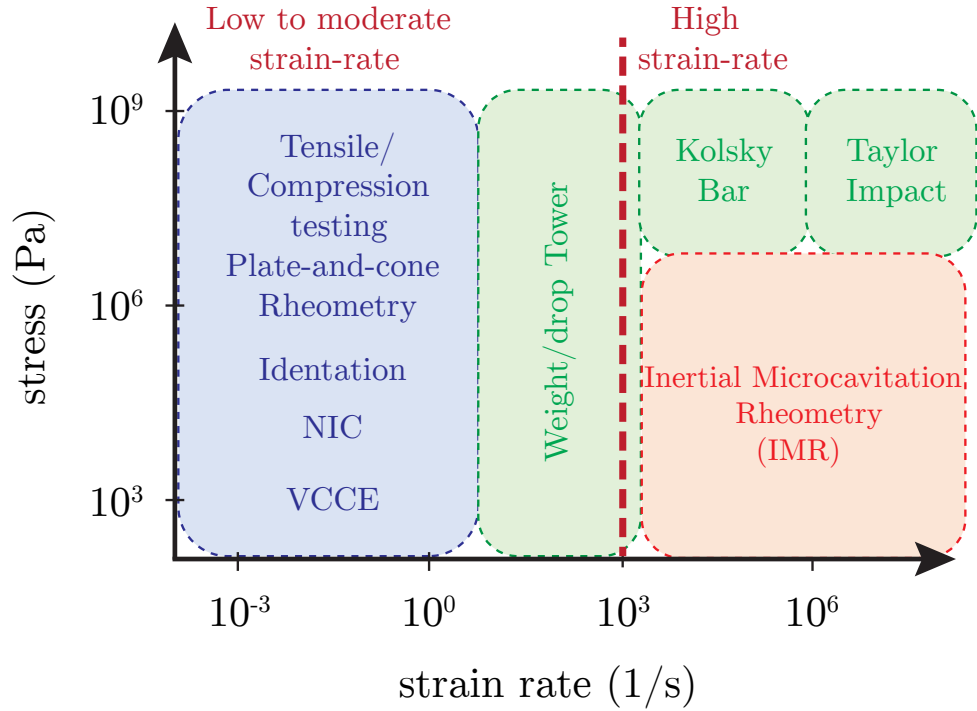


Figure 1.1: Illustration of material characterization techniques at different strain rates for different characteristic stress levels. The Inertial Microcavitation Rheometry (IMR) is a local microrheology technique used for the characterization of soft materials at high strain-rates.

a bubble. Cavitation in fluids has been the subject of extensive study for many decades, and the understanding of cavitation in fluids is quite mature [e.g., 20–27]. Cavitation can also occur in soft solids, such as biological tissues, but due to the nonlinear deformation of soft solid materials, the established theoretical foundation underlying the modeling of cavitation in fluids is not valid for soft solids. Unlike the case of fluids, the dynamics of cavitation in soft solids is strongly affected by their complex material behavior, such as nonlinear elasticity and rate-dependent rheological behavior. The bubble dynamics are a direct function of the soft material’s mechanical properties, and therefore, information about the mechanical response of the soft material at high strain rates may be extracted from the measured bubble dynamics, motivating cavitation-based rheometry.

Dynamic¹ cavitation forms the foundation of a new technique for material characterization called Inertial Microcavitation high strain-rate Rheometry (IMR) [28]. IMR belongs to a class of cavitation-based material characterization techniques [29]—including needle-induced cavitation [14–16] and volume-controlled cavity expansion [17–19]—and is unique in its ability to mechanically characterize soft materials at high strain rates greater than 10^3 s^{-1} . In IMR, an isolated, spherical microbubble is generated inside a soft material, using either a spatially-focused pulsed laser or focused ultrasound, to probe the mechanical behavior of the soft material at high strain rates ($> 10^3 \text{ s}^{-1}$). Then, since the bubble dynamics involve inhomogeneous deformation in the surrounding soft material, a cavitation modeling framework that accounts for the dominant physics including the constitutive behavior of the soft material is required. Using experimentally-measured high-speed time-lapse imaging of the bubble dynamics induced by cavitation in concert with the modeling framework provides a robust route to high-strain rate soft material characterization. Due to the short life-span of cavitation bubbles, the bubble dynamics induce high strain-rate deformation ($> 10^3 \text{ s}^{-1}$) in the surrounding material, and therefore, this technique enables the characterization of soft materials at high-strain rates—which is inaccessible using the aforementioned traditional techniques for high strain-rate mechanical characterization of hard materials.

1.2 Thesis contributions

Since the ability of IMR to provide accurate, quantitative material characterization at high strain rates relies upon the predictive capability of the cavitation modeling framework, the overarching motivation of this thesis is to develop robust numerical frameworks for modeling inertial microcavitation in soft solids that incorporate the appropriate constitutive behavior (e.g., nonlinear viscoelasticity, compressibility, and anisotropy) and enable the material characterization of a wide range of soft solids and biological tissues at high strain

¹Dynamic and inertial cavitation are used interchangeably throughout this thesis, and they represent the same phenomenon.

rates. The specific problems that this thesis addresses are summarized as follows.

1.2.1 Modeling one-dimensional spherical bubble dynamics using finite elements

We have developed a finite-element-based approach for modeling the dynamics of spherical bubbles in soft viscoelastic media which can be applied to both laser-induced and ultrasound-induced cavitation. The FE-based approach addresses two aspects of the constitutive response of the surrounding material: material compressibility and complex rheological behavior. Regarding material compressibility, we establish a Mach-number-type threshold beyond which material compressibility has a strong effect and influences the resulting bubble dynamics. We verify this threshold by performing fittings to a stiff polyacrylamide gel and compare against the Keller-Miksis-based approach. Regarding complex rheological behavior, we propose a non-Newtonian, power-law version of the quadratic Kelvin-Voigt (qKV) rheological behavior for the surrounding material and apply the model to a soft polyacrylamide gel over a range of cavitation amplitudes to obtain a set of material parameters for soft PA gel at high strain-rates.

1.2.2 Material characterization of fixed porcine brain tissue

We apply IMR to characterize the material behavior of fixed porcine gray matter at high strain rates. Because the bulk brain tissue is not optically clear, LIC experiments were performed in thin slices of brain tissue, which were constrained between two parallel glass coverslips. Due to the glass constraints, the one-dimensional finite-element framework is no longer able to provide an accurate description of the physics in a thin, confined layer because boundary effects are present. For this reason, we present an axisymmetric finite-element approach to account for nonlinear interactions between the bubble and the two rigid walls. We then utilize the finite-element framework and perform material characterization in fixed

porcine gray matter tissue. Because simulations based on the Keller-Miksis equation are more computationally efficient and allow the incorporation of more complex physics inside the bubble, we discuss an extended Keller-Miksis equation that accounts for thin layers by utilizing the concept of Bjerknes forces between interacting bubbles and the method of images. Finally, we compare the two methods by providing error estimates on the fitting parameters when the modified Keller-Miksis equation is used for material characterization instead of the axisymmetric finite-element method.

1.2.3 Predicting the onset of non-spherical instability

Recent experiments and numerical simulations have shown that spherical symmetry can often break down during inertial cavitation, even in nominally homogeneous and isotropic soft matter [30–34]. Compared to water, cavitation in a soft viscoelastic solid material can produce distinctively different complex (non-spherical) instability patterns, including bubble surface wrinkles and creases. Wrinkles and creases can significantly degrade the performance of any soft matter system and can be especially harmful when occurring in soft biological tissues, generating unwanted collateral damage in eye laser surgeries [9] or histotripsy procedures [5–7, 35, 36]. Yet, these cavitation-related instabilities have remained largely unstudied for soft, nonlinear solids. To this end, we present three-dimensional, finite-element-based simulations using Abaqus to model the bubble dynamics due to cavitation and explore the onset of non-spherical bubble shapes during cavitation. We account for large deformations, strain stiffening effects, and material viscosity to describe the behavior of the surrounding material. We apply a small geometric imperfection on the initial geometry and perform post-buckling-type simulations. This numerical framework is capable of accounting for non-spherical behavior to predict the transition from spherical to non-spherical bubbles during inertial microcavitation. With nonlinear, three dimensional post-buckling-type simulations, we verify the linear stability theory developed by Yang et al. [37]. Finally, we examine the effect of the maximum stretch, material viscosity, and surface tension on the

stability of bubble dynamics.

1.2.4 The role of material anisotropy on bubble dynamics

A goal of IMR is to enable the mechanical characterization of soft tissues at high strain rates and, in particular, study the mechanical behavior of biological materials such as tissues, collagen etc. So far, regarding biological tissue, we have applied in IMR in fixed porcine gray matter. However, as other works [38, 39] have documented, other parts of porcine brain tissue, such as white matter or specific regions of gray matter, are known to be anisotropic. Yet, the existing numerical framework for material characterization in IMR accounts only for isotropic material behavior. To this end, we present a finite-element based numerical framework that accounts for material anisotropy in the surrounding material by implementing the compressible version of the quadratic standard reinforcing model—a nonlinear hyperelasticity model for transversely isotropic materials. We utilize the three-dimensional finite-element capability and examine the effect of anisotropy on the bubble dynamics in both ultrasound-induced and laser-induced cavitation.

1.3 Thesis outline

This thesis is outlined around the above four contributions. Chapter 2 presents our finite-element-based numerical simulation capability for inertial microcavitation of spherical bubbles that enables appreciable compressibility to be accounted for and more complex viscoelastic constitutive laws to be incorporated. We validate our method by performing material characterization of stiff and soft polyacrylamide gels. Chapter 3 presents a robust finite-element numerical framework for inertial microcavitation in confined geometries that enables the interactions between the bubble and the two rigid walls within a soft material to be accurately captured and leads to an accurate estimation of the material parameters of

fixed porcine gray matter at high strain rates. Chapter 4 presents a three-dimensional, finite-element-based study that is capable of accounting for non-spherical behavior to predict the transition from spherical to non-spherical bubbles during the dynamics and makes comparisons against the results of a linear stability analysis. Chapter 5 presents the effect of material anisotropy on bubble dynamics during inertial microcavitation. We close in Chapter 6 with a summary of the major contributions of the thesis and future research directions.

Chapter 2

Modeling inertial microcavitation in soft materials: The role of material behavior in bubble dynamics

Note: A version of this chapter has been submitted to *Soft Matter*. Data and figures have been used with all co-authors' consent.

Anastasia Tzoumaka, Jin Yang, Selda Buyukozturk, Christian Franck, and David L. Henann.
“Modeling high strain-rate microcavitation in soft materials: The role of material behavior in bubble dynamics.”

2.1 Introduction

Soft materials—such as biological tissues and gels used as tissue phantoms—undergoing high strain-rate deformation play a crucial role in a number of important engineering problems—e.g., modeling head injuries due to blunt impact and blast [1–4], estimating tissue damage during histotripsy procedures [5–7], and understanding the physics involved in laser-based surgical procedures [8, 9]. Therefore, there is a need for mechanical characterization of

soft materials at high strain rates ($> 10^3 \text{ s}^{-1}$). While low-to-moderate strain-rate (10^{-4} – 10^2 s^{-1}) testing techniques—such as tension-compression experiments, indentation [13], needle-induced cavitation [14, 15], and volume-controlled cavity expansion [17–19]—have been successfully deployed to understand the mechanical behavior of soft materials, applying traditional high-strain-rate material characterization techniques—e.g., Kolsky bar [40] or plate-impact testing [41]—to soft materials remains challenging, due to their low impedance and slow shear wave speeds.

To address this challenge, significant effort in recent years has gone towards developing new characterization techniques capable of extracting information on the high strain-rate mechanical behavior of soft materials—including small-scale ballistic cavitation [16], laser-induced particle impact testing [42], shear impact testing [43], and Inertial Microcavitation Rheometry (IMR) [28, 44–47], which is the focus of the present work. IMR belongs to a class of cavitation-based material characterization techniques [29]—including needle-induced cavitation [14–16] and volume-controlled cavity expansion [17–19]—and is unique in its ability to mechanically characterize soft materials at high strain rates greater than 10^3 s^{-1} . In IMR, a single spherical microbubble is generated inside a soft material using either a spatially-focused pulsed laser [28, 48] or high-amplitude focused ultrasound [45, 49, 50], inducing high strain-rate ($> 10^3 \text{ s}^{-1}$) deformation in the surrounding material. The subsequent radius versus time behavior of the spherical bubble is measured using high-speed imaging. Then, since the bubble dynamics involves inhomogeneous deformation in the surrounding soft material, a cavitation modeling framework that accounts for the dominant physics, including the viscoelastic constitutive behavior of the soft material [28, 44, 51, 52], is required. By fitting predictions of the cavitation modeling framework to the experimentally measured bubble dynamics, nonlinear viscoelastic material parameters for a given constitutive model may be inferred, and the IMR approach has been successfully applied to measuring the viscoelastic mechanical properties of polyacrylamide [28, 44, 48], agarose [45, 47], and collagen [4] gels at high strain rates. Moreover, similar efforts

involving cavitation induced by a pulsed laser have been applied to fibrin and polyethylene glycol (600) diacrylate gels[53] as well as to polydimethylsiloxane gels [54] aided by laser-absorbing seed particles.

To obtain accurate estimates of high strain-rate material properties for soft materials using IMR, a high fidelity description of the dominant physics must be incorporated into the cavitation modeling framework. Approaches based on the Rayleigh-Plesset (RP) equation [20, 21] and its extensions are commonly used to describe cavitation dynamics in liquids and soft solids. In particular, the Keller-Miksis (KM) equation [23] extends the RP equation to account for slight compressibility and stress wave propagation in the surrounding material and has been used in many studies of cavitation in soft solids [28, 44–49, 51, 52]. However, use of both the RP and the KM equations places several limitations on the types of constitutive behavior for the surrounding material that may be included in the cavitation modeling framework. First, since the RP equation assumes the surrounding material to be incompressible and the KM equation only accounts for slight compressibility, predictions obtained using the RP or the KM equation can become inaccurate when the bubble dynamics involves violent collapse, which is accompanied by appreciable material compressibility. Second, the implementation of the viscoelastic, distortional material response into the RP/KM framework requires the integration of mathematical expressions, which are challenging to evaluate in closed form for sufficiently complex constitutive models. A number of nonlinear viscoelastic models have been implemented in the KM framework [44, 51, 52] in an effort to capture the mechanical behavior of soft materials; however, doing so remains mathematically intractable for many of the finite-deformation constitutive models utilized in the literature for soft solids [19, 55, 56]. Therefore, the RP/KM framework places limitations on the breadth of material behavior that may be considered to characterize the high strain-rate response of soft materials.

The purpose of this chapter is to address the limitations that the use of the RP/KM equation places on IMR. We develop a numerical framework for modeling the cavitation

dynamics of a single spherical bubble using the finite-element (FE) method, which enables the implementation of general constitutive models for the surrounding soft material. With this numerical tool, we incorporate a description of the compressibility of the surrounding material, which leads to a numerical framework that describes the bubble dynamics more accurately when studying violently collapsing bubbles. Furthermore, we consider more complex viscoelastic constitutive descriptions of the surrounding material, including nonlinear elastic and power-law viscous behavior, which enables a higher fidelity characterization of soft materials at high strain rates.

The remainder of this chapter is organized as follows. In Section 2.2, we summarize the continuum field equations governing the dynamics of spherical bubbles in soft viscoelastic media, including the nonlinear, finite-deformation viscoelastic constitutive models considered in our cavitation modeling framework. In Section 2.3, we describe the KM-based and FE-based modeling frameworks for both laser-induced and ultrasound-induced cavitation. Section 2.4 compares FE-based and KM-based simulations of bubble dynamics as bubble collapse becomes increasingly violent and proposes a criteria for when the bubble dynamics involves material compressibility beyond that accounted for in the KM-based approach, and Section 2.5 applies FE-based simulations in IMR to obtain improved high-strain-rate properties for polyacrylamide gels—specifically, using the laser-induced cavitation data of Yang et al. [44] and Buyukozturk et al. [48]. Finally, we close with some concluding remarks in Section 2.6.

2.2 Continuum framework

In this section, we summarize the continuum field equations governing the dynamics of spherical bubbles in soft viscoelastic media. We assume (1) that the bubble and the motion of the surrounding material remain spherically symmetric during the cavitation process and (2) that the surrounding material is a homogeneous, infinite, isotropic viscoelastic medium.

2.2.1 Kinematics

We denote the undeformed radius of the spherical bubble as R_0 and the referential radial coordinate in the surrounding material as r_0 , where r_0 is measured from the center of the bubble. The reference configuration for the surrounding material is identified by the material points $\{r_0 | R_0 \leq r_0 < \infty\}$. The time-dependent bubble radius in the deformed configuration is $R(t)$, and we assume a spherically symmetric motion, in which material points r_0 deform to spatial points $r = r(r_0, t)$ (see Fig. 2.1). The deformation gradient tensor $\mathbf{F}$ in the spherical coordinate system is

$$[\mathbf{F}] = \begin{bmatrix} \frac{\partial r}{\partial r_0} & 0 & 0 \\ 0 & \frac{r}{r_0} & 0 \\ 0 & 0 & \frac{r}{r_0} \end{bmatrix}, \quad (2.1)$$

with the volume ratio given by $J = \det\{\mathbf{F}\} > 0$.¹ For a spherically symmetric motion, the only non-zero component of the displacement vector is the radial component $u_r(r_0, t) = r(r_0, t) - r_0$. The left Cauchy-Green deformation tensor is $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$, the distortional deformation gradient tensor is $\bar{\mathbf{F}} = J^{-1/3}\mathbf{F}$, the distortional left Cauchy-Green deformation tensor is $\bar{\mathbf{B}} = J^{-2/3}\mathbf{B}$, and the distortional first principal stretch invariant is $\bar{I}_1 = \text{tr } \bar{\mathbf{B}}$.

The referential descriptions of the radial components of the velocity and acceleration vectors are $\dot{u}_r(r_0, t)$ and $\ddot{u}_r(r_0, t)$, respectively, where the superposed dot denotes the material time derivative. The spatial descriptions of the radial components of the velocity and acceleration vectors are $v_r(r, t)$ and $a_r(r, t)$, and the components of the spatial velocity gradient tensor $\mathbf{L}$ in the spherical coordinate system are

$$[\mathbf{L}] = \begin{bmatrix} \frac{\partial v_r}{\partial r} & 0 & 0 \\ 0 & \frac{v_r}{r} & 0 \\ 0 & 0 & \frac{v_r}{r} \end{bmatrix}. \quad (2.2)$$

¹*Notation:* We write $\text{tr } \mathbf{A}$, $\det\{\mathbf{A}\}$, $\text{dev } \mathbf{A}$, $\text{sym } \mathbf{A}$, and $\text{skw } \mathbf{A}$ to denote the trace, determinant, deviatoric part, symmetric part, and skew part of a tensor $\mathbf{A}$, respectively.

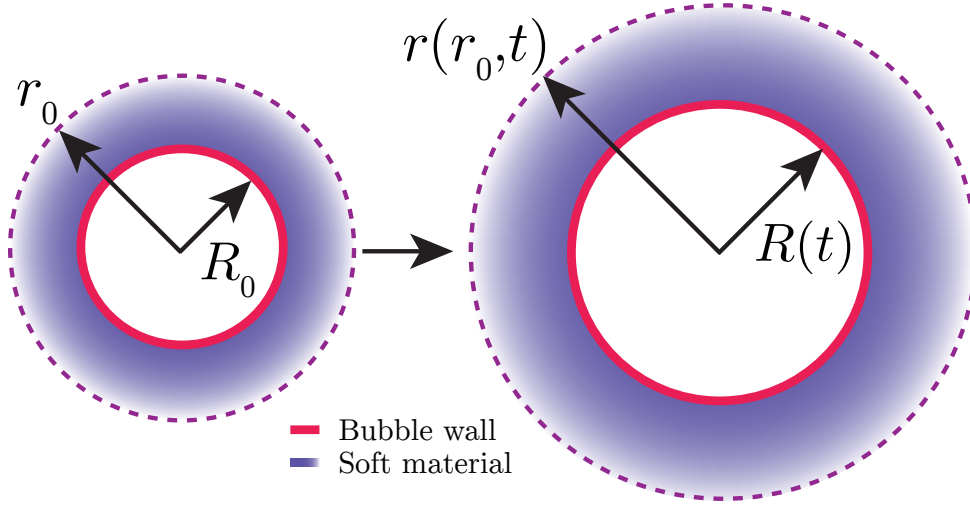


Figure 2.1: Schematic representation of the reference and deformed configurations. The radial coordinates in the reference and deformed configurations are denoted as r_0 and r , and the undeformed and deformed bubble radii are R_0 and R .

Finally, the stretching and spin tensors are $\mathbf{D} = \text{sym} \mathbf{L}$ and $\mathbf{W} = \text{skw} \mathbf{L}$, respectively.

2.2.2 Viscoelastic constitutive models for the surrounding material

Previous works [28, 44, 45, 47] have demonstrated that in order to accurately describe the nonlinear behavior of soft materials at high strain-rates, one must take into account both hyperelastic and viscous material behavior. In this subsection, we discuss the library of finite-deformation viscoelastic constitutive models that we have considered in our cavitation modeling framework, which fall into two categories. The first category is based on a nonlinear generalization of the Kelvin-Voigt (KV) model of linear viscoelasticity and consists of a hyperelastic response in parallel with a Newtonian rheological response, as illustrated in the rheological schematic of Fig. 2.2(a). We refer to this category of models as KV-type models. The second category is motivated by the standard linear solid model of linear viscoelasticity and consists of a hyperelastic response in parallel with a finite-deformation Maxwell element, as illustrated in Fig. 2.2(b). We refer to nonlinear generalizations of the standard linear solid model as Standard Nonlinear Solids (SNS), or SNS-type models.

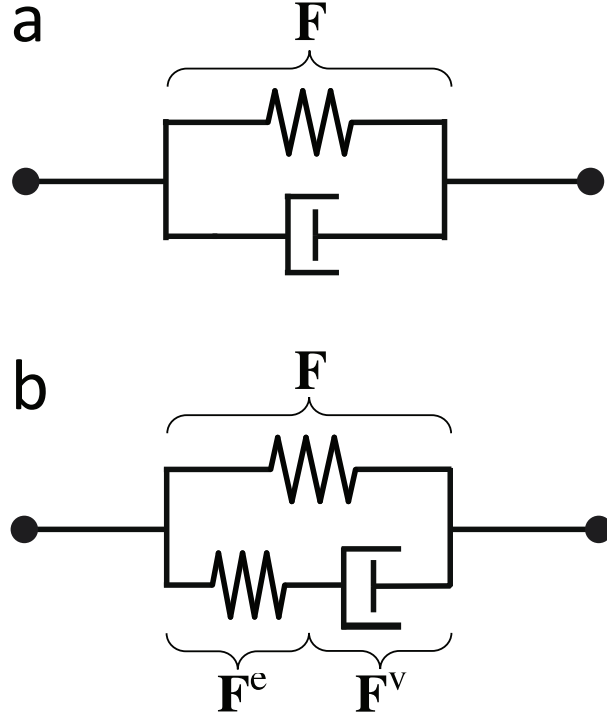


Figure 2.2: Rheological schematics of the nonlinear, finite-deformation viscoelastic modeling approaches for (a) KV-type models and (b) SNS-type models.

KV-type models

Motivated by the Kelvin-Voigt model of linear viscoelasticity, we consider a finite-deformation viscoelastic model consisting of a hyperelastic response (top branch in Fig. 2.2(a)) in parallel with a Newtonian viscous response (bottom branch in Fig. 2.2(a)). The contributions to the Cauchy stress due to the hyperelastic response and the viscous response are denoted as σ^e and σ^v , respectively, and the total Cauchy stress is $\sigma = \sigma^e + \sigma^v$.

The referential free-energy density describing the hyperelastic response ψ is taken to depend upon the invariants $\bar{I}_1$ and J for slightly-compressible, isotropic materials. In order to account for strain-stiffening effects in the surrounding material, we consider a quadratic-law distortional hyperelastic response [44, 57] with free-energy density given by

$$\psi = \frac{G}{2} \left[(\bar{I}_1 - 3) + \frac{\alpha}{2} (\bar{I}_1 - 3)^2 \right] + \frac{K}{2} (J - 1)^2, \quad (2.3)$$

where G is the ground-state shear modulus, α is a dimensionless parameter that describes large-deformation strain-stiffening behavior, and K is the ground-state bulk modulus. The first terms represent the free energy due to distortional (constant volume) deformation, and the last term in (2.3) represents the free energy due to purely volumetric deformation. Note that when $\alpha = 0$, (2.3) reduces to the free-energy density for a slightly-compressible Neo-Hookean material. Regarding compressibility, the ground-state bulk modulus K may be given through the longitudinal wave speed c and the ground-state shear modulus G by $K = \rho_0 c^2 - (4/3)G$, where ρ_0 is the referential mass density. For many soft viscoelastic materials, $K \gg G$, but instead of idealizing the material as incompressible, we treat the material as slightly compressible and examine the effect of compressibility on bubble dynamics in Section 2.4. The associated hyperelastic contribution to the Cauchy stress is then

$$\boldsymbol{\sigma}^e = 2J^{-1} \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} = J^{-1} G [1 + \alpha(\bar{I}_1 - 3)] \text{dev } \bar{\mathbf{B}} + K(J - 1) \mathbf{1}. \quad (2.4)$$

Neglecting volumetric viscosity, the viscous contribution to the Cauchy stress for Newtonian rheological behavior is

$$\boldsymbol{\sigma}^v = 2\mu \text{dev}(\mathbf{D}), \quad (2.5)$$

where μ is the shear viscosity. We note that it can be difficult to incorporate more complex rheological behavior into KV-type models. In particular, rheological constitutive equations involving shear-thinning, power-law behavior or a yield stress exhibit a strain-rate dependent viscosity that diverges as the strain-rate approaches zero, which is problematic in the modeling of bubble dynamics since the strain rate approaches zero far from the bubble as $r \rightarrow \infty$. Complex rheological behavior may be more readily incorporated in SNS-type models.

The total Cauchy stress is then

$$\boldsymbol{\sigma} = \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^v = J^{-1} G [1 + \alpha(\bar{I}_1 - 3)] \text{dev } \bar{\mathbf{B}} + K(J - 1) \mathbf{1} + 2\mu \text{dev}(\mathbf{D}). \quad (2.6)$$

When $\alpha > 0$, the model accounts for strain-stiffening, and we refer to this model as the *quadratic-law Kelvin-Voigt (qKV) model*. The qKV model involves the following four material parameters: the ground-state shear modulus G , the strain-stiffening parameter α , the bulk modulus K , and the shear viscosity μ . When strain-stiffening is neglected, $\alpha = 0$, and three material parameters remain. We refer to this model as the *Neo-Hookean Kelvin-Voigt (NH-KV) model*.

SNS-type models

SNS-type constitutive modeling approaches are based on a decomposition of the material response into an equilibrium hyperelastic response and a dissipative, non-equilibrium contribution. With reference to the schematic in Fig. 2.2(b), the top branch corresponds to the equilibrium hyperelastic response of the material, and the bottom branch stands for the non-equilibrium, rate-dependent material response, represented by the finite-deformation Maxwell element in parallel with the equilibrium branch.

The referential free-energy density ψ is additively decomposed into equilibrium and non-equilibrium contributions: $\psi = \psi^{\text{eq}} + \psi^{\text{neq}}$, and the equilibrium hyperelastic response is specified through the free-energy density ψ^{eq} , which we take to be given through (2.3), so that the corresponding equilibrium contribution to the Cauchy stress σ^{eq} is in the same form as given in (2.4).

The non-equilibrium, rate-dependent material response is accounted for by the finite-deformation Maxwell element (Fig. 2.2, bottom branch). For the non-equilibrium branch, the deformation gradient $\mathbf{F}$ is multiplicatively decomposed into elastic and viscous parts: $\mathbf{F} = \mathbf{F}^{\text{e}}\mathbf{F}^{\text{v}}$, where $\mathbf{F}^{\text{e}}$ is the non-equilibrium elastic distortion with $J^{\text{e}} = \det \mathbf{F}^{\text{e}} > 0$, and $\mathbf{F}^{\text{v}}$ is the viscous distortion with $J^{\text{v}} = \det \mathbf{F}^{\text{v}} = 1$. The right polar decomposition of $\mathbf{F}^{\text{e}}$ is $\mathbf{F}^{\text{e}} = \mathbf{R}^{\text{e}}\mathbf{U}^{\text{e}}$, where $\mathbf{U}^{\text{e}}$ is the elastic right stretch tensor and $\mathbf{R}^{\text{e}}$ is the elastic rotation tensor, and the elastic logarithmic (Hencky) finite-strain tensor is defined as $\mathbf{E}^{\text{e}} = \ln \mathbf{U}^{\text{e}}$. We take

the non-equilibrium free-energy density to be given by

$$\psi^{\text{neq}}(\mathbf{E}^e) = G_1 |\text{dev } \mathbf{E}^e|^2, \quad (2.7)$$

where G_1 is the non-equilibrium shear modulus. The stress conjugate to the elastic logarithmic strain is referred to as the Mandel stress and is given by

$$\mathbf{M}^e = \frac{\partial \psi^{\text{neq}}}{\partial \mathbf{E}^e} = 2G_1 \text{dev } \mathbf{E}^e, \quad (2.8)$$

and the non-equilibrium contribution to the Cauchy stress is given through the Mandel stress by $\boldsymbol{\sigma}^{\text{neq}} = J^{e-1} \mathbf{R}^e \mathbf{M}^e \mathbf{R}^{e\top}$. We note that since the non-equilibrium free-energy density (2.7) only involves the deviatoric part of the elastic logarithmic strain, the consequent Mandel stress $\mathbf{M}^e$ and non-equilibrium contribution to the Cauchy stress $\boldsymbol{\sigma}^{\text{neq}}$ are both deviatoric as well, and the volumetric material response is due to the equilibrium response alone. The total Cauchy stress is then given by $\boldsymbol{\sigma} = \boldsymbol{\sigma}^{\text{eq}} + \boldsymbol{\sigma}^{\text{neq}}$ [58].

Neglecting viscous spin, i.e., assuming $\mathbf{W}^v = \text{skw}(\dot{\mathbf{F}}^v \mathbf{F}^{v-1}) = \mathbf{0}$, the evolution of $\mathbf{F}^v$ is given by $\dot{\mathbf{F}}^v = \mathbf{D}^v \mathbf{F}^v$, where $\mathbf{D}^v$ is the symmetric viscous stretching tensor. Finally, it remains to specify a constitutive equation for $\mathbf{D}^v$. Defining the non-equilibrium equivalent shear stress $\bar{\tau} = \sqrt{\mathbf{M}^e : \mathbf{M}^e / 2}$, we adopt the following constitutive equation for $\mathbf{D}^v$:

$$\mathbf{D}^v = \dot{\gamma}^v \frac{\mathbf{M}^e}{2\bar{\tau}} \quad \text{with} \quad \dot{\gamma}^v = \dot{\gamma}_0 \left(\frac{\bar{\tau}}{G} \right)^{1/m}, \quad (2.9)$$

where $\dot{\gamma}^v$ denotes the equivalent shear viscous strain rate. The dimensionless material parameter $m > 0$ represents a strain-rate sensitivity exponent, and the parameter $\dot{\gamma}_0 > 0$ is a reference shear strain rate. The equilibrium ground-state shear modulus G is included in (2.9) as a reference shear stress for dimensional consistency. For $0 < m < 1$, the rheological behavior of the non-equilibrium response is shear thinning, and for $m = 1$, the rheological behavior is Newtonian. For the non-Newtonian case, two material parameters are required to

describe the rheological response, m and $\dot{\gamma}_0$, while for the Newtonian case, only one material parameter is required: the constant shear viscosity $\mu = G/\dot{\gamma}_0$. In this case, (2.9) takes the form

$$\mathbf{D}^v = \frac{\mathbf{M}^e}{2\mu}. \quad (2.10)$$

The constitutive modeling approach described above accounts for viscoelastic behavior, strain-stiffening, non-Newtonian rheological behavior, and compressibility and involves the following six material parameters: the equilibrium shear modulus G , the strain-stiffening parameter α , the bulk modulus K , the non-equilibrium shear modulus G_1 , the strain-rate sensitivity exponent m , and the reference strain rate $\dot{\gamma}_0$. We note that in the limit that the non-equilibrium shear modulus is much greater than the equilibrium shear modulus, i.e., $G_1/G \rightarrow \infty$, the distortional part of the non-equilibrium elastic response becomes nearly rigid, i.e., $\mathbf{F}^e \rightarrow J^e \mathbf{1}$, and an SNS-type model effectively reduces to a KV-type model. As pointed out in the preceding section, utilizing power-law rheological behavior in a KV-type model can be problematic, due to the diverging viscosity as the strain-rate approaches zero. Thus, taking $G_1 \gg G$ but finite in an SNS-type model may be used to attain a KV-like model with non-Newtonian rheological behavior. When the strain rate is finite, the stiff non-equilibrium response does not affect the total response of the material, but as the strain rate approaches zero (e.g., far from the bubble as $r \rightarrow \infty$), the stiff non-equilibrium response serves to regularize the diverging viscosity. Based on our tests, taking $G_1 = 1000G$ is sufficiently high to yield predictions of bubble dynamics that are independent of G_1 , and when this idealization is made, the aforementioned six parameters reduces to a set of five parameters, $\{G, \alpha, K, m, \dot{\gamma}_0\}$. We refer to this model as the *non-Newtonian qKV model*.

2.2.3 Balance of linear momentum

The balance of linear momentum in the absence of external body forces requires that $\text{div } \boldsymbol{\sigma} = \rho \mathbf{a}$, where $\text{div}(\bullet)$ is the spatial divergence operator, $\rho = J^{-1}\rho_0$ is the spatial mass

density, and $\mathbf{a}$ is the spatial description of the acceleration vector. Due to the idealization of spherical symmetry and the isotropy of the surrounding material, the only non-trivial component of the linear momentum balance equations is the radial component:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{2}{r}(\sigma_{rr} - \sigma_{\theta\theta}) = \rho a_r, \quad (2.11)$$

where we have recognized that $\sigma_{\theta\theta} = \sigma_{\phi\phi}$ due to spherical symmetry.

At this stage, we note that the pressure associated with the Cauchy stress, $p = -(1/3)\text{tr } \boldsymbol{\sigma}$, represents the “gauge” pressure—that is, the difference between the absolute pressure and the ambient atmospheric pressure p_∞ . Then, the traction boundary conditions may be expressed in terms of the gauge pressures inside the bubble and in the far-field. The radial component of the traction boundary condition at the bubble wall ($r = R$) is

$$\sigma_{rr}|_{r=R} = -(p_b(t) - p_\infty) + \frac{2\gamma}{R}, \quad (2.12)$$

where $p_b(t)$ is the absolute pressure inside the bubble, $(p_b(t) - p_\infty)$ is the gauge pressure inside the bubble, and γ is the surface tension of the bubble interface. The evolution of the time-dependent absolute pressure inside the bubble $p_b(t)$ is governed by the physics of the bubble contents, which we assume to be a homobaric mixture of water vapor and non-condensable gas [24, 27, 28, 59]. The mathematical description of bubble contents and resulting evolution equation for $p_b(t)$ are described in detail in Estrada et al. [28] and briefly summarized in Appendix A.

In the far-field, the stress approaches a state of hydrostatic pressure: $\boldsymbol{\sigma}|_{r \rightarrow \infty} = -p_f(t)\mathbf{1}$, where $p_f(t)$ is the time-dependent, far-field gauge pressure, which is prescribed for a given problem. For example, as discussed in Section 2.3.3, in laser-induced cavitation, the far-field absolute pressure simply remains the ambient atmospheric pressure, so that the far-field gauge pressure is zero. In ultrasound-induced cavitation, $p_f(t)$ is a time-varying, acoustic forcing [45, 49, 50]. Therefore, the far-field boundary condition for the radial stress

component is

$$\sigma_{rr}|_{r \rightarrow \infty} = -p_f(t). \quad (2.13)$$

2.3 Modeling of bubble dynamics

In this section, we describe two approaches for modeling the dynamics of spherical bubbles in soft viscoelastic media. The first is based on the Keller-Miksis (KM) equation [23], which reduces the balance of linear momentum partial differential equation (PDE) (2.11) to an ordinary differential equation (ODE) and enables an efficient computation of the bubble dynamics under the idealization of slight compressibility and for relatively simple constitutive descriptions of the surrounding material. In the second approach, the governing differential equation (2.11) is solved in the surrounding material using the finite-element method, which does not restrict the constitutive description of the surrounding material and allows for a higher fidelity description of the physics but is more computationally costly.

2.3.1 Keller-Miksis equation

The balance of linear momentum is reduced to an ODE by first assuming ideal incompressibility in the surrounding material ($J = 1$) and then integrating (2.11) over r from $r = R$ to $r \rightarrow \infty$ and applying the traction boundary conditions (2.12) and (2.13). The result is the Rayleigh-Plesset (RP) equation [20, 21] governing the evolution of the bubble radius $R(t)$:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho} \left(p_b - p_\infty - p_f - \frac{2\gamma}{R} + S \right), \quad (2.14)$$

where the stress integral S is given by

$$S = \int_R^\infty \frac{2}{r} (\sigma_{rr} - \sigma_{\theta\theta}) dr. \quad (2.15)$$

We note that, in (2.14), because ideal incompressibility has been assumed, the spatial mass density ρ is constant and equal to the referential mass density ρ_0 .

However, the assumption of ideal incompressibility fails to account for energy transfer through radial acoustic emission from the bubble to the far field, which has a significant impact on the bubble dynamics. Keller and Miksis [23] extended the RP approach to account for slight compressibility of the surrounding material by incorporating the material's finite longitudinal wave speed c into the description of the far-field, resulting in the following modified evolution equation for the bubble radius $R(t)$:

$$\begin{aligned} \left(1 - \frac{\dot{R}}{c}\right) R \ddot{R} + \frac{3}{2} \left(1 - \frac{\dot{R}}{3c}\right) \dot{R}^2 \\ = \frac{1}{\rho} \left(1 + \frac{\dot{R}}{c}\right) \left(p_b - p_\infty - p_f - \frac{2\gamma}{R} + S\right) + \frac{1}{\rho} \frac{R}{c} \overline{\left(p_b - p_f - \frac{2\gamma}{R} + S\right)}, \end{aligned} \quad (2.16)$$

where the definition of the stress integral S given in (2.15) remains unchanged. The KM equation is accurate to first order in the ratio of the bubble wall velocity to the longitudinal wave speed in the surrounding material $\dot{R}/c$ [25]. Note that in the limit of ideal incompressibility, $\dot{R}/c \rightarrow 0$, and the KM equation (2.16) reduces to the RP equation (2.14).

The constitutive behavior of the surrounding material enters both the RP and KM equations through the stress integral (2.15). For certain constitutive models, the stress integral may be straightforwardly calculated analytically. For example, for the qKV model, in which the Cauchy stress is given by (2.6), under the assumption of ideal incompressibility ($J = 1$), the stress integral is [44]

$$\begin{aligned} S = \frac{(3\alpha - 1)G}{2} \left[5 - \left(\frac{R_0}{R}\right)^4 - \frac{4R_0}{R} \right] - \frac{4\mu\dot{R}}{R} \\ + 2\alpha G \left[\frac{27}{40} + \frac{1}{8} \left(\frac{R_0}{R}\right)^8 + \frac{1}{5} \left(\frac{R_0}{R}\right)^5 + \left(\frac{R_0}{R}\right)^2 - \frac{2R}{R_0} \right]. \end{aligned} \quad (2.17)$$

Note that when strain-stiffening is neglected ($\alpha = 0$), the stress integral reduces to that for

the NH-KV model [51]. However, for more rheologically complex constitutive models, such as the SNS-type models of Section 2.2.2, closed-form, analytical expressions for the stress integral may not be obtainable, complicating the application of the KM equation.

In subsequent sections, to obtain numerical solutions for bubble dynamics using the KM equation when the surrounding material is described by a constitutive model possessing a closed-form stress integral, (2.16) is evolved forward in time in conjunction with the governing equations for the bubble contents (Appendix A) using a variable-step, variable-order solver (MATLAB ode15s).

2.3.2 Full-field solution using finite elements

The KM equation (2.16) is widely used to describe spherical bubble dynamics arising during inertial cavitation; however, as pointed out in Section 2.3.1, there are several limitations. First, it cannot account for moderate compressibility that can arise during more violent bubble collapse [44]. Second, calculation of the stress integral (2.15) can be intractable for complex constitutive models. Therefore, while the KM equation provides an efficient route for predicting bubble dynamics in certain scenarios, these limitations motivate the need for a full-field description of the surrounding material, in which the dependence on the radial coordinate r is retained and not integrated out.

