FI-Calculus and Representation Stability

by

Kaya Ferendo

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Date ______________ ______________________________
Tom Goodwillie, Advisor

Recommended to the Graduate Council

Date ______________ ______________________________
Dan Abramovich, Reader

Date ______________ ______________________________
Michael Ching, Reader

Approved by the Graduate Council

Date ______________ ______________________________
Thomas A. Lewis
Dean of the Graduate School

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Contents

CHAPTER 1

Introduction

1.1 Outline

We describe a flavor of functor calculus for functors $\mathsf{FI} \to \mathcal{V}$, which we call FI-objects, where FI is the category of finite sets and injections and $\mathcal{V}$ is a stable presentable ∞-category. At the outset, our study into this FI-calculus was inspired by an analogy to Weiss' orthogonal calculus, but we soon realized that FI-calculus is in fact a homotopical extension of the ideas of representation stability.

In Chapter 2 we define, for $n \in \mathbb{N}$, n-excisive FI-objects as those sending certain $n+1$-cubes to limit cubes, prove that our definition of an n-excisive FI-object is equivalent to the criterion that the FI-object be "presented in degree at most n," (a characterization analogous to one for representation stable FI-modules) and note that every FI-object admits a universal approximation by an n-excisive FI-object, giving rise to a Taylor tower. We introduce the ∞-category of "formal Taylor towers," which we call "excision sequences," define an analytic FI-object as one which is an iterated limit of excisive FI-objects, define a category of "convergent excision sequences," and show that the ∞-categories of analytic FI-objects and of convergent excision sequences are equivalent:

Theorem 17. *The Taylor tower functor* $\mathbf{P}$ *determines an equivalence*

$$\mathbf{P} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}} : \lim$$

In Chapter 3, we define an n-homogeneous FI-object as an n-excisive FI-object whose universal $n-1$-excisive approximation vanishes. We show, in analogy to other functor calculi, that the ∞-category of n-homogeneous FI-objects is equivalent to the ∞-category of $\mathfrak{S}_n$-objects. For an FI-object E, we call the $\mathfrak{S}_n$-object corresponding to the n-homogeneous layers of the Taylor tower of E the nth Taylor coefficient of E. A priori, the Taylor coefficients of E together form a symmetric sequence; we show that this symmetric sequence extends to an FI-object. We define an operation $\boldsymbol{\Delta}$ on FI-objects which behaves like a derivative and use it to show that the Taylor tower of a finitely supported FI-object is trivial, a result we summarize with the slogan, "an analytic FI-object is determined by its germ at infinity."

We then provide conditions under which an excision sequence (and hence in particular an analytic FI-object) can be recovered from its FI-object of Taylor coefficients; these results generalize a theorem of Sam and Snowden in [SS15] which has been generalized in a different fashion by Patzt and Wiltshire-Gordon in [PW19].

More specifically, we define "tame" excision sequences – more general than finitely presented FI-objects – and show that the aggregate Taylor coefficient functor restricts to an equivalence on tame excision sequences:

Corollary 46. *We have an equivalence*

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{coTame}}$$

We formulate a yet weaker condition we call "self-tameness" which still ensures that an excision sequence can be recovered from its Taylor coefficients.

Corollary 52. *We have an equivalence*

$$\mathrm{core}\,\mathbf{C} : \mathrm{core}\,\mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}} \simeq \mathrm{core}\,\mathsf{FI}\mathcal{V}^{\mathrm{selfcoTame}}$$

We observe that for many choices of $\mathcal{V}$ of interest – for example any $\mathbb{Q}$-linear ∞-category, and in particular $\mathcal{S}p^{\mathbb{Q}}$, the ∞-category of rational chain complexes – *all* excision sequences are tame, so that in such contexts an excision sequence can always be recovered from its Taylor coefficients.