We maintain the assumption of spherical symmetry, and thus, the only differential equation under consideration is the radial component of the balance of linear momentum (2.11), which we solve using the finite-element (FE) method. There are several practical matters that must be addressed to obtain FE solutions. First, in full-field calculations, the far-field must be truncated at a finite value of the radial coordinate. We denote the undeformed radius of the outer boundary after truncation as $R_{0,\infty}$ and its deformed counterpart as R_∞ . Throughout, we take $R_{0,\infty} = 200R_0$, which we have verified to be sufficiently large so as to accurately approximate an infinite domain. Second, in full-field simulations, radial acoustic waves emitted from the bubble to the far field will reflect off of the outer boundary and

refocus on the bubble, leading to incorrect numerical predictions. To mitigate this effect, a “quiet” boundary is implemented at the outer boundary [60], in which a normal traction of $\sigma_{rr}|_{r=R_\infty} = -\rho_0 c v_r|_{r=R_\infty}$ is applied at $r = R_\infty$. As a final point, in UIC, it is impractical to apply the acoustic forcing $p_f(t)$ at the outer boundary due to the presence of the quiet boundary. Moreover, such an approach would require simulating the additional time that it takes for the acoustic pulse to travel inward from the outer boundary to the bubble. Instead, in FE simulations of UIC, the acoustic forcing is applied directly at the bubble wall, which makes direct comparisons of FE and KM simulations straightforward. As a result of these three adjustments to the problem definition, for FE simulations, the boundary conditions, (2.12) and (2.13), are modified as follows:

$$\begin{cases} \sigma_{rr}|_{r=R} = -(p_b(t) - p_\infty - p_f(t)) + \frac{2\gamma}{R}, \\ \sigma_{rr}|_{r=R_\infty} = -\rho_0 c v_r|_{r=R_\infty}. \end{cases} \quad (2.18)$$

Equations (2.11) and (2.18) represent the strong form of the balance of linear momentum. In order to recast the balance of linear momentum in a variational formulation, we introduce a scalar test field $\eta(r)$. The weak form is obtained by multiplying (2.11) by the test field, integrating over the domain, applying the divergence theorem, and using the boundary conditions (2.18). The result is

$$\begin{aligned} - \int_R^{R_\infty} \left(\sigma_{rr} \frac{d\eta}{dr} + 2 \frac{\sigma_{\theta\theta}}{r} \eta \right) (4\pi r^2 dr) - \rho_0 c (v_r \eta)|_{r=R_\infty} (4\pi R_\infty^2) \\ + \left(p_b - p_\infty - p_f - \frac{2\gamma}{R} \right) \eta|_{r=R} (4\pi R^2) = \int_R^{R_\infty} \rho a_r \eta (4\pi r^2 dr). \end{aligned} \quad (2.19)$$

Equation 2.19 is then used as the basis for a conventional explicit dynamic finite-element formulation. Briefly, the undeformed domain is spatially discretized using n_{Elem} equally-sized finite elements, which are each $(R_{0,\infty} - R_0)/n_{\text{Elem}}$ in length. We utilize one-dimensional, linear, two-noded finite elements, in which the radial displacement field u_r is interpolated inside each element using linear shape functions. A standard Galerkin approach is employed

in that the test field is interpolated by the same shape functions. The resulting finite-element equations of motion are explicitly integrated in time using the explicit central difference scheme with a lumped mass matrix computed via the row-sum method. An adaptive time increment is employed, in which the time increment is chosen to be one third of the stable time increment calculated based on the smallest deformed element length, and the bubble pressure p_b is explicitly updated within each finite-element time increment by solving the governing equations for the physics of the bubble contents (Appendix A). We close this section by reiterating that the FE-based approach does not place restrictions on the constitutive model used to describe the surrounding material, and that we may straightforwardly incorporate any complex constitutive model for the surrounding material, including the SNS-type models of Section 2.2.2, into the cavitation modeling framework.

2.3.3 Initial conditions in laser-induced and ultrasound-induced cavitation bubbles

In this work, we consider both laser-induced cavitation (LIC) and ultrasound-induced cavitation (UIC). When simulating LIC, we initialize numerical simulations starting at the time that the bubble is at its maximum radius ($R = R_{\max}$). This is done because the initial growth phase of the bubble involves laser breakdown of the liquid and associated plasma physics, which is not accounted for in the cavitation modeling framework. When the bubble is at its maximum radius, we idealize the system as being in thermodynamic equilibrium with the surroundings, though not in mechanical equilibrium. The driving force for the subsequent cavitation dynamics is the gauge pressure inside the bubble modified by the Laplace pressure ($p_b - p_\infty - 2\gamma/R$), and in LIC, the far-field gauge pressure remains zero ($p_f = 0$). For KM-based numerical solutions, the bubble radius and velocity are initialized as $R(t = 0) = R_{\max}$ and $\dot{R}(t = 0) = 0$, respectively. For FE-based numerical simulations, the initial displacement field is prescribed based on the incompressible solution,

i.e., $u_r(r_0, t = 0) = (r_0^3 - R_0^3 + R_{\max}^3)^{1/3} - r_0$,² and the initial velocity field is zero everywhere, i.e., $\dot{u}_r(r_0, t = 0) = 0$. For both approaches, the bubble contents are initialized as discussed in Appendix A.

Next, for UIC, we simulate the bubble dynamics from the initial growth phase through the first collapse and subsequent rebound cycles as the bubble approaches its undeformed radius. The bubble is initially in mechanical equilibrium with the surroundings. Therefore, for KM-based simulations, the initial conditions are $R(t = 0) = R_0$ and $\dot{R}(t = 0) = 0$, and for FE-based simulations, the initial conditions are $u_r(r_0, t = 0) = \dot{u}_r(r_0, t = 0) = 0$. The initial conditions for the bubble contents for both approaches are given in Appendix A. For UIC, the driving force for the bubble dynamics is the time-dependent far-field gauge pressure $p_f(t)$, which we take to be given in the following form:

$$p_f(t) = \begin{cases} p_A \left(\frac{1 + \cos[\omega(t - \delta)]}{2} \right)^n & \text{if } |t - \delta| \leq \frac{\pi}{\omega}, \\ 0 & \text{if } |t - \delta| > \frac{\pi}{\omega}. \end{cases} \quad (2.20)$$

This half-cycle waveform for the far-field gauge pressure is based on representative experimental acoustic pulses and has been extensively used in prior studies of UIC [5–7, 35, 45, 49, 50]. In this study, motivated by the experiments of Wilson et al. [49], the frequency $f = \omega/2\pi$ of the half-cycle pulse is taken to be 1 MHz. The time delay δ is simply $\delta = \pi/\omega$, so that the waveform begins at $t = 0$, and the dimensionless shape parameter n is chosen to be $n = 3.7$ as in prior studies to best represent experimental acoustic pulses. Finally, the peak tensile pressure $p_A < 0$ is varied in the simulations of the subsequent section in order to consider a range of consequent bubble amplitudes.

²Based on our tests, for the values of $R_{\max}$ and R_0 and the material parameters considered in this work, FE simulations using an initial condition based on the incompressible solution and an initial condition based on the compressible, equilibrium displacement field yield indistinguishable results for the bubble dynamics, so the incompressible displacement field is used for the initial condition for expedience.

Table 2.1: Parameters used in the studies of Sections 2.4 and 2.5

Parameter	Value	Parameter	Value
ρ_0	1060 kg/m ³	c	1430 m/s
p_∞	101.3 kPa	γ	5.6×10^{-2} N/m

2.4 Effect of material compressibility on bubble dynamics

In this section, we compare FE-based and KM-based simulations of bubble dynamics for both LIC and UIC. The purpose of this exercise is twofold: (1) to verify our FE-based simulation approach and (2) to establish a threshold for when the bubble dynamics involves material compressibility beyond that accounted for in the KM-based approach. Recall that the KM-based approach accounts only for slight far-field compressibility, while the FE-based approach accounts for compressibility in the full field, using an equation of state stemming from the term involving J in the free-energy density function (2.3). In this section, we utilize the NH-KV model, so that solutions may be obtained using both approaches, and the quantitative values used for ρ_0 , c , p_∞ , and γ are summarized in Table 2.1. We begin by considering LIC for several different values of $R_{\max}/R_0$, while keeping the material parameters of the NH-KV model, G and μ , fixed and equal to $G = 10$ kPa, $\mu = 0.03$ Pa·s. Figure 2.3(a) shows radius versus time curves in the case of LIC for $R_{\max}/R_0 = 2, 4$, and 7 , where $R_0 = 22.5$ μm . For LIC, the normalized radius is $R^* = R/R_{\max}$, and the normalized time is $t^* = (t/R_{\max})\sqrt{p_\infty/\rho_0}$. The FE-based calculations are presented as solid lines, and the KM-based calculations are presented as dashed lines. We can see that for the case of $R_{\max}/R_0 = 2$, for which the bubble collapse is the least violent, the two approaches yield nearly identical results, verifying the FE-based simulation approach. As $R_{\max}/R_0$ increases and bubble collapse becomes more violent, the two approaches begin to give rise to increasingly different results, indicating that material compressibility not accounted for in the KM-approach becomes more important for more violent bubble collapse.

As an alternate approach to probing the role of material compressibility, we consider

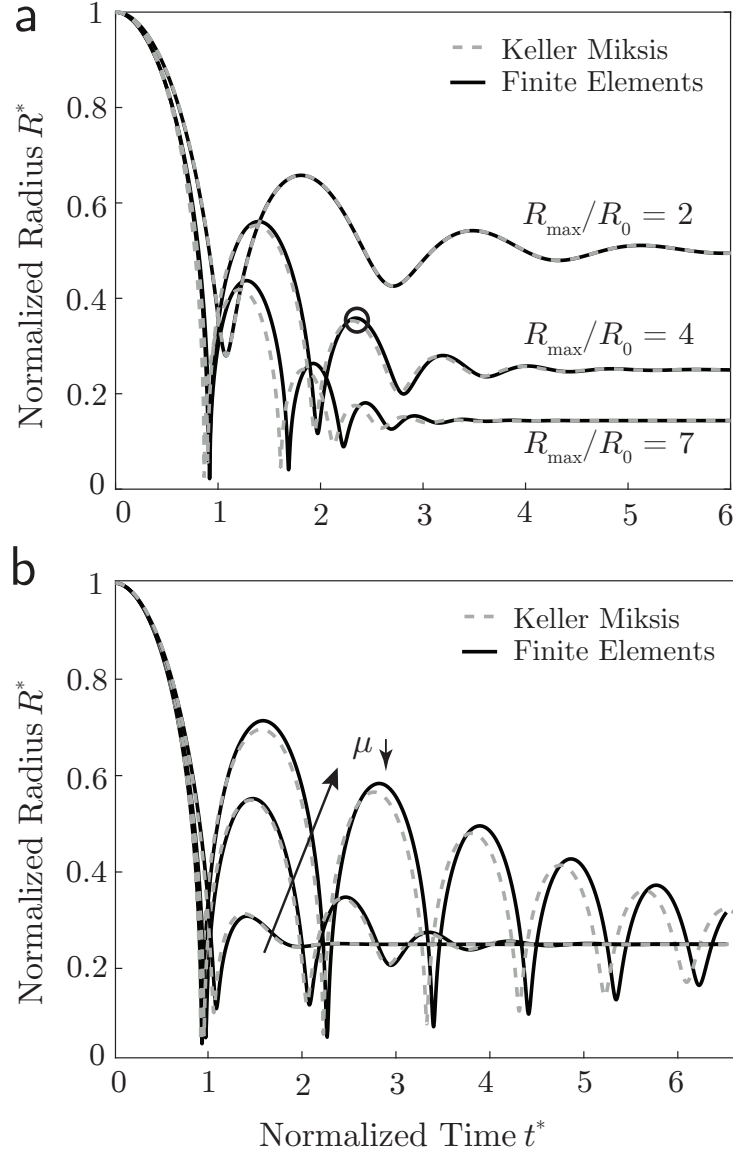


Figure 2.3: Effect of (a) the radius ratio $R_{\max}/R_0$ and (b) the viscosity of the surrounding material μ on normalized radius versus time curves for LIC. FE-based simulations are shown as solid lines, and KM-based simulations are shown as dashed lines. The open circle in (a) illustrates the time t_c used in (2.21) to calculate ΔR_{RMS} .

LIC for three different values of the viscosity, $\mu = 0.003, 0.032$, and 0.1 Pa·s, for a fixed radius ratio of $R_{\max}/R_0 = 4$, while maintaining $G = 10$ kPa and $R_0 = 22.5$ μm . Figure 2.3(b) shows the corresponding radius versus time curves, calculated using both FE-based and KM-based simulations. For the highest value of the viscosity, the bubble dynamics are more damped and less violent, and consequently, the two approaches give very similar results.

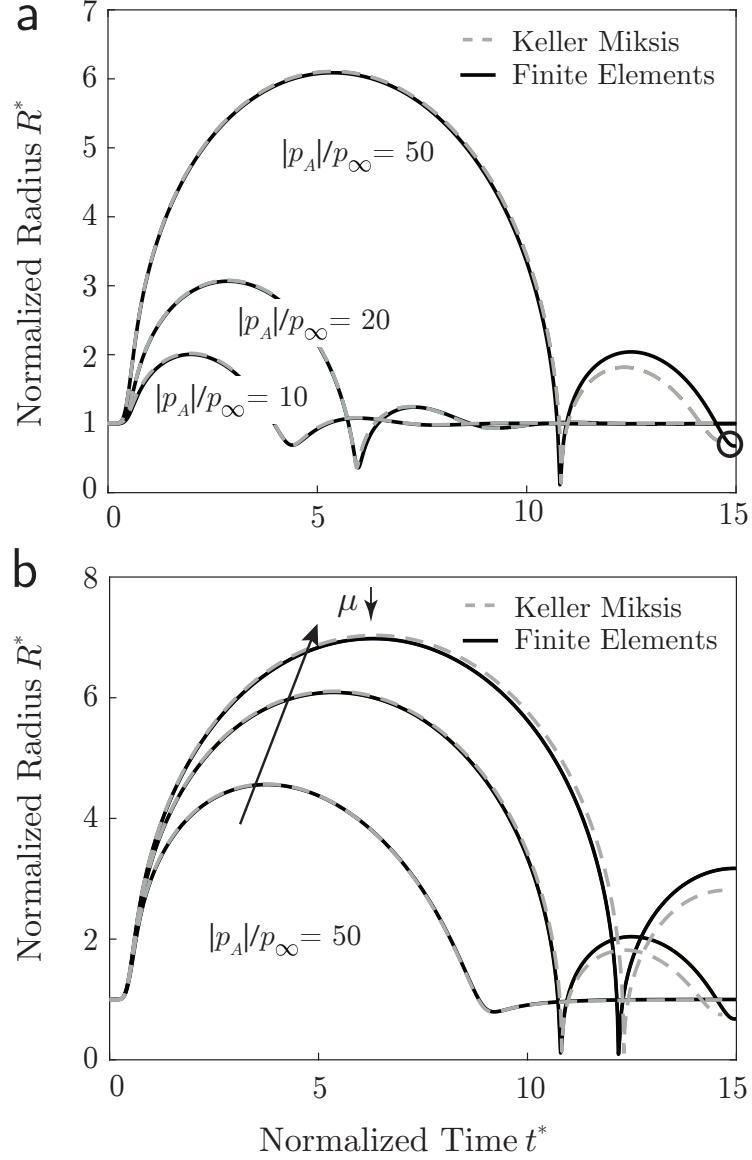


Figure 2.4: Effect of (a) the peak tensile pressure $|p_A|/p_\infty$ and (b) the viscosity of the surrounding material μ on normalized radius versus time curves for UIC. FE-based simulations are shown as solid lines, and KM-based simulations are shown as dashed lines. The open circle in (a) illustrates the time t_c used in (2.21) to calculate ΔR_{RMS} .

As the viscosity is decreased, bubble collapse becomes more violent, and the discrepancy between the two approaches grows. In general, for violent collapse, the compressibility of the surrounding material is more important, giving rise to the discrepancy between the two approaches.

Similar observations are made for the case of UIC. Figure 2.4(a) shows normalized

radius versus time curves for UIC for peak tensile pressures of $|p_A|/p_\infty = 10, 20$, and 50, where $G = 10$ kPa, $\mu = 0.03$ Pa·s, and $R_0 = 10$ μ m. As the peak tensile pressure increases, the cavitation amplitude increases, larger radius ratios ($R_{\max}/R_0$) are attained, and the consequent collapse is more violent. As for LIC, as the radius ratio increases, the discrepancy between the FE-based and KM-based simulations grows. Figure 2.4(b) shows normalized radius versus time curves for three different values of the viscosity, $\mu = 0.003, 0.032$, and 0.1 Pa·s, for a fixed peak tensile pressure of $|p_A|/p_\infty = 50$, while maintaining $G = 10$ kPa and $R_0 = 10$ μ m. Again, as the viscosity decreases, larger radius ratios are attained, the bubble collapse becomes more violent, and there is an increase in the discrepancy between the two simulation approaches.

To quantify the discrepancy between FE-based and KM-based simulations, we utilize the root-mean-square of the difference between the normalized radius history calculated using FE-based and KM-based approaches:

$$\Delta R_{\text{RMS}} = \sqrt{\frac{1}{t_c} \int_0^{t_c} (R_{\text{FE}}^*(t^*) - R_{\text{KM}}^*(t^*))^2 dt^*}, \quad (2.21)$$

where $R_{\text{FE}}^*(t^*)$ and $R_{\text{KM}}^*(t^*)$ are the FE-calculated and KM-calculated normalized radius histories, and t_c is a characteristic normalized time for each case that corresponds to the second rebound time in LIC and the second collapse time in UIC (illustrated with open circle markers in Figs. 2.3(a) and 2.4(a), respectively). Further, for each case, we define the maximum Mach number as $\max_t(\dot{R}(t))/c$, where c is the longitudinal wave speed in the surrounding material. Figures 2.5(a) and (b) then show plots of ΔR_{RMS} versus the maximum Mach number for LIC and UIC, respectively. In Fig. 2.5(a), nine different combinations of radius ratio $R_{\max}/R_0$ and viscosity μ are included for LIC, each represented by a symbol, where the viscosity is denoted by the symbol type, and the radius ratio is denoted by the shading of the connecting lines. As observed in Fig. 2.3, as the radius ratio $R_{\max}/R_0$ increases or as the viscosity μ decreases, bubble collapse becomes more violent. As a

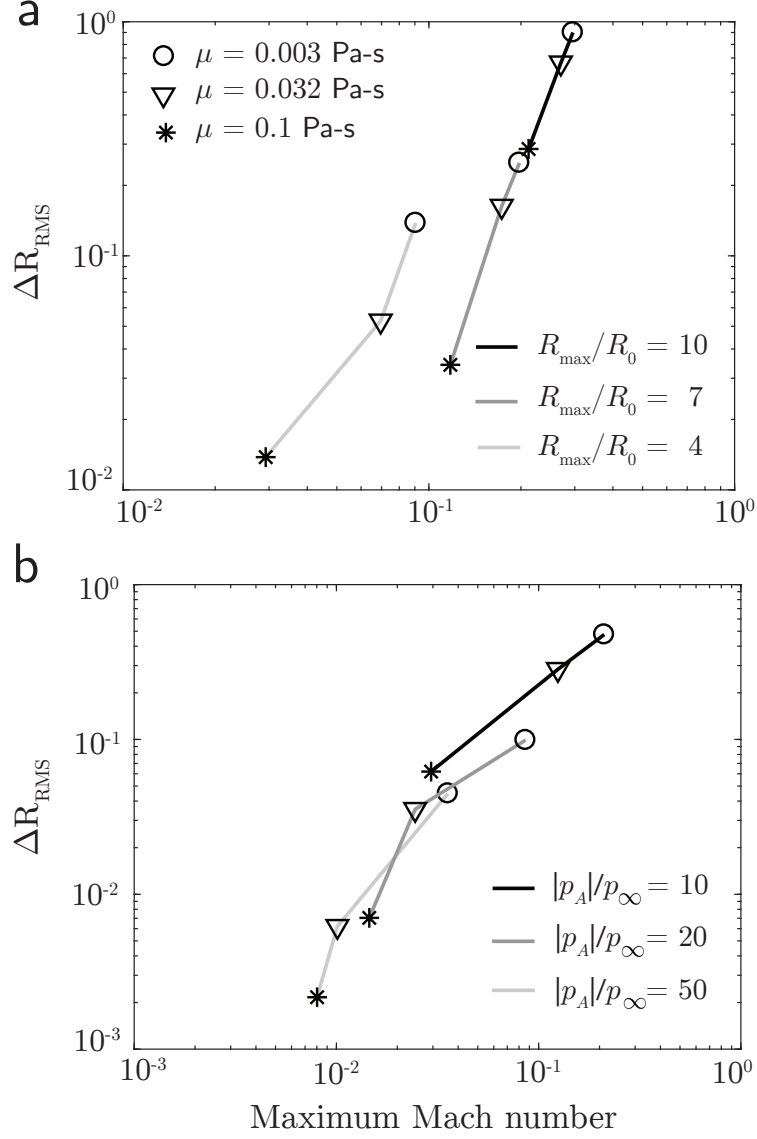


Figure 2.5: Plots of ΔR_{RMS} versus the maximum Mach number for (a) LIC and (b) UIC. We observe that the discrepancy between the FE-based and KM-based simulations increases as the maximum Mach number increases across all cases considered.

result, the consequent maximum Mach number and the discrepancy between FE-based and KM-based calculations, quantified by ΔR_{RMS} , both increase. Similarly, in Fig. 2.5(b), nine different combinations of peak tensile pressure $|p_A|/p_\infty$ and viscosity μ are included for UIC and denoted in the same manner as in Fig. 2.5(a). Again, we observe that as the peak tensile pressure $|p_A|/p_\infty$ increases or as the viscosity μ decreases, bubble collapse becomes more violent, and the maximum Mach number and ΔR_{RMS} both increase. For both LIC

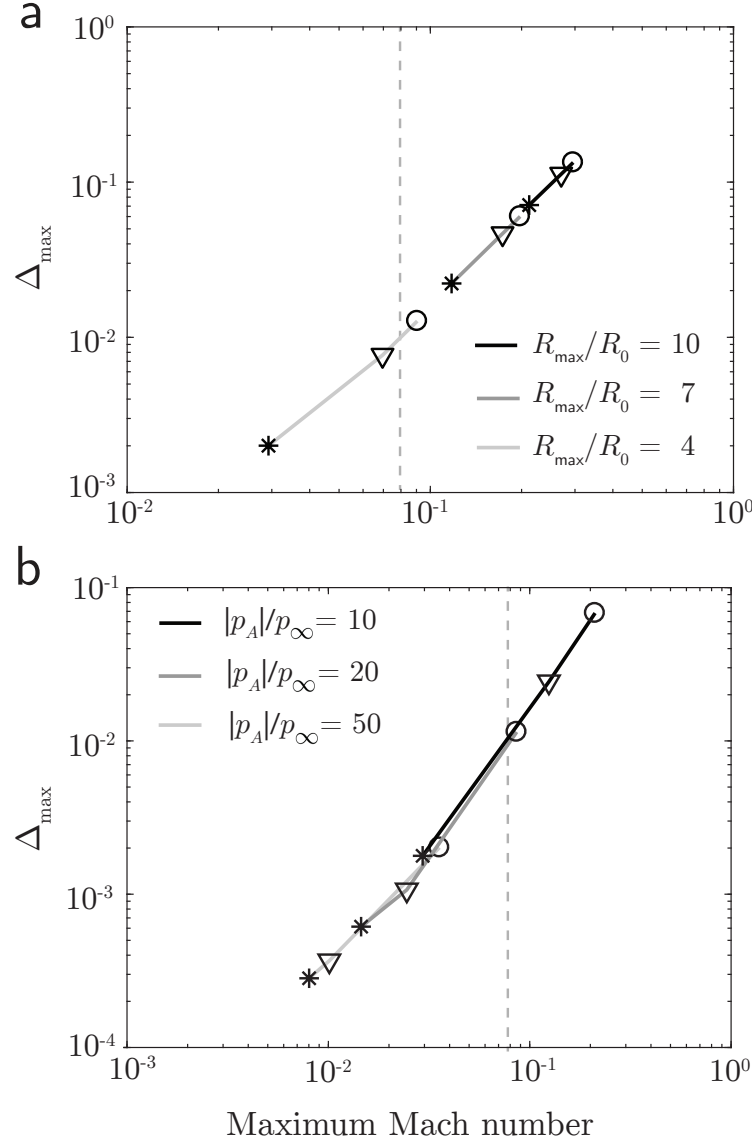


Figure 2.6: Plots of $\Delta_{\max}$ versus the maximum Mach number for (a) LIC and (b) UIC. We observe that the maximum volumetric strain magnitude $\Delta_{\max}$ increases as the maximum Mach number increases across all cases considered. In both LIC and UIC, for maximum Mach numbers less than about 0.08 (vertical dashed gray lines), $\Delta_{\max}$ remains below 0.01.

and UIC, a rough collapse in ΔR_{RMS} versus maximum Mach number is observed across the cases considered. We have confirmed that both FE-based and KM-based simulations are sufficiently refined both temporally and spatially, so that the discrepancy ΔR_{RMS} is attributable to the different approaches used to account for material compressibility in the two methods.

To make the connection to the compressibility of the surrounding material more concrete, we consider the volumetric strain field $\Delta = \ln J$, calculated using the FE approach. The maximum volumetric strain magnitude in the surroundings occurs at the bubble wall $r = R$ at first collapse in all cases of both LIC and UIC considered in this section and increases as bubble collapse becomes more violent. We define the maximum volumetric strain magnitude as $\Delta_{\max} = \max_t(|\Delta(r = R, t)|)$. Figures 2.6(a) and (b) then show plots of $\Delta_{\max}$ versus the maximum Mach number for LIC and UIC, respectively, calculated using the FE approach. The different cases of LIC and UIC are denoted in the same manner as in Fig. 2.5. For both LIC and UIC, a collapse is observed across the cases considered. For maximum Mach numbers less than about 0.08 (denoted by the vertical dashed gray lines in Fig. 2.6), $\Delta_{\max}$ remains below 0.01, and material compressibility can be regarded as small. Correspondingly, the discrepancy between FE-based and KM-based simulations is minimal. But for maximum Mach numbers greater than about 0.08, $\Delta_{\max}$ is greater than 0.01, and material compressibility plays a more significant role in the bubble dynamics. Consequently, ΔR_{RMS} grows, and FE-based simulations become more accurate, due to a higher fidelity description of the physics of material compressibility. These observations suggest a maximum Mach number of 0.08 may be used as a threshold for when bubble dynamics involves material compressibility beyond that accounted for in the KM-based approach. We note that this threshold complements the experimental observations of Yang et al. [44] for polyacrylamide gel, who reported a transition from viscoelastic material behavior to more complex behavior at a maximum Mach number of 0.08, based on measurements of the radial positions of tracked material points in the surrounding material.

2.5 Characterizing the high-strain-rate behavior of polyacrylamide gels

Next, we utilize FE-based simulations in Inertial Microcavitation Rheometry (IMR) to obtain improved high-strain-rate properties for soft materials—a process that we refer to as FE-IMR. First, in Section 2.5.1, we revisit the experimental LIC data of Yang et al. [44] for violent bubble collapse in a stiff polyacrylamide gel and apply FE-IMR that accounts for material compressibility to obtain higher fidelity properties. Then, in Section 2.5.2, we consider the experimental data of Buyukozturk et al. [48] for particle-assisted LIC in a soft polyacrylamide gel over a range of cavitation amplitudes and use FE-IMR to characterize the high-strain-rate behavior, including non-Newtonian rheological behavior.

2.5.1 Stiff polyacrylamide gel

First, we consider the LIC experiments of Yang et al. [44] on polyacrylamide (PA) gels and focus on their “stiff” PA gel with a ground-state shear modulus of $G = 2.77$ kPa, determined through quasi-static testing. Ten LIC experiments were carried out on the stiff PA gel, yielding mean maximum and equilibrium radii of 330.26 ± 30.36 μm and 46.52 ± 4.28 μm , respectively, corresponding to a radius ratio of $R_{\text{max}}/R_0 = 7.1$. The normalized radius $R^* = R/R_{\text{max}}$ versus normalized time $t^* = (t/R_{\text{max}})\sqrt{p_\infty/\rho_0}$ history for a representative LIC experiment in stiff PA gel from Yang et al. [44] is shown in Fig. 2.7 as square symbols. Throughout Section 2.5, following Estrada et al. [28], we use the values of ρ_0 , c , p_∞ , and γ summarized in Table 2.1.

In applying KM-based IMR (KM-IMR) to LIC data for stiff PA, Yang et al. [44] made two key observations. First, Yang et al. [44] showed that KM-IMR cannot provide an adequate description of the physics beyond the first collapse for violently collapsing bubbles, which are marked by a maximum Mach number of greater than 0.08. This was experimentally observed in their work based on measurements of the kinematics of tracked

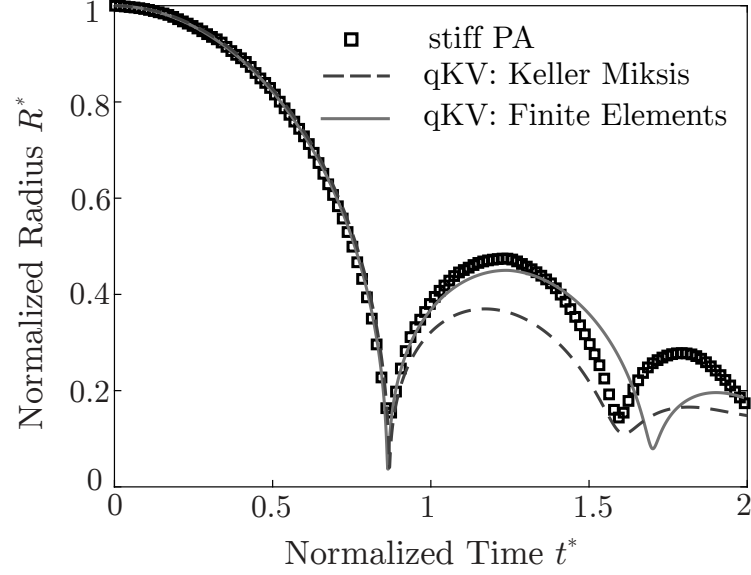


Figure 2.7: Comparison of experimental data (symbols), numerical KM-IMR results (dashed line), and numerical FE-IMR results (solid line) for the normalized radius versus time history for stiff PA gel. For KM-IMR, only the first collapse was used to obtain best-fitted material parameters, while FE-IMR utilizes the bubble radius history through the third collapse ($0 < t^* \lesssim 2$). Both approaches use the qKV model, and FE-IMR gives $\alpha = 0.48$ and $\mu = 0.158$ Pa·s as the best-fitted material parameters, while KM-IMR gives $\alpha = 0.48$ and $\mu = 0.186$ Pa·s.

material points in the surrounding material, and the results of Section 2.3.1 in the present work establish that compressibility begins to matter beyond this point, and that KM-IMR does not adequately account for compressibility beyond a maximum Mach number of 0.08. Thus, only the first collapse in the normalized radius versus time history was utilized in Yang et al. [44] to obtain best-fitted material parameters for PA gels. Second, Yang et al. [44] concluded that for sufficiently large radius ratios, accounting for strain stiffening effects is crucial to improve the quality of the fitted material parameters for PA gels. Therefore, the quadratic Kelvin-Voigt (qKV) model, which accounts for strain stiffening, was utilized, and a strain-stiffening parameter of $\alpha = 0.48$ and a viscosity of $\mu = 0.186$ Pa·s were obtained based on least-squares data fitting, while maintaining the ground-state shear modulus to be equal to the quasi-static value of $G = 2.77$ kPa. The corresponding normalized radius versus time curve, fitted to the first collapse, is included in Fig. 2.7 as a dashed line.

In the present work, we revisit violently-collapsing LIC bubbles and apply FE-IMR to the normalized radius versus time data to obtain refined high-strain-rate material properties. To do so, since FE-IMR accounts for material compressibility, we broaden the portion of the normalized radius versus time history used for fitting beyond the first collapse to include the history through the third collapse ($t^* \lesssim 2$ in Fig. 2.7). Moreover, we consider a more general constitutive model and utilize the non-Newtonian qKV model discussed in Section 2.2.2. As in Yang et al. [44], we take the equilibrium shear modulus to be its quasi-static value $G = 2.77$ kPa, and as discussed in Section 2.2.2, we take $G_1 = 1000G$. The bulk modulus is given through the longitudinal wave speed by $K = \rho_0 c^2 - (4/3)G$, leaving three adjustable material parameters: the strain-stiffening parameter α , the strain-rate sensitivity exponent m , and the reference strain rate $\dot{\gamma}_0$. We then perform least-squares fitting between experimental radius versus time curves for stiff PA gel and curves computationally obtained using FE-IMR with the adjustable parameters $\{\alpha, m, \dot{\gamma}_0\}$. As a result of this procedure, we find that the best-fitted value of the rate-sensitivity exponent m is one, meaning that the rheological behavior of the stiff PA gel is Newtonian. Recalling that for the Newtonian case, the constant shear viscosity is $\mu = G/\dot{\gamma}_0$, the best fitted parameters determined using FE-IMR are $\alpha = 0.48$ and $\mu = 0.158$ Pa·s. The corresponding normalized radius versus time curve is included in Fig. 2.7 as a solid line.

Comparing the fitted bubble radius histories for FE-IMR and KM-IMR in Fig. 2.7, we observe that both FE-IMR and KM-IMR capture the first collapse very well. However, beyond the first collapse, KM-IMR underestimates the height of the second rebound, while FE-IMR manages to capture it well. Although there is some time delay in the second collapse predicted by FE-IMR, overall, it captures the bubbles dynamics in stiff PA gel well, yielding a more precise estimation of the material parameters. Comparing the material parameters determined using FE-IMR and KM-IMR, we note that the value of the fitted strain-stiffening parameter α does not change, while the value of the fitted viscosity decreases from 0.186 Pa·s to 0.158 Pa·s. This may be understood as follows. As observed

in Fig. 2.3(a), KM-based simulations predict that first collapse occurs slightly faster than corresponding FE-based simulations, due to the differences in how the two methods account for material compressibility. To compensate for this, when fitting the experimental bubble dynamics during first collapse, KM-IMR overestimates the viscosity, slowing down the predicted collapse dynamics to be in line with the experimental data but leading to the subsequent rebounds being overdamped. The ability of FE-IMR to more precisely capture the second rebound stems from the fact that FE-IMR accounts for the compressibility of the surrounding material, which slightly delays the collapse time as shown in Section 2.4. Thus, we conclude that accounting for material compressibility is important when dealing with violently collapsing bubbles, and doing so leads to higher fidelity estimates of material properties—in particular, the viscosity—of soft materials when applying IMR to experimental data.

2.5.2 Soft polyacrylamide gel

Next, we consider the LIC experiments of Buyukozturk et al. [48] for a “soft” PA gel with a ground-state shear modulus of $G = 461 \pm 4$ Pa.³ In Buyukozturk et al. [48], three types of micron-sized nucleation seed particles (glass, stainless steel, and paramagnetic-coated polyethylene beads) were dilutely dispersed in the soft PA gel. Scanning Electron Microscope (SEM) images of glass and paramagnetic-coated polyethylene seed particles are reproduced from Buyukozturk et al. [48] in Fig. 2.8(a). In each LIC experiment, the laser pulse was focused on a single seed particle, and varying laser energies were utilized across different experiments to achieve a range of radius ratios ($4 \lesssim R_{\max}/R_0 \lesssim 9$). LIC experiments without seed particles were also carried out. Schematics of the cavitation focal point in the gel specimen both without and with seed particles are included in Fig. 2.8(a).

³We note that this is a slightly different PA gel than the soft PA gel considered in Yang et al. [44], which had a ground-state shear modulus of $G = 570$ Pa.

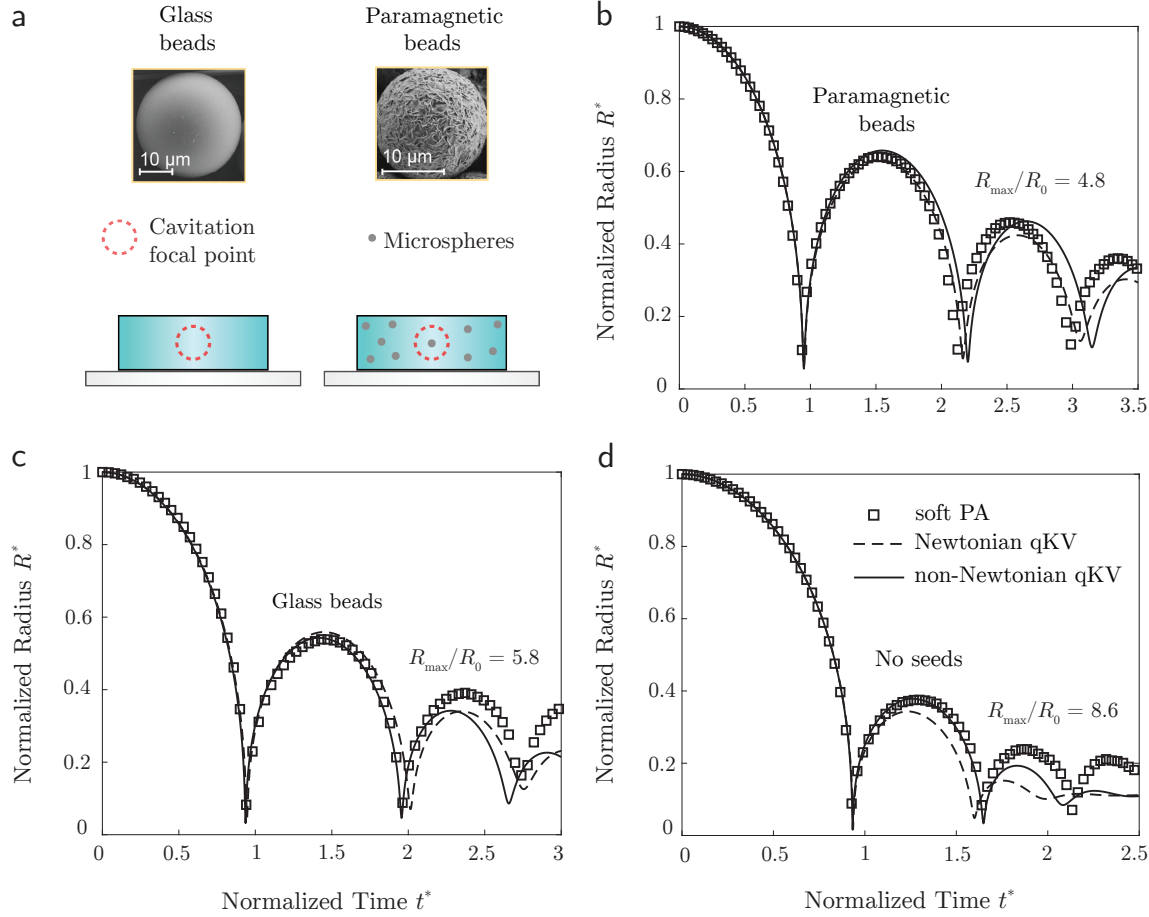


Figure 2.8: Application of FE-IMR to soft PA gel. (a) SEM images of glass and paramagnetic-coated polyethylene seed particles from Buyukozturk et al. [48] and schematics of the cavitation focal point both without and with seed particles. (b-d) Comparison of experimental data (symbols), numerical FE-IMR results using the Newtonian qKV model (dashed lines), and numerical FE-IMR results using the non-Newtonian model (solid lines) for normalized radius versus time histories for soft PA gel. Different radius ratios were achieved in Buyukozturk et al. [48] using (b) paramagnetic-coated polyethylene seed particles ($R_{\text{max}}/R_0 = 4.8$), (c) glass seed particles ($R_{\text{max}}/R_0 = 5.8$), and (d) no seed particles ($R_{\text{max}}/R_0 = 8.6$). The best-fitted material parameters for the Newtonian qKV model were $\alpha = 0.75$ and $\mu = 0.0646 \text{ Pa}\cdot\text{s}$, and the best-fitted material parameters for the non-Newtonian qKV model were $\alpha = 0.75$, $\dot{\gamma}_0 = 6046 \text{ s}^{-1}$ and $m = 0.9$.

The normalized radius $R^* = R/R_{\text{max}}$ versus normalized time $t^* = (t/R_{\text{max}})\sqrt{p_{\infty}/\rho_0}$ histories for representative LIC experiments from Buyukozturk et al. [48] in soft PA gel using paramagnetic-coated polyethylene seed particles, glass seed particles, and no seed particles are shown in Figs. 2.8(b), (c), and (d), respectively, as square symbols. All three of the representative cases correspond to a nominal laser energy of $117.8 \mu\text{J}$. The resultant maximum

bubble radii for the representative cases were $R_{\max} = 271.63, 239.14,$ and $199.21 \mu\text{m}$ for paramagnetic-coated polyethylene seed particles, glass seed particles, and no seed particles, respectively, and the corresponding equilibrium radii were $R_0 = 56.28, 41.19,$ and $23.26 \mu\text{m}$, corresponding to radius ratios of $R_{\max}/R_0 = 4.8, 5.8,$ and 8.6 . Thus, through the use of seed particles, different deformation amplitudes and rates were achieved in the same material, and the aim of this section is to apply FE-IMR to collectively fit the experimental data of Fig. 2.8 and extract a set of material parameters for soft PA gel at high strain rates.

To apply FE-IMR, we begin with the Newtonian version of the qKV model discussed in Section 2.2.2 and used for the stiff PA gel in Section 2.5.1. We take the equilibrium shear modulus to be the quasi-static value of $G = 461 \text{ Pa}$, leaving two adjustable parameters: the strain-stiffening parameter α and the shear viscosity μ . For fitting, we utilize the three normalized radius versus time histories in Fig. 2.8, each through their respective third collapse point: $t^* \lesssim 3.0$ in Fig. 2.8(b), $t^* \lesssim 2.7$ in Fig. 2.8(c), and $t^* \lesssim 2.1$ in Fig. 2.8(d). Then, we perform least-squares fitting between the three experimental histories and the corresponding histories calculated using FE-IMR with one set of adjustable parameters $\{\alpha, \mu\}$. We find that the best fitted parameters for the qKV model determined using FE-IMR are $\alpha = 0.75$ and $\mu = 0.0646 \text{ Pa}\cdot\text{s}$. We note that these values are comparable to the best-fit values determined in Buyukozturk et al. [48] using KM-IMR with the qKV model (see Fig. 6(c) of Buyukozturk et al. [48]). The best-fit normalized radius versus time curves for the qKV model are included in Figs. 2.8(b)-(d) as dashed lines. Overall, the fit is reasonable, especially for first collapse in all cases and across the full history for the case of paramagnetic-coated polyethylene seed particles in Fig. 2.8(b), for which the radius ratio $R_{\max}/R = 4.8$ is the smallest of the cases considered. For the largest radius ratio of $R_{\max}/R = 8.6$ corresponding to the case of no seed particles in Fig. 2.8(d), the best-fit predictions of the qKV model are overdamped for the second and third collapses.

Next, to obtain refined high-strain-rate properties for the soft PA gel, we apply FE-IMR using the non-Newtonian qKV model and follow a similar fitting procedure as described

in the preceding paragraph. We take the equilibrium shear modulus to be the quasi-static value of $G = 461$ Pa and the non-equilibrium shear modulus to be $G_1 = 1000G$, leaving three adjustable parameters: $\{\alpha, m, \dot{\gamma}_0\}$. Again, we perform collective least-squares fitting of FE calculations to the experimental histories of Figs. 2.8(b)-(d) through the third collapse. The resulting best estimates of the material parameters of the non-Newtonian qKV model are $\alpha = 0.75$, $\dot{\gamma}_0 = 6046 \text{ s}^{-1}$ and $m = 0.9$, and the corresponding best-fit normalized radius versus time curves are included as solid lines in Figs. 2.8(b)-(d). The value of the stiffening parameter α is the same as the value determined for the Newtonian qKV model, while a best-fit value of the rate-sensitivity exponent of $m = 0.9$ is determined. Therefore, the primary difference between predictions of the Newtonian and non-Newtonian versions of the qKV model is the shear-thinning rheological behavior accounted for in the non-Newtonian version. We observe that the overall quality of the fitting is improved in Fig. 2.8. The first collapse as well as the following rebound and second collapse are captured across all three radius ratios with paramagnetic beads giving the best match across all the seed conditions. Therefore, by incorporating non-Newtonian, shear-thinning rheological behavior in the constitutive description of the surrounding material, an improved set of high-strain-rate material parameters may be determined.

2.6 Concluding remarks

In this chapter, we have developed a finite-element-based approach for modeling the cavitation dynamics of spherical bubbles in soft viscoelastic media, which may be applied to both laser-induced and ultrasound-induced cavitation. Compared to the standard approach that utilizes the Keller-Miksis equation [28, 44, 45, 47], the FE-based approach can account for more complex constitutive behavior of the surrounding material. In this chapter, we focused on two aspects of the behavior of the surrounding material: compressibility and non-Newtonian rheological behavior. Regarding material compressibility, we observed that

for both LIC and UIC, the discrepancy between FE-based and KM-based simulations grows with the maximum Mach number, and that for maximum Mach numbers greater than about 0.08, the bubble dynamics involves material compressibility beyond that accounted for in the KM-based approach. Therefore, under this condition, FE-IMR may be used to obtain higher fidelity material properties. This was demonstrated by revisiting the LIC data of Yang et al. [44] for a stiff PA gel and applying FE-IMR to the normalized radius versus time history beyond first collapse to obtain a refined estimate of the material parameters of the qKV model. Regarding rheological behavior, we incorporated a non-Newtonian, power-law version of the qKV model into the FE-IMR framework and applied the model to the particle-assisted LIC data of Buyukozturk et al. [48] for a soft PA gel over a range of cavitation amplitudes to obtain a set of material parameters for soft PA gel at high strain rates. The trade-off of the FE-based approach compared to the KM-based approach is the additional computational expense required to solve the balance of linear momentum differential equation in the surrounding material. Therefore, the FE-IMR is complementary to the KM-IMR, since KM-IMR may be used to efficiently estimate material properties using simple models, which may be further refined using FE-IMR. The FE-based and KM-based codes used in this work have been made available to the community via GitHub (<https://github.com/HenannResearchGroup>).

Our modeling framework accounts for the dynamics of a single spherical bubble, which places limitations on when FE-IMR and KM-IMR may be applied. Specifically, the versions of FE-IMR and KM-IMR described in this chapter may only be applied to experimental $R(t)$ data in isotropic materials under situations in which the bubble remains spherically symmetric during the cavitation event in order to accurately estimate material parameters. First, dynamic cavitation in soft viscoelastic materials can lead to complex, non-spherical instability patterns, including bubble surface wrinkles and creases as shown by Yang et al. [37, 47]. Second, cavitation in anisotropic soft materials is expected to lead to non-spherical bubble shapes. Accounting for the effects of non-spherical bubble dynamics is beyond the

capacity of spherical bubble modeling and requires full, three-dimensional modeling of the bubble dynamics, which will be addressed in Chapters 4 and 5.

Chapter 3

Characterizing the mechanical behavior of fixed porcine brain tissue at high strain rates

3.1 Introduction

Traumatic brain injury (TBI) is a cause of disability or even death in children and adults [61] and is common among soldiers exposed to explosive blast in combat [62]. Computational head models [1–3] can be used to provide insight on injury prevention, diagnosis, and treatment. Yet, computational models require accurate material properties for brain tissue in order to be quantitatively predictive [63]. Due to the difficulty in characterizing human brain tissue, to understand how brain tissue behaves under varied loading conditions and impacts, porcine brain has been widely used as a proxy by many researchers [64–68]. Therefore, understanding how porcine brain tissue behaves under different loading modes, strain amplitudes, and strain rates can help improve the predictive capability of computational head models under varied types of blunt-impact and blast loadings.

Due to the prevalence of TBI arising from blunt impacts and blasts, an accurate description of the high strain-rate behavior of brain tissue is crucial for predictive computational heading modeling. However, characterizing brain tissue at high strain rates ($> 10^3 \text{ s}^{-1}$) and under large deformations remains a significant challenge. Several works have made contributions towards this goal. For example, Rashid et al. [67] used tensile testing to characterize the large-deformation behavior of porcine brain tissue within the strain-rate range $30\text{-}90 \text{ s}^{-1}$, and Pervin and Chen [69] implemented Kolsky-bar testing to characterize the mechanical behavior of different regions of bovine brain tissue at a higher strain-rate range of $1000\text{-}3000 \text{ s}^{-1}$. Magnetic Resonance Elastography (MRE) has been utilized by Guertler et al. [68] and has been successfully used to determine the linear viscoelastic behavior of *in vivo* and *ex vivo* porcine brain over a range of frequencies ($50\text{-}350 \text{ s}^{-1}$). MRE is unique in its ability to perform *in vivo* characterization but is limited to the linear, small-deformation regime. While such measurements have contributed much to our current understanding of the constitutive behavior of porcine brain tissue, the focus has remained on the moderate strain-rate regime ($50 - 3000 \text{ s}^{-1}$), and characterization of porcine brain tissue at higher strain rates ($> 10^3 \text{ s}^{-1}$) will lead to improved computational head modeling and understanding of TBI.

To obtain a broad characterization of porcine brain tissue, spanning from quasi-static to high strain rates, each of these experimental techniques can be used to obtain information in their respective strain-rate regimes, and there is an opportunity to use the Inertial Cavitation Rheometry (IMR) characterization approach to contribute material property data in the high strain-rate regime ($> 10^3 \text{ s}^{-1}$). To this end, the purpose of this chapter is to apply IMR to mechanically characterize porcine brain tissue. More specifically, in this study, we focus on fixed porcine gray matter, and we extract the material properties at high strain rates ($> 10^3 \text{ s}^{-1}$). To do so requires specialized experimental protocols because bulk brain tissue is not optically transparent. For this reason, laser-induced cavitation (LIC) experiments were carried out in thin slices of brain tissue constrained between parallel glass coverslips. As

a result of this experimental modification, the one-dimensional finite-element framework discussed in Chapter 3 is no longer able to provide an accurate description of the physics in a thin, confined layer. It is appreciated in the literature that when cavitation happens near an impedance-mismatched interface [70–72], the boundary can affect the bubble dynamics and, in some cases, lead to non-spherical behavior. Therefore, it is crucial to build a robust numerical framework that will enable the interactions between the bubble and the rigid walls within a soft material to be accurately captured and lead to an accurate estimation of the material parameters of fixed porcine gray matter at high strain rates.

Thus, in this chapter, we present an axisymmetric finite-element approach, which is a high fidelity numerical tool that allows us to account for nonlinear interactions between the bubble and the two rigid walls when the size of the bubble approaches the size of the thin layer. Then, we apply the finite-element framework to mechanically characterize the gray matter region of fixed porcine brain tissue during microcavitation at high strain rates. Next, because the axisymmetric finite-element approach is computationally more expensive, we present a modified version of the Keller-Miksis equation, which utilizes the idea of the Bjerknes force [73, 74] and the method of images to account for the interactions between the bubble and the two rigid walls. This approach represents a first-order approximation, since we account only for leading-order interactions between the bubble and the two rigid walls, and we expect that this approach will become inaccurate as the size of the bubble approaches the size of the gap thickness. For this reason, we computationally examine different thicknesses of the thin layer of material and compare the two numerical methods to determine the limits at which the modified Keller-Miksis equation cannot accurately resolve the bubble dynamics, and thus, the axisymmetric finite-element method is needed to accurately extract the material parameters of fixed porcine brain tissue.

The remainder of this chapter is organized as follows. We provide an overview of the experimental methods and results in Section 3.2. In Section 3.3, we discuss the axisymmetric finite-element modeling approach for LIC in a thin layer, and the axisymmetric version

of IMR is applied to the high strain-rate characterization of fixed porcine gray matter in Section 3.4. In Section 3.5, we introduce the modified version of the Keller-Miksis equation that accounts for thin-layer confinement and assess it against corresponding axisymmetric, finite-element simulations. We close with some concluding remarks in Section 3.6.

3.2 LIC experiments in porcine brain tissue

All LIC experiments described in this chapter were conducted by Elizabeth Bremer at the University of Wisconsin-Madison under the supervision of Professor Christian Franck using a similar LIC setup as described in Estrada et al. [28]. Here, we describe some of the key details of the experiments for completeness. Porcine brain tissue from adult pigs is used as the soft matter in the experiments. The brains from adult pigs were obtained from the Department of Animal Sciences & Dairy Sciences at the University of Wisconsin-Madison. The bulk samples of porcine brain tissue are not optically clear like the gel materials considered in the Chapter 2. As a result, this causes light scattering which prevents clearly capturing the bubble dynamics with the high-speed camera. The remedy here is to cut the samples into thinner slices. In doing so, the light scattering is reduced, which allows the bubble dynamics during the LIC event to be resolved.

In order to cut the material in thin slices, extra care needs to be taken. An illustration of the necessary steps to follow in order to prepare thin samples that are optically clear, is shown in Fig. 3.1. The process is as follows. First, the dura mater—i.e., thin outer membrane that “envelopes” the brain—was removed, and each brain was sliced into 5 mm thickness coronal slices. Next, to obtain thinner slices that are optically clear and from distinct anatomical regions of interest, an 8 mm diameter tissue biopsy punch was used to punch out samples from a given region of interest. This results in cylindrical tissue cores that are about 8 mm in diameter and about 5 mm in thickness. The tissue cores were then embedded into an ultra-low-melt agarose gel inside a modified syringe (modified by cutting the end off to

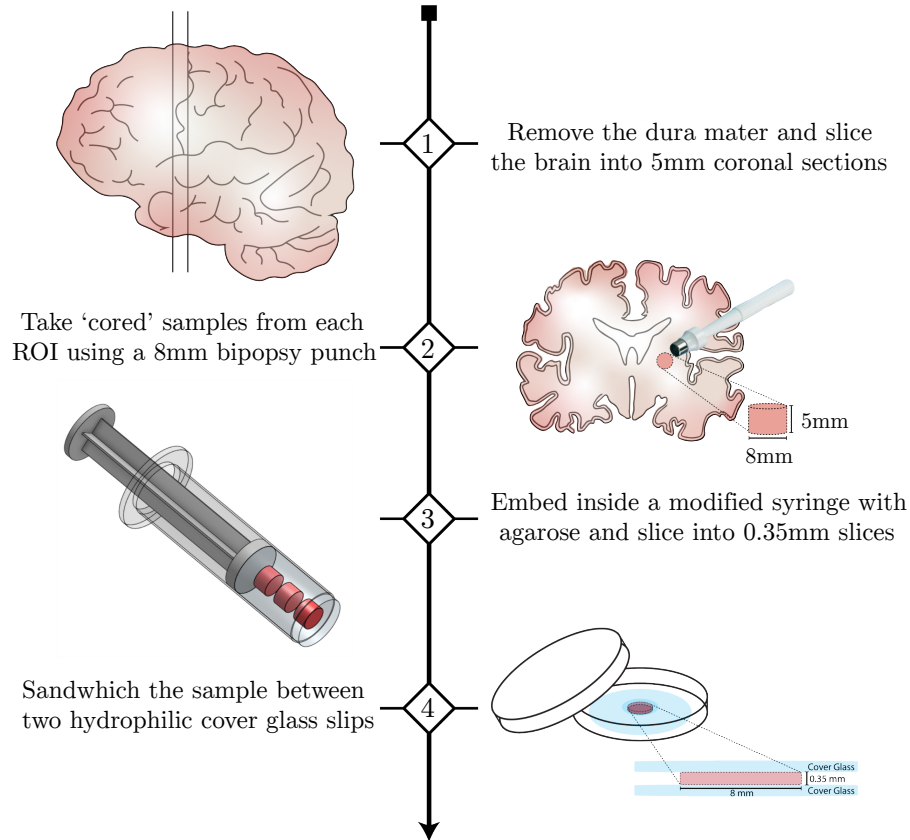


Figure 3.1: Illustration of the experimental protocol followed to prepare optically clear samples for LIC experiments in porcine brain tissue. (Courtesy of Elizabeth Bremer)

provide a smooth, flat cutting edge). After the gel polymerizes in ice and in the 4°C fridge, it was sliced using a syringe pump to obtain samples that are 350 μm thick. Slicing can be performed while the sample is submerged in phosphate-buffered saline (PBS) solution to aid in reducing the surface tension between the blade and the sample (mostly for fresh samples). After slicing, the agarose gel was removed from around the tissue slice, which was then sandwiched between two plasma-treated (hydrophilic) glass coverslips inside of PBS, so that the tissue remains hydrated (enough to keep the tissue hydrated while maintaining fixed tissue placement between the coverslips).

The established experimental process for LIC [28] may then be applied to the 350 μm thick slices of porcine brain tissue. In this study, we focus on LIC experiments carried out in fixed porcine gray matter, as illustrated in Fig. 3.2(a). The history of the cavitation event

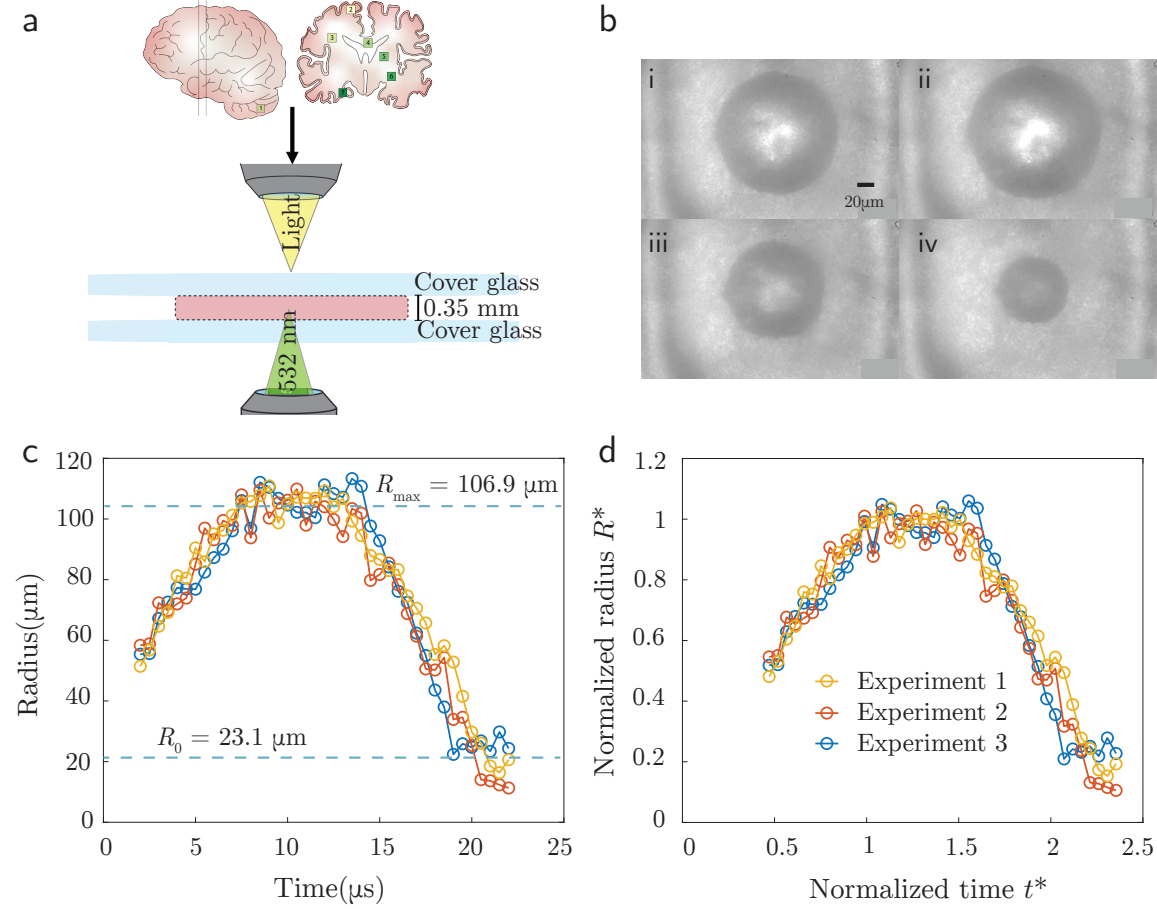


Figure 3.2: (a) Illustration of the experimental setup for LIC in a $350 \mu\text{m}$ thick slice of fixed porcine brain tissue. The material is enclosed between two glass coverslips, and a Q-switched 532 nm Nd:YAG laser is used to generate the bubble. (b) Images from the lifespan of a single LIC bubble within a thin layer of a fixed porcine brain tissue: (i) the bubble during the growth phase, (ii) the bubble at its maximum size, (iii) the bubble during the collapse phase, and (iv) the bubble at its equilibrium size. (c) Experimentally obtained radius versus time curves for three different LIC experiments. The mean maximum radius ($R_{\max}$) and equilibrium radius (R_0) are shown with dashed lines. (d) Normalized radius versus normalized time histories for the three different LIC experiments. The radius is normalized as $R^* = R/R_{\max}$, and the time is normalized as $t^* = (t/R_{\max})\sqrt{p_{\infty}/\rho_0}$.

was recorded using a Phantom v2511 high speed camera (Vision Research, Wayne, NJ). In Fig. 3.2(b), snapshots of a representative LIC experiment in fixed porcine gray matter are shown as follows: (i) the bubble during the initial growth phase, (ii) the maximum radius achieved, (iii) the bubble during the collapse phase, and (iv) the equilibrium bubble radius. These high-speed video frames recorded during LIC are then analyzed to determine the bubble radius at every time point. To determine the bubble radius, discrete points along

the bubble surface were chosen, and the centroid and radius of the bubble were determined using the circle fitting algorithm developed by Taubin [75]. In Fig. 3.2(c), the radius versus time histories for three different LIC experiments in a $350\mu\text{m}$ thick layer of fixed porcine brain are shown. Data from the initial stage of growth at short time is omitted, and due to difficulties in segmenting the bubble after first collapse, only the first growth and collapse phases of the bubble dynamics are shown. The sets of symbols and solid lines represent the three experiments, and the two horizontal dashed lines denote the mean equilibrium radius (on bottom), which is equal to $R_0 = 23.1\mu\text{m}$, and the mean maximum radius R_{max} achieved (on top) which is equal to $R_{\text{max}} = 106.9\mu\text{m}$. In Fig. 3.2(d), we see the radius versus time curve in non-dimensional form, in which the radius is normalized as $R^* = R/R_{\text{max}}$, and the time is normalized as $t^* = (t/R_{\text{max}})\sqrt{p_\infty/\rho_0}$, where $p_\infty = 101.3\text{ kPa}$ is the ambient pressure and $\rho_0 = 1000\text{ kg/m}^3$ is the referential mass density. We observe that all three experimental data curves overlay well, which indicates the repeatability of the experiments.

The major difference between these LIC experiments and prior LIC experiments in gel materials [4, 28, 44, 47] is the presence of the geometric constraint imposed by the glass coverslips. This necessitates an extended modeling framework in order to apply IMR for the purpose of material characterization.

3.3 Axisymmetric finite-element modeling of LIC in a thin layer

The presence of the glass coverslips introduces confinement of the thin layer of porcine brain tissue and breaks the assumption of spherical symmetry utilized in IMR as described in Chapter 2. However, the geometry of a cavitation bubble between two confining planes possesses axisymmetry, which permits a two-dimensional idealization. Therefore, to account for the thin-layer confinement, we consider an initially spherical cavity within a thin layer of soft material and perform nonlinear, axisymmetric simulations of bubble dynamics in LIC

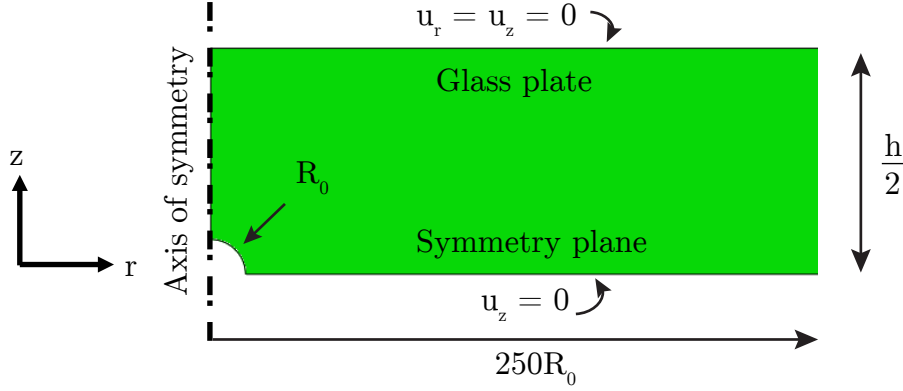


Figure 3.3: Illustration of the two-dimensional axisymmetric finite-element configuration. We simulate the one fourth of the original geometry. The top surface is the glass plate boundary where, we account for a no-slip boundary condition.

using finite-element simulations in Abaqus [76]. An illustration of the undeformed geometry used in the finite-element simulations is shown in Fig. 3.3. Following the experiments, the undeformed bubble radius is equal to $R_0 = 23.1 \mu\text{m}$, and the thickness of the thin layer is $h = 350 \mu\text{m}$. We assume that the bubble is centered along the midplane of the thin layer at a distance $z = h/2$ from the top surface, so that the problem possesses a symmetry plane along the midplane of the thin layer. For computational efficiency, we simulate the top half of the layer geometry, as shown in Fig. 3.3. The far-field radius of the simulation domain is taken to be $R_{0,\infty} = 250R_0$, which we have verified is sufficiently large so that no effects of the far-field boundary are present in our simulations. A total of 4601 Abaqus-CAX4H two-dimensional axisymmetric elements (four-node, two-dimensional, axisymmetric, constant-pressure elements) were used to model the surrounding material. Mesh convergence studies were carried out to verify that this mesh density was sufficient to provide accurate simulation results. In order to account for the infinite-sized domain along the r -direction, we used infinite elements (CINAX4) within Abaqus along a thin strip on the outer edge of the far-field radius (right side of the domain in Fig. 3.3).

The constitutive behavior of the surrounding solid material is assumed to be described by the *Neo-Hookean Kelvin-Voigt (NH-KV) model* (2.3), which is implemented in Abaqus using a UMAT subroutine. For all our simulations, the adjustable material parameters

associated with the NH-KV model are the shear modulus G and viscosity μ . The ground-state bulk modulus K is given by $K = \rho_0 c^2 - (4/3)G$, where c is the longitudinal wave speed and is equal to $c = 1430$ m/s, while ρ_0 is the referential mass density and is equal to $\rho_0 = 1000$ kg/m³.

Regarding boundary conditions, the axis of symmetry passes through the center of the bubble along the z -direction, and $u_r = 0$ for nodes along this edge. For nodes along the symmetry plane, $u_z = 0$, and we apply a no-slip boundary condition on the top surface, in which $u_r = u_z = 0$, to represent the glass coverslip during the experiments. To model the bubble, we utilize the built-in fluid-filled-cavity constraint in Abaqus [76], which assumes isothermal ideal gas behavior for the bubble contents and neglects the effects of heat and mass transfer both inside the bubble and with the surroundings. This is a simplification compared to the spherical IMR approach described in Chapter 2, which utilizes the description of the bubble contents given in Appendix A. However, as shown by Murakami [77], the high fidelity description of the bubble contents involving heat and mass transfer is most important to accurately capture the bubble dynamics during the rebounds after the first collapse, and since we focus entirely on experimental data from first collapse in this chapter, this simplification is expected to have a secondary effect. The pressure inside the bubble is then simply $p_b = (p_\infty + 2\gamma/R_0)(V_0/V)$, where $p_\infty = 101.325$ kPa is the ambient atmospheric pressure, $2\gamma/R_0$ is the Laplace pressure in the undeformed configuration, $\gamma = 0.056$ N/m is the surface tension, $V_0 = (4/3)\pi R_0^3$ is the undeformed bubble volume, and V is the volume of the deformed bubble. We account for surface tension at the bubble wall in axisymmetric finite-element simulations by utilizing the method of Henann and Bertoldi [78].

We simulate laser-induced cavitation (LIC) conditions. Because the initial conditions for LIC involve a bubble that is out of equilibrium with the surrounding solid, special care must be taken in achieving the initial conditions for LIC, and a detailed description of the application of LIC initial conditions is given in Appendix B. Briefly, initialization for LIC

simulations occurs over several analysis steps in Abaqus [76]. First, the bubble is quasi-statically inflated to the maximum bubble radius $R_{\max}$ by adding mass to the bubble, and then viscoelastic relaxation of the surroundings is allowed to complete while the bubble radius is held fixed. Next, the added mass is dynamically removed from the bubble over a timescale much smaller than the characteristic collapse time $R_{\max} \sqrt{\rho_0/p_\infty}$, so that the bubble radius remains approximately $R_{\max}$ but the bubble pressure drops to $p_b = (p_\infty + 2\gamma/R_0)(R_0/R_{\max})^3$, which are the target initial conditions for LIC. The subsequent history is simulated using a dynamic analysis step in Abaqus [76].

3.4 High-strain rate characterization of fixed porcine gray matter

In order to perform material characterization, for the reasons mentioned in Chapter 3, we ignore the growth phase and consider only the data points from the time when the bubble radius has reached its maximum size through first collapse. Shifting time, so that $t = 0$ corresponds to the time when the bubble radius is $R_{\max}$, the radius versus time data for the three experiments discussed in Section 3.2 are shown as symbols in both dimensional and dimensionless forms in Figs. 3.4(a) and (b), respectively.

We then utilize the axisymmetric finite-element approach described in Section 3.3 to apply the IMR process to fixed porcine gray matter. To this end, we perform least-square fittings between the experimental $R(t)$ data and axisymmetric numerical solutions using the NH-KV model, in which both G and μ are treated as adjustable material parameters. In our simulations, while the geometry is not spherically symmetric due to the constraint of the thin layer, the bubble remains nearly spherical during first collapse. Since the view from the high-speed camera (Fig. 3.2) is along the z -axis in the context of Fig. 3.3, we utilize the radius of the bubble along the r -direction of the symmetry plane in Fig. 3.3 as the numerically calculated $R(t)$, which is then collectively fit to the three experimental

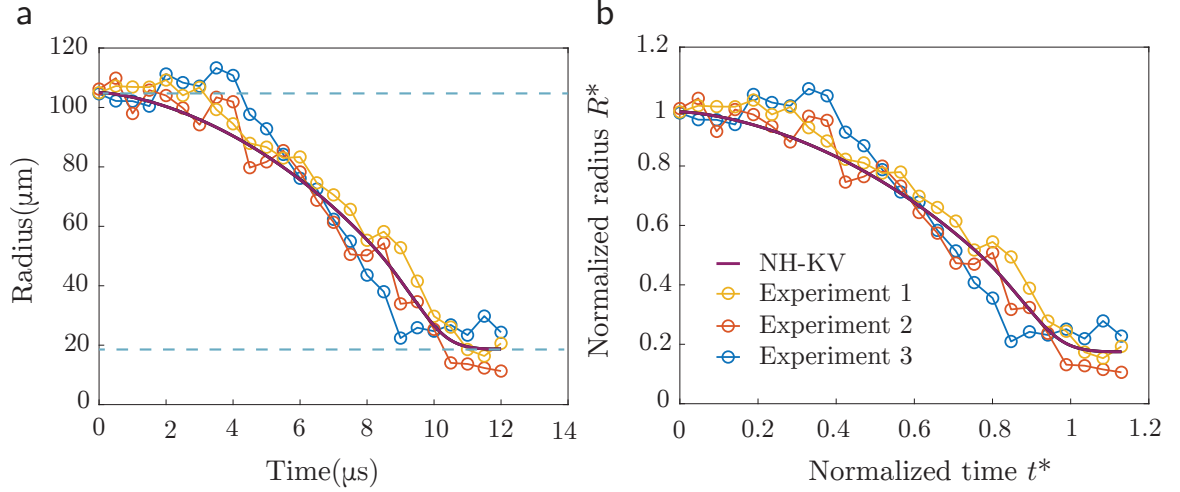


Figure 3.4: Application of axisymmetric, finite-element-based IMR to characterization of fixed porcine gray matter. Comparison of experimental data (symbols) and best-fit numerical results using the NH-KV model for the (a) non-normalized and (b) normalized radius versus time histories. The best-fit parameters of the NH-KV model are $G = 110$ kPa and $\mu = 0.05$ Pa·s.

data-sets. The solid purple lines in Fig. 3.4 show the best-fit numerical results using the NH-KV model. The best-fit material parameters obtained from the fitting procedure are $G = 110$ kPa and $\mu = 0.05$ Pa·s. We can see that the numerically calculated results can capture the experimental data quite well. It is important to note here that no re-entrant jets were observed during the collapse phase in these simulations and that the bubble maintained a nearly spherical shape.

Comparing to the results for polyacrylamide gel from Chapter 3, we note that the fitted shear modulus for fixed porcine brain tissue is notably higher ($O(100\text{kPa})$ compared to $O(1\text{kPa})$), so it is reasonable to ask whether the fitted parameters are in line with expectations. First, in general, fixed tissue is stiffer than fresh tissue, so a stiffer elastic response is to be expected. Next, the shear modulus of the NH-KV model represents a lumped shear modulus in which relaxation mechanisms associated with long characteristic relaxation times are frozen and contribute to increasing the stiffness—i.e., a nonlinear generalization of the storage modulus. Since the characteristic timescale of the LIC experiments is the collapse time which is approximately $10\text{ }\mu\text{s}$, the characteristic strain rate for the material near the bubble

wall is on the order of 10^5 s^{-1} . In the MRE experiments of Guertler et al. [68], the storage modulus for fresh porcine brain tissue at a frequency of 300 s^{-1} was approximately 5 kPa, and in the Kolsky-bar tests of Pervin and Chen [69], the initial shear modulus of fixed bovine brain tissue may be estimated to be on the order of $O(10 \text{ kPa})$ at a strain rate of 10^3 s^{-1} . Since the characteristic strain rate in the LIC experiments is several orders of magnitude higher, a shear modulus of order $O(100 \text{ kPa})$ is in line with existing data for brain tissue in the literature. Therefore, the IMR process may be used to extract the material behavior of porcine brain tissue in a strain-rate range that was previously inaccessible.

3.5 Discussion

While the axisymmetric finite-element framework provides a robust tool for material characterization in confined geometries since it takes into account nonlinear interactions between the bubble and the boundary, it is more computationally expensive than the one-dimensional approaches described in Chapter 2. A second drawback is a practical limitation imposed by Abaqus [76], which only allows the bubble contents to be modeled as an ideal gas without the effects of heat and mass transfer inside the bubble. While we expect that heat and mass transfer within the bubble is a secondary effect during first collapse [77], modeling the bubble contents as a mixture of water vapor and non-condensable gas would enable a more precise estimation of material properties and potentially extend the validity of simulations beyond first collapse. For these reasons, in this section, we explore the possibility of an extended version of the Keller-Miksis equation 2.16 that accounts for the thin layer confinement. To do so, we first discuss the concept of Bjerknes forces [73, 74] that take into account the interaction between two spherical bubbles. Then, we utilize this concept of bubble interactions and assume image bubbles interacting with a primary bubble in order to replicate the thin layer confinement.

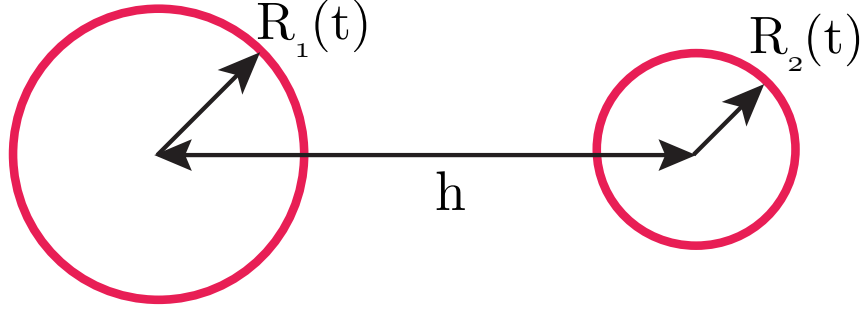


Figure 3.5: Illustration of two interacting bubbles with different radii $R_1(t)$ and $R_2(t)$ separated by a distance h .

3.5.1 Extended Keller-Miksis equation for thin layers

Bjerknes Forces

We utilize the concept of two interacting bubbles with their centers separated by a distance h during inertial microcavitation, schematically shown in Fig. 3.5. The theoretical model for the interaction between two bubbles takes into account the Bjerknes forces that have been previously studied in the literature [73, 74, 79, 80]. A brief summary of the theoretical estimation of the Bjerknes interaction forces between two spherical bubbles and its incorporation into the Keller-Miksis equation (2.16) is given below.

First, we calculate the change in the pressure field induced in the surroundings by a single isolated spherical bubble with time-dependent radius $R(t)$. We idealize the surrounding material as incompressible, so that the radial velocity field is given by

$$v_r = \frac{R^2 \dot{R}}{r^2}, \quad (3.1)$$

and $\rho = \rho_0$. Then, (i) using the velocity field (3.1) and the constitutive equation for the NH-KV material (2.6) (with $\alpha = 0$ and $J = 1$) in the radial component of the linear momentum balance (2.11), (ii) integrating over r from a position r to the far-field ($r \rightarrow \infty$), and (iii) applying the far-field boundary condition that the absolute pressure approaches p_∞ as $r \rightarrow \infty$,

the following expression for the absolute pressure field may be obtained:

$$p = p_{\infty} + \rho \left(\frac{R^2 \ddot{R} + 2R\dot{R}^2}{r} - \frac{1}{2} \frac{R^4 \dot{R}^2}{r^4} \right) + G \left(\frac{5}{2} - \frac{2}{3} \lambda^2 - 2\lambda^{-1} + \frac{1}{6} \lambda^{-4} \right), \quad (3.2)$$

where $\lambda = r/r_0$ and $r_0 = (r^3 + R_0^3 - R^3)^{1/3}$. The terms multiplied by the density ρ represent the contribution to the pressure field arising due to inertial effects, and between these terms, the convective term is of order $O(r^{-4})$ and decays more rapidly as r increases. The terms multiplied by the shear modulus G represent the contribution to the pressure arising due to Neo-Hookean hyperelasticity, and likewise, it may be shown that these terms decay rapidly in r . Therefore, these more rapidly decaying terms are neglected, and the leading order alteration to the ambient pressure due to a cavitation bubble at a distance h from the center of the bubble is

$$\delta p = \frac{\rho}{h} (R^2 \ddot{R} + 2R\dot{R}^2). \quad (3.3)$$

Then, returning to the two-bubble scenario (Fig. 3.5), the bubbles are assumed to only interact through the respective pressures that each impose on the other. The Bjerknes pressure of the second bubble acting on the first bubble at a distance h is

$$\delta p_{21} = \frac{\rho}{h} (R_2^2 \ddot{R}_2 + 2R_2 \dot{R}_2^2), \quad (3.4)$$

and similarly, the Bjerknes pressure of the first bubble acting on the second bubble at a distance h is

$$\delta p_{12} = \frac{\rho}{h} (R_1^2 \ddot{R}_1 + 2R_1 \dot{R}_1^2). \quad (3.5)$$

Finally, it is assumed that the bubbles remain spherical, so that their dynamics can be described by the coupled Keller-Miksis equations, where the respective Bjerknes pressures

are incorporated as modifications to the far-field pressure for each bubble:

$$\left\{ \begin{array}{l} \left(1 - \frac{\dot{R}_1}{c}\right) R_1 \ddot{R}_1 + \frac{3}{2} \left(1 - \frac{\dot{R}_1}{3c}\right) \dot{R}_1^2 \\ \quad = \frac{1}{\rho} \left(1 + \frac{\dot{R}_1}{c} + \frac{R_1}{c} \frac{d}{dt}\right) \left(p_{b1} - p_\infty - \frac{2\gamma}{R_1} + S_1\right) \underbrace{- \frac{1}{h} (R_2^2 \ddot{R}_2 + 2R_2 \dot{R}_2^2)}_{\text{coupling terms}}, \\ \left(1 - \frac{\dot{R}_2}{c}\right) R_2 \ddot{R}_2 + \frac{3}{2} \left(1 - \frac{\dot{R}_2}{3c}\right) \dot{R}_2^2 \\ \quad = \frac{1}{\rho} \left(1 + \frac{\dot{R}_2}{c} + \frac{R_2}{c} \frac{d}{dt}\right) \left(p_{b2} - p_\infty - \frac{2\gamma}{R_2} + S_2\right) \underbrace{- \frac{1}{h} (R_1^2 \ddot{R}_1 + 2R_1 \dot{R}_1^2)}_{\text{coupling terms}}. \end{array} \right. \quad (3.6)$$

The higher order terms involving bubble interactions and compressibility—i.e., terms involving $(\dot{R}_1/c)(R_2/h)$ in the first equation and $(\dot{R}_2/c)(R_1/h)$ in the second equation—are neglected in the above formulation. The terms S_1 and S_2 in (3.6) are the stress integrals for each bubble for a NH-KV material:

$$\left\{ \begin{array}{l} S_1 = -\frac{G}{2} \left[5 - \left(\frac{R_{10}}{R_1}\right)^4 - \frac{4R_{10}}{R_1} \right] - \frac{4\mu \dot{R}_1}{R_1}, \\ S_2 = -\frac{G}{2} \left[5 - \left(\frac{R_{20}}{R_2}\right)^4 - \frac{4R_{20}}{R_2} \right] - \frac{4\mu \dot{R}_2}{R_2}, \end{array} \right. \quad (3.7)$$

where R_{10} and R_{20} are the undeformed radii of the two bubbles. Finally, following the preceding sections, in what follows, we continue to assume that the bubble contents are described by an isothermal ideal gas. Thus, the internal pressures in the first and second bubbles, which we denote as p_{b1} and p_{b2} , respectively, are given by

$$\left\{ \begin{array}{l} p_{b1} = \left(p_\infty + \frac{2\gamma}{R_{10}}\right) \left(\frac{R_{10}}{R_1}\right)^3, \\ p_{b2} = \left(p_\infty + \frac{2\gamma}{R_{20}}\right) \left(\frac{R_{20}}{R_2}\right)^3. \end{array} \right. \quad (3.8)$$

However, since the physics of each bubble's contents is uncoupled from the other, it would

be straightforward to incorporate the description of the bubble contents involving heat and mass transfer from Appendix B here.