In Chapter 4 we concern ourselves with the case when $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$ and seek to show that representation stability for FI-modules is an emanation of FI-calculus. We have the following theorem:

Theorem 54. *For some $n \in \mathbb{N}$, let E be an n-excisive FI-object taking values in rational spectra with finitely generated homology groups. Then the FI-modules $H_i(E)$ are representation stable.*

Combining this result with Corollary 46, we calculate an explicit dictionary allowing us to read off the representations appearing in the stable part of a representation-stable FI-module from its coefficient FI-module by proceeding one homogeneous layer at a time:

Corollary 57. *For E an n-homogeneous rational FI-object with $\mathbf{C}_n E \cong V(\mu)$ for $\mu \vdash n$, $E|_{\mathsf{FI}_{\geq 2n}} \cong V(\mu)_\bullet|_{\mathsf{FI}_{\geq 2n}}$.*

Our results therefore suggest that a larger family of rational FI-objects – the analytic ones – deserve consideration under the mantle of representation stability even when their homology FI-modules fail to be representation stable, since their behavior is nonetheless controlled by the same functor calculus phenomena and is still recorded by their Taylor coefficient FI-objects.

Additionally, because n-excisive FI-objects which are eventually concentrated in a particular homological dimension may not be concentrated in that dimension on sufficiently small sets, we observe that FI-calculus illuminates the existence of "good" pre-stable behavior involving the interaction of homology groups in different dimensions in the pre-stable range. This suggests searching for such good behavior in the pre-stable ranges of real-world FI-objects of interest as a way of extending downwards the lower bounds on their "good" behavior.

1.2 Review of presentable stable ∞-categories

An ∞-category $\mathcal{J}$ is *finite* if its classifying space $B\mathcal{J}$ is equivalent to a finite CW-complex. A stable ∞-category $\mathcal{V}$ is one which is both complete and cocomplete and such that for $\mathcal{I}, \mathcal{J}$ finite ∞-categories and for every functor

$$F : \mathcal{I} \times \mathcal{J} \to \mathcal{V}$$

the canonical morphism

$$\operatorname*{colim}_{j \in \mathcal{J}} \lim_{i \in \mathcal{I}} F(i,j) \to \lim_{i \in \mathcal{I}} \operatorname*{colim}_{j \in \mathcal{J}} F(i,j)$$

is invertible. This characterization, stated in the framework of derivators, is due to Moritz Rahn and Michael Shulman and proven in [RS21b]. We similarly have that for every

$$G : \mathcal{I} \to \mathcal{V}$$

and for every $X \in \mathcal{V}$, the canonical morphisms

$$\operatorname*{colim}_{i \in \mathcal{I}} \mathcal{V}(X, G(i)) \to \mathcal{V}\Big(X, \operatorname*{colim}_{i \in \mathcal{I}} G(i)\Big)$$
$$\mathcal{V}\Big(\lim_{i \in \mathcal{I}} G(i), X\Big) \to \lim_{i \in \mathcal{I}} \mathcal{V}(G(i), X)$$

are invertible.

The traditional definition, found, for example, in [Lur17], is that an ∞-category $\mathcal{V}$ is stable if it is complete, cocomplete, the canonical morphism from the initial object to the terminal object is invertible (i.e. $\mathcal{V}$ is pointed), and fiber and cofiber squares coincide, i.e. that a square diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d] & B \arrow[d] \\ 0 \arrow[r] & C \end{tikzcd}$$

is a pullback if and only if it is a pushout. Note that a morphism in a stable ∞-category is an isomorphism if and only if either its fiber or its cofiber is contractible (or, equivalently, both its fiber and its cofiber are contractible).

Note that given two objects $X, Y \in \mathcal{V}$ in a stable ∞-category, the morphism object $\mathcal{V}(X, Y)$ carries the structure of a spectrum. For further details, see [Lur17, Chapter 1].

The condition that $\mathcal{V}$ be presentable plays no explicit role in the paper except to ensure that $\mathcal{V}$ admit the necessary limits and colimits and to allow us to use the adjoint functor theorem for presentable ∞-categories. It is satisfied in all examples of interest.

We call a sub-∞-category of an arbitrary ∞-category *reflective* if the inclusion functor admits a left adjoint, which we call the *reflection* functor. The dual notion is called a *coreflective* sub-∞-category. Given $\mathcal{V}' \subseteq \mathcal{V}$ a reflective sub-∞-category of a presentable ∞-category $\mathcal{V}$, the collection of objects $X \in \mathcal{V}$ such that for all $Y \in \mathcal{V}'$

$$\mathcal{V}(X, Y) \cong 0$$

is called the *left orthogonal complement* of $\mathcal{V}'$ and is a coreflective sub-∞-category of $\mathcal{V}$. The dual notion is called the *right orthogonal complement* of a reflective sub-∞-category. If $\mathcal{V}$ is stable, left and right orthogonal complement are, up to equivalence, inverse operations.

1.3 Notation

Throughout, $\mathcal{S}$ refers to the ∞-category of spaces, $\mathcal{S}p$ to that of spectra, and $\mathbb{S}$ to the sphere spectrum. Fix $\mathcal{V}$ a stable presentable ∞-category. Given $X \in \mathcal{V}$ and an ∞-groupoid $Y \in \mathcal{S}$, denote by $Y \otimes X$ the colimit of the functor $Y \to \mathcal{V}$ that is constantly X. The functor

$$- \otimes X : \mathcal{S} \to \mathcal{V}$$

is a left adjoint so it extends canonically along Σ^∞_+ to a left adjoint

$$- \otimes X : \mathcal{S}p \to \mathcal{V}$$

from $\mathcal{S}p$, the stabilization of $\mathcal{S}$, and we use the same notation to describe this extended functor. Right adjoint to $- \otimes X$ is the enrichment of $\mathcal{V}$ in $\mathcal{S}p$: $\mathcal{V}(X, -) : \mathcal{V} \to \mathcal{S}p$.

Dually, for $Y \in \mathcal{S}$ and $X \in \mathcal{V}$, we denote by $Y \pitchfork X$ the limit of the functor $Y \to \mathcal{V}$ that is constantly X, so that we have a functor

$$- \pitchfork X : \mathcal{S}^{\mathrm{op}} \to \mathcal{V}$$

We call a functor $\mathsf{FI} \to \mathcal{V}$ an FI-*object* of $\mathcal{V}$ and we denote the category of such functors $\mathsf{FI}\mathcal{V}$. We sometimes conflate a set and its cardinality; e.g. when we compare two sets with symbols such as $\leq$, we are really comparing their cardinalities. For Φ a property, we denote by FI_Φ the full subcategory of FI spanned by those sets satisfying Φ; $\mathsf{FI}_{\leq n}$ is a typical example. We write $\mathsf{FI}_\Phi\mathcal{V}$ for the evident functor category.

When two ∞-categories are canonically equivalent up to an insignificant level of ambiguity, we sometimes conflate them. As an example, we denote by $\mathfrak{S}_n$ both the category with sole object the set $\{1, \ldots, n\}$ and morphisms bijections and the subcategory of FI spanned by all sets with cardinality n.

For $\mathcal{C}$, $\mathcal{D}$ small ∞-categories and $\mathcal{E}$ a presentable ∞-category, when there is a canonical functor $\mathcal{C} \to \mathcal{D}$, we write

$$\mathrm{Lan}_{\mathcal{C}}^{\mathcal{D}} : Fun(\mathcal{C}, \mathcal{E}) \to Fun(\mathcal{D}, \mathcal{E})$$

for left Kan extension, leaving the functor $\mathcal{C} \to \mathcal{D}$ implicit. Similarly, we denote right Kan extension by

$$\mathrm{Ran}_{\mathcal{C}}^{\mathcal{D}} : Fun(\mathcal{C}, \mathcal{E}) \to Fun(\mathcal{D}, \mathcal{E})$$

CHAPTER 2

Excision and Taylor Towers

2.1 Excisive FI-objects

Definition 1. For $n \in \mathsf{FI}$, we define the n-cube category to be $\mathsf{FI}_{/n}$, equivalently the powerset lattice of n. We define a *standard cube* to be a diagram in FI determined by finite sets $S \subseteq S'$ in which the vertices are sets T such that $S \subseteq T \subseteq S'$ and the morphisms are the inclusions. We say that a standard cube determined by $S \subseteq S'$ is a *standard n-cube* if $S' \setminus S \cong n$. We say that an FI-object E is *n-excisive* if it sends each standard $n+1$-cube to a limit diagram (often called a Cartesian cube) in $\mathcal{V}$. We denote the ∞-category of n-excisive FI-objects in $\mathcal{V}$ with the notation $\mathrm{Exc}_n\mathcal{V}$.

Remark 2. Some readers may feel more comfortable calling n-excisive FI-objects "n-polynomial" or "polynomial of degree n". This would be sensible nomenclature. We use the term "excisive" to be in accord with the terminology of Goodwillie calculus.