Here, we have discussed the problem of two interacting spherical bubbles, but this approach may be applied to any number of interacting spherical bubbles [79, 80]. For N interacting bubbles, the dynamics of a given bubble i is governed by

$$\begin{aligned} & \left(1 - \frac{\dot{R}_i}{c}\right) R_i \ddot{R}_i + \frac{3}{2} \left(1 - \frac{\dot{R}_i}{3c}\right) \dot{R}_i^2 \\ &= \frac{1}{\rho} \left(1 + \frac{\dot{R}_i}{c} + \frac{R_i}{c} \frac{d}{dt}\right) \left(p_{bi} - p_\infty - \frac{2\gamma}{R_i} + S_i\right) - \sum_{j=1, j \neq i}^N \frac{2}{h_{ij}} (R_j^2 \ddot{R}_j + 2R_j \dot{R}_j^2), \quad (3.9) \end{aligned}$$

where h_{ij} is the distance between the centers of bubbles i and j .

Thin-layer modeling

Returning to the problem of cavitation in a thin layer, to apply the Keller-Miksis approach, we need to account for the finite thickness of the samples. To do so, we construct an approximate problem using the method of images, consisting of an infinite sequence of image bubbles arranged along a line and each spaced a distance h apart (Fig. 3.6). The radius of each image bubble is identical to the radius of the real bubble $R(t)$. Such a problem possesses symmetry planes along the midplane between each pair of adjacent bubbles, which remain flat during the dynamics and mimic the glass plates acting as geometric constraints in the real problem.

Since the real bubble and the image bubbles have the same radius $R(t)$, they will all have the same dynamics. Applying (3.9) to the problem of a bubble in a thin layer interacting with a sequence of image bubbles, we obtain the following modified Keller-Miksis equation,

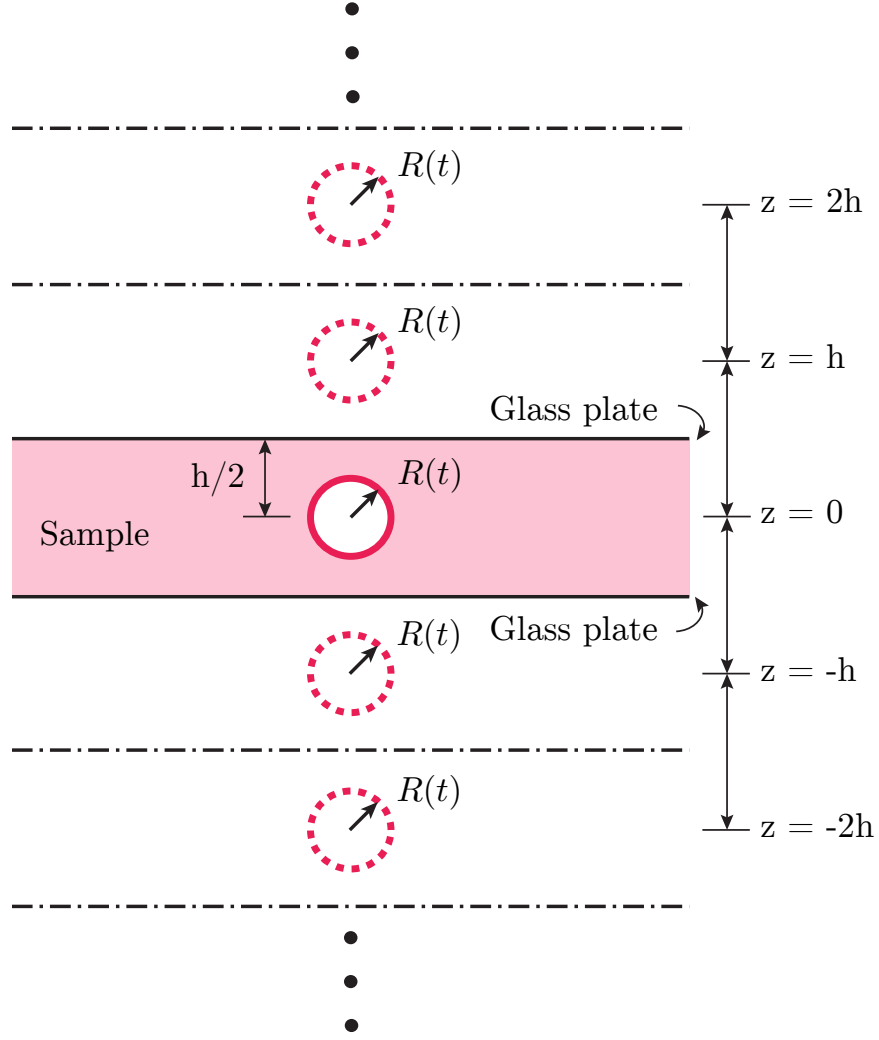


Figure 3.6: Illustration of an infinite sequence of image bubbles that satisfy all the boundary conditions for the thin layer. The first ($z=h$) image satisfies the rigid interface ($z= h/2$) boundary condition ($u= 0$), but results in a net stress at ($z= -h/2$). That stress is eliminated by the second image source ($z= -h$) which, however, violates the rigid interface boundary condition ($z= h/2$). This necessitates the next image ($z= -2h$), and so on. The degree of boundary condition violation decreasing with every pair of images because they are further removed from those boundaries.

which takes into account the thin layer correction:

$$\begin{aligned}
 & \left(1 - \frac{\dot{R}}{c}\right) R \ddot{R} + \frac{3}{2} \left(1 - \frac{\dot{R}}{3c}\right) \dot{R}^2 \\
 &= \frac{1}{\rho} \left(1 + \frac{\dot{R}}{c} + \frac{R}{c} \frac{d}{dt}\right) \left(p_b - p_\infty - \frac{2\gamma}{R} + S\right) - \underbrace{\sum_{n=1}^{\infty} \frac{2}{nh} (R^2 \ddot{R} + 2R \dot{R}^2)}_{\text{coupling terms}}, \quad (3.10)
 \end{aligned}$$

where we sum all the coupling terms that account for all image bubbles interactions, and the factor of 2 takes into account the contribution of each pair of image bubbles above and below the real bubble at a distance nh . We note that the interaction terms in (3.10) form an infinite geometric series which is a non-convergent series. This is likely due to physics that was neglected in the formulation of (3.10)—namely, the effect of time-delay arising due to the finite sound speed in the interactions of the real bubble with the image bubbles [79]. In practice, we truncate the sum in (3.10) at $n = 1$, so that only one image bubble at a distance h above and below the real bubble is accounted for.

3.5.2 Comparison between the two methods

To this point, we have presented two numerical methods to account for the thin layer confinement. The first method is the axisymmetric finite-element approach of Section 3.3, which is a high fidelity numerical tool that enables us to account for nonlinear interactions between the bubble and the two rigid walls during inertial microcavitation. The second approach is the modified Keller-Miksis equation 3.10, which incorporates the concept of the Bjerknes forces from image bubbles to account for the interactions between the bubble and the two rigid walls. Although the modified Keller-Miksis equation allows for the incorporation of a mixture of water-vapor and non-condensable gas inside the cavity, which could potentially improve material parameter estimation, it takes into account only first-order interactions between the bubble and the rigid walls (i.e., we only account only for $R(t)/h$ terms in (3.10)). Moreover, interactions in the Keller-Miksis framework are truncated at one pair of image bubbles. Despite these issues, because the Keller-Miksis-based approach is more computationally efficient and allows for the incorporation of more complex physics inside the bubble, it is worth exploring the circumstances under which the reduced-order, Keller-Miksis-based approach may be used. We expect that the ability of the reduced-order approach to accurately predict the bubble dynamics will degrade as the maximum bubble diameter becomes comparable to the gap thickness, since nonlinear interactions occur, and

the purpose of this study is to examine the cases under which the modified Keller-Miksis equation produces accurate results in confined geometries.

We perform corresponding numerical simulations using both the axisymmetric finite-element approach and the modified Keller-Miksis approach. For the modified Keller-Miksis equation (3.10), we truncate interactions at one a pair of image bubbles at distance $z = h$. We obtain radius time histories using both numerical methods and examine the effect of different gap thicknesses, different radius ratios ($R_{\max}/R_0$), and different material viscosities on the resulting bubble dynamics. For all the baseline simulations the initial undeformed bubble radius is $R_0 = 18.35 \mu\text{m}$. The values of the gap thickness we tested are $h = 500 \mu\text{m}$ and $350 \mu\text{m}$. The different radius ratios we tested are $R_{\max}/R_0 = 3.2, 4, 4.6$ and 5.9 , while the values of viscosity considered are $\mu = 0.005, 0.015, 0.035$, and $0.06 \text{ Pa}\cdot\text{s}$. We model the bubble contents as an isothermal, ideal gas in both methods, so that comparisons may be made, and neglect the effect of surface tension. We continue to assume a NH-KV model for the surrounding material and keep the shear modulus fixed throughout the simulations and equal to $G = 110 \text{ kPa}$.

For each combination of gap thickness, radius ratio, and viscosity, we measure the root-mean-square difference (ΔR_{RMS}^*) between the average non-dimensional finite-element radius history and the non-dimensional modified Keller-Miksis radius history from $t^* = 0$ until the normalized time at first collapse t_c , defined as follows:

$$\Delta R_{\text{RMS}}^* = \sqrt{\frac{1}{t_c} \int_0^{t_c} \left(R_{\text{FE}}^*(t^*) - R_{\text{KM,layer}}^*(t^*) \right)^2 dt^*}. \quad (3.11)$$

In (3.11), the average non-dimensional radius in the finite-element calculations is defined as $R_{\text{FE}}^* = (1/R_{\max})(3V/4\pi)^{1/3}$, where V is the deformed volume of the bubble. In Figs. 3.7 and 3.8(right), we present contour plots of the RMS difference for the two different gap thicknesses. For each contour plot, we plot ΔR_{RMS}^* as a function of the different radius ratios and viscosities. The results for a gap thickness of $h = 500 \mu\text{m}$ are shown in Fig. 3.7. The

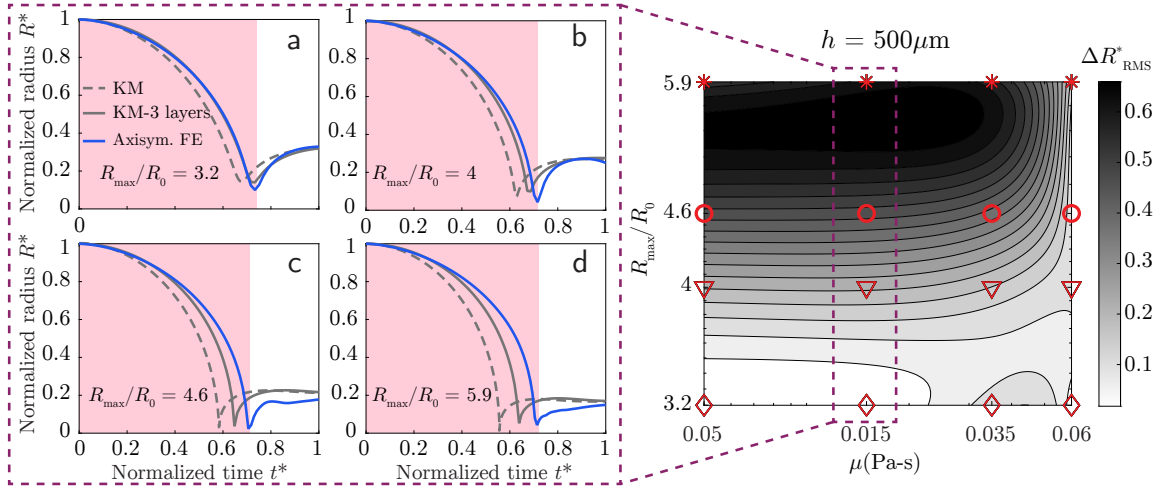


Figure 3.7: Radius versus time curves for different radius ratios (a) $R_{\max}/R_0 = 3.2$, (b) $R_{\max}/R_0 = 4$, (c) $R_{\max}/R_0 = 4.6$ and (d) $R_{\max}/R_0 = 5.9$ and a fixed material viscosity of $\mu = 0.015$ Pa-s for a gap thickness of $h = 500 \mu\text{m}$. The blue solid lines are results from the axisymmetric finite-element simulations, the gray solid lines are results from the modified Keller-Miksis equation with two image bubbles (3 layers), and the dashed gray lines are results from the conventional Keller-Miksis equation. (Right) Contour plot of the ΔR_{RMS}^* difference between the axisymmetric finite-element approach and the modified Keller-Miksis solution with two image bubbles. The difference is plotted as a function of the radius ratios $R_{\max}/R_0$ and material viscosity μ .

first observation is that for small radius ratios (e.g., $R_{\max}/R_0 = 3.2$) both approaches give similar results. However, as the radius ratio $R_{\max}/R_0$ increases, the difference between the two methods increases. This is also evident from the radius versus time history plots in Figs. 3.7(a-d) for a fixed viscosity $\mu = 0.015$ Pa-s, where we overlay the axisymmetric finite-element solution (blue solid line) along with the modified Keller-Miksis equation solution (gray solid line) and the conventional Keller-Miksis equation solution (gray dashed line). Results from the conventional Keller-Miksis equation (dashed gray line) are included only for visualization purposes to illustrate that the modification to the Keller-Miksis equation does improve predictions in all cases, and these results are not used in the calculation of ΔR_{RMS}^* on the right of Fig. 3.7. That ΔR_{RMS}^* grows with increasing $R_{\max}/R_0$ is to be expected since for higher $R_{\max}$ and fixed R_0 , the ratio $h/R_{\max}$ becomes smaller, meaning that nonlinear interactions will have a greater effect on the bubble dynamics. Examining the effect of viscosity, we can see that for the low radius ratio cases, the difference ΔR_{RMS}^*

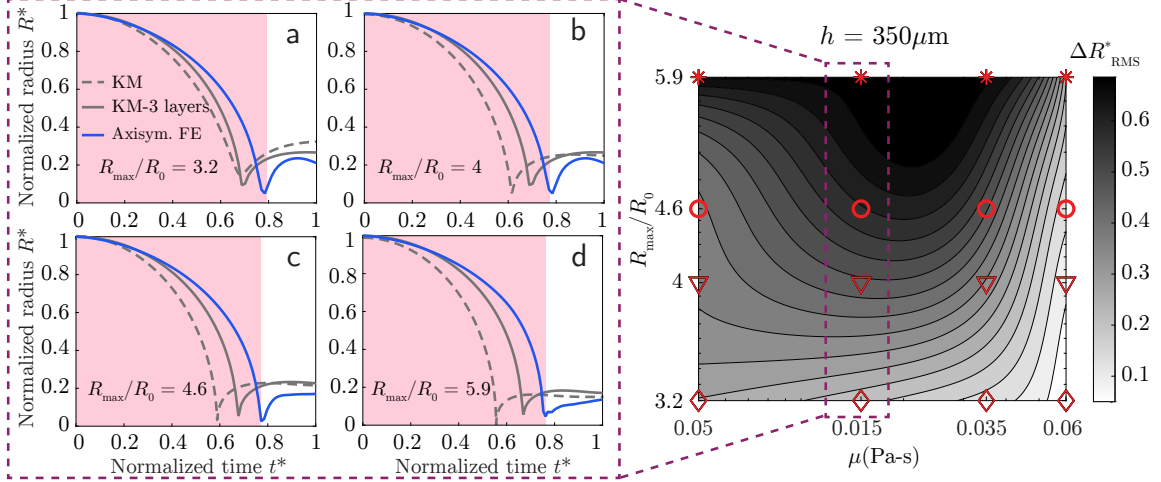


Figure 3.8: Radius versus time curves for different radius ratios (a) $R_{\max}/R_0 = 3.2$, (b) $R_{\max}/R_0 = 4$, (c) $R_{\max}/R_0 = 4.6$ and (d) $R_{\max}/R_0 = 5.9$ and a fixed material viscosity of $\mu = 0.015$ Pa-s for a gap thickness of $h = 350 \mu\text{m}$. The blue solid lines are results from the axisymmetric finite-element simulations, the gray solid lines are results from the modified Keller-Miksis equation with two image bubbles (3 layers), and the dashed gray lines are results from the conventional Keller-Miksis equation. (Right) Contour plot of ΔR_{RMS}^* difference between the axisymmetric finite-element approach and the modified Keller-Miksis solution with two image bubbles. The difference is plotted as a function the radius ratios $R_{\max}/R_0$ and material viscosity μ .

is relatively low, and for all radius ratios, the difference ΔR_{RMS}^* generally decreases with increasing viscosity.

Similar observations are made when examining the contour plot for a smaller gap thickness. In Fig. 3.8(right), we present a contour plot of the difference ΔR_{RMS}^* between the two approaches for a gap thickness $h = 350 \mu\text{m}$. The overall difference between the two methods increases further compared to Fig. 3.7 for larger $h/R_{\max}$. Looking at the radius versus time histories in Figs. 3.8 (a-d) for a fixed viscosity $\mu = 0.015$ Pa-s, we can see that even for low values of $R_{\max}/R_0$ the difference is higher compared to the larger gap thickness case. We conclude again that for higher μ and low material stretch $R_{\max}/R_0$ the error between the two methods is at its smallest but not negligible for this case of gap thickness.

To further quantify the differences between the two methods, we perform fittings on the finite-element results using the modified Keller-Miksis approach. More specifically, for

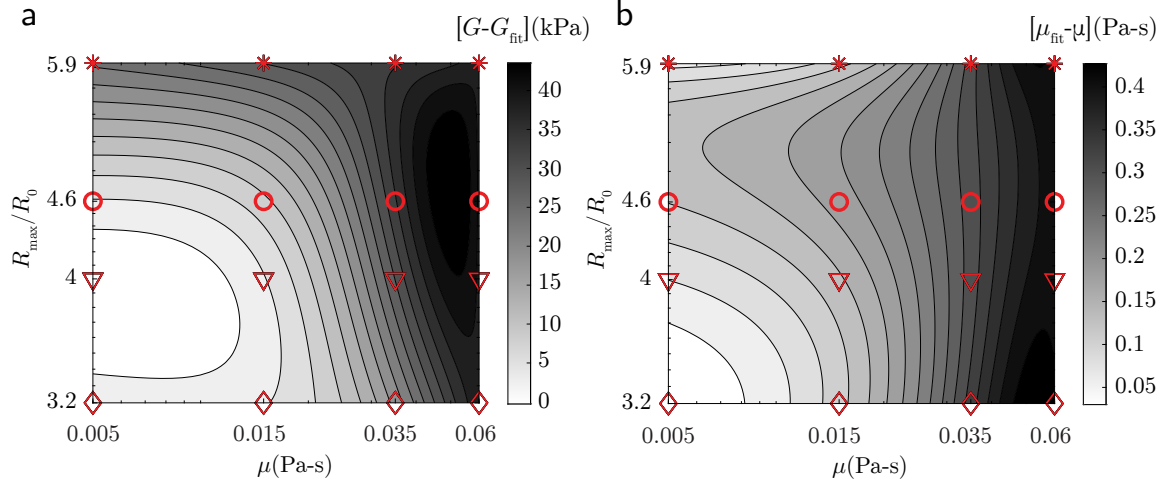


Figure 3.9: Contour plots of the differences in the fitted material parameters, (a) $G - G_{\text{fit}}$ and (b) $\mu_{\text{fit}} - \mu$, for a layer thickness of $h = 500 \mu\text{m}$ obtained using the modified Keller-Miksis approach.

each gap thickness, we treat the finite-element results as experimental data with unknown material properties, and we fit this data using the modified Keller-Miksis framework. The aim of this study is to assess the goodness of the fitting when the modified Keller-Miksis equation is used for material characterization. We treat both G and μ as fitting parameters, and we fit only the first collapse. We examine a total of 16 different cases for each gap thickness, where we account for four different radius ratios R_{max}/R_0 and four different values of viscosity μ . In Fig. 3.9, we present the results of the fitting for a gap thickness of $h = 500 \mu\text{m}$. Each of the red symbols on the contour plots represents each of the different finite-element simulations. We present the results of the best fitted G_{fit} and μ_{fit} in two different contour plots. In Fig. 3.9(a), we present contour plots of the difference between the known value of shear modulus $G = 110 \text{ kPa}$ and the best fitted shear modulus G_{fit} , while in Fig. 3.9(b), we present contour plots of the difference between the best fitted viscosity μ_{fit} and the known viscosity μ for each case. It is important to look at both contour plots in Fig. 3.9 to get the best fitted material properties ($G_{\text{fit}}, \mu_{\text{fit}}$) from the modified Keller-Miksis approach for each case. Furthermore, we observe that for small to medium radius ratios, $R_{\text{max}}/R_0 = 3.2$ and 4, and low viscosity, the modified Keller-Miksis equation predicts fairly

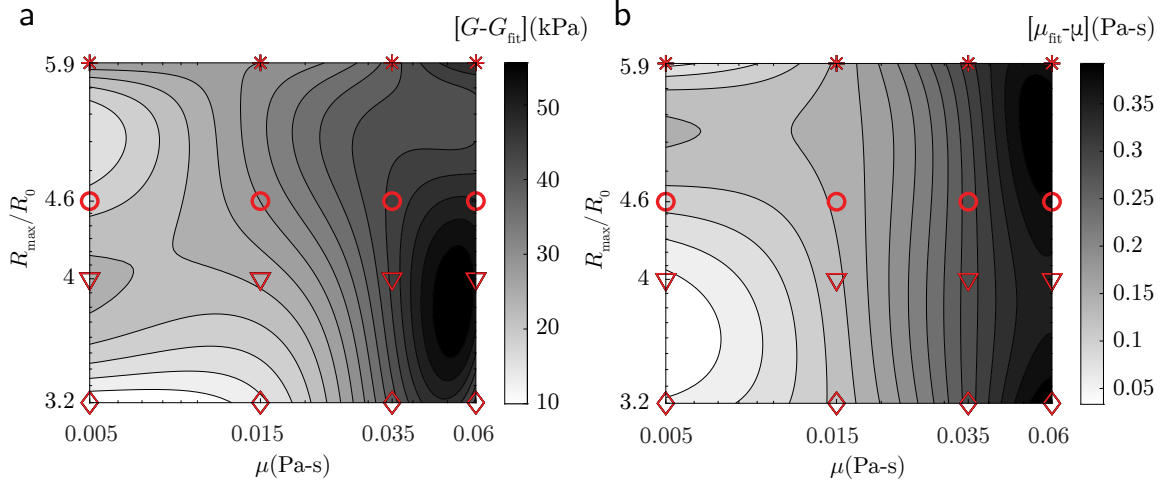


Figure 3.10: Contour plots of the differences in the fitted material parameters, (a) $G - G_{\text{fit}}$ and (b) $\mu_{\text{fit}} - \mu$, for a layer thickness of $h = 350 \mu\text{m}$ obtained using the modified Keller-Miksis approach.

well the values of G and μ . However, as we move to higher radius ratios, the error of the estimated parameters G and μ increases, which is to be expected since for higher radius ratios, the $h/R_{\max}$ ratio becomes smaller and thus the effect of the boundary becomes more prominent. Regarding the effect of viscosity, we observe from the contour plots that for higher values of viscosity μ , the modified Keller-Miksis approach does not provide good estimates of the fitted G and μ mainly because for higher μ , the behavior of the bubble in the axisymmetric finite-element simulations is overdamped.

In Fig. 3.10, we present the results of the fitting for a gap thickness of $h = 350 \mu\text{m}$. Examining the effect of a lower gap thickness of $h = 350 \mu\text{m}$, we observe similar trends as in the case of $h = 500 \mu\text{m}$. In the contour plot for $G - G_{\text{fit}}$ in Fig. 3.9(a) and the contour plot for $\mu_{\text{fit}} - \mu$ in Fig. 3.9(b), we conclude that for low radius ratio, $R_{\max}/R_0 = 3.2$, and low viscosity, the modified Keller-Miksis approach does a reasonable job capturing the values of G and μ , while as the radius ratio or viscosity is increased, the error of the estimated parameters G and μ increases. Moreover, for $R_{\max}/R_0 > 6$ and $\mu = 0.015 \text{ Pa}\cdot\text{s}$, we begin to observe re-entrant jets forming during the first collapse for $h/R_{\max} = 350 \mu\text{m}$, and the assumption of nearly spherical behavior begins to break down. Overall, as expected, the

differences for the lower gap thickness of $h = 350\mu\text{m}$ are greater than the corresponding cases for the gap thickness of $h = 500\mu\text{m}$. Overall, we conclude that for smaller radius ratios ($R_{\text{max}}/R_0 = 3.2$)¹, the modified Keller-Miksis approach may be used to efficiently obtain a rough estimate of the material parameters, but the axisymmetric finite-element approach must still be used to obtain precise estimates for the material parameters, especially if the radius ratio is large or the bubble dynamics are overdamped.

3.6 Concluding remarks

In this chapter, we applied the Inertial Microcavitation Rheometry (IMR) protocol to characterize the high strain-rate behavior of fixed porcine gray matter tissue using experimental data from LIC. Because bulk samples of porcine brain tissue are not optically clear like soft gels, e.g., polyacrylamide gels, the experiments utilized a protocol that minimizes light scattering and allows for resolvable cavitation events in fixed porcine brain tissue slices with thicknesses of $100 - 350\mu\text{m}$. This enables high-speed imaging of bubble dynamics to be carried out; however, the thin material slices impose constraints which affect the bubble dynamics, necessitating an extension of the theoretical cavitation framework from Chapter 2. For this reason, we extended our numerical framework to account for the thin layer confinement and developed an axisymmetric finite-element approach, which is a high fidelity numerical tool that allows us to account for nonlinear interactions between the bubble and the two rigid walls when the size of the bubble approaches the size of the thin layer. Next, we applied the axisymmetric, finite-element framework to extract the material properties of fixed porcine gray matter. Moreover, since we can neither model the bubble contents as a mixture of water-vapor and non-condensable gas nor account for heat and mass transfer inside the bubble due to limitations imposed by Abaqus, we present an extended

¹For reference, for the case with the $R_{\text{max}}/R_0 = 3.2$ and the lowest viscosity, the maximum Mach number was around 0.02, so while collapse is not violent, it is still inertial.

version of the Keller-Miksis equation that accounts for the thin layer confinement. We compare the two methods by providing error estimates for different radius ratios and material viscosities. Finally, we assess the goodness of the fitted material parameters determined using the extended Keller-Miksis equation by treating the FE-based results as experimental data and perform fittings using the extended Keller-Miksis equation, as if the later was to be used for material characterization. In this way, we showed that the extended Keller-Miksis equation may be used if the radius ratio is sufficiently small, but for larger radius ratios, the axisymmetric finite-element approach remains the preferred approach to accurately extract material properties.

Chapter 4

Predicting the onset of non-spherical instability during inertial microcavitation in soft solids

Note: Portions of this chapter are published in *Physical Review E*. Data and figures have been used with all co-authors' consent.

Jin Yang, Anastasia Tzoumaka, Kazuya Murakami, Eric Johnsen, David L. Henann, and Christian Franck. "Predicting complex nonspherical instability shapes of inertial cavitation bubbles in viscoelastic soft matter." Physical Review E 104 (2021): 045108.

4.1 Introduction

Much of our current understanding of inertial cavitation relies on the idealization that the cavitation bubble remains nominally spherical through its life cycle. However, recent experiments and numerical simulations have shown that spherical symmetry can often break down during inertial cavitation, even in nominally homogeneous and isotropic soft matter [30–34]. While the basic physics of these cavitation-related instabilities are relatively well

understood for bubbles in Newtonian fluids, such as the classical Rayleigh-Taylor (RT) instability [81] or parametric instabilities, which arise due to the accumulation of non-spherical perturbations over many oscillation periods [31, 34, 82, 83], they have remained largely unresolved for soft, nonlinear solids.

Compared to water, cavitation in a soft viscoelastic solid material can produce distinctively different complex (non-spherical) instability patterns, including bubble surface wrinkles and creases. While wrinkles are smooth undulations that can generate significant surface strains along the bubble-material interface, creases, which are defined by their topologically sharp features and extreme local curvatures [84, 85], can produce even higher surface strains, which can push a material past its inherent breaking point leading to significant material damage or fracture [86–88]. This type of strain and stress amplification is a hallmark feature of the presence of instabilities in a solid, not typically observed in a fluid. Perhaps most importantly, these kinds of surface instabilities, e.g., wrinkles and creases, can significantly degrade the performance of any soft matter system, and can be especially harmful when occurring in soft biological tissues generating unwanted collateral damage in eye laser surgeries [9] or histotripsy procedures [5–7, 35, 36].

In a recent effort, Yang et al. [37] developed a theoretical framework for predicting the onset of non-spherical instabilities during inertial cavitation in soft, nonlinear viscoelastic solids by explicitly accounting for all pertinent nonlinear interactions between the cavitation bubble and solid surroundings. Using this theoretical framework, they were able to predict the onset of non-spherical instability of a cavitation bubble in a soft material, and comparison against high-resolution experimental images from laser-induced cavitation (LIC) events in a polyacrylamide (PA) gel showed good agreement. Interestingly, and consistent with experimental findings, the instability analysis predicts the emergence of various dynamic instability shapes for circumferential bubble stretch ratios greater than one in contrast to most quasi-static investigations.

While the theoretical framework of Yang et al. [37] was shown to be consistent with

experimental observations of the onset of dynamic non-spherical instabilities in PA gel, it has not been verified in nonlinear, three-dimensional post-buckling-type simulations of bubble dynamics. The purpose of this chapter is to carry out such an analysis. We present a three-dimensional, finite-element-based study that is capable of accounting for non-spherical behavior to predict the transition from spherical to non-spherical bubbles during the dynamics. More specifically, we apply a small non-spherical perturbation to the initial, spherical bubble geometry and then perform simulations of LIC in a soft solid described by the quadratic-law Kelvin-Voigt model discussed in Chapter 2. Different radius ratios ($R_{\max}/R_0$) and material parameters of the surrounding material are considered to probe the generality of theoretical instability framework.

This chapter is organized as follows. We present an overview of the governing equations in Section 4.2.1, the linear stability analysis of Yang et al. [37] in Section 4.2.2, and our methodology for nonlinear, dynamic, post-buckling-type finite-element analyses in Section 4.2.3. We then examine the onset of non-spherical bubble behavior during dynamic cavitation in stiff polyacrylamide gels using nonlinear simulations in Section 4.3, including the effects of the radius ratio in Section 4.3.1 and the effect of material viscosity and surface tension in Section 4.3.2 on the stability of the bubble dynamics, and comparisons are made between the nonlinear simulation results and the predictions of the linear stability analysis. In Section 4.3.3, we highlight the effect of non-spherical deformation in inducing localized amplification of strain and stress near the bubble wall, and in Section 4.4, we close with some concluding remarks.

4.2 Continuum framework

4.2.1 Three-dimensional governing equations

We begin by summarizing the three-dimensional governing equations used in the nonlinear finite-element simulations. We consider a reference undeformed configuration $\mathcal{B}_0$ and

denote material points within $\mathcal{B}_0$ as $\mathbf{X}$. The reference body $\mathcal{B}_0$ then undergoes a motion $\mathbf{x} = \chi(\mathbf{X}, t)$ to the deformed body $\mathcal{B}$ at each time t . The deformation gradient is given by $\mathbf{F} = \nabla \chi$, such that the ratio of referential to deformed volume is strictly greater than zero, i.e., $J = \det \mathbf{F} > 0$. The left Cauchy-Green deformation tensor is $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$, the distortional deformation gradient tensor is $\bar{\mathbf{F}} = J^{-1/3}\mathbf{F}$, the distortional left Cauchy-Green deformation tensor is $\bar{\mathbf{B}} = J^{-2/3}\mathbf{B}$, and the distortional first principal stretch invariant is $\bar{I}_1 = \text{tr} \bar{\mathbf{B}}$. The spatial velocity gradient tensor is $\mathbf{L} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, and the stretching and spin tensors are $\mathbf{D} = \text{sym} \mathbf{L}$ and $\mathbf{W} = \text{skw} \mathbf{L}$, respectively.

Throughout this chapter, we use the nonlinear quadratic-law Kelvin-Voigt (qKV) viscoelastic material. As discussed in Section 2.2.2, the hyperelastic response of the qKV model accounts for strain-stiffening effects, and the referential free-energy density is given by

$$\psi = \frac{G}{2} \left[(\bar{I}_1 - 3) + \frac{\alpha}{2} (\bar{I}_1 - 3)^2 \right] + \frac{K}{2} (J - 1)^2, \quad (4.1)$$

where G is the ground-state shear modulus, α is a dimensionless material parameter characterizing large-deformation, strain-stiffening effects, and K is the ground-state bulk modulus. In our finite-element simulations, we take K to be finite but with $K \gg G$ so that the surrounding material is slightly compressible. The viscous response of the qKV model is described by Newtonian rheological behavior with a constant viscosity μ . Thus, the total Cauchy stress for the qKV model is written as

$$\boldsymbol{\sigma} = J^{-1} G [1 + \alpha(\bar{I}_1 - 3)] \text{dev} \bar{\mathbf{B}} + K (J - 1) \mathbf{1} + 2\mu \text{dev}(\mathbf{D}). \quad (4.2)$$

The qKV model has been numerically implemented in Abaqus [76] using a user-material (UMAT) subroutine, and all three-dimensional finite-element calculations are carried out using Abaqus [76].

Finally, the balance of linear momentum in the absence of external body forces requires

that

$$\nabla \cdot \boldsymbol{\sigma} = \rho \mathbf{a}, \quad (4.3)$$

where $\nabla \cdot (\bullet)$ is the spatial divergence operator, ρ is the spatial mass density, and $\mathbf{a}$ is the acceleration vector.

4.2.2 Linear perturbation analysis

In this section, we present the theoretical framework for predicting the onset of non-spherical instabilities during inertial cavitation in soft, nonlinear viscoelastic solids developed by Yang et al. [37].

Base state

The theoretical framework is based on perturbations of spherically symmetric and incompressible base states of deformation, described by the motion $r = (r_0^3 + R(t)^3 - R_0^3)^{1/3}$, where r is the current radial coordinate, r_0 is the referential radial coordinate, $R(t)$ is the time-dependent bubble radius, and R_0 is the undeformed bubble radius. The time-dependent bubble radius of the base state $R(t)$ is governed by the Rayleigh-Plesset (RP) equation [20, 21]:

$$R\ddot{R} + \frac{3}{2}\dot{R}^2 = \frac{1}{\rho} \left(p_b - p_\infty - \frac{2\gamma}{R} + S \right), \quad (4.4)$$

where $\rho = \rho_0$ is the constant mass density of the surrounding viscoelastic material, p_b is the internal bubble pressure, p_∞ is the far-field pressure, which is assumed to be atmospheric as there are no external driving forces in LIC, γ is the surface tension between the gaseous bubble phase and the surrounding medium, and S is given through the integral of deviatoric

Cauchy stress components over the surroundings, which for the qKV model is given by [44]

$$S = \frac{(3\alpha - 1)G}{2} \left[5 - \left(\frac{R_0}{R} \right)^4 - \frac{4R_0}{R} \right] - \frac{4\mu\dot{R}}{R} + 2\alpha G \left[\frac{27}{40} + \frac{1}{8} \left(\frac{R_0}{R} \right)^8 + \frac{1}{5} \left(\frac{R_0}{R} \right)^5 + \left(\frac{R_0}{R} \right)^2 - \frac{2R}{R_0} \right]. \quad (4.5)$$

Perturbation

In the base state, the only non-zero component of displacement is the radial component $u_r = r - r_0$. We then consider a displacement perturbation of the form $\{\tilde{u}_r, \tilde{u}_\theta, \tilde{u}_\phi\}$, which is added to the base-state displacement field. The resulting deformation gradient is $\mathbf{F} = \mathbf{F}_0 + \mathbf{d}$, where $\mathbf{F}_0$ is the base-state deformation gradient, which can be expressed referentially as

$$[\mathbf{F}_0] = \begin{bmatrix} \frac{r_0^2}{(r_0^3 + R^3 - R_0^3)^{2/3}} & 0 & 0 \\ 0 & \frac{(r_0^3 + R^3 - R_0^3)^{1/3}}{r_0} & 0 \\ 0 & 0 & \frac{(r_0^3 + R^3 - R_0^3)^{1/3}}{r_0} \end{bmatrix}, \quad (4.6)$$

so that $\det(\mathbf{F}_0) = 1$ and $\mathbf{d}$ is the perturbation of the deformation gradient, which is expressed spatially as

$$[\mathbf{d}] = \begin{bmatrix} \frac{\partial \tilde{u}_r}{\partial r} & \frac{1}{r} \frac{\partial \tilde{u}_r}{\partial \theta} - \frac{\tilde{u}_\theta}{r} & \frac{1}{r \sin \theta} \frac{\partial \tilde{u}_r}{\partial \phi} - \frac{\tilde{u}_\phi}{r} \\ \frac{\partial \tilde{u}_\theta}{\partial r} & \frac{1}{r} \frac{\partial \tilde{u}_\theta}{\partial \theta} + \frac{\tilde{u}_r}{r} & \frac{1}{r \sin \theta} \frac{\partial \tilde{u}_\theta}{\partial \phi} - \frac{\tilde{u}_\phi \cot \theta}{r} \\ \frac{\partial \tilde{u}_\phi}{\partial r} & \frac{1}{r} \frac{\partial \tilde{u}_\phi}{\partial \theta} & \frac{1}{r \sin \theta} \frac{\partial \tilde{u}_\phi}{\partial \phi} + \frac{\tilde{u}_r}{r} + \frac{\tilde{u}_\theta \cot \theta}{r} \end{bmatrix}. \quad (4.7)$$

Since the surrounding material is assumed to be incompressible, the incremental deformation gradient $\mathbf{d}$ must satisfy $\text{tr}(\mathbf{d}) = 0$.

Motivated by their experimental observations, Yang et al. [37] focused on a non-spherical,

isochoric displacement perturbation in the following form:

$$\begin{cases} \tilde{u}_r = \frac{a(t)R^2}{r^2} Y_l^m(\theta, \phi) & (l \geq |m| > 0) \\ \tilde{u}_\theta = 0 \\ \tilde{u}_\phi = 0 \end{cases} \quad (4.8)$$

where $\{r, \theta, \phi\}$ are the current radial, polar, and azimuthal angular coordinates, and $a(t)$ is the time-dependent perturbation magnitude at the bubble wall, which is assumed to be much smaller than R . Perturbations of this form are related to the normalized spherical harmonic functions $Y_l^m(\theta, \phi) = \cos(m\phi)P_l^m(\cos\theta)$, where $P_l^m(\cos\theta)$ are the normalized associated Legendre polynomials, m is the fixed wavenumber in the azimuthal angular coordinate, and the integers m and l satisfy $|m| \leq l$.