Remark 3. Call an n-cube $J : \mathsf{FI}_{/n} \to \mathsf{FI}$ *semi-standard* if there exist sets S, T and a function $f : T \to n$ such that $J(x) = S \sqcup f^{-1}(x)$ for all $x \subseteq n$. We view these semi-standard cubes as homologs of strongly coCartesian cubes of Goodwillie calculus. An FI-object is n-excisive if and only if it sends all semi-standard $n+1$-cubes to Cartesian cubes. We do not use this fact, so we omit the proof in the interest of brevity.

Recollection 4. Recall that the *total fiber* of an n-cube $J : \mathsf{FI}_{/n} \to \mathcal{C}$ for $\mathcal{C}$ a pointed ∞-category is the fiber of the canonical morphism

$$J(\emptyset) \to \lim_{\emptyset \neq S \subseteq n} J(S)$$

We denote the total fiber of J by $\operatorname{tofib} J$ or $\operatorname{tofib}_{S \subseteq n} J(S)$. Recall that the dual notion is called the *total cofiber* of J. Recall also that for $J : \mathsf{FI}_{/k \sqcup m} \to \mathcal{C}$ we have an isomorphism

$$\operatorname*{tofib}_{S \subseteq k \sqcup m} J(S) \cong \operatorname*{tofib}_{T \subseteq k} \left(\operatorname*{tofib}_{T' \subseteq m} J(T \sqcup T') \right)$$

Recall that when $f : X \to Y$ is a morphism in a stable ∞-category $\mathcal{C}$, $\operatorname{cofib} f \cong \Sigma \operatorname{fib} f$, so given an n-cube $J : \mathsf{FI}_{/n} \to \mathcal{C}$, we can regard n as the disjoint union of its singleton subsets and apply the preceding result repeatedly to obtain that $\operatorname{tocofib} J \cong \Sigma^n \operatorname{tofib} J$. For more details, see [MV15, Proposition 5.5.4]. Since in a stable ∞-category a morphism is an isomorphism if and only if its fiber is contractible, we have that in a stable ∞-category, an n-cube is Cartesian if and only if it is coCartesian. We therefore could have defined n-excisive FI-objects to be those sending semi-standard (or standard) $n+1$-cubes to coCartesian cubes, and we will make use of this characterization.

Corollary 5. *For $m \geq n$, $\mathrm{Exc}_n \mathcal{V} \subseteq \mathrm{Exc}_m \mathcal{V}$.*

Proof. For $E \in \mathrm{Exc}_n \mathcal{V}$ it is enough to verify that the total fiber of the image under

E of any standard $m+1$-cube is 0. By Recollection 4, this is equivalent to the total fiber of an $m-n$-cube of total fibers of the images of standard $n+1$-cubes, each of which is 0 by assumption. □

Proposition 6. *The full sub-∞-category* $\mathrm{Exc}_n\mathcal{V}$ *of* $\mathsf{FI}\mathcal{V}$ *is reflective. We denote its reflection functor* $\mathbf{P}_n$.

Proof. Limits commute with Cartesian cubes and filtered colimits commute with coCartesian cubes, which are Cartesian cubes because $\mathcal{V}$ is stable. It follows that $\mathrm{Exc}_n\mathcal{V}$ is closed under limits and filtered colimits in $\mathsf{FI}\mathcal{V}$. The result follows from the adjoint functor theorem [Lur09, Corollary 5.5.2.9]. □

Definitions 7. We say that an FI-object is *excisive* if it is n-excisive for some $n \in \mathbb{N}$. We denote by $\mathsf{FI}\mathcal{V}^{\mathrm{Anly}}$ the reflective subcategory of $\mathsf{FI}\mathcal{V}$ generated by the excisive FI-objects and we call its objects *analytic*.

Definition 8. Given $n \in \mathsf{FI}$ and $X \in \mathcal{V}$, we call FI-objects isomorphic to those of the form $\mathsf{FI}(n,-)\otimes X$ *representable*. For brevity, we denote $F_{n,X} \stackrel{\mathrm{def}}{=} \mathsf{FI}(n,-)\otimes X$. Recall that all FI-objects are iterated colimits of representable FI-objects. When $\mathcal{V} = \mathcal{S}p$ and $X = \mathbb{S}$, we simply write F_n for $F_{n,\mathbb{S}}$.

Proposition 9. *We have that for all* $X \in \mathcal{V}$ *and* $n \in \mathsf{FI}$, $F_{n,X} \in \mathrm{Exc}_n\mathcal{V}$.