Then, using the perturbed displacement field (4.8), various linearized kinematic quantities may be calculated. Detailed derivations are discussed in Yang et al. [37] and for brevity, not repeated here. Several results of the kinematic linearization are the spatial description of the left Cauchy-Green deformation tensor

$$[\mathbf{B}] \approx \begin{bmatrix} \frac{(r^3+R_0^3-R^3)^{4/3}}{r^4} - \frac{4aR^2(r^3+R_0^3-R^3)^{1/3}Y_l^m}{r^4} & \frac{aR^2}{r(r^3+R_0^3-R^3)^{2/3}} \frac{\partial Y_l^m}{\partial \theta} & \frac{aR^2}{r \sin \theta (r^3+R_0^3-R^3)^{2/3}} \frac{\partial Y_l^m}{\partial \phi} \\ \frac{aR^2}{r(r^3+R_0^3-R^3)^{2/3}} \frac{\partial Y_l^m}{\partial \theta} & \frac{r^2}{(r^3+R_0^3-R^3)^{2/3}} + \frac{2aR^2 r^2 Y_l^m}{(r^3+R_0^3-R^3)^{5/3}} & 0 \\ \frac{aR^2}{r \sin \theta (r^3+R_0^3-R^3)^{2/3}} \frac{\partial Y_l^m}{\partial \phi} & 0 & \frac{r^2}{(r^3+R_0^3-R^3)^{2/3}} + \frac{2aR^2 r^2 Y_l^m}{(r^3+R_0^3-R^3)^{5/3}} \end{bmatrix}, \quad (4.9)$$

the spatial description of the radial velocity field

$$v_r(r, \theta, \phi, t) \approx \frac{R^2 \dot{R}}{r^2} + \frac{\dot{a}R^2 + 2aR\dot{R}}{r^2} Y_l^m, \quad (4.10)$$

the rate of deformation tensor $\mathbf{D}$

$$[\mathbf{D}] \approx \begin{bmatrix} -\frac{2R^2\dot{R}}{r^3} - \frac{2(\dot{a}R^2+2aR\dot{R})}{r^3}Y_l^m & \frac{(\dot{a}R^2+2aR\dot{R})}{2r^3}\frac{\partial Y_l^m}{\partial\theta} & \frac{(\dot{a}R^2+2aR\dot{R})}{2r^3\sin\theta}\frac{\partial Y_l^m}{\partial\phi} \\ \frac{(\dot{a}R^2+2aR\dot{R})}{2r^3}\frac{\partial Y_l^m}{\partial\theta} & \frac{R^2\dot{R}}{r^3} + \frac{(\dot{a}R^2+2aR\dot{R})}{r^3}Y_l^m & 0 \\ \frac{(\dot{a}R^2+2aR\dot{R})}{2r^3\sin\theta}\frac{\partial Y_l^m}{\partial\phi} & 0 & \frac{R^2\dot{R}}{r^3} + \frac{(\dot{a}R^2+2aR\dot{R})}{r^3}Y_l^m \end{bmatrix}, \quad (4.11)$$

and the spatial description of the radial acceleration field

$$a_r(r, \theta, \phi, t) \approx \frac{2R\dot{R}^2 + R^2\ddot{R}}{r^2} - \frac{2R^4\dot{R}^2}{r^5} + \frac{(\ddot{a}R^2 + 4\dot{a}R\dot{R} + 2a\ddot{R}^2 + 2aR\ddot{R})}{r^2}Y_l^m - \frac{(4\dot{a}R^4\dot{R} + 8aR^3\dot{R}^2)}{r^5}Y_l^m. \quad (4.12)$$

Dynamics of the perturbation magnitude

The perturbed displacement field is governed by the momentum balance equation (4.3). The radial component of the momentum balance equation is

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{1}{r \sin \theta} \frac{\partial \sigma_{r\phi}}{\partial \phi} + \frac{2}{r} (\sigma_{rr} - \sigma_{\theta\theta}) + \frac{\cot \theta}{r} \sigma_{r\theta} = \rho a_r, \quad (4.13)$$

where we have recognized that $\sigma_{\theta\theta} = \sigma_{\phi\phi}$. The traction boundary condition at the bubble wall ($r = R + aY_l^m$) is

$$\boldsymbol{\sigma} \mathbf{n}|_{r=R+aY_l^m} = -p_b \mathbf{n} - \gamma (\nabla_{\mathcal{S}} \cdot \mathbf{n}) \mathbf{n}, \quad (4.14)$$

where $[\mathbf{n}] = [-1, aY_{l,\theta}^m/R, aY_{l,\phi}^m/(R\sin\theta)]^\top$ is the linearized outward unit normal vector on the perturbed bubble wall, and $\nabla_{\mathcal{S}} \cdot (\bullet)$ is the surface divergence operator in the deformed configuration. After linearization [89], the radial component of (4.14) is

$$\sigma_{rr}|_{r=R+aY_l^m} = -p_b + \frac{2\gamma}{R} + \frac{(l+2)(l-1)}{R^2} aY_l^m. \quad (4.15)$$

In the far-field, the stress approaches a state of hydrostatic pressure: $\boldsymbol{\sigma}|_{r \rightarrow \infty} = -p_\infty \mathbf{1}$, so that the radial component of the far-field boundary condition is

$$\sigma_{rr}|_{r \rightarrow \infty} = -p_\infty. \quad (4.16)$$

Using (4.9) and (4.11), we insert the stress components (4.2) and the radial acceleration (4.12) into the momentum balance (4.13), integrate over r from the current, perturbed bubble wall ($R + aY_l^m$) to the far-field ($r \rightarrow \infty$) and apply the radial boundary conditions (4.15) and (4.16). After collecting the $O(1)$ terms, we recover the Rayleigh-Plesset governing equation for the dynamics of the unperturbed bubble radius (4.4). Collecting the $O(a)$ terms, we obtain the governing, second-order differential equation for the bubble-wall perturbation magnitude, a :

$$\ddot{a} + \eta \dot{a} - \xi a = 0, \quad (4.17)$$

where parameters η and ξ are defined as:

$$\eta = \frac{3\dot{R}}{R} + \frac{4\mu}{\rho R^2} + \frac{l(l+1)\mu}{3\rho R^2} \quad (4.18)$$

and

$$\begin{aligned} \xi = & -\frac{\ddot{R}}{R} + \frac{4\mu\dot{R}}{\rho R^3} + \frac{2l(l+1)\mu\dot{R}}{3\rho R^3} - \frac{(l+2)(l-1)\gamma}{\rho R^3} \\ & - \frac{2GR_0}{\rho R^3} \left(1 + \frac{R_0^3}{R^3}\right) - \frac{Gl(l+1)}{\rho(R^2 + RR_0 + R_0^2)} \\ & - \frac{2\alpha G}{\rho R^2} \frac{(R - R_0)^2}{RR_0} \left(1 + \frac{R_0}{R}\right)^3 \left(2 - \frac{2R_0}{R} + \frac{3R_0^2}{R^2} - \frac{R_0^3}{R^3} + \frac{R_0^4}{R^4}\right) \\ & - \frac{\alpha Gl(l+1)(R - R_0)^2}{5\rho RR_0(R^2 + RR_0 + R_0^2)} \left(10 + \frac{6R_0}{R} + \frac{3R_0^2}{R^2} + \frac{R_0^3}{R^3}\right). \end{aligned} \quad (4.19)$$

Eigenvalues of the dynamical system

The second order ordinary differential equation for $a(t)$ (4.17) can be rewritten as the following dynamical system:

$$\frac{d}{dt} \begin{bmatrix} a \\ \dot{a} \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ \xi & -\eta \end{bmatrix} \begin{bmatrix} a \\ \dot{a} \end{bmatrix}. \quad (4.20)$$

It is straightforward to see that $\mathbf{a}^* := \{a = 0, \dot{a} = 0\}$ is always an equilibrium point of (4.20), and its stability depends on the sign of the eigenvalues of the dynamical system (4.20). The eigenvalues are of the form

$$\zeta_{1,2} = -\frac{\eta}{2} \pm \frac{\sqrt{\Delta}}{2} \quad \text{with} \quad \Delta = \eta^2 + 4\xi, \quad (4.21)$$

which may be classified in the following cases:

- **Case (i):** If $\xi > 0$, the eigenvalues are real and of opposite sign, and therefore, $\mathbf{a}^*$ is a *saddle* point;
- **Case (ii):** If $\xi < 0$, $\eta < 0$, and $\Delta > 0$, the eigenvalues are real and positive, and $\mathbf{a}^*$ is a *source*-type instability point;
- **Case (iii):** If $\xi < 0$, $\eta < 0$, and $\Delta < 0$, the eigenvalues are complex with positive real part, and $\mathbf{a}^*$ is an *unstable spiral*-type instability;
- **Case (iv):** If $\xi < 0$, $\eta > 0$, and $\Delta < 0$, the eigenvalues are complex with negative real part, and $\mathbf{a}^*$ is a *stable spiral*;
- **Case (v):** If $\xi < 0$, $\eta > 0$, and $\Delta > 0$, the eigenvalues are real and negative, and $\mathbf{a}^*$ is a *sink*.

Examining the differential relation (4.17) and the associated dynamical system, we may express the criteria for when perturbations a grow either in terms of the eigenvalues of the

dynamical system $\zeta_{1,2}$ or the parameters η and ξ . The eigenvalues of the dynamical system are related to the rate at which perturbations grow, and when the real part of either eigenvalue is positive, perturbations can grow rather than decay. In order to calculate the most positive growth rate, we find the real part of each eigenvalue and then obtain the maximum positive value among them:

$$\text{Growth Rate} = \max(\text{Re}(\zeta_1), \text{Re}(\zeta_2)). \quad (4.22)$$

Perturbations can grow and solutions are unstable whenever this growth rate is positive.

In terms of η and ξ , we find that perturbations a can grow if $\eta < 0$ or $\xi > 0$ and decay if $\eta > 0$ and $\xi < 0$. Thus, non-spherical instabilities can be quantitatively predicted via the signs of the two parameters η and ξ , which are affected by the bubble dynamics base state $\{R, \dot{R}, \ddot{R}\}$, the surrounding material's constitutive properties—namely, the material's viscosity μ , shear modulus G , and strain stiffening parameter α —as well as the surface tension γ . Specifically, the first term in η explicitly accounts for inertial effects during cavitation, and the second and third terms depend on the viscosity of the surrounding material. The first three terms in ξ also account for inertial effects whereas the fourth term accounts for surface tension effects. All other terms in ξ depend on the shear modulus G and the large deformation strain stiffening parameter α of the material. In addition, η and ξ are also affected by the instability shape mode l .

Interpreting (4.17)-(4.19) physically, we find that material viscosity and the nonlinear strain stiffening behavior of the solid along with surface tension always tend to stabilize the bubble against non-spherical deformations when the instability shape mode $l > 2$, which can be deduced directly from the relative (positive and negative) contributions in the derived stability parameters η and ξ . Increasing the viscosity μ , shear modulus G , strain stiffening parameter α , and surface tension γ of the surrounding material tend to prevent non-spherical instabilities. However, when the bubble size is much greater than the characteristic length, γ/G , the effect of surface tension becomes negligible. When the bubble approaches the final equilibrium radius ($R \rightarrow R_0$, $\dot{R} \rightarrow 0$, and $\ddot{R} \rightarrow 0$), we find that $\eta|_{t \rightarrow \infty} > 0$ and $\xi|_{t \rightarrow \infty} < 0$, so

that a static spherical bubble will be stable under all perturbation modes.

4.2.3 Nonlinear dynamic post-buckling analysis

In order to assess the linear perturbation analysis and its criteria for the onset of non-spherical instability, we perform nonlinear, three-dimensional post-buckling-type simulations of bubble dynamics in laser-induced cavitation (LIC) using finite-element simulations in Abaqus [76]. We consider an isolated, spherical cavity, and following the experimental observations of Yang et al. [37] (see Fig. 4.1(a)), we focus on the instability mode shape corresponding to $l = m = 8$. The undeformed cavity radius is equal to $R_0 = 50 \mu\text{m}$, while the far-field radius of the simulation domain is $R_{0,\infty} = 300R_0$, which is sufficiently large to assure that no effects of the far-field boundary are present in our simulations. To study the onset of non-spherical instability, we follow a standard approach in nonlinear post-buckling analyses and apply a small geometrical imperfection to the undeformed geometry, which is achieved using the `*imperfection` command within the Abaqus input file. Specifically, a radial perturbation of the following form is applied to the nodes along the bubble wall:

$$\delta_r(\theta, \phi) = a_0 Y_l^m(\theta, \phi), \quad (4.23)$$

where $Y_l^m(\theta, \phi)$ is the normalized $(l, m)^{\text{th}}$ spherical harmonic function and a_0 is the magnitude of the perturbation at the bubble wall. In Fig. 4.1(a), we see a snapshot of a single bubble in a stiff polyacrylamide gel during an LIC experiment from Yang et al. [37], in which the $l = m = 8$ instability mode shape is observed and outlined in red, while in Fig. 4.1(b), we see the undeformed geometry in the finite-element analysis after the application of the imperfection corresponding to the $l = m = 8$ instability mode shape, which is used throughout our nonlinear dynamic post-buckling simulations. The magnitude of the geometric perturbation a_0 used in this finite-element study is equal to $0.1 \mu\text{m}$, or $0.002R_0$.

For computational efficiency, we simulate one eighth of the spherical bubble geometry,

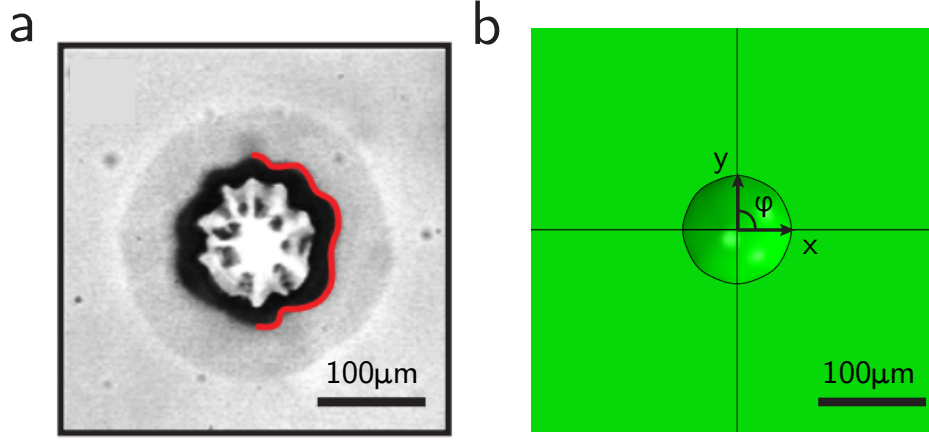


Figure 4.1: (a) Snapshot of a single bubble in a stiff polyacrylamide gel during the second collapse event of an LIC experiment from Yang et al. [37]. The red line is overlaid onto the original image to visualize the shape of the bubble wall, which approximately corresponds to the instability mode shape $l = m = 8$. (b) Initial geometry in the finite-element analysis, in which a geometric imperfection corresponding to the $l = m = 8$ mode shape has been applied. In the finite-element simulations, one eighth of the bubble geometry is considered, but for visualization purposes, one half of the bubble geometry corresponding to a slice along the x - y -plane is shown.

in which symmetry boundary conditions are applied on the x - y , x - z , and y - z planes in Fig. 4.1(b). Such an idealization does preclude some spherical harmonic instability mode shapes but is consistent with the $l = m = 8$ mode shape of the geometric perturbation. A total of 112,409 Abaqus-C3D8H three-dimensional elements (eight-node, three-dimensional, hexahedral, constant-pressure elements) were used to model the surrounding material. Mesh convergence studies were carried out to verify that this mesh density was sufficient to provide accurate simulation results. In order to account for the infinite-sized domain, we used infinite elements (CIN3D8) within Abaqus along a very thin strip on the outer layer of the far-field radius. The constitutive behavior of the surrounding solid material is described by the qKV model (4.2) and implemented using a UMAT subroutine. For all our finite-element simulations, we use the elastic material properties—namely, G and α —for stiff polyacrylamide gel, which are given in Table 4.1. We take the ground-state bulk modulus K to be given through the longitudinal wave speed c by $K = \rho_0 c^2 - (4/3)G$, where ρ_0 is the referential mass density. The values used for c and ρ_0 are also included in Table 4.1. For the viscosity,

Table 4.1: Baseline material properties for the qKV model used in the nonlinear dynamic post-buckling analyses

G (kPa)	α	μ (Pa·s)	c (m/s)	ρ_0 (kg/m ³)
2.77	0.48	0.0186	1430	1060

we take a low baseline value of $\mu = 0.0186$ Pa·s to highlight the effect of non-spherical instability in our finite-element simulations. In Section 4.3.2, we also consider a higher viscosity value from Yang et al. [44].

To model the bubble, we utilize the built-in fluid-filled-cavity constraint in Abaqus [76], which assumes isothermal ideal gas behavior for the bubble contents and neglects the effects of heat and mass transfer both inside the bubble and with the surroundings. Our baseline simulations neglect surface tension, and therefore, the pressure inside the bubble is $p_b = p_\infty(V_0/V)$, where p_∞ is the ambient atmospheric pressure, $V_0 = (4/3)\pi R_0^3$ is the undeformed bubble volume, and V is the volume of the deformed, non-spherical bubble. As in prior chapters, we continue to take $p_\infty = 101.3$ kPa. While our baseline simulations neglect the surface tension of the bubble interface, we return in Section 4.3.2 to consider the effect of non-zero surface tension. We account for surface tension at the bubble wall by utilizing the method of Wang and Henann [90]. Then, to simulate laser-induced cavitation (LIC) conditions in Abaqus, we utilize the process introduced in Chapter 3 and described in detail in Appendix B.

Based on the finite-element results, we numerically calculate the mean radius, radial velocity, and radial acceleration of the non-spherical bubble as well as the magnitude of the bubble perturbations. For each case and at each time-point, we extract the displacement components u_x, u_y and u_z from Abaqus for each node on the bubble surface and calculate the corresponding deformed coordinates in the spherical polar coordinate system, $\{r, \theta, \phi\}$. The nodal coordinates represent discrete points within the deformed surface of the bubble $r(\theta, \phi)$. We calculate the deformed area of the one-eighth section of the non-spherical

bubble as

$$A = \int_0^{\pi/2} \int_0^{\pi/2} (r(\theta, \phi))^2 \sin \theta d\theta d\phi$$

and the mean bubble radius as

$$\bar{r} = \frac{1}{A} \int_0^{\pi/2} \int_0^{\pi/2} (r(\theta, \phi))^3 \sin \theta d\theta d\phi. \quad (4.24)$$

The root-mean-square (RMS) value of the magnitude of the bubble perturbation is defined as

$$a_{\text{RMS}} = \left(\frac{1}{A} \int_0^{\pi/2} \int_0^{\pi/2} [r(\theta, \phi) - \bar{r}]^2 (r(\theta, \phi))^2 \sin \theta d\theta d\phi \right)^{1/2} \quad (4.25)$$

Moreover, we extract the Cartesian components of the nodal velocity $\{v_x, v_y, v_z\}$ and the nodal acceleration $\{a_x, a_y, a_z\}$ for each node on the bubble surface and at each time-point and calculate the corresponding radial components of velocity v_r and acceleration a_r . Then, the mean radial velocity and acceleration are

$$\bar{v}_r = \frac{1}{A} \int_0^{\pi/2} \int_0^{\pi/2} v_r(\theta, \phi) (r(\theta, \phi))^2 \sin \theta d\theta d\phi, \quad (4.26)$$

$$\bar{a}_r = \frac{1}{A} \int_0^{\pi/2} \int_0^{\pi/2} a_r(\theta, \phi) (r(\theta, \phi))^2 \sin \theta d\theta d\phi. \quad (4.27)$$

4.3 Results and discussion

4.3.1 Effect of radius ratio

We begin with a set of baseline LIC simulations, in which the radius ratio R_{max}/R_0 is varied and a small perturbation of the initial geometry of the form (4.23) is assumed. We examine three different radius ratios $R_{\text{max}}/R_0 = 3, 4.5$, and 6.5 with an initial bubble radius $R_0 = 50 \mu\text{m}$, and the material properties are given in Table 4.1. Figures 4.2(a-c) show the corresponding mean radius $\bar{r}$ versus time curves for the three radius ratios considered,

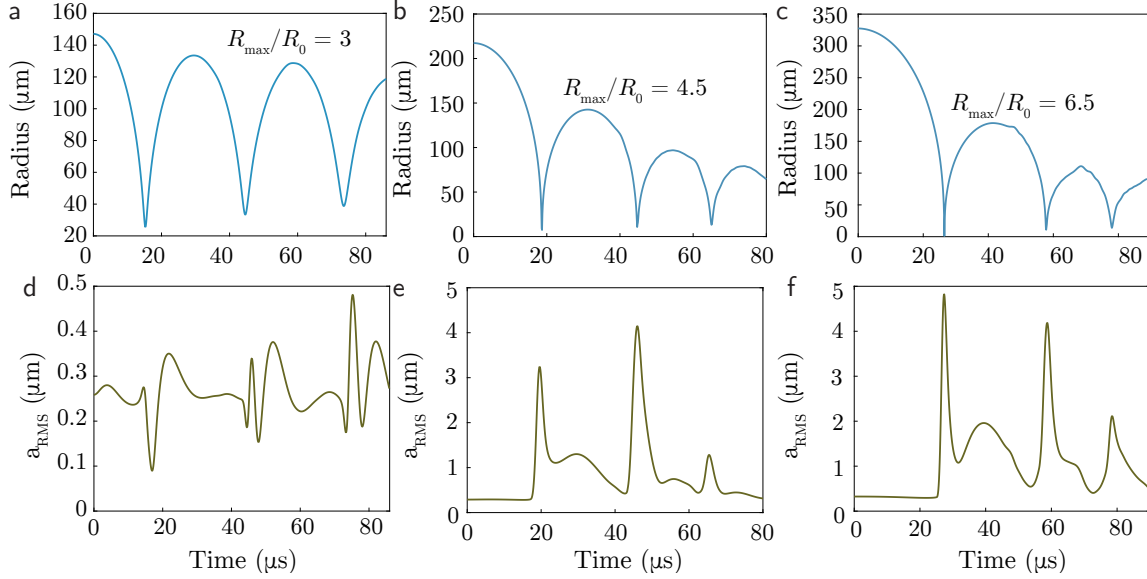


Figure 4.2: Numerically computed mean radius versus time curves for (a) $R_{\text{max}}/R_0 = 3$, (b) $R_{\text{max}}/R_0 = 4.5$, and (c) $R_{\text{max}}/R_0 = 6.5$. In (d)-(f) we plot histories of the magnitude of the perturbation a_{RMS} for each case of R_{max}/R_0 .

calculated using the integral in (4.24). Due to the low baseline value of the viscosity, collapse tends to be more violent, which tends to lead to more non-spherical behavior of the bubble. For each case of R_{max}/R_0 , we also calculate the magnitude of the perturbations, a_{RMS} , using (4.25), and present the results in Figs. 4.2(d-f). The magnitude of the perturbation a_{RMS} informs how the perturbations grow throughout the lifespan of the bubble. We observe that for the lowest bubble ratio of $R_{\text{max}}/R_0 = 3$, the perturbation magnitude does not grow significantly and remains within the range of 0.1-0.5 μm , while for the two higher bubble ratios, growth of the perturbation magnitude occurs around the first and second collapse events.

Next, we compare results from the nonlinear dynamic post-buckling analysis against the linear perturbation analysis, presented in Section 4.2.2. We utilize our finite element results by extracting the histories of $\bar{r}$, $\bar{v}_r$, and $\bar{a}_r$ for each case. We then use these kinematic quantities in place of R , $\dot{R}$, and $\ddot{R}$ in (4.18) and (4.19) to calculate the histories of the quantities ξ and η from the linear stability theory for the case of $l = m = 8$. In Figs. 4.3(a,c and e) and Figs. 4.3(b,d and f), we see plots of the time histories of the bubble radius $\bar{r}$ along

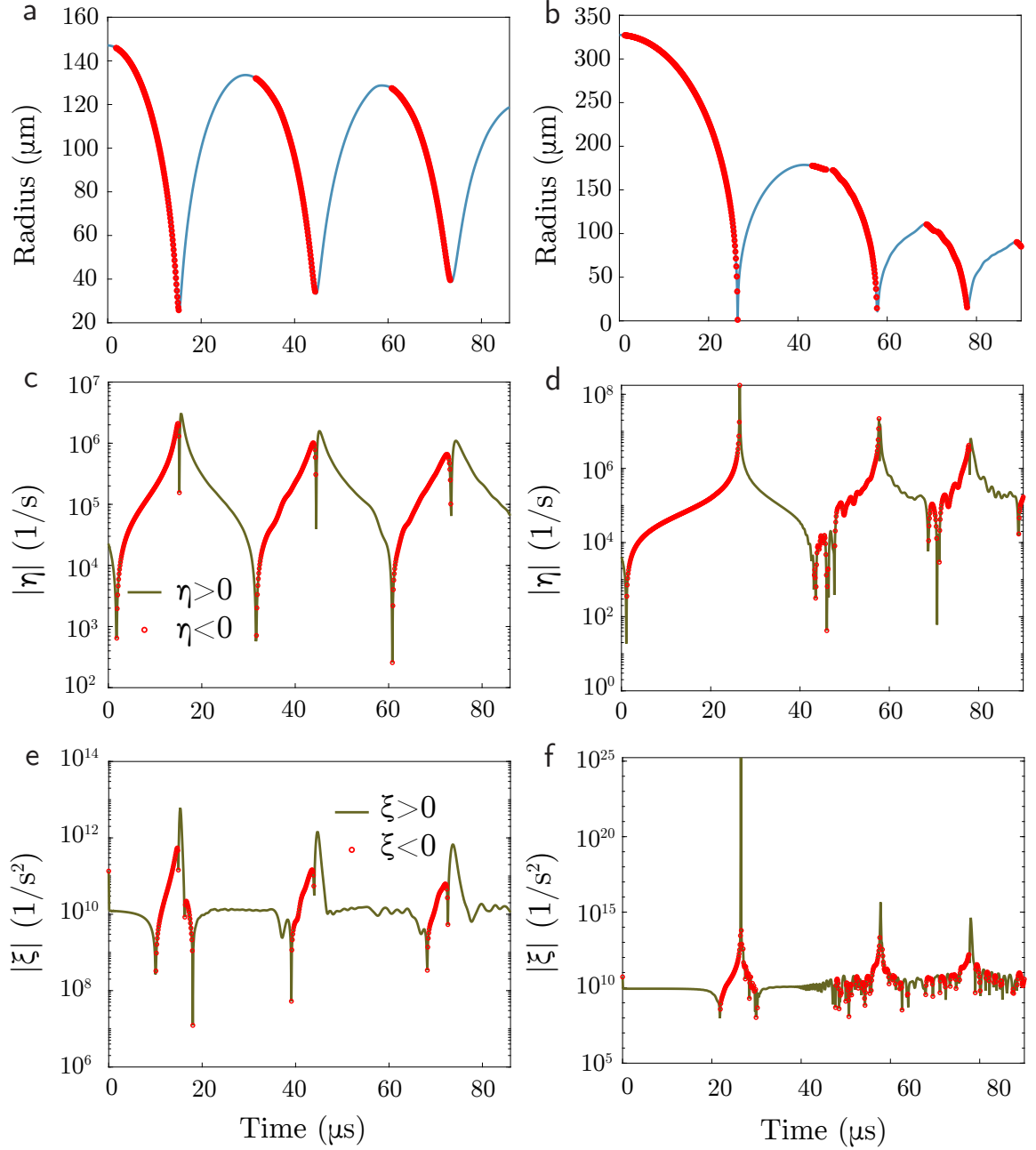


Figure 4.3: Numerically computed mean radius versus time curves for (a) $R_{\text{max}}/R_0 = 3$ and (b) $R_{\text{max}}/R_0 = 6.5$. Time histories of (c,d) η and (e,f) ξ are numerically computed using simulation parameters for $R_{\text{max}}/R_0 = 3$ and $R_{\text{max}}/R_0 = 6.5$, respectively, for $l = m = 8$. Red dotted regions denote situations in which $\eta < 0$ in (c,d), $\xi < 0$ in (e,f), and $\eta < 0$ and $\xi < 0$ in (a,b).

with plots of ξ and η , corresponding to $R_{\text{max}}/R_0 = 3$ and 6.5 , respectively. Situations in which $\eta < 0$ are marked as red dotted regions in Figs. 4.3(c and d), and similarly, situations in which $\xi < 0$ are marked as red dotted regions in Figs. 4.3(e and f). In Figs. 4.3(a and b),

the red dotted regions correspond to conditions under which non-saddle-point instabilities are predicted (i.e., $\eta < 0$ and $\xi < 0$). It is evident from the numerical results in Figs. 4.3(a-f) that the linear stability analysis predicts that non-spherical shape instabilities in LIC occur during the collapse phases. This is to be expected because during the collapse phases, inertial effects are dominant since the bubble wall has a large negative radial velocity and acceleration, causing η and ξ to take negative signs as shown in Figs. 4.3(c-d) and (e-f). Taking a closer look at our numerical findings, we observe that our numerical results are in qualitative agreement with the onset of non-spherical deformation predicted by the linear stability analysis and the experimental findings of Yang et al. [37]. However, in our three-dimensional finite-element simulations, non-spherical instabilities are predicted for the case of $R_{\max}/R_0 = 3$ (Fig. 4.3(a)), while very little growth of the perturbation is observed (Fig. 4.2(d)). Moreover, for the case of $R_{\max}/R_0 = 6.5$, the onset of non-spherical instability is predicted in Fig. 4.3(b) at a earlier time than the growth in perturbation magnitude is observed in Fig. 4.2(f). The rationale for this discrepancy is examined next.

Although the linear stability theory can predict the circumstances under which the spherical bubble can become unstable, in a dynamic problem like inertial microcavitation, where the time scales are small, perturbations might not have sufficient time to grow. If perturbations do not have sufficient time to grow, then unstable behavior may not be observed in nonlinear, post-buckling-type simulations, even if the linear stability theory predicts instability. To account for the finite time over which perturbations can grow, we calculate the growth rate of perturbations (4.22), which is based on the eigenvalues $\zeta_{1,2}$ of the dynamical system (4.21) and given by the most positive real part of either eigenvalue ζ_1 and ζ_2 at each time step. Figures 4.4 (a-c) show the growth rate histories for the three different radius ratios $R_{\max}/R_0 = 3, 4.5$ and 6.5 , where the growth rate has been non-dimensionalized by the characteristic collapse time $R_{\max}\sqrt{\rho_0/p_\infty}$. The solid lines denote positive growth rates for all three cases, while time-spans that correspond to negative growth rates are omitted. We

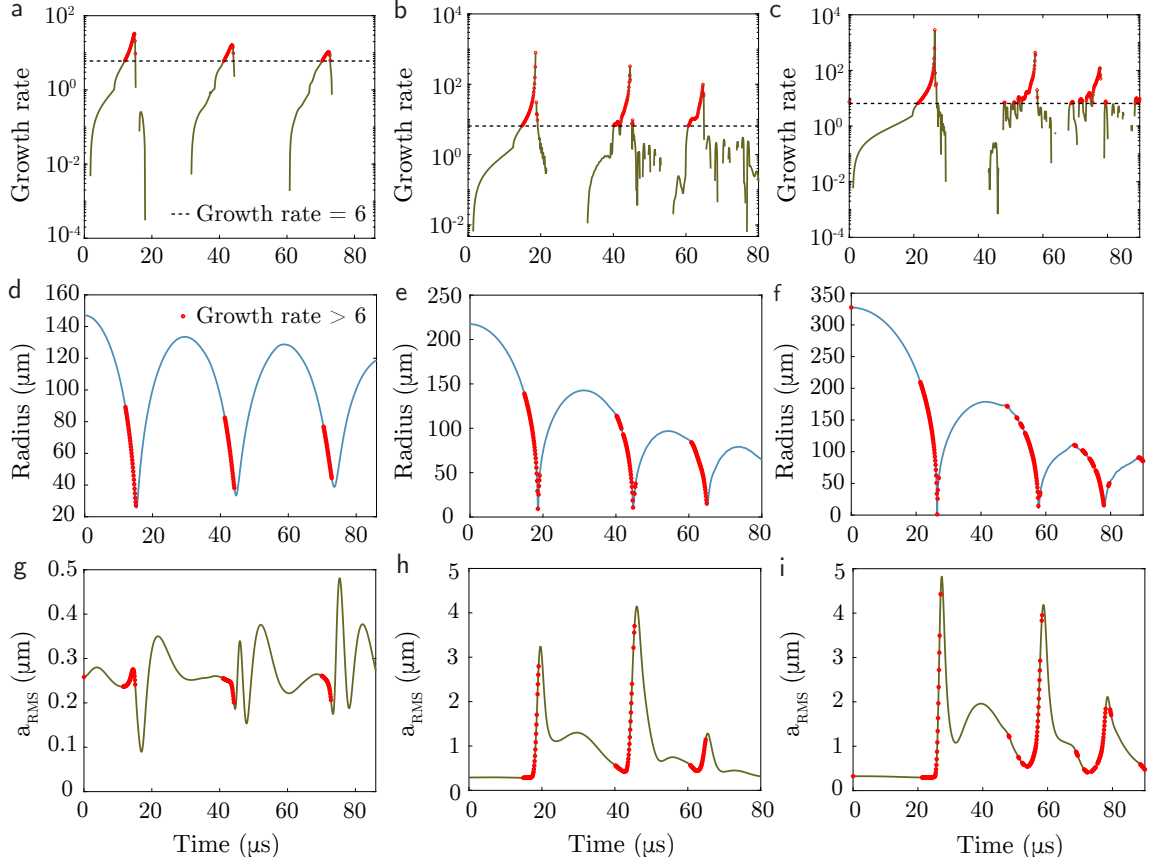


Figure 4.4: Numerically computed time histories of the non-dimensionalized growth rate for (a) $R_{\max}/R_0 = 3$, (b) $R_{\max}/R_0 = 4.5$, and (c) $R_{\max}/R_0 = 6.5$. The horizontal dashed lines denote the threshold for the growth rate under which instabilities do not have sufficient time to grow. (d-f) Numerically computed radius versus time curves for the three radius ratios and (g-i) time histories of the RMS value of the perturbation magnitude. The red dotted regions mark the portions of the histories corresponding to a growth rate greater than the threshold.

observe that at early times, instability is predicted by the linear stability analysis, but the non-dimensional growth rate is quite small $O(10^{-2})$, so that the characteristic time for growth of the perturbation is 100 times the characteristic collapse time. For this reason, growth of the perturbation magnitude is not observed in the nonlinear simulations at early times. However, when the non-dimensional growth rate reaches $O(1)$ - $O(10)$, growth of the perturbation is expected to be sufficiently fast so that it may be observed in the history of the perturbation magnitude. Using the calculated growth rate histories (Figs. 4.4(a-c)) in conjunction with the calculated perturbation magnitude histories across all three radius ratios (Figs. 4.4(g-i)),

we seek a critical growth rate that corresponds to the onset of growth in a_{RMS} , representing the threshold for the growth rate under which instabilities do not have sufficient time to grow and cause unstable behavior and potentially localized deformation. Based on the simulation results, we estimate the threshold for the non-dimensional growth rate to be ~ 6 , and the horizontal dashed lines in Figs. 4.4(a-c) denote this threshold. Figures 4.4(d-f) show the mean radius versus time curves, and Figs. 4.4(g-i) show the RMS value of the magnitude of the bubble surface perturbation. Red dotted regions in Fig. 4.4 mark the portions of the histories corresponding to a growth rate greater than the threshold. From Figs. 4.4(a-c) and (g-i), we can conclude that a critical growth rate serves as an effective threshold for predicting the onset of non-spherical bubble shapes. This is especially clear for the higher radius ratios ($R_{\text{max}}/R_0 = 4.5$ and 6.5), where the sharp growth in a_{RMS} corresponds closely to the critical growth rate being attained. The situation is less clear for the lowest radius ratio ($R_{\text{max}}/R_0 = 3$), which only shows a very small increase in the perturbation magnitude upon first collapse when the critical growth rate is attained. Nonetheless, a threshold based on a critical, positive growth rate, rather than any positive growth rate, shows promise for predicting the onset of non-spherical instability during dynamic cavitation.

Effect of the non-spherical mode shape l on the normalized growth rate: In our analysis, following Yang et al. [37], we have focused on the instability mode shape corresponding to $l = m = 8$. Here, we examine the effect of the non-spherical mode shape l on the normalized growth rate. We utilize the time histories of displacement, acceleration, and velocity from the finite-element simulations for a radius ratio $R_{\text{max}}/R_0 = 4.5$, when the $l = m = 8$ mode shape is used to perturb the initial geometry. In Fig. 4.5, the time histories of the normalized growth rate are plotted for different values of the mode shape l . We observe that the growth rates for all values of the mode shape reach the same maximum growth rate, while the growth rate threshold is met at approximately the same time for mode shapes 3 through 10. Therefore, the predicted onset of non-spherical instability is only weakly

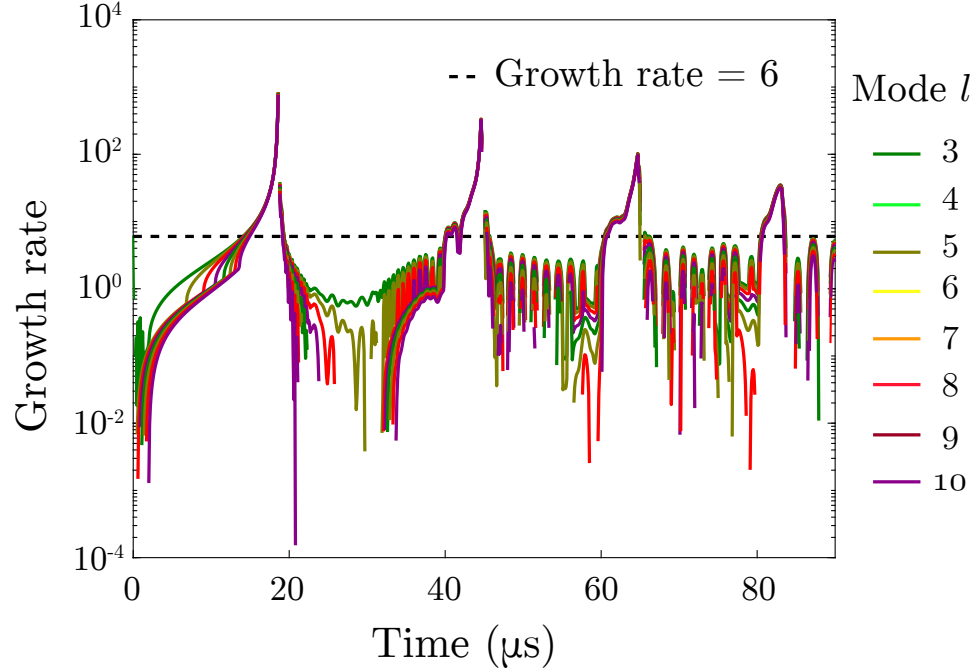


Figure 4.5: Numerically computed time histories of the non-dimensionalized growth rate for different non-spherical mode shape numbers. The dashed line denotes the previously obtained growth rate threshold.

dependent upon the mode shape l , and the mode shape that is observed in the non-spherical deformed bubble is likely determined by imperfections in the undeformed geometry.

4.3.2 Effect of viscosity and surface tension on the bubble stability

We wish to test whether a threshold based on a critical growth rate is capable of predicting the onset of non-spherical instability across a variety of cavitation conditions. Examining equations (4.17)-(4.19), we observe that material viscosity and surface tension tend to stabilize the bubble against non-spherical deformations. This can also be deduced directly from the relative (positive and negative) contributions in the parameters η and ξ . To this end, we examine the effect of viscosity and surface tension on the bubble stability using nonlinear, post-buckling-type simulations.

Material Viscosity

We first investigate the effect of material viscosity on the bubble stability, following the same procedure as described in the preceding section. In this analysis, we consider two different values of viscosity ($\mu = 0.0186 \text{ Pa}\cdot\text{s}$ and $\mu = 0.186 \text{ Pa}\cdot\text{s}$), and we keep the radius ratio, $R_{\max}/R_0 = 4.5$, fixed throughout this section. All other parameters are maintained at their baseline values. Specifically, the initial bubble radius is $R_0 = 50 \mu\text{m}$, and all material properties besides viscosity are given in Table 4.1. Based on the nonlinear finite-element results, we calculate the histories of $\bar{r}$, $\bar{v}_r$, and $\bar{a}_r$, calculate the histories of the parameters ξ and η , and identify portions of the time history in which unstable behavior is predicted based on the linear stability analysis. Figure 4.6 shows the results from these finite-element analyses. Figures 4.6(a, c, and e) correspond to $\mu = 0.0186 \text{ Pa}\cdot\text{s}$, and Figs. 4.6(b, d, and f) correspond to $\mu = 0.186 \text{ Pa}\cdot\text{s}$. We plot the mean radius versus time curves in Figs. 4.6(a and b) and the time histories of η and ξ in Figs. 4.6(c and d) and (e and f), respectively. The red marks in (a) and (b) denote the regions where $\eta < 0$ and $\xi < 0$ and non-saddle-point instabilities are predicted by the linear stability analysis. Situations in which $\eta < 0$ are marked as red dotted regions in Figs. 4.6(c and d), and similarly, situations in which $\xi < 0$ are marked as red dotted regions in Figs. 4.6(e and f). For the low viscosity case in Figs. 4.6(a, c, and e), we can see that the linear stability analysis predicts unstable behavior during each collapse portion of the bubble history. On the other hand, for the higher value of viscosity in Figs. 4.6(b, d, and f), unstable behavior is predicted only during first collapse, and overdamped behavior is observed thereafter. This is to be expected since high values of viscosity slow down the bubble dynamics and enhance the stability of spherical bubbles. We note that the onset of unstable behavior predicted by the linear stability analysis occurs early during collapse portions of the history as observed in the previous section, which motivates consideration of the growth rate in predicting the onset of non-spherical bubble shapes.

The histories of the growth rate of perturbations based on the linear stability analysis are plotted in Figs. 4.7(a and b) for the two values of viscosity. As in the preceding section, the

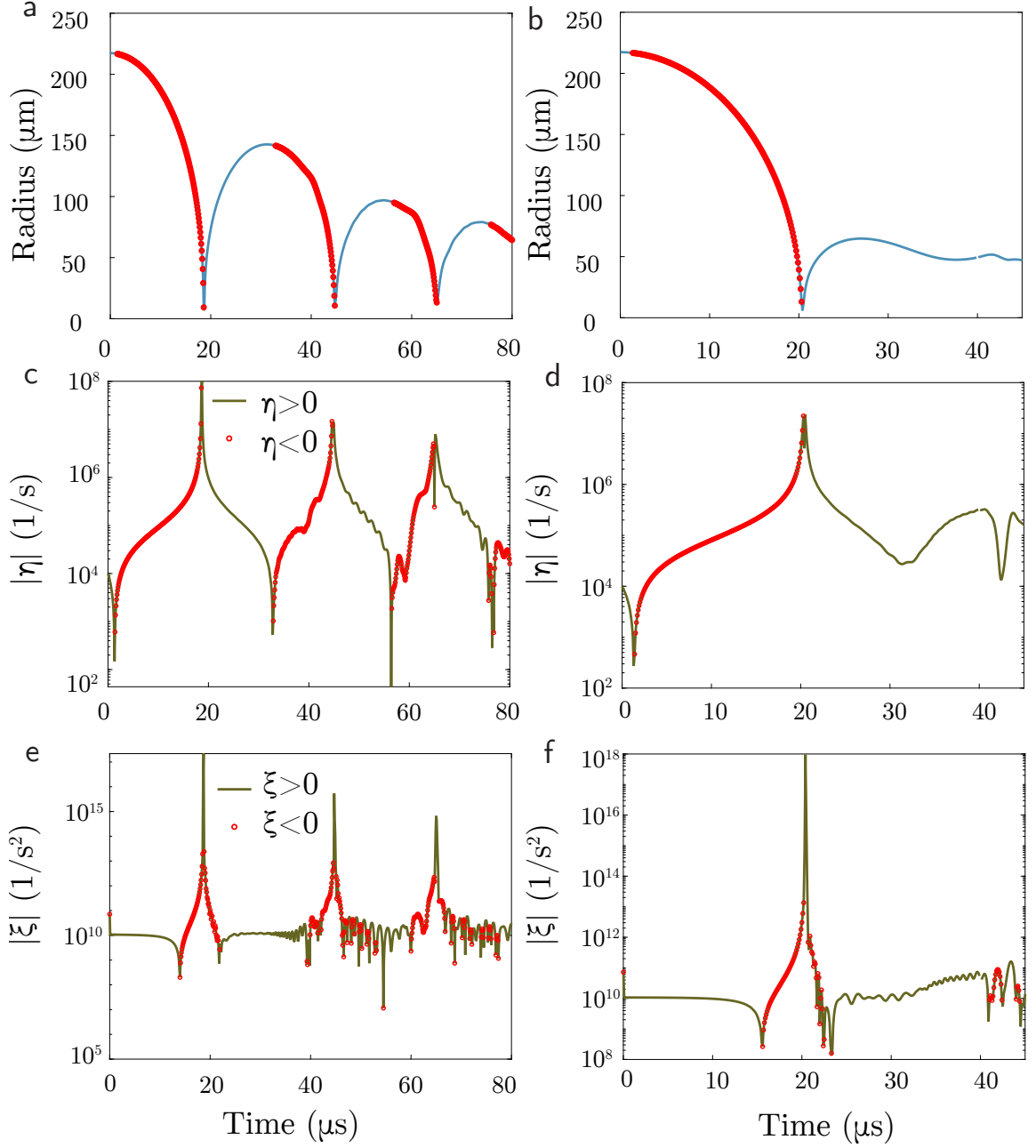


Figure 4.6: Numerically computed mean radius versus time curves for $R_{\max}/R_0 = 4.5$ and (a) $\mu = 0.0186 \text{ Pa}\cdot\text{s}$ and (b) $\mu = 0.186 \text{ Pa}\cdot\text{s}$. Time histories of (c,d) η and (e,f) ξ are numerically computed using simulation parameters for $\mu = 0.0186 \text{ Pa}\cdot\text{s}$ and $\mu = 0.186 \text{ Pa}\cdot\text{s}$, respectively. Red dotted regions denote situations in which $\eta < 0$ in (c,d), $\xi < 0$ in (e,f), and $\eta < 0$ and $\xi < 0$ in (a,b).

solid lines denote positive growth rates, and time-spans that correspond to negative growth rates are omitted. The threshold for the non-dimensional growth rate of 6 is denoted by the horizontal dashed lines, and the red dotted regions mark the portions of the histories

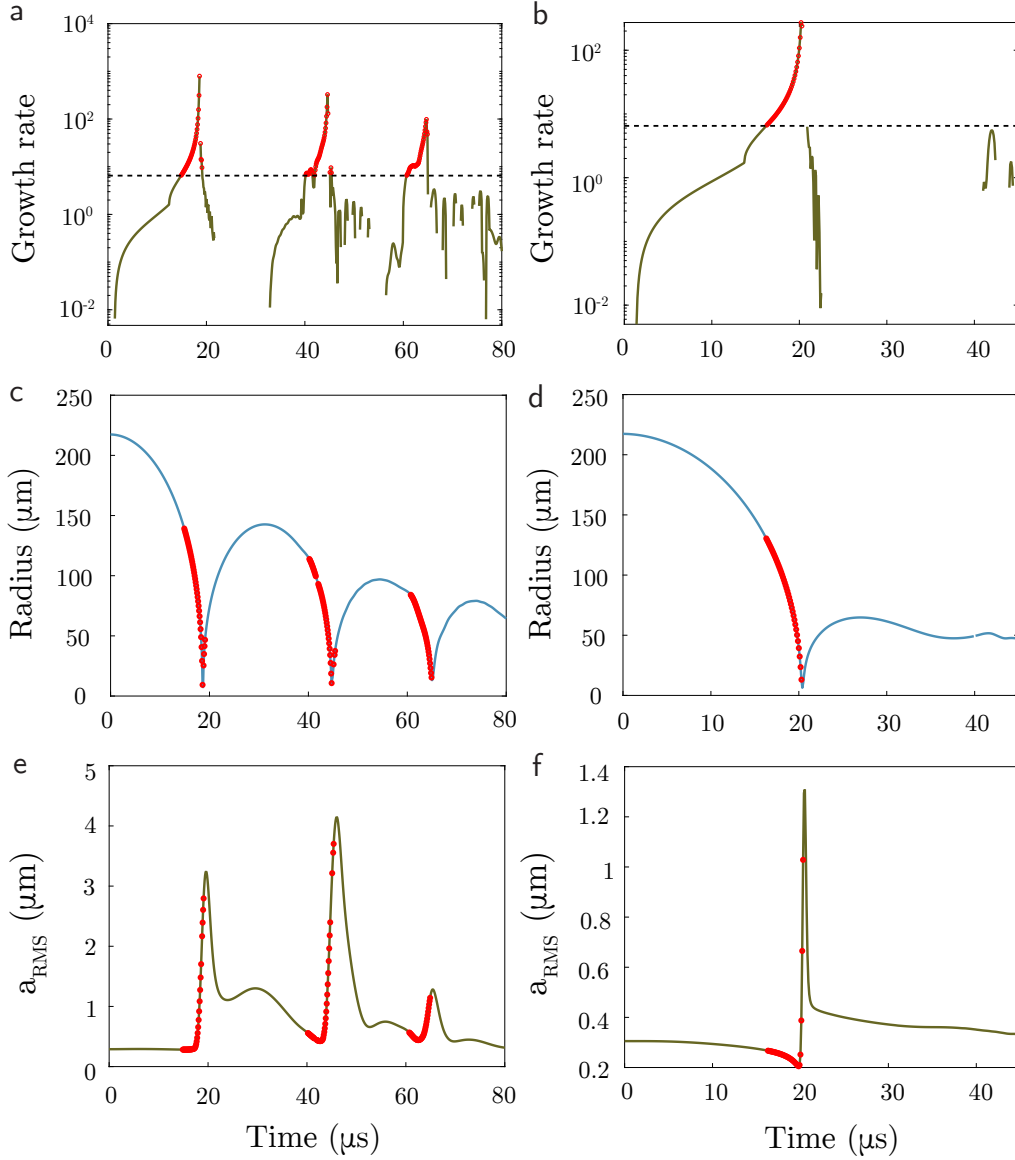


Figure 4.7: Numerically computed time histories of the non-dimensionalized growth rate for $R_{\text{max}}/R_0 = 4.5$ and two different viscosities: (a) $\mu = 0.0186 \text{ Pa}\cdot\text{s}$ and (b) $\mu = 0.186 \text{ Pa}\cdot\text{s}$. The horizontal dashed lines denote the threshold for the growth rate under which instabilities do not have sufficient time to grow. (c,d) Numerically computed radius versus time curves for each viscosity and (e,f) time histories of the RMS value of the perturbation magnitude. The red dotted regions mark the portions of the histories corresponding to a growth rate greater than the threshold.

corresponding to a growth rate greater than the threshold. Figures 4.7(c and d) show the histories of the mean radius, and Figs. 4.7(e and f) show the histories of the perturbation magnitude a_{RMS} for both values of the viscosity. In these figures, the red dotted regions

denote the portions of the histories identified in Figs. 4.7(a and b), in which the threshold growth rate is exceeded. We observe that for the low viscosity case, the threshold is exceeded during each collapse event in the mean radius history, while for the high viscosity case, the threshold growth rate is only exceeded during first collapse. Moreover, examining Figs. 4.7(e and f), we can see that the conditions in which the critical growth rate is exceeded correspond well with sharp growth in a_{RMS} in both cases. Therefore, the threshold based on a critical, positive growth rate continues to accurately predict the onset of non-spherical, dynamic instability as the viscosity is changed.

Surface tension

Next, we investigate the effect of surface tension on the bubble stability. To account for surface tension at the bubble wall in finite-element simulations, we utilize the method of Wang and Henann [90] in Abaqus. The undeformed bubble radius is $R_0 = 50 \mu\text{m}$, the radius ratio is $R_{\text{max}} = 4.5R_0$, and all material properties are given in Table 4.1. To quantify the surface tension γ , we utilize the elastocapillary number (EC), which is defined as

$$\text{EC} = \frac{\gamma}{GR_0} \quad (4.28)$$

and consider two finite values of surface tension corresponding to $\text{EC} = 0.5$ and 5 in addition to a baseline case with no surface tension. The value of $\text{EC} = 0.5$ corresponds to $\gamma = 0.0693 \text{ N/m}$, which is close to the value of surface tension utilized in the preceding chapters ($\gamma = 0.056 \text{ N/m}$). Figure 4.8(a) shows the mean radius versus time curves for each case, and Fig. 4.8(b) shows the calculated RMS value of the magnitude of the bubble surface perturbation. Here, we focus only on the first collapse portion of the bubble dynamics. The red dotted regions mark the portions of the histories corresponding to a growth rate greater than the previously established non-dimensional threshold, which is ~ 6 . We observe that for both $\text{EC} = 0.5$ and $\text{EC} = 5$, the threshold growth rate is exceeded during first collapse,

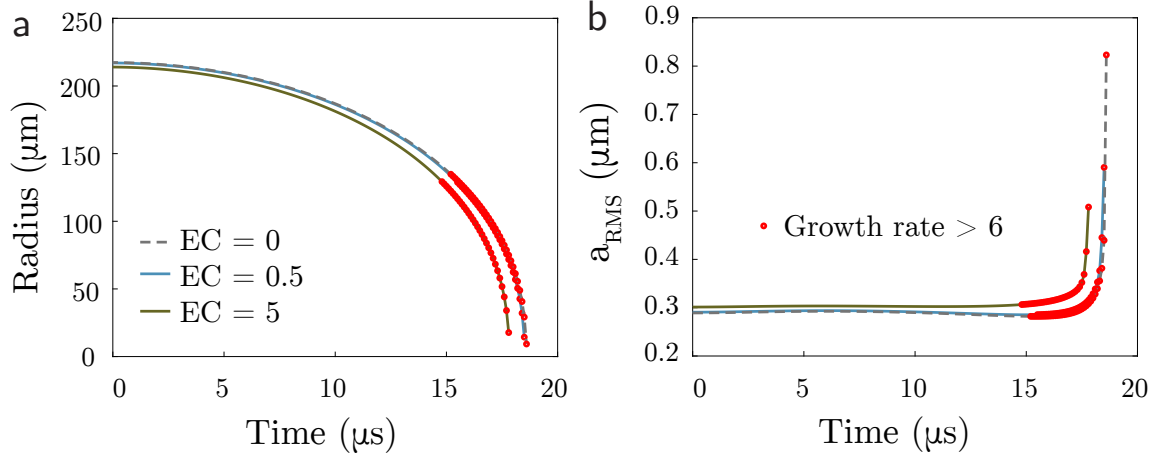


Figure 4.8: (a) Numerically computed radius versus time curves for three elastocapillary numbers (EC) and (b) time histories of the RMS value of the perturbation magnitude. The red dotted regions mark the portions of the histories corresponding to a growth rate greater than the threshold.

which is accompanied by an increase in a_{RMS} , suggesting that the threshold based on a critical, positive growth rate continues to accurately predict the onset of non-spherical, dynamic instability as the surface tension is changed. We note that, for the case of $\text{EC} = 0.5$, which is representative of the cases considered in prior chapters, the presence of surface tension has little to no effect on the dynamics or the stability of the bubble.

4.3.3 Strain and stress localization

Finally, to demonstrate characteristic strain and stress fields induced by non-spherical deformation, we construct a quasi-static finite-element model of an isolated three-dimensional spherical cavity within an infinite material using Abaqus [76]. Here we focus on the quasi-static response due to non-spherical deformation at the bubble wall. For an undeformed bubble radius of $45 \mu\text{m}$, a displacement boundary condition is imposed on the bubble wall surface using a user-displacement (UDISP) subroutine, which consists of the sum of a spherically symmetric deformation and a spherical harmonic perturbation (4.8). We perform a quasi-static simulation corresponding to a deformed bubble radius of $72 \mu\text{m}$, a perturbation amplitude of $1.6386 \mu\text{m}$, and shape mode $l = m = 8$.

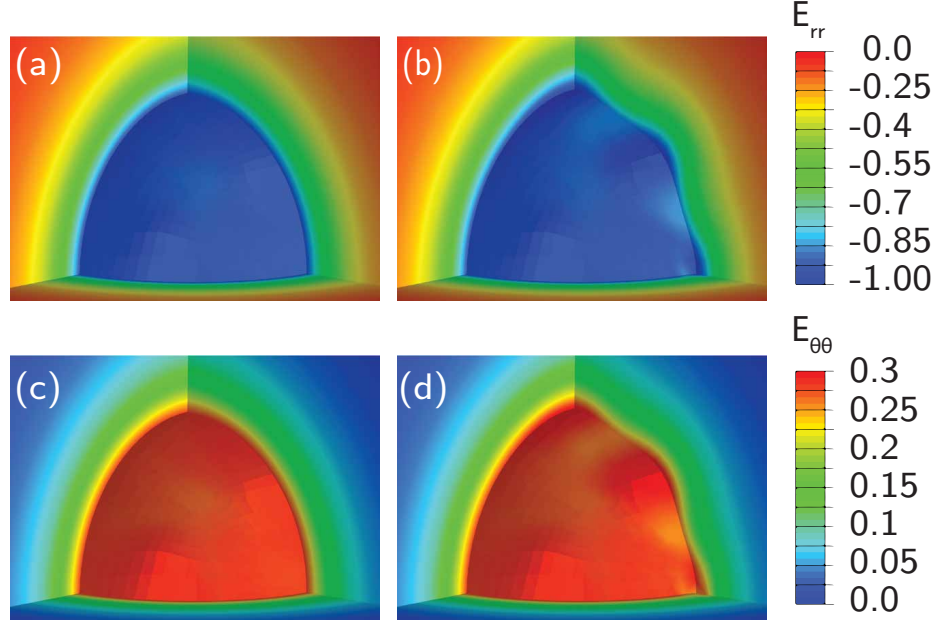


Figure 4.9: Contours of the radial component of the logarithmic strain, E_{rr} , for (a) the spherical bubble and (b) the perturbed bubble. Contours of the circumferential component of the logarithmic strain, $E_{\theta\theta}$, for (c) the spherical bubble and (d) the perturbed bubble. The deformed radius of the spherical bubble is $72 \mu\text{m}$, and the perturbation amplitude in (b) and (d) is $1.6386 \mu\text{m}$.

In Figs. 4.9(a and b), we plot contours of the radial component of the logarithmic strain, E_{rr} , while in Figs. 4.9(c and d), we plot contours of the circumferential component of the logarithmic strain $E_{\theta\theta}$. In Figs. 4.10(a and b) we plot contours of the radial component of the Cauchy stress, σ_{rr} , while in Figs. 4.10(c and d), we plot contours of the circumferential component of the Cauchy stress, $\sigma_{\theta\theta}$. We include both the spherically symmetric case and the perturbed case, illustrating the amplification of stress and strain due to non-spherical deformation, notably in the circumferential components of stress and strain. Specifically, surface instabilities can cause more than a 25% increase in the local radial strain, and more than a 30% stress increase along the bubble-gel interface.

By conducting three-dimensional finite-element simulations, we find that non-spherical instabilities near the bubble wall can induce strain and stress amplification in soft materials, which might lead to a thin damage layer developing near the bubble wall. It is also

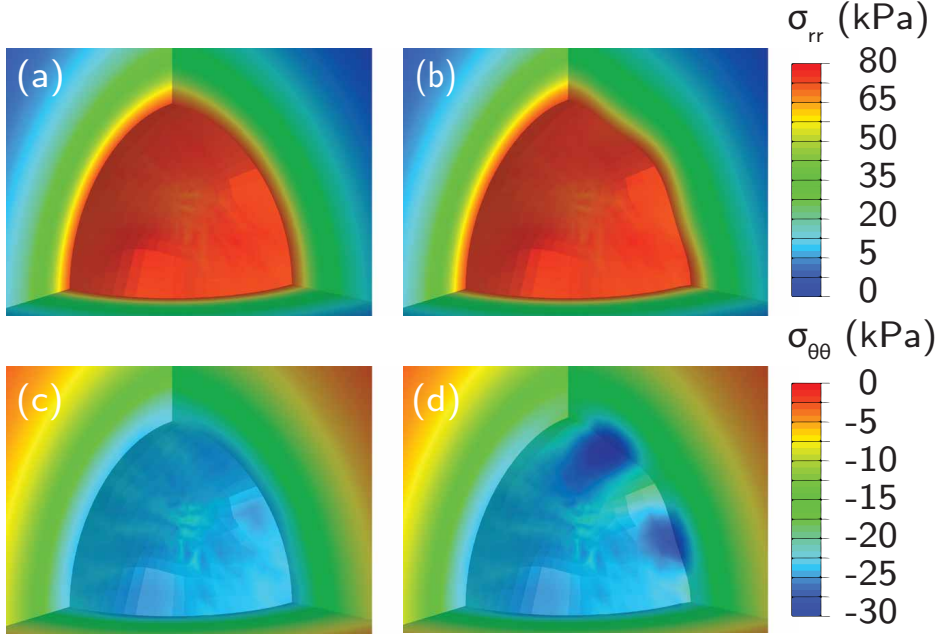


Figure 4.10: Contours of the radial component of the Cauchy stress, σ_{rr} , for (a) the spherical bubble and (b) the perturbed bubble. Contours of the circumferential component of the Cauchy stress, $\sigma_{\theta\theta}$, for (c) the spherical bubble and (d) the perturbed bubble. The deformed radius of the spherical bubble is $72 \mu\text{m}$, and the perturbation amplitude in (b) and (d) is $1.6386 \mu\text{m}$.

conceivable that within such a layer the material could experience inelastic deformation, fracture, or significant strain softening [86–88]. Addressing appropriate material damage models is beyond the scope presented here, and future studies aimed at resolving the intricate mechanics and physics near the bubble wall during these high strain-rate, inertially dominated deformations are necessary.

4.4 Concluding remarks

In this chapter, we carried out nonlinear, three-dimensional post-buckling-type simulations of bubble dynamics during LIC in order to determine the conditions under which non-spherical instabilities of an originally spherical bubble arise and to make comparisons against the theoretical linear stability analysis of Yang et al. [37]. To do so, we utilized three-dimensional finite-element simulations in Abaqus [76] to model the bubble dynamics during

LIC, accounting for large deformations, strain-stiffening effects, and viscoelastic behavior. To examine under which circumstances the bubble loses its sphericity, we introduced a small geometric imperfection on the initial bubble geometry based on the spherical harmonic function (4.23). We then examined the stability of the bubble by performing finite-element simulations of LIC for different radius ratios and compared the results with the linear stability analysis. Because this is a dynamic problem and the time scales are small, we computed the history of the growth rate of the perturbations based on the linear stability analysis for each case and utilized this quantity to inform whether the perturbation will have sufficient time to grow during the bubble oscillation. We observed that if the non-dimensionalized growth rate is below a threshold of about 6, perturbations will not have sufficient time to grow, and thus, the bubble will maintain its spherical shape. We found that such a threshold that is based on a finite, positive growth rate does a better job of predicting the onset of non-spherical instability than a threshold based on any positive growth rate, which is the conventional approach in assessing instability under quasi-static conditions. We also considered cases of LIC for different material viscosities and bubble wall surface tensions at a fixed radius ratio of $R_{\max}/R_0 = 4.5$ to test the generality of the growth-rate-based criteria and found that a critical value of the growth rate continued to accurately predict the onset of non-spherical bubble shapes in the nonlinear finite-element simulations.

Being able to predict the onset and evolution of instabilities and their associated material strain fields not only provides unprecedented insight into the cavitation dynamics in a soft, nonlinear solid, but it also provides a quantitative means of interpreting bubble dynamics relevant to a wide array of engineering and medical applications as well as natural phenomena.

Chapter 5

The role of material anisotropy in inertial microcavitation

5.1 Introduction

Inertial Microcavitation Rheometry (IMR) has been successfully applied to determine the material properties of various soft materials, such as polyacrylamide [28, 44, 48], agarose [45, 47], and collagen [4] gels as well as fixed porcine brain tissue as discussed in Chapter 3. The goal of building IMR was to enable the mechanical characterization of soft tissues at high strain rates and, in particular, study the mechanical behavior of biological materials such as tissues, collagen, etc. In Chapter 3, we studied the mechanical behavior of fixed porcine brain in gray matter and we found that this part of the brain can still be described using an isotropic material. However, as other works [38, 39] have documented, other parts of the porcine brain tissue, such as white matter or specific regions of gray matter, are known to be anisotropic. Yet, the existing numerical framework for material characterization in IMR accounts only for isotropic material behavior. Thus extending the current numerical framework to account for material anisotropy will allow us to mechanical characterize a variety of soft tissues.

The purpose of this chapter is to present a finite-element-based study that accounts for material anisotropy in the surrounding medium and explores the effect of material anisotropy on bubble dynamics in both ultrasound-induced and laser-induced cavitation. More specifically, we utilize three-dimensional simulations in Abaqus [76] and describe the surrounding material as a transversely isotropic hyperelastic material using a compressible version of the quadratic standard reinforcing model [38, 91–94]. In this study, we examine the effect of anisotropy on bubble dynamics in both UIC and LIC for different material stretches at high strain rates. Studying the effect of material anisotropy in bubble dynamics is of importance since it will add to our current understanding of material anisotropy at high-strain rates in soft tissues and help guide future experimental endeavors aimed towards extracting information on anisotropic material behavior from inertial microcavitation experiments.

This chapter is organized as follows. The quadratic standard reinforcing model is reviewed in Section 5.2.1, and the methodology for three-dimensional finite-element simulations of UIC and LIC in anisotropic material is discussed in Section 5.2.2. Then, results of computational parametric studies of the effects of driving amplitude and material anisotropy on bubble dynamics in UIC and LIC are presented in Section 5.3.1 and 5.3.2, respectively. Finally, we close with some concluding remarks in Section 5.4.

5.2 Methods

We account for both hyperelastic and viscous effects, since most biological tissues exhibit strongly nonlinear viscoelastic behavior. We assume a nonlinear Kelvin-Voigt rheological representation (Section 2.2.2), where the equilibrium branch is a transversely isotropic nonlinear hyperelastic contribution, and the non-equilibrium branch is a Newtonian viscous contribution. We then perform three-dimensional finite-element simulations of both UIC and LIC to elucidate the effect of anisotropic material behavior on cavitation dynamics.

5.2.1 Transversely isotropic hyperelastic material model

The Holzapfel, Gasser, and Ogden (HGO) model [95, 96] is a well known hyperelastic material model in the literature that accounts for material anisotropy and was originally developed for modeling of arterial walls. The model was built to account for anisotropy arising due to two families of collagen fibers embedded in an isotropic matrix. Despite the fact that the HGO model was developed for arterial walls, it has proven an excellent choice for modeling anisotropic behavior in a variety of biological tissues. In particular, since many tissues (including brain tissue) have one preferred fiber orientation and are transversely isotropic materials, prior works have simplified the HGO model to account for only one family of fibers for use in finite-element modeling of the brain [2, 39, 97]. Here, in the interest of minimizing the number of additional material parameters introduced, we consider the case of one family of perfectly aligned fibers, which involves only one additional material parameter to describe a transversely isotropic hyperelastic response. This hyperelastic model is often called the *quadratic standard reinforcing model* in the literature [38, 91–94] and goes back to Triantafyllidis and Abeyaratne [91]. This hyperelastic model is then utilized within a nonlinear Kelvin-Voigt framework.

First, we summarize the quadratic standard reinforcing hyperelasticity model. The deformation gradient tensor is denoted as $\mathbf{F}$ with $J = \det \mathbf{F}$ the volume ratio. The right and left Cauchy-Green deformation tensors are $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$ and $\mathbf{B} = \mathbf{F} \mathbf{F}^\top$; the distortional deformation gradient tensor is $\bar{\mathbf{F}} = J^{-1/3} \mathbf{F}$; the distortional right and left Cauchy-Green deformation tensors are $\bar{\mathbf{C}} = J^{-2/3} \mathbf{C}$ and $\bar{\mathbf{B}} = J^{-2/3} \mathbf{B}$; and the distortional first and second principal stretch invariants are $\bar{I}_1 = \text{tr} \bar{\mathbf{C}} = \text{tr} \bar{\mathbf{B}}$ and $\bar{I}_2 = (1/2)[(\text{tr} \bar{\mathbf{C}})^2 - \text{tr}(\bar{\mathbf{C}}^2)] = (1/2)[(\text{tr} \bar{\mathbf{B}})^2 - \text{tr}(\bar{\mathbf{B}}^2)]$. We then introduce a unit vector $\mathbf{n}$ denoting the fiber direction in the reference configuration (Fig. 5.1) and two additional pseudo-invariants to account for the anisotropic response of the material: $\bar{I}_4 = \mathbf{n} \cdot \bar{\mathbf{C}} \mathbf{n}$ and $\bar{I}_5 = \mathbf{n} \cdot \bar{\mathbf{C}}^2 \mathbf{n}$. The free-energy density for a transversely isotropic material may then be represented as $\psi = \hat{\psi}(\bar{I}_1, \bar{I}_2, J, \bar{I}_4, \bar{I}_5)$. The quadratic standard reinforcing model assumes the following simple form for the free-energy

density function:

$$\psi = \frac{G}{2} (\bar{I}_1 - 3) + \frac{K}{2} (J - 1)^2 + \frac{\theta}{2} (\bar{I}_4 - 1)^2, \quad (5.1)$$

where G is an isotropic contribution to the ground-state shear modulus, K is the ground-state bulk modulus, and θ is a parameter quantifying the additional reinforcement from the fibers. The two first terms of the free-energy function (5.1) represent a nearly-incompressible Neo-Hookean model and account for the isotropic behavior of the material, while the anisotropic behavior is captured through the last term of (5.1), where θ is a measure of how anisotropic the material is and physically represents the additional stiffness provided by the fibers. This simple form of the reinforcing model is capable of capturing transversely isotropic material behavior with one new parameter θ . Using the free-energy function for the equilibrium branch (5.1), the associated hyperelastic contribution to the Cauchy stress is

$$\boldsymbol{\sigma}^e = 2J^{-1} \frac{\partial \psi}{\partial \mathbf{B}} \mathbf{B} = J^{-1} G \text{dev} \bar{\mathbf{B}} + K(J - 1) \mathbf{1} + 2J^{-1} \theta (\bar{I}_4 - 1) \left(\bar{\mathbf{F}} \mathbf{n} \otimes \bar{\mathbf{F}} \mathbf{n} - \frac{1}{3} \bar{I}_4 \mathbf{1} \right), \quad (5.2)$$

where $\text{dev} \bar{\mathbf{B}}$ is the deviatoric part of the distortional left Cauchy-Green tensor.

Lastly, to account for viscoelasticity, it remains to specify the viscous contribution to the Cauchy stress. The stretching tensor is given by $\mathbf{D} = (1/2)(\dot{\mathbf{F}}\mathbf{F}^{-1} + (\dot{\mathbf{F}}\mathbf{F}^{-1})^\top)$, and the non-equilibrium contribution to the Cauchy stress is $\boldsymbol{\sigma}^v = 2\mu \text{dev}(\mathbf{D})$, where μ is the viscosity. Thus, the total Cauchy stress in the Kelvin-Voigt type rheological model is the sum of the stresses in the equilibrium and the non-equilibrium branches:

$$\begin{aligned} \boldsymbol{\sigma} &= \boldsymbol{\sigma}^e + \boldsymbol{\sigma}^v \\ &= J^{-1} G \text{dev} \bar{\mathbf{B}} + K(J - 1) \mathbf{1} + 2J^{-1} \theta (\bar{I}_4 - 1) \left(\bar{\mathbf{F}} \mathbf{n} \otimes \bar{\mathbf{F}} \mathbf{n} - \frac{1}{3} \bar{I}_4 \mathbf{1} \right) + 2\mu \text{dev}(\mathbf{D}). \end{aligned} \quad (5.3)$$

The value of the isotropic contribution to the shear modulus G used in this study is based

Table 5.1: Material properties used for the study of the effect of anisotropy on bubble dynamics

G (kPa)	K (GPa)	μ (Pa·s)	ρ_0 (kg/m ³)
15.09	2.167	0.1	1000

on the the value of the shear modulus for the stiff PA gel in determined in Yang et al. [44] using the NH-KV model. Table 5.1 summarizes the material parameters used in this chapter. In what follows, the values of G , K , and μ remain fixed and equal to the values presented in Table 5.1. Finally, three values of the fiber reinforcement parameter θ have been chosen in such a way so that $\theta/G = 0.1$, 0.3 , and 3 .

5.2.2 Finite-element modeling

We perform nonlinear, three-dimensional, finite-element simulations of inertial microcavitation in Abaqus [76]. The undeformed cavity radius is $R_0 = 50 \mu\text{m}$, while the far-field radius of the surrounding material is $R_{0,\infty} = 300R_0$, so that an infinite medium is approximated, and we do not have far-field boundary effects. We simulate one eighth of the spherical bubble geometry as in Chapter 5.2.2 for computational efficiency as illustrated in Fig. 5.1, which does not restrict the bubble dynamics for a transversely isotropic material. A total of 52,156 three-dimensional constant pressure elements (C3D8H) are used to model the surrounding material except from a very thin layer at the far-field radius, where we use infinite elements (CIN3D8) to account for a non-reflective, far-field boundary. A total of 1,159 elements are used on that region.

The constitutive behavior of the surrounding solid material is described by the model of Section 5.2.1, which is a Kelvin-Voigt-type, nonlinear viscoelastic model, in which material anisotropy is incorporated by utilizing the quadratic standard reinforcing hyperelasticity model for the equilibrium spring. The anisotropic model was implemented in Abaqus using a user-material subroutine (UMAT). The fiber orientation is assigned along the x -direction

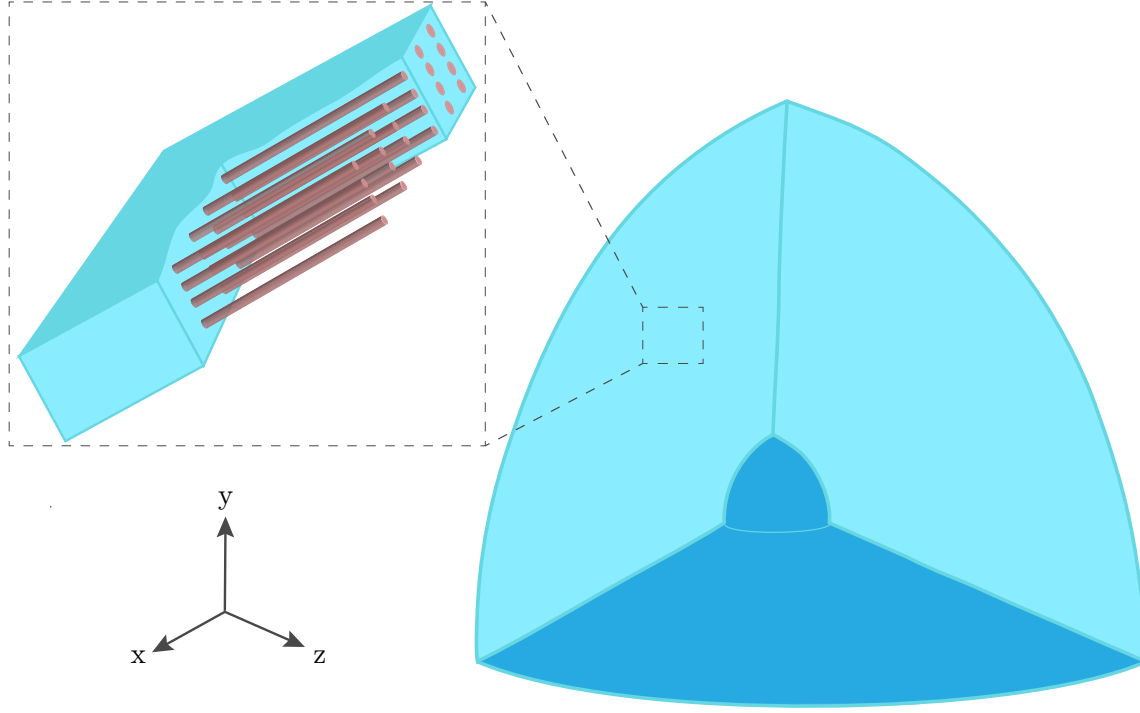


Figure 5.1: Schematic of the finite-element configuration for one eighth of the spherical bubble geometry. The fiber direction is illustrated in the schematic and is aligned with the x -axis.

as seen in Fig. 5.1.

We assume symmetry boundary conditions along each plane, x - y , y - z and x - z , in our computational domain, and we simulate both UIC and LIC conditions. A detailed description of the implementation of the driving conditions for UIC and LIC in Abaqus may be found in Appendix B. We model the interior of the bubble as an ideal and isothermal gas using a fluid-filled cavity constraint in Abaqus. In this chapter, we neglect surface tension, so that the pressure inside the bubble is $p_b = p_\infty(V_0/V)$, where $p_\infty = 101.3$ kPa is the ambient pressure, $V_0 = (4/3)\pi R_0^3$ is the undeformed bubble volume, and V is the deformed bubble volume.

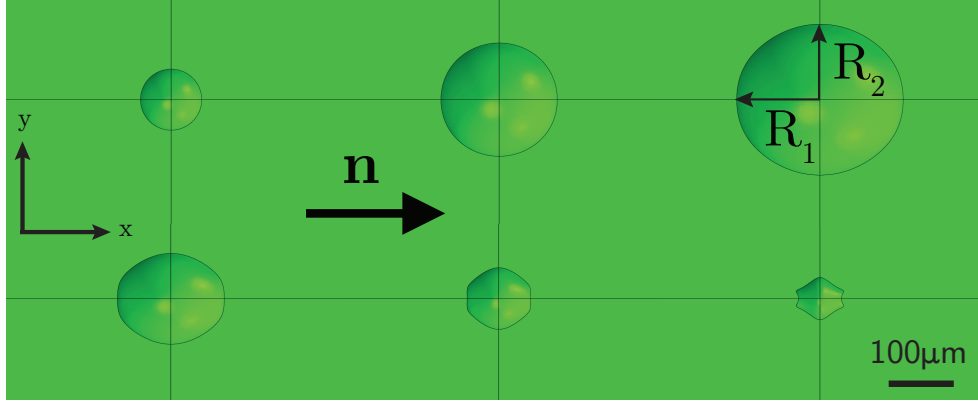


Figure 5.2: Spatiotemporal evolution of a single bubble within a soft anisotropic solid during UIC. The surrounding solid is modeled as an anisotropic Neo-Hookean Kelvin-Voigt material with anisotropy parameter $\theta/G = 3$, and the peak tensile pressure in UIC is $|p_A|/p_\infty = 10$.

5.3 Results and discussion

5.3.1 Ultrasound-induced cavitation

We begin by numerically examining the effect of material anisotropy on the bubble dynamics during UIC. As discussed in Appendix B, the driving force for cavitation in UIC is the time-dependent, far-field gauge pressure $p_f(t)$, which for this study is given by (B.5), in which $p_A < 0$ is the peak tensile pressure, representing the magnitude of the driving force. Figure 5.2 shows the spatiotemporal evolution of a single bubble during UIC for a baseline case in which $\theta/G = 3$ and $|p_A|/p_\infty = 10$. The direction of the undeformed fibers is aligned with the vector $\mathbf{n}$, and at $t = 0$, the spherical bubble is in mechanical and thermodynamic equilibrium with its surroundings. As the bubble grows, due to the driving far-field gauge pressure $p_f(t)$, it becomes ellipsoidal in shape, and we denote with R_1 the radius along the x -direction and with R_2 the radius along the y -direction. The presence of the fibers stiffens the material along the fiber direction $\mathbf{n}$, and for the portion of the bubble boundary where the fibers are aligned with the circumferential direction, the stiffening fibers resist growth in the hoop stretch and, hence, growth of the bubble radius. Therefore, because the radius R_2 in Fig. 5.2 is perpendicular to the direction of the fibers, we expect R_2 to grow less than R_1 ,

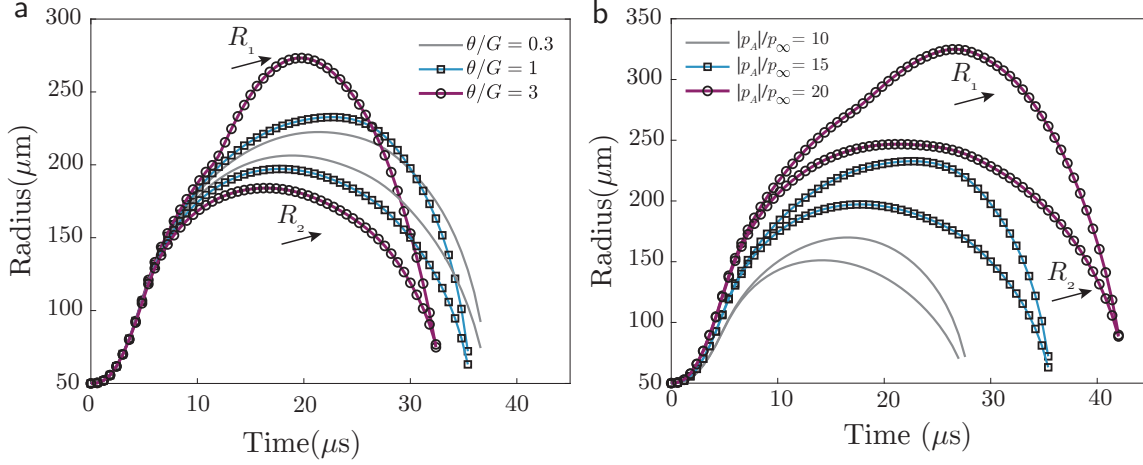


Figure 5.3: (a) Radius versus time plots for $|p_A|/p_\infty = 15$ and three different values of the anisotropy parameter θ/G . (b) Radius versus time plots for $\theta/G = 1$ and three different values of the peak tensile pressure $|p_A|/p_\infty$. The major axis of the ellipsoidal bubble is denoted as R_1 and the minor axis as R_2 .

which is observed in Fig. 5.2.

To better understand how anisotropy affects the bubble dynamics, we measure the histories of the two radii, R_1 and R_2 , in finite-element simulations from $t = 0$ until the first collapse, and we examine the effect of the anisotropy parameter θ on the two radii. Figure 5.3(a) shows plots of the radius versus time response for the two radii, R_1 and R_2 , for a fixed value of $|p_A|/p_\infty = 15$ and for three different values of $\theta/G = 0.3, 1$ and 3 . We can see from these plots that for a given θ/G , R_1 and R_2 are similar during the initial stage of growth. However, as the bubble continues to grow, the two radii begin to deviate from each other, and they grow to different maximum radii and at different rates. This further confirms our previous observation that the bubble is no longer spherical but rather becomes an ellipsoid with R_1 being the major axis and R_2 the minor axis. Furthermore, we observe that as the ratio of θ/G increases, the aspect ratio of the bubble increases since R_1 grows to a greater maximum radius. Moreover, we can see that for higher θ/G ratio, the bubble grows and collapses faster compared to lower ratios of θ/G due to the increased stiffness.