Proof. Because $-\otimes X : \mathcal{S} \to \mathcal{V}$ is a left adjoint, it suffices to show that the functors

$$\mathsf{FI}(S,-) : \mathsf{FI} \to \mathcal{S}$$

send standard $n+1$-cubes to coCartesian $n+1$-cubes.

We will use the following fact. Let $f : \mathsf{FI}_{/n} \to \mathsf{FI}_{/m}$ be a functor which preserves meets and let $g : \mathsf{FI}_{/m} \to \mathcal{S}$ be the functor sending the subsets of m to themselves understood as discrete spaces. Then gf is a coCartesian cube if

$$f(n) = \bigcup_{i \in n} f\left(n \setminus \{i\}\right)$$

For $S \subseteq T \subseteq S'$ and $S \subseteq T' \subseteq S'$,

$$\mathsf{FI}(n, T \cap T') \cong \mathsf{FI}(n, T) \cap \mathsf{FI}(n, T')$$

Each subset of S' of cardinality n is a subset of $S' \setminus \{i\}$ for some $i \in S' \setminus S$ exactly when $n < |S' \setminus S|$ – i.e. when the standard cube in question is a standard m-cube for $m > n$. $\square$

Theorem 10. *We have an equivalence of categories*

$$\mathrm{Lan}_{\mathsf{FI}_{\leq n}}^{\mathsf{FI}} : \mathsf{FI}_{\leq n}\mathcal{V} \simeq \mathrm{Exc}_n\mathcal{V} : \mathrm{Res}_{\mathsf{FI}}^{\mathsf{FI}_{\leq n}}$$

Proof. For $n, k \in \mathbb{N}$, denote by $\mathrm{Exc}_{n,\leq k}\mathcal{V}$ the full sub-∞-category of $\mathsf{FI}_{\leq k}\mathcal{V}$ spanned by functors sending all standard $n+1$-cubes in $\mathsf{FI}_{\leq k}$ to Cartesian $n+1$-cubes in $\mathcal{V}$. Because $\mathsf{FI} \cong \mathrm{colim}_{k \in \mathbb{N}} \mathsf{FI}_{\leq k}$, we have

$$\mathsf{FI}\mathcal{V} \cong \lim_{k \in \mathbb{N}^{\mathrm{op}}} \mathsf{FI}_{\leq k}\mathcal{V}$$

where the inverse limit is taken over the restriction functors $\mathrm{Res}_{\mathsf{FI}_{\leq k+1}}^{\mathsf{FI}_{\leq k}}$, and because every standard $n+1$-cube in FI lies in $\mathsf{FI}_{\leq k}$ for some $k \in \mathbb{N}$, we also have that

$$\mathrm{Exc}_n\mathcal{V} \cong \lim_{k \in \mathbb{N}^{\mathrm{op}}} \mathrm{Exc}_{n,\leq k}\mathcal{V}$$

Because $\mathsf{FI}_{\leq k} \to \mathsf{FI}_{k+1}$ is fully faithful, $\mathrm{Lan}_{\mathsf{FI}_{\leq k}}^{\mathsf{FI}_{\leq k+1}}$ is right inverse to $\mathrm{Res}_{\mathsf{FI}_{\leq k+1}}^{\mathsf{FI}_{\leq k}}$.

For the other composition, let $n \geq k$ and $E \in \mathrm{Exc}_{n,\leq k+1}\mathcal{V}$. The counit

$$\underset{S \subsetneq k+1}{\mathrm{colim}}\, E(S) \cong \mathrm{Lan}_{\mathsf{FI}_{\leq k}}^{\mathsf{FI}_{\leq k+1}} \mathrm{Res}_{\mathsf{FI}_{\leq k+1}}^{\mathsf{FI}_{\leq k}} E(k+1) \stackrel{\varepsilon_{k+1}}{\to} E(k+1) \cong \underset{S \subsetneq k+1}{\mathrm{colim}}\, E(S)$$

is an isomorphism, with the last isomorphism following from Recollection 4. $\square$

Observation 11. It follows that $\mathrm{Exc}_n\mathcal{V}$ is a coreflective sub-∞-category of $\mathsf{FI}\mathcal{V}$ (an easier way to see the result is the adjoint functor theorem). We denote the coreflection functor $\mathbf{Q}_n$. Note that

$$\mathbf{Q}_n E \cong \mathrm{Lan}_{\mathsf{FI}_{\leq n}}^{\mathsf{FI}} \mathrm{Res}_{\mathsf{FI}}^{\mathsf{FI}_{\leq n}} E$$

Note that for all $E \in \mathsf{FI}\mathcal{V}$,

$$E \cong \operatorname*{colim}_{n \in \mathbb{N}} \mathbf{Q}_n E$$

since for all $m \geq n$, $\mathbf{Q}_m E(n) \to E(n)$ is an isomorphism.