We observe similar effects when we consider different magnitudes of the peak tensile pressure in UIC. Figure 5.3(b) shows plots of the radius versus time response for the two

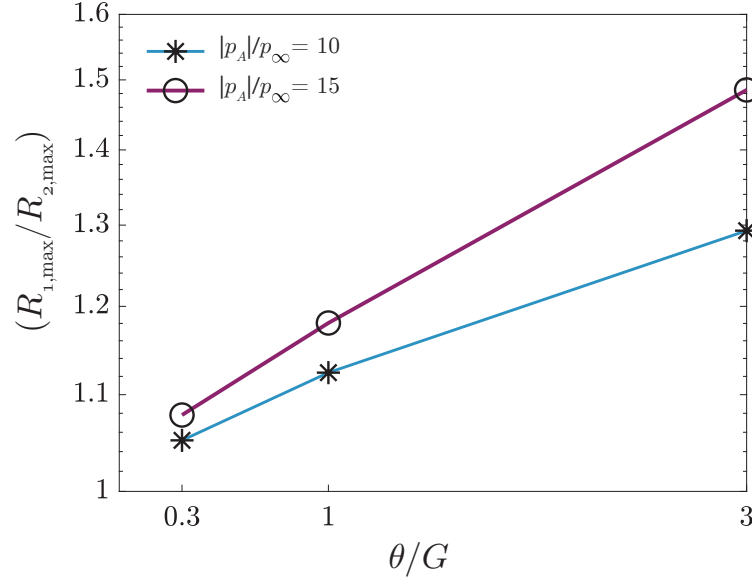


Figure 5.4: Effect of the anisotropy parameter θ/G on the ratio $R_{1,max}/R_{2,max}$ for two different values of the peak tensile pressure $|p_A|/p_\infty$. $R_{1,max}$ denotes the maximum radius achieved along the major axis, while $R_{2,max}$ denotes the maximum radius achieved along the minor axis. The ratio $R_{1,max}/R_{2,max}$ increases monotonically as θ/G and p_A/p_∞ increase.

radii, R_1 and R_2 , for a fixed value of the anisotropy parameter $\theta/G = 1$ and for three different values of the peak tensile pressure $|p_A|/p_\infty = 10, 15$, and 20 . Taking a closer look at the R_1 versus time and the R_2 versus time plots in Fig. 5.3(b), we see that increasing $|p_A|$ leads to the bubble growing larger overall and thus collapsing slower. Since θ/G is fixed in this case, both R_1 and R_2 increase for higher $|p_A|/p_\infty$ values. Moreover, we can see that for higher values of $|p_A|/p_\infty$, the R_1 versus time curve becomes markedly asymmetric when comparing the growth and collapse portions of the history. Finally, another interesting observation in Figs. 5.3(a and b) is that despite the different growth rates observed for R_1 and R_2 in each case, the two radii always reach the point of first collapse at the same time.

To summarize these results, Fig. 5.4 shows the ratio of the maximum radius $R_{1,max}$ to the maximum radius $R_{2,max}$ as a function of the non-dimensional anisotropy parameter θ/G for two different values of the peak tensile pressure $|p_A|/p_\infty$. We can see that as the peak tensile pressure increase, the ratio $R_{1,max}/R_{2,max}$ always increases, which means that larger bubbles tend to be more ellipsoidal with a higher aspect ratio. As expected, this is also true

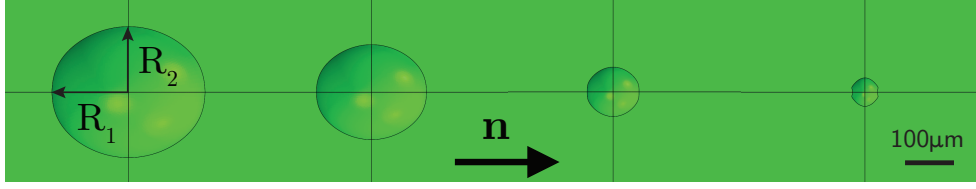


Figure 5.5: Spatiotemporal evolution of a single bubble within a soft anisotropic solid during LIC. The surrounding solid is modeled as an anisotropic Neo-Hookean Kelvin-Voigt material with anisotropy parameter $\theta/G = 3$, and the target radius ratio in LIC is $R_{\max}/R_0 = 3$.

as the anisotropy parameter θ/G increases, since for higher θ/G , the ratio $R_{1,\max}/R_{2,\max}$ increases.

5.3.2 Laser-induced cavitation

Finally, we numerically examine the effect of material anisotropy on bubble dynamics in laser-induced cavitation. The driving conditions for LIC are applied as described in Appendix B, and the spatiotemporal evolution of a single cavitation bubble under LIC conditions is shown in Fig. 5.5 for a target radius ratio of $R_{\max}/R_0 = 3$ and an anisotropy parameter of $\theta/G = 3$, illustrating the history from the maximum bubble volume through first collapse. Here, we perform a parametric study for different radius ratios $R_{\max}/R_0$ and different anisotropy parameters θ/G aiming to examine the role of anisotropy in the bubble dynamics.

Similar observations follow in LIC as in the case of UIC. Figure 5.6(a) shows the radius history of the major axis R_1 and the minor axis R_2 for two different values of the non-dimensional anisotropy parameter θ/G . Within our numerical framework in Abaqus, the method for initializing LIC simulations is to inflate the bubble to the maximum target radius. The target radius ratio for the two different cases of θ/G in Fig. 5.6(a) was $R_{\max}/R_0 = 3$. However, anisotropy affects the maximum radius achieved in both R_1 and R_2 , with R_1 growing to a higher value than R_2 , as can be seen at $t = 0$ in Fig. 5.6(a). This is to be expected since R_2 is perpendicular to the direction of the fibers. Similar observations are made when we try a higher value of radius ratio. Figure 5.6(b) shows the radius histories of

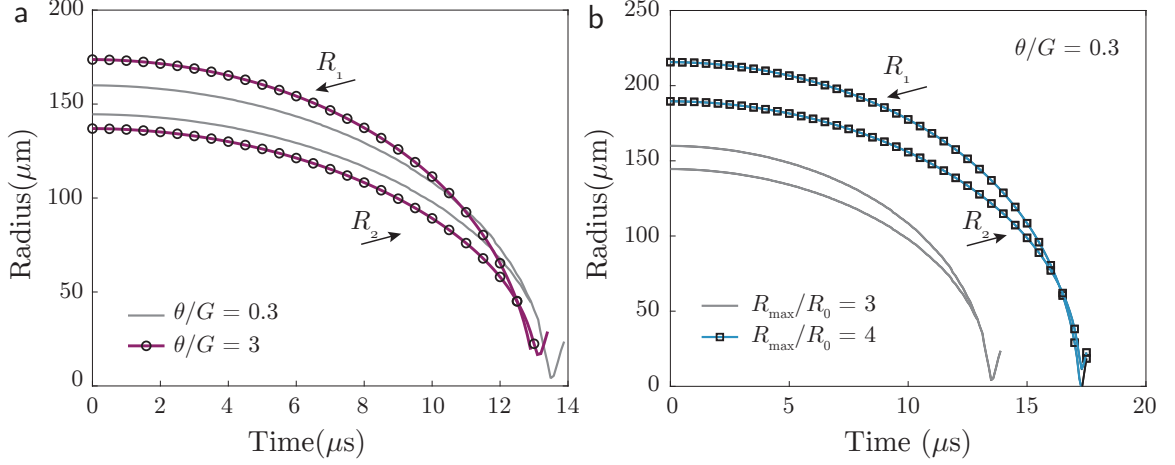


Figure 5.6: (a) Radius versus time plots for $R_{\text{max}}/R_0 = 3$ and two different values of the anisotropy parameter θ/G . (b) Radius versus time plots for $\theta/G = 0.3$ and two different values of the radius ratio R_{max}/R_0

R_1 and R_2 for fixed θ/G and two radius ratios, $R_{\text{max}}/R_0 = 3$ and 4. The effect of anisotropy is clearly more prominent in this case for the higher value of R_{max}/R_0 , and we observe an increased deviation between $R_{1,\text{max}}$ and $R_{2,\text{max}}$. Finally, it is interesting to note that while $R_{1,\text{max}}$ and $R_{2,\text{max}}$ begin at different maximum radii, they each reach first collapse at the same time, which is consistent with previous observations in UIC.

5.4 Concluding remarks

In this chapter, we have implemented a compressible version of the quadratic standard reinforcing model within a three-dimensional finite-element framework, which may be applied to simulating both ultrasound-induced and laser-induced cavitation. This constitutive model incorporates only one additional material parameter describing the anisotropy of the material and enables a straightforward investigation of the role of material anisotropy on bubble dynamics in both UIC and LIC. We performed parametric studies for different driving amplitudes in both UIC and LIC and different levels of anisotropy and found that for higher maximum radius ratios the effect of anisotropy is more prominent. Therefore, in principle, the bubble aspect ratio during its dynamics in UIC or LIC may be used to quantify

the anisotropy of the material and extract the material parameter θ . With recent advances in orthogonal axis imaging of bubble dynamics [98], we expect that the simulation capability presented in this chapter may be used in concert with future experimental data to achieve anisotropic IMR.

Chapter 6

Conclusions and outlook

6.1 Concluding remarks

Inertial Microcavitation Rheometry (IMR) is a promising tool for characterizing the mechanical behavior of soft solids at high strain rates. Building a robust numerical framework that allows for the accurate determination of the mechanical properties of soft biological tissues and gels used as tissue phantoms is of high importance. Such an endeavor demands a modeling framework in which a wide variety of complex constitutive models, material compressibility, material anisotropy, and non-spherical behavior can be incorporated. The work in this thesis was aimed at addressing these needs. The main conclusions and contributions of this thesis are summarized as follows:

- A finite-element-based approach for modeling spherically symmetric cavitation dynamics in both ultrasound-induced and laser-induced cavitation was developed. This method enables the consideration of violently collapsing bubbles, which involve compressible behavior that affects the bubble dynamics. We examined the effect of compressibility on bubble dynamics by comparing against corresponding simulation results using the Keller-Miksis approach, and we applied our method to provide improved material parameter estimations for a stiff polyacrylamide gel. Moreover, the

finite-element-based approach, unlike Rayleigh-Plesset-based approaches, does not place any limit on the nonlinear viscoelastic constitutive models that may be employed to describe the soft material. Thus, we proposed a strain-stiffening non-Newtonian Kelvin-Voigt type model. We then applied the model to the particle-assisted LIC data of Buyukozturk et al. [48] for a soft PA gel over a range of cavitation amplitudes to obtain a set of material parameters for soft PA gel at high strain rates.

- An axisymmetric finite-element-based approach was developed to resolve the bubble dynamics in thin layers. This is a high fidelity tool that accounts for nonlinear interactions between the bubble and the boundaries. We then applied the approach to mechanically characterize fixed porcine gray matter tissue at high-strain rates. The extracted material properties correspond to a strain-rate regime that was previously inaccessible to existing characterization techniques in the literature. Since the FE-based approach is computationally more expensive compared to the Keller-Miksis equation, we discussed an extended Keller-Miksis approach which represents a first-order correction to the spherically symmetric Keller-Miksis equation accounting for the constraint due to the thin layer. We compared the two methods for different gap thicknesses, different viscosities, and radius ratios, and we found that for low radius ratios and low viscosity the extended Keller-Miksis approach can provide reasonable estimates of material properties.
- To study the onset of non-spherical instabilities that arise from an originally spherical bubble, a three-dimensional, nonlinear, post-buckling-type framework was developed and used to determine the conditions under which non-spherical instabilities arise during cavitation dynamics. The framework incorporates large deformations, strain-stiffening effects, and viscoelastic behavior that are important in the cavitation dynamics of soft materials. Using the material properties for a stiff polyacrylamide gel since there is experimental evidence of unstable behavior in those materials [37],

we compared our numerical results against theoretical predictions from a linear stability analysis [37] and found qualitative agreement. Because cavitation is a dynamic problem and the time scales are small, we calculated the growth rate of non-spherical instability which informs whether perturbations have sufficient time to grow. We established a non-dimensional growth-rate-based threshold, which we found to be around 6. We then examined the effect of viscosity and surface tension and found that a critical value of the growth rate continued to quantitatively agree with the onset of non-spherical bubble shapes in the nonlinear finite-element simulations.

- Finally, a three-dimensional finite-element-based framework that accounts for material anisotropy was presented. We described the surrounding solid as a transversely isotropic hyperelastic material using a compressible version of the quadratic standard reinforcing model [38, 91, 91–94] and accounted for material viscosity using a nonlinear Kelvin-Voigt-type viscoelastic model. We considered both UIC and LIC conditions, and we examined the effect of anisotropy on bubble dynamics. We found that high radius ratios and high values of the anisotropy parameter greatly affect the bubble spherical shape, and in high anisotropic cases, we observe evidence of the onset of bubble jetting and unstable behavior.

6.2 Future directions

There remain many directions for future research work in the modeling of bubble dynamics in soft solids. Several directions are highlighted below.

- Since the FE-based approach of Chapter 2 does not place restrictions on the constitutive model used to describe the surrounding material, additional material physics, such as damage and plasticity, may be considered. For example, past studies that incorporate damage models into a RP-based framework [53, 87] have only considered simple damage models, due to the mathematical restrictions involved in evaluating

the stress integral (2.15). Thus, future work that implements continuum damage mechanics models into FE-IMR may be used to more accurately model the evolution of damage during inertial microcavitation.

- The current cavitation modeling frameworks employ the “cold medium” assumption, which idealizes the surrounding material as isothermal. Several previous works account for temperature evolution in the surroundings [50, 59, 99], and the FE-IMR approach of Chapter 2 may be straightforwardly extended to solve the balance of energy equation for the evolution of the temperature field in the surroundings and potentially further refine the estimation of material parameters.
- The FE-based approach accounts for slight compressibility using a simple equation of state. Future work can incorporate more complex equations of state, e.g., the Tait equation of state, to further probe the role of significant volume change during violent collapse. Moreover, the FE-based approach can be used to evaluate other extensions of the Rayleigh-Plesset-based approach beyond the Keller-Miksis equation that account for material compressibility [25], such as the Gilmore equation [100], in order to test whether the 0.08 Mach-number-based threshold applies to these approaches as well.
- In this thesis, we provide estimated material parameters for fixed porcine brain tissue at high strain rates. However, the material behavior of brain tissue is spatially heterogeneous, and future work will use cavitation-based rheometry to characterize different regions of porcine brain tissue. Moreover, future studies will apply IMR in all chosen anatomical regions and for both fixed and fresh porcine brain tissue at high strain rates ($> 10^3 \text{ s}^{-1}$) in order to quantify the difference between fixed and fresh constitutive behavior for porcine brain tissue at high strain rates.
- The current FE-based approach in confined geometries models the interior of the bubble as an isothermal ideal gas. While this idealization is sufficient for extracting material parameters from the first collapse, since heat and mass transfer is a secondary

effect [77] in this regime, including the high fidelity description of the bubble contents involving heat and mass transfer would be important for accurate material parameter estimation when the full radius history is used. An extended KM approach would be one avenue to achieve this. However, the extended KM approach presented in Chapter 4 becomes inaccurate when applied to larger radius ratios. To circumvent this limitation, the concept of time delay in the interaction forces [79] should be incorporated into the numerical framework. This will improve the estimation of pressure fluctuations in a slightly compressible material and provide more accurate predictions. Thus, future work would apply the extended KM approach taking into account multiple layers while incorporating the concept of time delay.

- In this thesis, we presented a three-dimensional finite-element framework that is able to predict the onset of unstable behavior during inertial microcavitation when we considered the instability mode shape $m = l = 8$. However, other types of non-spherical instabilities can occur during inertial microcavitation. For example, axisymmetric instabilities may also appear during inertial cavitation in heterogeneous media or in the vicinity of impedance mismatched boundaries [70, 101]. The critical conditions for predicting the onset of these and other non-spherical instabilities could be the subject of future work.
- Many biological tissues are transversely isotropic. Thus, material anisotropy is an important aspect of the constitutive behavior of soft biological tissues. As future work, the FE-based framework presented in Chapter 5 may be applied to high strain-rate characterization of soft solids that are transversely isotropic. Alternatively, one can envision developing an expedited modeling approach for an axisymmetric geometry using a Keller-Miksis-based approach for a system of bubble radii over a discrete number of azimuthal angles with symmetry assumed in the polar direction. Then, recent experimental work has developed a two-axes orthogonal viewing system

for imaging laser-induced cavitation events along two perpendicular directions that highlights asphericity of the bubble [98]. In concert with two-axes experimental data, the FE-based approach may be used to characterize transversely isotropic materials.

- The three-dimensional finite-element framework presented in Chapters 3 and 4 assumes that the bubble contents behave as a homobaric ideal gas, which is a sufficient assumption when focusing only on the first collapse. To extend the three-dimensional modeling, future work will apply the three-dimensional finite-element framework including a full-field description of the physics of the bubble interior.

Appendix A

Physics of the bubble contents

In this appendix, we summarize the equations governing the physics of the bubble contents. The modeling approach follows the literature on the physics of laser-induced cavitation bubbles [24, 27, 28, 59], and a more detailed discussion may be found in Estrada et al. [28]. We idealize the bubble contents as a homobaric, two-phase mixture consisting of water vapor and non-condensable gas, which are both modeled as ideal gases. The (dimensionless) vapor mass fraction field $k(r, t)$ and the temperature field $T(r, t)$ inside the bubble ($0 \leq r \leq R$) evolve according to

$$\frac{\partial k}{\partial t} + v_m \frac{\partial k}{\partial r} = \frac{1}{\rho_m r^2} \frac{\partial}{\partial r} \left(\rho_m r^2 D \frac{\partial k}{\partial r} \right) \quad (\text{A.1})$$

and

$$\frac{\kappa}{\kappa - 1} \frac{p_b}{T} \left(\frac{\partial T}{\partial t} + v_m \frac{\partial T}{\partial r} \right) = \dot{p}_b + \frac{1}{r^2} \frac{\partial}{\partial r} \left(r^2 K \frac{\partial T}{\partial r} \right) + \frac{\kappa}{\kappa - 1} \frac{p_b}{T} \frac{\mathcal{R}_v - \mathcal{R}_g}{\mathcal{R}} D \frac{\partial k}{\partial r} \frac{\partial T}{\partial r}, \quad (\text{A.2})$$

respectively, where

$$v_m(r, t) = \frac{1}{\kappa p_b} \left[(\kappa - 1) K \frac{\partial T}{\partial r} - \frac{1}{3} r \dot{p}_b \right] + \frac{\mathcal{R}_v - \mathcal{R}_g}{\mathcal{R}} D \frac{\partial k}{\partial r} \quad (\text{A.3})$$

is the mixture velocity field, $\rho_m = p_b / \mathcal{R} T$ is the mixture density field, $\mathcal{R} = k \mathcal{R}_v + (1 - k) \mathcal{R}_g$ is the mixture gas constant field, and $K(T) = AT + B$ is the temperature-dependent thermal

Table A.1: Parameters describing the physics of the bubble contents

Parameter	Value	Parameter	Value
D	$24.2 \times 10^{-6} \text{ m}^2/\text{s}$	κ	1.4
$\mathcal{R}_v$	$0.462 \text{ kJ/kg}\cdot\text{K}$	$\mathcal{R}_g$	$0.287 \text{ kJ/kg}\cdot\text{K}$
A	$5.3 \times 10^{-5} \text{ W/m}\cdot\text{K}^2$	B	$1.17 \times 10^{-2} \text{ W/m}\cdot\text{K}$
p_{ref}	$1.17 \times 10^8 \text{ kPa}$	T_{ref}	5200 K
T_∞	298.15 K		

conductivity of the mixture [26]. Finally, the ODE governing the evolution of the absolute pressure inside the bubble $p_b(t)$ is

$$\dot{p}_b = \frac{3}{R} \left[-\kappa p_b \dot{R} + (\kappa - 1) K(T|_{r=R}) \frac{\partial T}{\partial r} \Big|_{r=R} + \kappa p_b \frac{\mathcal{R}_v}{\mathcal{R}(k|_{r=R})} \frac{D}{1 - k|_{r=R}} \frac{\partial k}{\partial r} \Big|_{r=R} \right]. \quad (\text{A.4})$$

The constant parameters associated with the bubble contents are the binary diffusion coefficient D , the specific heats ratio of the mixture κ , the gas constants of the water vapor and non-condensable gas, $\mathcal{R}_v$ and $\mathcal{R}_g$, and the empirical constants describing the temperature-dependence of the thermal conductivity of the mixture, A and B , which are summarized in Table A.1.

The boundary conditions for the PDEs governing the temperature field (A.2) and the vapor mass fraction field (A.1) at the bubble center ($r = 0$) are $\partial T / \partial r|_{r=0} = \partial k / \partial r|_{r=0} = 0$. It is assumed that the surrounding material remains isothermal at an ambient temperature of T_∞ (i.e., the “cold medium” assumption), so that the temperature boundary condition at the bubble wall is $T|_{r=R} = T_\infty$. Further, at the bubble wall, the partial pressure of the water vapor is taken to be equal to its saturation pressure, leading to the following boundary condition for the vapor mass fraction:

$$k|_{r=R} = \left[1 + \frac{\mathcal{R}_v}{\mathcal{R}_g} \left(\frac{p_b}{p_{v,\text{sat}}(T_\infty)} - 1 \right) \right]^{-1}, \quad (\text{A.5})$$

where $p_{v,\text{sat}}(T) = p_{\text{ref}} \exp(-T_{\text{ref}}/T)$ is the temperature-dependent saturation pressure of the

water vapor with empirical constants p_{ref} and T_{ref} [59]. Quantitative values for p_{ref} , T_{ref} , and T_{∞} used in this study are included in Table A.1.

Regarding the initial conditions for the bubble contents, for both LIC and UIC, we assume that the bubble contents are in thermal equilibrium at $t = 0$. Accordingly, the initial temperature field is spatially constant and given by $T(r, t = 0) = T_{\infty}$. Likewise, the water vapor partial pressure field is taken to be spatially constant and given through its saturation pressure at the ambient temperature, so that the initial vapor mass fraction field $k(r, t = 0)$ is also spatially constant and given by (A.5). Finally, following the arguments in Estrada et al. [28], for LIC, the initial bubble pressure is

$$p_b(t = 0) = p_{v,\text{sat}}(T_{\infty}) + \left(p_{\infty} + \frac{2\gamma}{R_0} - p_{v,\text{sat}}(T_{\infty}) \right) \left(\frac{R_0}{R_{\text{max}}} \right)^3. \quad (\text{A.6})$$

For UIC, the initial bubble pressure is $p_b(t = 0) = p_{\infty} + 2\gamma/R_0$.

Prior to obtaining numerical solutions, the governing equations for the bubble contents are non-dimensionalized as discussed in Barajas and Johnsen [59] and Estrada et al. [28]. We define a characteristic length scale $R_c = R_{\text{max}}$ for LIC and $R_c = R_0$ for UIC and a characteristic velocity $v_c = \sqrt{p_{\infty}/\rho_0}$. We then non-dimensionalize the governing equations using the characteristic length scale R_c , the characteristic velocity v_c , the far-field pressure p_{∞} , and the far-field temperature T_{∞} . The resulting dimensionless quantities as summarized in Table A.2. Furthermore, we express the PDEs for the vapor mass fraction $k(r, t)$ and the temperature $T(r, t)$ inside the bubble, (A.1) and (A.2), in terms of a dimensionless radial coordinate $y = r/R(t)$ on the fixed domain $0 \leq y \leq 1$. Finally, in lieu of the temperature field, following Prosperetti et al. [26], we utilize the dimensionless field variable

$$\theta = \frac{1}{K(T_{\infty})} \int_1^{T/T_{\infty}} K(T_{\infty} \hat{\theta}) d\hat{\theta}. \quad (\text{A.7})$$

Then, the non-dimensionalized PDEs governing the vapor mass fraction field $k(y, t)$ and

Table A.2: Dimensionless quantities involved in the physics of the bubble contents

Dimensional quantity	Dimensionless quantity	Name
t	$t^* = t v_c / R_c$	Time
R	$R^* = R / R_c$	Bubble-wall radius
p_b	$p_b^* = p_b / p_\infty$	Bubble pressure
D	$Fo = D / (v_c R_c)$	Mass Fourier number
T	$T^* = T / T_\infty$	Temperature
K	$K^* = K / K(T_\infty)$	Mixture thermal conductivity
$K(T_\infty) T_\infty$	$\chi = K(T_\infty) T_\infty / (p_\infty v_c R_c)$	Lockhart-Martinelli number
$p_{v,sat}(T_\infty)$	$p_{v,sat}^* = p_{v,sat}(T_\infty) / p_\infty$	Vapor saturation pressure

the field $\theta(y, t)$ inside the bubble ($0 \leq y \leq 1$) are

$$\begin{cases} \frac{\partial k}{\partial t^*} + \frac{v_m^* - \dot{R}^* y}{R^*} \frac{\partial k}{\partial y} = \frac{Fo}{R^{*2}} \frac{1}{\rho_m y^2} \frac{\partial}{\partial y} \left(\rho_m y^2 \frac{\partial k}{\partial y} \right), \\ \frac{\kappa}{\kappa - 1} \frac{p_b^*}{K^* T^*} \left(\frac{\partial \theta}{\partial t^*} + \frac{v_m^* - \dot{R}^* y}{R^*} \frac{\partial \theta}{\partial y} \right) = \dot{p}_b^* + \frac{\chi}{R^{*2}} \frac{1}{y^2} \frac{\partial}{\partial y} \left(y^2 \frac{\partial \theta}{\partial y} \right) \\ \quad + \frac{\kappa}{\kappa - 1} \frac{p_b^*}{K^* T^*} \frac{Fo}{R^{*2}} \frac{\mathcal{R}_v - \mathcal{R}_g}{\mathcal{R}} \frac{\partial k}{\partial y} \frac{\partial \theta}{\partial y}, \end{cases} \quad (A.8)$$

with the dimensionless mixture velocity field inside the bubble $v_m^*(y, t)$ given by

$$v_m^* = \frac{1}{\kappa p_b^*} \left[(\kappa - 1) \frac{\chi}{R^*} \frac{\partial \theta}{\partial y} - \frac{1}{3} R^* y \dot{p}_b^* \right] + \frac{Fo}{R^*} \frac{\mathcal{R}_v - \mathcal{R}_g}{\mathcal{R}} \frac{\partial k}{\partial y}. \quad (A.9)$$

Finally, the ODE governing the evolution of the dimensionless bubble pressure $p_b^*(t)$ is

$$\dot{p}_b^* = \frac{3}{R^*} \left[-\kappa p_b^* \dot{R}^* + (\kappa - 1) \frac{\chi}{R^*} \frac{\partial \theta}{\partial y} \Big|_{y=1} + \kappa p_b^* \frac{\mathcal{R}_v}{\mathcal{R}(k|_{y=1})} \frac{Fo}{1 - k|_{y=1}} \frac{1}{R^*} \frac{\partial k}{\partial y} \Big|_{y=1} \right]. \quad (A.10)$$

Regarding the boundary conditions for the PDEs (A.8), at the bubble center, we have that $\partial \theta / \partial y|_{y=0} = \partial k / \partial y|_{y=0} = 0$. At the bubble wall, using (A.7), the temperature boundary condition $T|_{r=R} = T_\infty$ becomes $\theta|_{y=1} = 0$, and the wall boundary condition for the vapor

mass fraction (A.5) becomes

$$k|_{y=1} = \left[1 + \frac{\mathcal{R}_v}{\mathcal{R}_g} \left(\frac{p_b^*}{p_{v,\text{sat}}^*} - 1 \right) \right]^{-1}. \quad (\text{A.11})$$

Then, regarding the spatial discretization of the PDEs (A.8), the spatial derivatives are calculated using second-order central differences on a grid of $N + 1$ equidistant points inside the bubble [26]. In this study, we use $N = 500$. The method of time evolution for the governing equations for the bubble contents differs between the KM-based approach and the FE-based approach. For KM-based numerical solutions, the discretized governing equations for the bubble contents are evolved forward in time in conjunction with the KM equation (2.16) using a variable-step, variable-order solver (MATLAB `ode15s`). For FE-based numerical solutions, within a FE time increment, the displacement field in the surroundings, and hence the bubble radius, is updated explicitly using the explicit central difference scheme, and then, the discretized governing equations for the bubble contents are evolved forward in time using the fourth-order Runge-Kutta method. When the governing equations for the bubble contents require a stable time step that is smaller than the stable time increment required for the FE update, the discretized governing equations for the bubble contents are evolved forward in time using a number of substeps, in which the bubble radius is linearly interpolated over the course of the FE time increment.

Appendix B

Driving conditions for laser-induced and ultrasound-induced cavitation in Abaqus

B.1 Laser-induced cavitation

For LIC, as discussed in Chapter 2, we begin our numerical simulations at the time when the bubble has reached its maximum radius $R_{\max}$, and it is in thermodynamic equilibrium but not mechanical equilibrium with the surrounding solid. Taking that into account, to achieve the initial condition for LIC in Abaqus [76], we first perform several initialization steps.

First, for any simulation involving non-zero surface tension, an initialization step is required in which the surface tension is quasi-statically ramped from zero to the target value γ using the method of Henann and Bertoldi [78] for axisymmetric simulations or the method of Wang and Henann [90] for three-dimensional simulations. In both axisymmetric and three-dimensional simulations in Abaqus, we make the assumption that the interior of the bubble is an isothermal ideal gas. Therefore, if one were to increase the surface tension without increasing the mass of gas in the bubble, the bubble volume would decrease as the surface tension is increased. To counteract this, mass is added to the bubble during this step using the `*fluid flux` command in a static analysis step. Using the ideal gas equation of

state, the added mass may be calculated as $(2\gamma/R_0)V_0/\mathcal{R}_g T_\infty$, where $p_\infty = 101.3$ kPa is the ambient pressure, R_0 is the undeformed radius of the spherical bubble, $V_0 = (4/3)\pi R_0^3$ is the undeformed volume of the bubble, $\mathcal{R}_g = 0.287$ kJ/kg·K is the gas constant of air,¹ and $T_\infty = 298.15$ K is the ambient temperature. At the conclusion of this step, the radius of the spherical bubble remains R_0 , while the bubble pressure increases to $p_\infty + 2\gamma/R_0$.

Next, we quasi-statically inflate the bubble to the target maximum radius $R_{\max}$ by pumping mass into the bubble until the desired maximum radius is achieved. To calculate the necessary mass to be added to the bubble, we first note that, based on the ideal gas equation of state, the initial mass of gas in the bubble is

$$m_0 = \frac{(p_\infty + 2\gamma/R_0)V_0}{\mathcal{R}_g T_\infty}. \quad (\text{B.1})$$

After quasi-static inflation, the deformed volume of the bubble is $V_{\max} = (4/3)\pi R_{\max}^3$ and its internal pressure is $p_\infty + 2\gamma/R_{\max} + p_i$, where

$$p_i = \frac{(1-3\alpha)G}{2} \left[5 - \left(\frac{R_0}{R_{\max}} \right)^4 - \frac{4R_0}{R_{\max}} \right] - 2\alpha G \left[\frac{27}{40} + \frac{1}{8} \left(\frac{R_0}{R_{\max}} \right)^8 + \frac{1}{5} \left(\frac{R_0}{R_{\max}} \right)^5 + \left(\frac{R_0}{R_{\max}} \right)^2 - \frac{2R_{\max}}{R_0} \right] \quad (\text{B.2})$$

is the pressure required to quasi-statically inflate the bubble for a quadratic-law hyperelastic material. (Recall that for $\alpha = 0$, p_i is the required inflation pressure for a Neo-Hookean material.) Therefore, the total mass in the bubble after quasi-static inflation is

$$m_1 = \frac{(p_\infty + 2\gamma/R_{\max} + p_i)V_{\max}}{\mathcal{R}_g T_\infty}, \quad (\text{B.3})$$

¹The gas constant for a given ideal gas is given by $\mathcal{R}_g = \mathcal{R}_u/\text{MW}$, where $\mathcal{R}_u = 8.314$ J/mol·K is the universal gas constant, and MW is the molecular weight of the gas, which for air is 0.0289 kg/mol. In Abaqus, one gives $\mathcal{R}_u$ and MW, rather than $\mathcal{R}_g$.

and the mass added to the bubble during the inflation step is

$$\Delta m = m_1 - m_0 = \left[p_\infty \left(1 - \frac{R_0^3}{R_{\max}^3} \right) + \frac{2\gamma}{R_{\max}} \left(1 - \frac{R_0^2}{R_{\max}^2} \right) + p_i \right] \frac{V_{\max}}{\mathcal{R}_g T_\infty}. \quad (\text{B.4})$$

The added mass is specified in the Abaqus input file through the `*fluid flux` command in a static analysis step, where the mass flux is $\dot{m} = \Delta m / t_{\text{infl}}$ and $t_{\text{infl}} = 0.0045 \mu\text{s}$ is the inflation time. We note that in the simulations of Chapter 5 for LIC in anisotropic materials, we continue to estimate the added mass in this way. The resulting bubble shape is ellipsoidal with major and minor radii greater than and less than the target radius $R_{\max}$, respectively.

It is important to note that, because the surrounding material is modeled as a qKV viscoelastic material, the target maximum bubble radius will only be achieved after we let the surrounding material relax for a sufficient amount of time. For this reason, we perform another quasi-static analysis step, in which we set the `*fluid flux` equal to zero and apply no driving to the bubble. In our simulations, we let the surrounding material relax for $t_{\text{relax}} = 10\mu/G$, where μ/G is the characteristic relaxation time of the surrounding material.

Once the surrounding material has relaxed, and the target maximum radius is achieved, it remains to remove the extra mass added into the bubble during the first analysis step. Hence, we perform an analysis step, in which the added mass (B.4) is rapidly removed from the bubble. This is a dynamic analysis step, and we have to ensure that the mass is removed over a sufficiently short timescale, so that the bubble radius remains approximately unchanged at $R_{\max}$. To this end, we choose a deflation time of $t_{\text{defl}} = 0.0045 \mu\text{s}$, which is sufficiently short, so that the bubble volume is approximately preserved, while the pressure inside the bubble reaches the target initial condition of $p_b = (p_\infty + 2\gamma/R_0)(R_0/R_{\max})^3$. At this point, the bubble radius and pressure are at their target initial conditions, in which the system is out of mechanical equilibrium, and thus, we can proceed to perform the final analysis step, which is a dynamic analysis step simulating the LIC process. In this step, we set the `*fluid flux` equal to zero, so that the bubble remains a closed system with mass m_0 , and let the

bubble collapse, rebound, and oscillate until it reaches mechanical equilibrium, in which $R = R_0$.

B.2 Ultrasound-induced cavitation

Recall from Chapter 2 that in UIC, the bubble is in both mechanical and thermodynamic equilibrium with the surroundings in its initial state. Thus, $R(t = 0) = R_0$, and the driving force for cavitation is the application of a time-dependent, far-field gauge pressure:

$$p_f(t) = \begin{cases} p_A \left(\frac{1 + \cos[\omega(t - \delta)]}{2} \right)^n & \text{if } |t - \delta| \leq \frac{\pi}{\omega}, \\ 0 & \text{if } |t - \delta| > \frac{\pi}{\omega}, \end{cases} \quad (\text{B.5})$$

where $p_A < 0$ is the peak tensile pressure, $n = 3.7$ is a dimensionless shape parameter, $\omega = 2\pi f$ is the angular frequency of the half-cycle pulse (where $f = 1$ MHz), and $\delta = \pi/\omega$ is the time delay. This expression for the far-field gauge pressure is based on representative experimental acoustic pulses and has been successfully used to simulate UIC conditions accurately in the literature [5–7, 35, 45, 49, 50]. Thus, in order to simulate UIC in our three-dimensional finite-element simulations, we apply a time-dependent pressure on the bubble wall, which is the negative of the far-field pressure (B.5), as a distributed pressure load on the bubble wall surface. The interior of the bubble is modeled as an isothermal, ideal gas using a fluid-filled cavity constraint in Abaqus, in which $p_\infty = 101.3$ kPa is the ambient pressure, $T_\infty = 298.15$ K is the ambient temperature, $MW = 0.0289$ kg/mol is the molecular weight of air, $\mathcal{R}_u = 8.314$ J/mol·K is the universal gas constant, and $\mathcal{R}_g = \mathcal{R}_u/MW$ is the gas constant of air.

Appendix C

Effect of experimental sampling frequency on fitted parameters

Laser-induced cavitation (LIC) as well as ultrasound-induced cavitation (UIC) are rapid phenomena, and as a result, obtaining high-resolution experimental data is always a challenge. Thus, if we do not have sufficiently high resolution experimental data (enough fitting points), mechanical characterization of the surrounding material can be difficult and as a result not robust. In this appendix, we examine how the experimental sampling frequency can affect the resulting fitted parameters.

For the purposes of this study, we perform one-dimensional FE simulations, using the method of Chapter 2, with known material parameters and downsample the simulation results at different sampling frequencies. We then fit the downsampled FE-generated data using the Keller-Miksis approach for cases of both LIC and UIC. To begin with, we assume that the surrounding material behaves as a qKV material, which as discussed in Chapter 2.2.2, involves the adjustable parameters α and μ . For α , we consider four different values, $\alpha = 0.1, 0.3, 1$, and 3 , and for μ , we consider three values, $\mu = 0.001, 0.01$, and 0.1 . For LIC, we consider a fixed radius ratio of $R_{\max}/R_0 = 3$, while for UIC, we consider a fixed peak tensile pressure of $p_A = -3$ MPa, which results in a range of radius ratios from

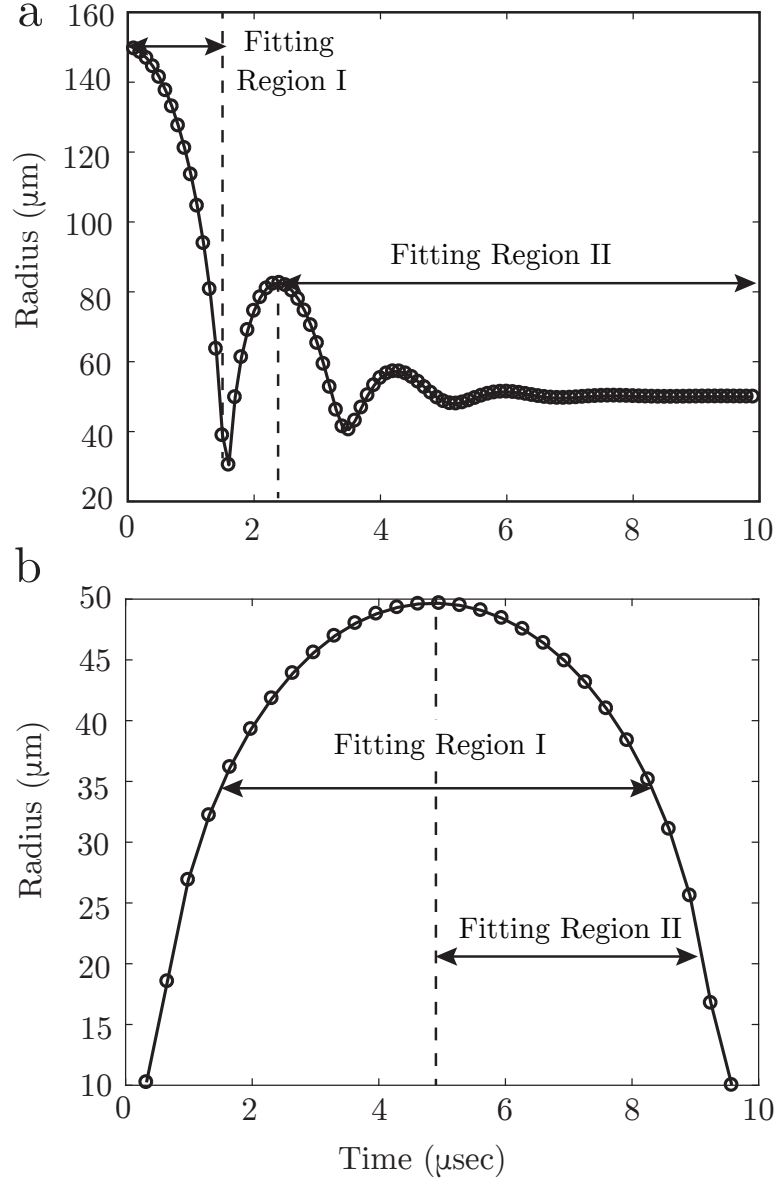


Figure C.1: Regions of the radius history used for the purpose of fitting in the sampling frequency study. In (a), we can see the two regions, in which we perform our fittings for LIC. To fit α , we use Region I, while to fit μ , we use Region II. In (b), we can see the two regions, in which we perform our fittings for UIC. In this case, we consider fittings of both α and μ using both regions.

$R_{\max}/R_0 = 3$ through $R_{\max}/R_0 = 5$ for the material parameters considered here.

The final step is to define the sampling frequencies for both LIC and UIC. For LIC we choose $\omega = 1$ MHz, 650 kHz, and 300 kHz, while for UIC we choose $\omega = 1$ MHz, 700 kHz, and 500 kHz.