2.2 Excision sequences

Definition 12. We define the ∞-category $\mathrm{ExSeq}\mathcal{V}$ of *excision sequences* in $\mathcal{V}$ to be

$$\mathrm{ExSeq}\mathcal{V} \overset{\mathrm{def}}{=} \lim \cdots \to \mathsf{FI}\mathcal{V} \overset{\mathbf{P}_1}{\to} \mathsf{FI}\mathcal{V} \overset{\mathbf{P}_0}{\to} \mathsf{FI}\mathcal{V}$$

so that an excision sequence is a collection of FI-objects $\{E_i\}_{i \in \mathbb{N}}$ (but we will often simply denote a given excision sequence with a single capital Latin letter, such as E) equipped with, for $m \geq n$, compatible isomorphisms $\mathbf{P}_n E_m \cong E_n$. The reflection morphisms give us a tower

$$\begin{array}{c} \vdots \\ \downarrow \\ E_n \\ \downarrow \\ \vdots \\ \downarrow \\ E_0 \end{array}$$

We have a functor $\mathbf{P} : \mathsf{FI}\mathcal{V} \to \mathrm{ExSeq}\mathcal{V}$ given by

$$(\mathbf{P}E)_i \overset{\text{def}}{=} \mathbf{P}_i E$$

For $E \in \mathsf{FI}\mathcal{V}$, we call $\mathbf{P}E$ the *Taylor tower* of E.

Definitions 13. We denote by $\mathrm{ExSeq}_n\mathcal{V}$ the full sub-∞-category of $\mathrm{ExSeq}\mathcal{V}$ spanned by excision sequences $\{E_i\}$ such that for all $m \geq n$, the morphism $E_m \to E_n$ is an isomorphism. We denote by

$$\mathbf{J}_n : \mathrm{ExSeq}\mathcal{V} \to \mathrm{ExSeq}_n\mathcal{V}$$

the coreflection functor. We say that an excision sequence is *convergent* if it is a colimit in $\mathrm{ExSeq}\mathcal{V}$ of a diagram taking values in $\bigcup_{n\in\mathbb{N}} \mathrm{ExSeq}_n\mathcal{V}$. We denote by $\mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}}$ the full sub-∞-category of $\mathrm{ExSeq}\mathcal{V}$ spanned by convergent excision sequences.

Lemma 14. *For $E \in \mathrm{ExSeq}\mathcal{V}$,*

$$\lim_{i\in\mathbb{N}^{\mathrm{op}}} (\mathbf{J}_n E)_i \cong \mathbf{Q}_n\left(\lim_{i\in\mathbb{N}^{\mathrm{op}}} E_i\right)$$

Lemma 15. *Let $\mathcal{E}$ be an arbitrary presentable ∞-category and let*

$$\mathcal{E}_0 \subseteq \mathcal{E}_1 \subseteq \cdots \subseteq \mathcal{E}_n \subseteq \cdots \subseteq \mathcal{E}$$

be an increasing sequence of presentable reflective sub-∞-categories with reflection functors L_n. Denote by $\mathcal{E}_\infty$ the full sub-∞-category of $\mathcal{E}$ spanned by limits in $\mathcal{E}$ of diagrams taking values in $\bigcup_{n\in\mathbb{N}} \mathcal{E}_n$. Then $\mathcal{E}_\infty$ is reflective with reflection functor

$$L_\infty \cong \lim_{n\in\mathbb{N}^{\mathrm{op}}} L_n$$

When $\mathcal{E}$ is stable, we obtain a dual theorem for presentable coreflective sub-∞-categories by taking left orthogonal complements.