We then perform fittings and obtain best-fit α_{fit} and μ_{fit} for each combination of input material parameters α and μ , for each sampling frequency, and for both LIC and UIC. For the fitting of α , we only use the first collapse in the $R(t)$ curve (see Fig. C.1(a) *Fitting Region I*) while we use the rest of the curve from the second peak onward for the fitting of μ (see C.1(a) *Fitting Region II*). The error in the fitted parameter α_{fit} is defined as

$$E_{\alpha} = \left| \frac{\alpha - \alpha_{\text{fit}}}{\alpha} \right|^2, \quad (\text{C.1})$$

and similarly, the error in the fitted viscosity μ_{fit} is defined as

$$E_{\mu} = \left| \frac{\mu - \mu_{\text{fit}}}{\mu} \right|^2. \quad (\text{C.2})$$

Figure C.2 shows the results of the fitting procedure for the parameter α in the case of LIC. We observe that the level of error in the estimation of the parameter α increases for lower sampling frequency, which is expected since the sampling frequency is directly associated with the number of fitting points. Another observation from this error map is that for the highest sampling frequency (Fig. C.2(a)) for combinations of low values of viscosity μ ($\mu \leq 0.008 \text{ Pa}\cdot\text{s}$) and α ($\alpha \leq 0.3$), the error in the fitting procedure will be large and more than 30%. This happens because for low values of α the curves are not distinguishable due to the fact that the strain stiffening effects are not strong, while for low values of μ the first collapse always happens at nearly the same characteristic time $R_{\text{max}} \sqrt{\rho_0/p_{\infty}}$, and in general in this region, elasticity is more dominant. For a medium sampling frequency (Fig. C.2(b)) for any value of μ less than $0.88 \text{ Pa}\cdot\text{s}$ and for $\alpha \leq 0.3$, the fitting procedure cannot estimate the parameter α well, while for the smallest sampling frequency C.2(c) the only region that gives a good estimation of α is for $\alpha \geq 1$ and any value of μ .

Next, Fig. C.3 shows the error maps for the fitting of μ in LIC. For this fitting procedure, we use the part of the curve that is shown in Fig. C.1(a) and is denoted as *Fitting Region II*, where we treat the first rebound of the curve as our initial maximum radius at time $t = 0$, and

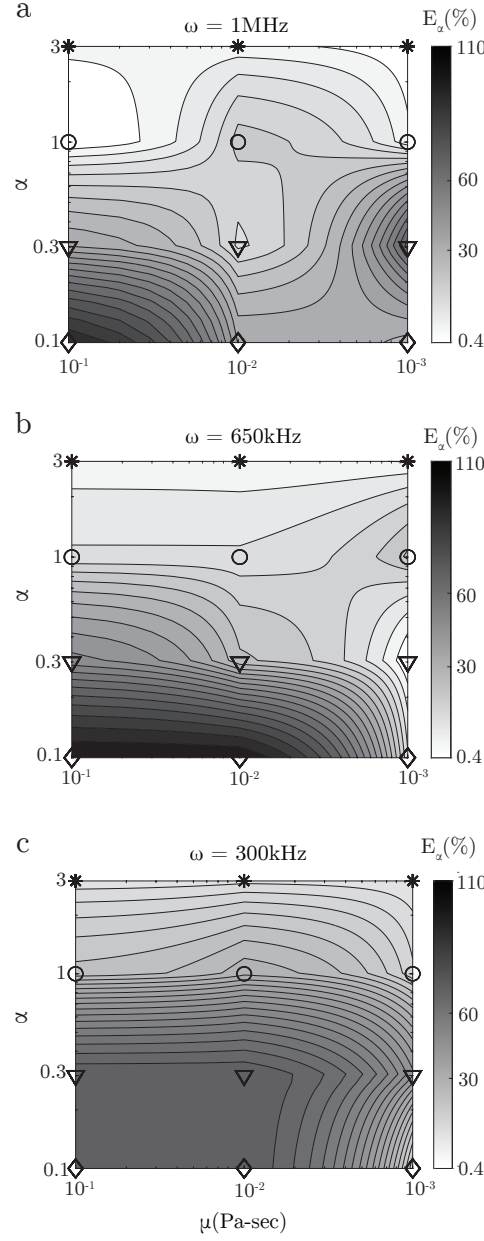


Figure C.2: Contour plots showing the error in the fitting of the parameter α in the case of LIC for three different sampling frequencies: (a) 1 MHz, (b) 650 kHz, and (c) 300 kHz. In these plots, we can see the error in the best-fit α_{fit} for different combinations of α and μ .

we start the fitting at that point until the end of the curve where the bubble is in equilibrium with its surroundings. The reason we distinguish between two different fitting regions is that the damping nature of the rebounds in the R versus t curve are strongly affected by viscous effects rather than elastic effects, and as a result there is more information regarding

μ within that region. In Figs. C.3(a), C.3(b), and C.3(c), we can see the error maps for μ for three different sampling frequencies. Again for the highest sampling frequency (Fig. C.3(a)), we can estimate μ very well in all regions of the parameter space except from the regions where $\mu \leq 0.01$ Pa·s and $0.3 < \alpha < 0.7$. This is something that we expect since for low values of μ the effect of viscosity is not present yet, and the material behaves more like an elastic solid (almost no damping) while for low values of α the strain stiffening effects are not that strong. Next, for the medium sampling frequency, for μ less than 0.8 Pa·s and for any α , the error in μ is more than 30%, while for the smallest sampling frequency, the only region with error less than 30% is for μ more than 0.03 Pa·s and any value of α .

Next, we perform least-square best fitting between FE results and KM results for the case of UIC. First, we use both growth and collapse in order to fit α and μ (see Fig. C.1(b), *Fitting Region I*). In Figs. C.4(a), C.4(b), and C.4(c), we can see the error maps for α for three different sampling frequencies. We can see from these three plots that for small α (less than 0.3) and for any viscosity μ the error in the fitting of α ($E_\alpha(\%)$) is more than 30%. Every other combination of α and μ gives a reasonable estimation of the α parameter. We make similar observations in Figs. C.5(a), C.5(b), and C.5(c), where we can see the error maps for the fitting of μ . We can see from these three plots that the only cases with error greater than 30% are for α less than 0.5 and for μ less than 0.08 Pa·s. A final observation is that, as we decrease the sampling frequency the level of error in both α and μ increases, but these combinations of α and μ in the contour plots still correspond to error less than 30%.

As a final test, we perform fittings for α and μ using *Fitting Region II* in Fig. C.1(b) (In this case, we shift the data so that the peak corresponds to $t = 0$.) We have again data sets for three different sampling frequencies. The results from the fitting procedure are shown in Figs. C.6 and C.7.

Figure C.6(a) shows the error map for the fitting in α ($E_\alpha(\%)$) for the highest sampling frequency (1 MHz). The regions with error less than 30% corresponds to $\alpha > 0.8$ and $0.001 < \mu < 0.08$ Pa·s as well as $\alpha > 0.2$ and $0.08 < \mu < 0.1$ Pa·s. As we decrease the

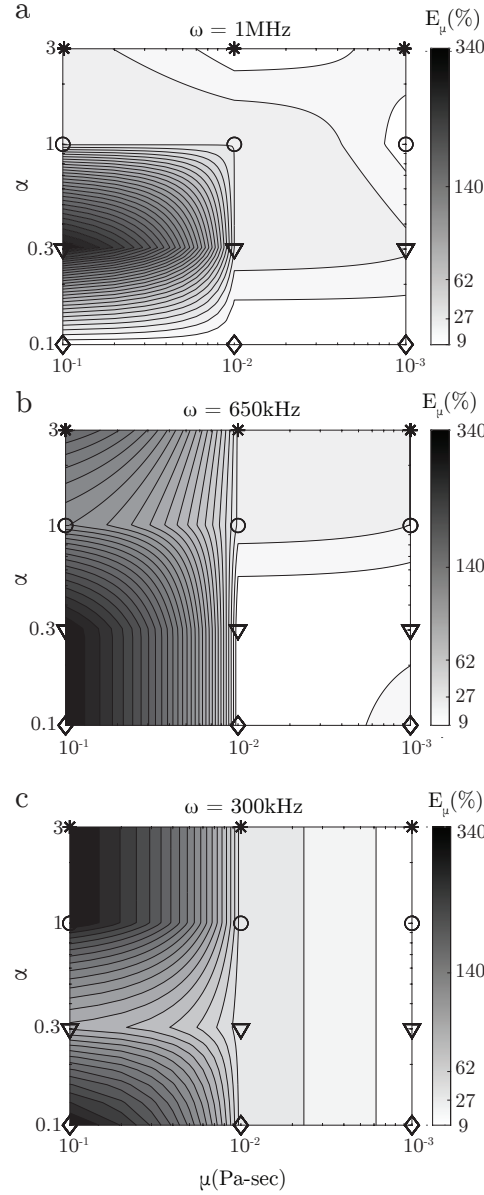


Figure C.3: Contour plots showing the error in the fitting of the viscosity μ in the case of LIC for three different sampling frequencies: (a) 1 MHz, (b) 650 kHz, and (c) 300 kHz. In these plots, we can see the error in the best-fit μ_{fit} for different combinations of α and μ .

sampling frequency in Figs. C.6(b) and C.6(c), the error does not increase monotonically as in the previous cases tested for UIC and LIC. In other words, some regions with error greater than 30% for the highest sampling frequency (1 MHz) correspond to an error less than 30% for the lower sampling frequencies. Similar trends can be observed in Figs. C.7(a), C.7(b), and C.7(c) for the error in μ , where the error does not increase monotonically as

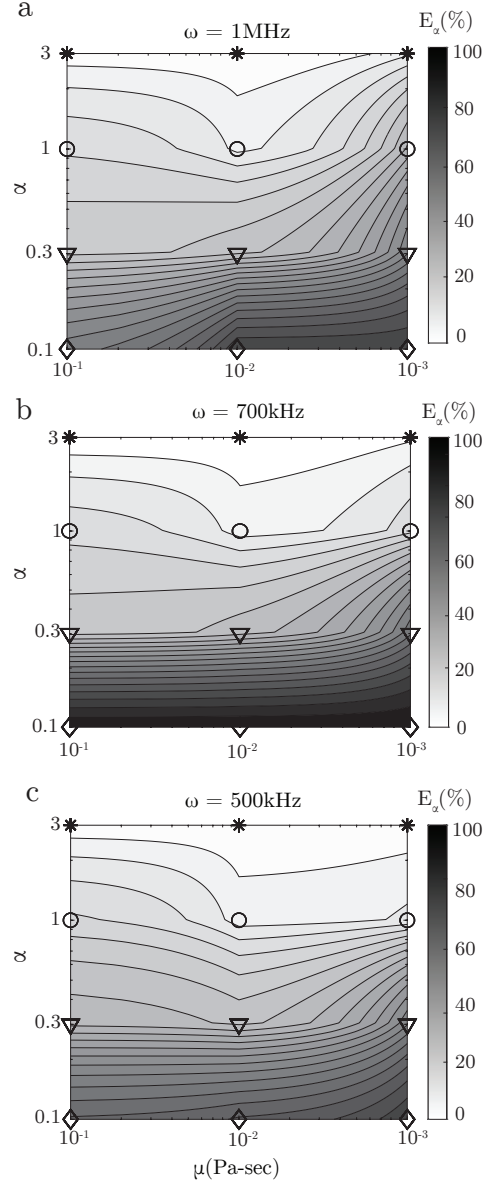


Figure C.4: Contour plots showing the error in the fitting of the parameter α in the case of UIC (Region I) for three different sampling frequencies: (a) 1 MHz, (b) 700 kHz, and (c) 500 kHz. In these plots, we can see the error in the best-fit α_{fit} for different combinations of α and μ .

well.

The fact that the error maps corresponding to *Fitting Region II* in UIC do not follow a monotonic increase in the error as we decrease the sampling frequency arises due to compressibility effects and the role of compressibility in the different fitting regions. As discussed in Section 2.4, compressibility of the surrounding material begins to matter during

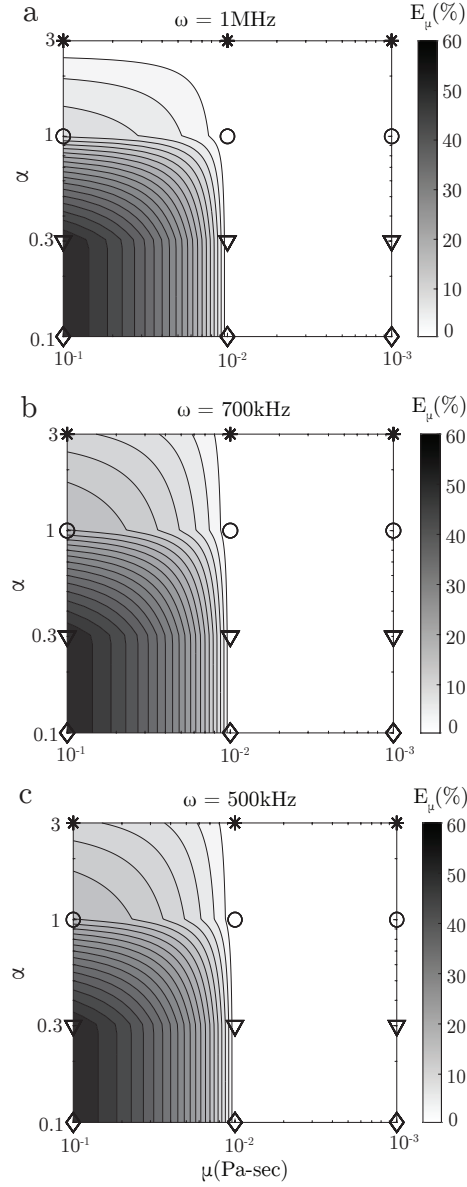


Figure C.5: Contour plots showing the error in the fitting of the viscosity μ in the case of UIC (Region I) for three different sampling frequencies: (a) 1 MHz, (b) 700 kHz, and (c) 500kHz. In these plots, we can see the error in the best-fit μ_{fit} for different combinations of α and μ .

bubble violent collapse. In UIC we have combinations of α and μ values that correspond to an R_{max} that is five times R_0 . Violent bubble collapse introduces a difference between the FE and the KM data because there is a small offset between the curves (recall Fig. 2.4). As a result, this difference due to bubble violent collapse is also present in the error maps here. Another contributor to the difference is the different parts of the curves we choose

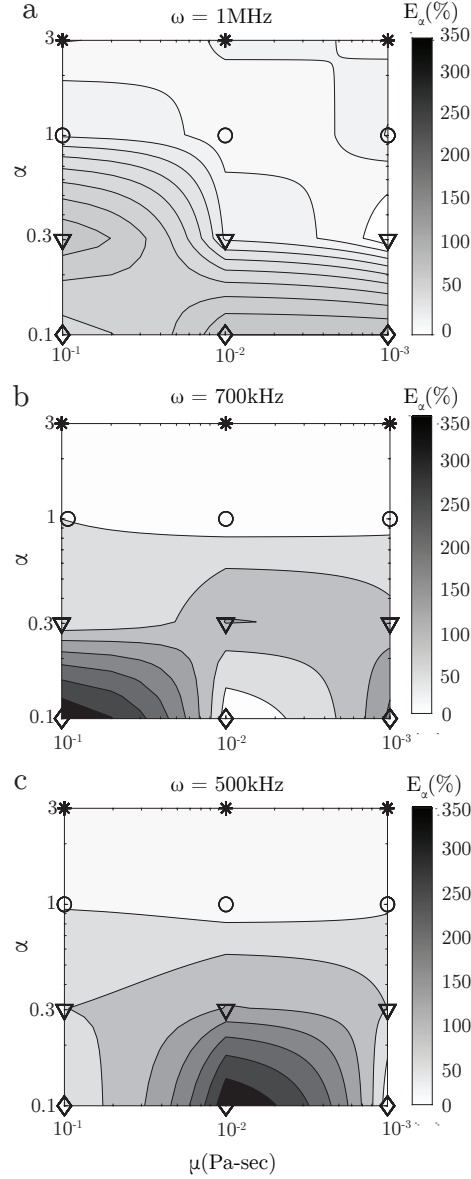


Figure C.6: Contour plots showing the error in the fitting of the parameter α in the case of UIC (Region II) for three different sampling frequencies: (a) 1 MHz, (b) 700 kHz, and (c) 500 kHz. In these plots, we can see the error in the best-fit α_{fit} for different combinations of α and μ .

to fit. We generate downsampled FE data using only the first collapse from the full UIC curve. Then we perform fittings between these FE data and data from KM that correspond to the the first collapse from LIC. However, we already know that during UIC we simulate the growth phase while in the LIC we assume the initial conditions at R_{max} , and as a result, the first collapse during UIC for the same parameters looks slightly different from the first

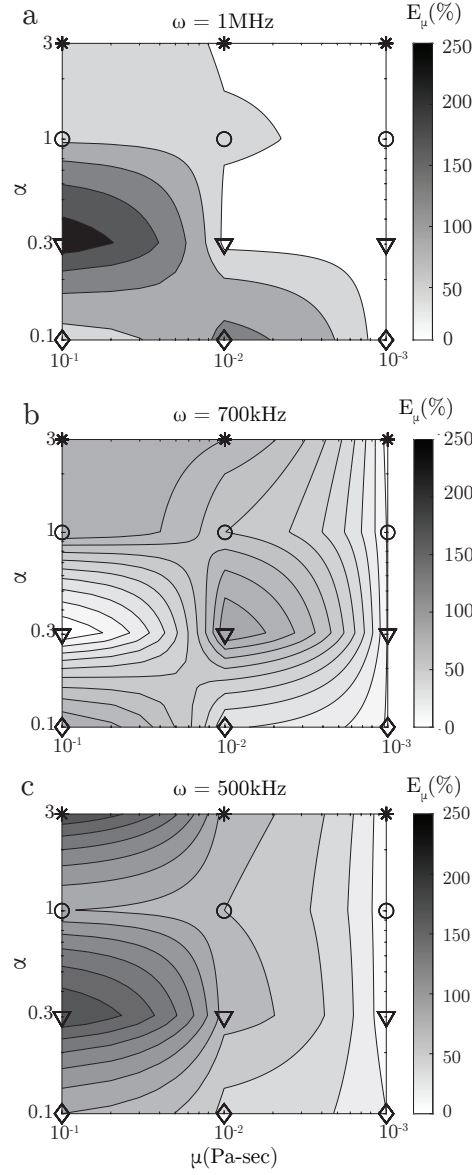


Figure C.7: Contour plots showing the error in the fitting of the viscosity μ in the case of UIC (Region II) for three different sampling frequencies: (a) 1 MHz, (b) 700 kHz, and (c) 500 kHz. In these plots, we can see the error in the best-fit μ_{fit} for different combinations of α and μ .

collapse during LIC due to the different manners in which simulations of LIC and UIC are initialized.

Bibliography

- [1] M. K. Nyein, A. M. Jason, L. Yu, C. M. Pita, J. D. Joannopoulos, D. F. Moore, and R. A. Radovitzky. In silico investigation of intracranial blast mitigation with relevance to military traumatic brain injury. *Proceedings of the National Academy of Sciences*, 107(48):20703–20708, 2010.
- [2] R. M. Wright, A. Post, B. Hoshizaki, and K. T. Ramesh. A multiscale computational approach to estimating axonal damage under inertial loading of the head. *Journal of Neurotrauma*, 30(2):102–118, 2013.
- [3] R. W. Carlsen, A. L. Fawzi, Y. Wan, H. Kesari, and C. Franck. A quantitative relationship between rotational head kinematics and brain tissue strain from a 2-D parametric finite element analysis. *Brain Multiphysics*, 2:100024, 2021.
- [4] J. B. Estrada, H. C. Cramer III, M. T. Scimone, S. Buyukozturk, and C. Franck. Neural cell injury pathology due to high-rate mechanical loading. *Brain Multiphysics*, 2:100034, 2021.
- [5] E. Vlaisavljevich, K. W. Lin, M. Warnez, R. Singh, L. Mancina, A. Putnam, E. Johnsen, C. Cain, and Z. Xu. Effects of tissue stiffness, ultrasound frequency, and pressure on histotripsy-induced cavitation bubble behavior. *Physics in Medicine and Biology*, 60: 2271–2292, 2015.
- [6] L. Mancina, E. Vlaisavljevich, Z. Xu, and E. Johnsen. Predicting tissue susceptibility

- to mechanical cavitation damage in therapeutic ultrasound. *Ultrasound in Medicine and Biology*, 43(7):1421–1440, 2017.
- [7] L. Mancia, E. Vlaisavljevich, N. Yousefi, M. Rodriguez, T. J. Ziemlewicz, F. T. Lee, D. L. Henann, C. Franck, Z. Xu, and E. Johnsen. Modeling tissue-selective cavitation damage. *Physics in Medicine & Biology*, 64(22):225001, 2019.
- [8] E.-A. Brujan and A. Vogel. Stress wave emission and cavitation bubble dynamics by nanosecond optical breakdown in a tissue phantom. *Journal of Fluid Mechanics*, 558: 281–308, 2006.
- [9] M. S. Hutson and X. Ma. Plasma and cavitation dynamics during pulsed laser microsurgery in vivo. *Physical Review Letters*, 99(15):158104, 2007.
- [10] S. Marra, K. Ramesh, and A. Douglas. Mechanical characterization of active poly(vinyl alcohol)–poly(acrylic acid) gel. *Materials Science and Engineering: C*, 14(1):25 – 34, 2001.
- [11] B. Johnson, D. Beebe, and W. Crone. Effects of swelling on the mechanical properties of a ph-sensitive hydrogel for use in microfluidic devices. *Materials Science and Engineering: C*, 24(4):575 – 581, 2004.
- [12] X. Zhao, N. Huebsch, D. J. Mooney, and Z. Suo. Stress-relaxation behavior in gels with ionic and covalent crosslinks. *Journal of Applied Physics*, 107(6):063509, 2010.
- [13] Y. Hu, J.-O. You, D. T. Auguste, Z. Suo, and J. J. Vlassak. Indentation: A simple, nondestructive method for characterizing the mechanical and transport properties of ph-sensitive hydrogels. *Journal of Materials Research*, 27(1):152–160, 2012.
- [14] J. A. Zimmerlin, N. Sanabria-DeLong, G. N. Tew, and A. J. Crosby. Cavitation rheology for soft materials. *Soft Matter*, 3(6):763–767, 2007.

- [15] S. Kundu and A. J. Crosby. Cavitation and fracture behavior of polyacrylamide hydrogels. *Soft Matter*, 5(20):3963–3968, 2009.
- [16] M. P. Milner and S. B. Hutchens. A device to fracture soft solids at high speeds. *Extreme Mechanics Letters*, 28:69–75, 2019.
- [17] S. Raayai-Ardakani, Z. Chen, D. R. Earl, and T. Cohen. Volume-controlled cavity expansion for probing of local elastic properties in soft materials. *Soft matter*, 15(3):381–392, 2019.
- [18] S. Raayai-Ardakani and T. Cohen. Capturing strain stiffening using volume controlled cavity expansion. *Extreme Mechanics Letters*, 31:100536, 2019.
- [19] S. Chockalingam, C. Roth, T. Henzel, and T. Cohen. Probing local nonlinear viscoelastic properties in soft materials. *Journal of the Mechanics and Physics of Solids*, 146:104172, 2021.
- [20] C. E. Brennen. Cavitation and bubble dynamics, 1995.
- [21] M. Plesset. The dynamics of cavitation bubbles. *Journal of Applied Mechanics*, 16:277–282, 1949.
- [22] H. G. Flynn. Cavitation dynamics. I. A mathematical formulation. *The Journal of the Acoustical Society of America*, 57(6):1379–1396, 1975.
- [23] J. Keller and M. Miksis. Bubble oscillations of large amplitude. *The Journal of the Acoustical Society of America*, 68(2):628–633, 1980.
- [24] R. Nigmatulin, N. Khabeev, and F. Nagiev. Dynamics, heat and mass transfer of vapour-gas bubbles in a liquid. *International Journal of Heat and Mass Transfer*, 24(6):1033–1044, 1981.
- [25] A. Prosperetti and A. Lezzi. Bubble dynamics in a compressible liquid. Part 1. First-order theory. *Journal of Fluid Mechanics*, 168:457–478, 1986.

- [26] A. Prosperetti, L. A. Crum, and K. W. Commander. Nonlinear bubble dynamics. *The Journal of the Acoustical Society of America*, 83(2):502–514, 1988.
- [27] I. Akhatov, O. Lindau, A. Topolnikov, R. Mettin, N. Vakhitova, and W. Lauterborn. Collapse and rebound of a laser-induced cavitation bubble. *Physics of Fluids*, 13(10):2805, 2001.
- [28] J. B. Estrada, C. Barajas, D. L. Henann, E. Johnsen, and C. Franck. High strain-rate soft material characterization via inertial cavitation. *Journal of the Mechanics and Physics of Solids*, 112:291–317, 2018.
- [29] C. W. Barney, C. E. Dougan, K. R. McLeod, A. Kazemi-Moridani, Y. Zheng, Z. Ye, S. Tiwari, I. Sacligil, R. A. Riggelman, S. Cai, J.-H. Lee, S. R. Peyton, G. N. Tew, and A. J. Crosby. Cavitation in soft matter. *Proceedings of the National Academy of Sciences*, 117(17):9157–9165, 2020.
- [30] M. P. Brenner, D. Lohse, and T. F. Dupont. Bubble shape oscillations and the onset of sonoluminescence. *Physical Review Letters*, 75(5):954, 1995.
- [31] F. Hamaguchi and K. Ando. Linear oscillation of gas bubbles in a viscoelastic material under ultrasound irradiation. *Physics of Fluids*, 27:113103, 2015.
- [32] S. J. Shaw. Nonspherical sub-millimeter gas bubble oscillations: Parametric forcing and nonlinear shape mode coupling. *Physics of Fluids*, 29:122103, 2017.
- [33] M. Guédra and C. Inserra. Bubble shape oscillations of finite amplitude. *Journal of Fluid Mechanics*, 857:681–703, 2018.
- [34] B. Saint-Michel and V. Garbin. Acoustic bubble dynamics in a yield-stress fluid. *Soft Matter*, 16(46):10405–10418, 2020.
- [35] E. Vlasisavljevich, K. W. Lin, A. Maxwell, M. Warnez, L. Mancina, R. Singh, A. Putnam, J. B. Fowlkes, E. Johnsen, C. Cain, and Z. Xu. Effects of ultrasound frequency

- and tissue stiffness on the histotripsy intrinsic threshold for cavitation. *Ultrasound in Medicine and Biology*, 41:1651–1667, 2015.
- [36] E. Vlaisavljevich, A. Maxwell, L. Mancia, E. Johnsen, C. Cain, and Z. Xu. Visualizing the histotripsy process: Bubble cloud–cancer cell interactions in a tissue-mimicking environment. *Ultrasound in Medicine and Biology*, 42(10):2466–2477, 2016.
- [37] J. Yang, A. Tzoumaka, K. Murakami, E. Johnsen, D. L. Henann, and C. Franck. Predicting complex nonspherical instability shapes of inertial cavitation bubbles in viscoelastic soft matter. *Physical Review E*, 104(4):045108, 2021.
- [38] R. M. Wright and K. T. Ramesh. An axonal strain injury criterion for traumatic brain injury. *Biomechanics and Modeling in Mechanobiology*, 11(1-2):245–260, 2012.
- [39] R. W. Carlsen and N. P. Daphalapurkar. The importance of structural anisotropy in computational models of traumatic brain injury. *Frontiers in Neurology*, 6:28, 2015.
- [40] D. Richler and D. Rittel. On the testing of the dynamic mechanical properties of soft gelatins. *Experimental Mechanics*, 54(5):805–815, 2014.
- [41] Y. Toyoda and Y. Gupta. Shockless and shock wave compression of ballistic gel to 1.3 GPa. *Journal of Applied Physics*, 116(15):153512, 2014.
- [42] D. Veysset, Y. Sun, J. Lem, S. Kooi, A. Maznev, S. Cole, R. Mrozek, J. Lenhart, and K. Nelson. High-strain-rate behavior of a viscoelastic gel under high-velocity microparticle impact. *Experimental Mechanics*, 60(9):1179–1186, 2020.
- [43] X. G. Tan, Y. Chen, and T. J. O’Shaughnessy. Material characterization and simulation for soft gels subjected to impulsive loading. *Journal of the Mechanical Behavior of Biomedical Materials*, 133:105293, 2022.
- [44] J. Yang, H. C. Cramer, and C. Franck. Extracting non-linear viscoelastic material

- properties from violently-collapsing cavitation bubbles. *Extreme Mechanics Letters*, 39:100839, 2020.
- [45] L. Mancia, J. Yang, J.-S. Spratt, J. R. Sukovich, Z. Xu, T. Colonius, C. Franck, and E. Johnsen. Acoustic cavitation rheometry. *Soft Matter*, 17:2931–2941, 2021.
- [46] J.-S. Spratt, M. Rodriguez, K. Schmidmayer, S. H. Bryngelson, J. Yang, C. Franck, and T. Colonius. Characterizing viscoelastic materials via ensemble-based data assimilation of bubble collapse observations. *Journal of the Mechanics and Physics of Solids*, 152:104455, 2021.
- [47] J. Yang, H. C. Cramer III, E. C. Bremer, S. Buyukozturk, Y. Yin, and C. Franck. Mechanical characterization of agarose hydrogels and their inherent dynamic instabilities at ballistic to ultra-high strain-rates via inertial microcavitation. *Extreme Mechanics Letters*, 51:101572, 2022.
- [48] S. Buyukozturk, J.-S. Spratt, D. Henann, T. Colonius, and C. Franck. Particle-assisted laser-induced inertial cavitation for high strain-rate soft material characterization. *Experimental Mechanics*, 62:1–14, 2022.
- [49] C. T. Wilson, T. L. Hall, E. Johnsen, L. Mancia, M. Rodriguez, J. E. Lundt, T. Colonius, D. L. Henann, C. Franck, and Z. Xu. Comparative study of the dynamics of laser and acoustically generated bubbles in viscoelastic media. *Physical Review E*, 99(4):043103, 2019.
- [50] L. Mancia, M. Rodriguez, J. Sukovich, Z. Xu, and E. Johnsen. Single-bubble dynamics in histotripsy and high-amplitude ultrasound: Modeling and validation. *Physics in Medicine & Biology*, 65(22):225014, 2020.
- [51] R. Gaudron, M. T. Warnez, and E. Johnsen. Bubble dynamics in a viscoelastic medium with nonlinear elasticity. *Journal of Fluid Mechanics*, 766:54–75, 2015.

- [52] M. Warnez and E. Johnsen. Numerical modeling of bubble dynamics in viscoelastic media with relaxation. *Physics of Fluids*, 27:063103, 2015.
- [53] J. C. Luo, H. Ching, B. G. Wilson, A. Mohraz, E. L. Botvinick, and V. Venugopalan. Laser cavitation rheology for measurement of elastic moduli and failure strain within hydrogels. *Scientific reports*, 10(1):1–13, 2020.
- [54] S. Tiwari, A. Kazemi-Moridani, Y. Zheng, C. W. Barney, K. R. McLeod, C. E. Dougan, A. J. Crosby, G. N. Tew, S. R. Peyton, S. Cai, and J.-H. Lee. Seeded laser-induced cavitation for studying high-strain-rate irreversible deformation of soft materials. *Soft Matter*, 16(39):9006–9013, 2020.
- [55] S. Reese and S. Govindjee. A theory of finite viscoelasticity and numerical aspects. *International Journal of Solids and Structures*, 35(26-27):3455–3482, 1998.
- [56] A. Kumar and O. Lopez-Pamies. On the two-potential constitutive modeling of rubber viscoelastic materials. *Comptes Rendus Mecanique*, 344(2):102–112, 2016.
- [57] J. K. Knowles. The finite anti-plane shear field near the tip of a crack for a class of incompressible elastic solids. *International Journal of Fracture*, 13(5):611–639, 1977.
- [58] L. Anand, N. M. Ames, V. Srivastava, and S. A. Chester. A thermo-mechanically coupled theory for large deformations of amorphous polymers. Part I: Formulation. *International Journal of Plasticity*, 25:1474–1494, 2009.
- [59] C. Barajas and E. Johnsen. The effects of heat and mass diffusion on freely oscillating bubbles in a viscoelastic, tissue-like medium. *The Journal of the Acoustical Society of America*, 141(2):908–918, 2017.
- [60] J. Lysmer and R. L. Kuhlemeyer. Finite dynamic model for infinite media. *Journal of the Engineering Mechanics Division*, 95(4):859–877, 1969.

- [61] M. Faul, L. Xu, M. M. Wald, V. Coronado, and A. M. Dellinger. Traumatic brain injury in the United States: national estimates of prevalence and incidence, 2002–2006. *Injury Prevention*, 16:A268–A268, 2010.
- [62] E. M. Martin, W. C. Lu, K. Helmick, L. French, and D. L. Warden. Traumatic brain injuries sustained in the Afghanistan and Iraq wars. *The American Journal of Nursing*, 108(4):40–47, 2008.
- [63] S. Chatelin, A. Constantinesco, and R. Willinger. Fifty years of brain tissue mechanical testing: from in vitro to in vivo investigations. *Biorheology*, 47(5-6):255–276, 2010.
- [64] K. B. Arbogast and S. S. Margulies. Material characterization of the brainstem from oscillatory shear tests. *Journal of Biomechanics*, 31(9):801–807, 1998.
- [65] M. Hrapko, J. A. W. van Dommelen, G. W. M. Peters, and J. S. H. M. Wismans. The mechanical behaviour of brain tissue: large strain response and constitutive modelling. *Biorheology*, 43(5):623–636, 2006.
- [66] F. Shen, T. E. Tay, J. Z. Li, S. Nigen, P. V. S. Lee, and H. K. Chan. Modified Bilston nonlinear viscoelastic model for finite element head injury studies. *Journal of Biomechanical Engineering*, 128(5):797–801, 2006.
- [67] B. Rashid, M. Destrade, and M. D. Gilchrist. Mechanical characterization of brain tissue in tension at dynamic strain rates. *Journal of the Mechanical Behavior of Biomedical Materials*, 33:43–54, 2014.
- [68] C. A. Guertler, R. J. Okamoto, J. L. Schmidt, A. A. Badachhape, C. L. Johnson, and P. V. Bayly. Mechanical properties of porcine brain tissue *in vivo* and *ex vivo* estimated by MR elastography. *Journal of Biomechanics*, 69:10–18, 2018.

- [69] F. Pervin and W. W. Chen. Dynamic mechanical response of bovine gray matter and white matter brain tissues under compression. *Journal of Biomechanics*, 42(6): 731–735, 2009.
- [70] E.-A. Brujan, K. Nahen, P. Schmidt, and A. Vogel. Dynamics of laser-induced cavitation bubbles near an elastic boundary. *Journal of Fluid Mechanics*, 433:251–281, 2001.
- [71] E.-A. Brujan, K. Nahen, P. Schmidt, and A. Vogel. Dynamics of laser-induced cavitation bubbles near elastic boundaries: influence of the elastic modulus. *Journal of Fluid Mechanics*, 433:283–314, 2001.
- [72] M. Rodriguez Jr. *Numerical Simulations of Bubble Dynamics Near Viscoelastic Media*. PhD thesis, University of Michigan, 2018.
- [73] R. Mettin, I. Akhatov, U. Parlitz, C. Ohl, and W. Lauterborn. Bjerknes forces between small cavitation bubbles in a strong acoustic field. *Physical review E*, 56(3):2924, 1997.
- [74] N. A. Pelekasis, A. Gaki, A. Doinikov, and J. A. Tsamopoulos. Secondary bjerknes forces between two bubbles and the phenomenon of acoustic streamers. *Journal of Fluid Mechanics*, 500:313–347, 2004.
- [75] G. Taubin. Estimation of planar curves, surfaces, and nonplanar space curves defined by implicit equations with applications to edge and range image segmentation. *IEEE Transactions on Pattern Analysis & Machine Intelligence*, 13(11):1115–1138, 1991.
- [76] Abaqus. Reference manuals. Dassault Systèmes Simulia Corp., 2018.
- [77] K. Murakami. *Spherical and Non-spherical Bubble Dynamics in Soft Matter*. PhD thesis, University of Michigan, 2020.

- [78] D. L. Henann and K. Bertoldi. Modeling of elasto-capillary phenomena. *Soft Matter*, 10(5):709–717, 2014.
- [79] H. Chen, Z. Lai, Z. Chen, and Y. Li. The secondary Bjerknes force between two oscillating bubbles in Kelvin-Voigt-type viscoelastic fluids driven by harmonic ultrasonic pressure. *Ultrasonics Sonochemistry*, 52:344–352, 2019.
- [80] H. Chen, Z. Chen, and Y. Li. Modulation of the secondary Bjerknes force in multi-bubble systems. *Ultrasonics Sonochemistry*, 61:104814, 2020.
- [81] G. I. Taylor. The instability of liquid surfaces when accelerated in a direction perpendicular to their planes. I. *Proceedings of the Royal Society of London. Series A. Mathematical and Physical Sciences*, 201(1065):192–196, 1950.
- [82] K. Murakami, R. Gaudron, and E. Johnsen. Shape stability of a gas bubble in a soft solid. *Ultrasonics Sonochemistry*, 67:105170, 2020.
- [83] R. Gaudron, K. Murakami, and E. Johnsen. Shape stability of a gas cavity surrounded by linear and nonlinear elastic media. *Journal of the Mechanics and Physics of Solids*, 143:104047, 2020.
- [84] S. Cai, K. Bertoldi, H. Wang, and Z. Suo. Osmotic collapse of a void in an elastomer: breathing, buckling and creasing. *Soft Matter*, 6(22):5770–5777, 2010.
- [85] M. Diab, T. Zhang, R. Zhao, H. Gao, and K.-S. Kim. Ruga mechanics of creasing: from instantaneous to setback creases. *Proceedings of the Royal Society A: Mathematical, Physical and Engineering Sciences*, 469(2157):20120753, 2013.
- [86] S. B. Hutchens, S. Fakhouri, and A. J. Crosby. Elastic cavitation and fracture via injection. *Soft matter*, 12(9):2557–2566, 2016.
- [87] P. Movahed, W. Kreider, A. D. Maxwell, S. B. Hutchens, and J. B. Freund. Cavitation-induced damage of soft materials by focused ultrasound bursts: A fracture-based

- bubble dynamics model. *The Journal of the Acoustical Society of America*, 140(2): 1374–1386, 2016.
- [88] S. Raayai-Ardakani, D. R. Earl, and T. Cohen. The intimate relationship between cavitation and fracture. *Soft matter*, 15(25):4999–5005, 2019.
- [89] M. Plesset. On the stability of fluid flows with spherical symmetry. *Journal of Applied Physics*, 25(1):96–98, 1954.
- [90] Y. Wang and D. L. Henann. Finite-element modeling of soft solids with liquid inclusions. *Extreme Mechanics Letters*, 9:147–157, 2016.
- [91] N. Triantafyllidis and R. Abeyaratne. Instabilities of a finitely deformed fiber-reinforced elastic material. *Journal of Applied Mechanics*, 50(1):149–156, 1983.
- [92] G. Y. Qiu and T. J. Pence. Remarks on the behavior of simple directionally reinforced incompressible nonlinearly elastic solids. *Journal of Elasticity*, 49(1):1–30, 1997.
- [93] J. Merodio and R. W. Ogden. Instabilities and loss of ellipticity in fiber-reinforced compressible non-linearly elastic solids under plane deformation. *International Journal of Solids and Structures*, 40(18):4707–4727, 2003.
- [94] X. Ning, Q. Zhu, Y. Lanir, and S. S. Margulies. A transversely isotropic viscoelastic constitutive equation for brainstem undergoing finite deformation. *Journal of Biomechanical Engineering*, 128(6):925–933, 2006.
- [95] G. A. Holzapfel, T. C. Gasser, and R. W. Ogden. A new constitutive framework for arterial wall mechanics and a comparative study of material models. *Journal of Elasticity*, 61(1):1–48, 2000.
- [96] T. C. Gasser, R. W. Ogden, and G. A. Holzapfel. Hyperelastic modelling of arterial layers with distributed collagen fibre orientations. *Journal of the Royal Society Interface*, 3(6):15–35, 2006.

- [97] C. Giordano, R. J. H. Cloots, J. A. W. van Dommelen, and S. Kleiven. The influence of anisotropy on brain injury prediction. *Journal of Biomechanics*, 47(5):1052–1059, 2014.
- [98] S. Buyukozturk. *Experimental methods for high strain-rate soft material characterization*. PhD thesis, Brown University, 2021.
- [99] V. Kamath, A. Prosperetti, and F. Egolfopoulos. A theoretical study of sonoluminescence. *The Journal of the Acoustical Society of America*, 94(1):248–260, 1993.
- [100] F. R. Gilmore. The growth or collapse of a spherical bubble in a viscous compressible liquid. Technical Report 26-4, California Institute of Technology Hydrodynamics Laboratory, 1952.
- [101] N. Bremond, M. Arora, S. M. Dammer, and D. Lohse. Interaction of cavitation bubbles on a wall. *Physics of Fluids*, 18:121505, 2006.