Proof. Let $\mathcal{E}'_\infty$ denote the intersection of all presentable reflective sub-∞-categories of $\mathcal{E}$ which contain the union of the $\mathcal{E}_i$ and with reflection functor L'_∞. Then $\mathcal{E}'_\infty$ is a presentable reflective sub-∞-category of $\mathcal{E}$ and is the closure under iterated limits of

$$\bigcup_{i\in\mathbb{N}} \mathcal{E}_i$$

It follows that the canonical natural transformation $L'_\infty(\mathrm{id} \to L_\infty)$ is an isomorphism. Then because for any $E \in \mathcal{E}$, $L_\infty E \in \mathcal{E}'_\infty$, it follows that $L_\infty \cong L'_\infty$ and hence $\mathcal{E}'_\infty = \mathcal{E}_\infty$. □

Corollary 16. *$\mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \subseteq \mathsf{FI}\mathcal{V}$ and $\mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}} \subseteq \mathrm{ExSeq}\mathcal{V}$ are reflective and coreflective respectively. We denote their reflection and coreflection functors $\mathbf{P}_\infty$ and $\mathbf{J}_\infty$ respectively.*

Theorem 17. *The Taylor tower functor $\mathbf{P}$ determines an equivalence*

$$\mathbf{P} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}} : \lim$$

Proof. First, note that because

$$\mathbf{P} : \mathsf{FI}\mathcal{V} \to \mathrm{ExSeq}\mathcal{V}$$

factors through $\mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}}$, so does its right adjoint

$$\lim_{i\in\mathbb{N}^{\mathrm{op}}} : \mathrm{ExSeq}\mathcal{V} \to \mathsf{FI}\mathcal{V}$$

so every analytic FI-object is the limit of a convergent excision sequence. Let

$$E \in \mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}}$$

We have

$$\mathbf{P}\left(\lim_{i\in\mathbb{N}^{\mathrm{op}}} E_i\right) \cong \mathbf{P}\left(\operatorname*{colim}_{n\in\mathbb{N}} \mathbf{Q}_n \lim_{i\in\mathbb{N}^{\mathrm{op}}} E_i\right)$$

$$
\begin{aligned}
&\cong \mathbf{P}\left(\operatorname*{colim}_{n\in\mathbb{N}} \lim_{i\in\mathbb{N}^{\mathrm{op}}} (\mathbf{J}_n E)_i\right) \\
&\cong \operatorname*{colim}_{n\in\mathbb{N}} \mathbf{P}\left(\lim_{i\in N} (\mathbf{J}_n E)_i\right) \\
&\cong \operatorname*{colim}_{n\in\mathbb{N}} (\mathbf{J}_n E)_i \\
&\cong E
\end{aligned}
$$

establishing that

$$\mathbf{P} \circ \left(\lim_{i\in\mathbb{N}^{\mathrm{op}}} -\right) \simeq \mathrm{id}_{\mathrm{ExSeq}\mathcal{V}^{\mathrm{Conv}}}$$

while

$$\left(\lim_{i\in\mathbb{N}^{\mathrm{op}}} -\right) \circ \mathbf{P} \simeq \mathrm{id}_{\mathsf{FI}\mathcal{V}^{\mathrm{Anly}}}$$

follows from Lemma 15. $\square$

CHAPTER 3

Taylor Coefficients

3.1 Homogeneous FI-objects

Definition 18. We say that $E \in \mathrm{Exc}_n\mathcal{V}$ is *n-homogeneous* if $\mathbf{P}_{n-1}E = 0$. We denote the full sub-∞-category of n-homogeneous FI-objects $\mathrm{Hmg}_n\mathcal{V}$. We define

$$\mathbf{D}_n \overset{\text{def}}{=} \mathrm{fib}\,(\mathbf{P}_n \to \mathbf{P}_{n-1}) : \mathsf{FI}\mathcal{V} \to \mathrm{Hmg}_n\mathcal{V}$$

We say that $\mathbf{D}_nE$ is the *nth layer* of the Taylor tower of E. More generally, we can speak of the nth layer of any excision sequence, and we denote this construction also by $\mathbf{D}_n$.

Definition 19. We say that $E \in \mathsf{FI}\mathcal{V}$ is *n-cohomogeneous* if E is in the image of $\mathrm{Lan}_{\mathfrak{S}_n}^{\mathsf{FI}}$. Equivalently, $E \in \mathsf{FI}\mathcal{V}$ is n-cohomogeneous when $E \in \mathrm{Exc}_n\mathcal{V}$ and $\mathbf{Q}_{n-1}E \cong 0$. We denote the category of n-cohomogeneous FI-objects $\mathrm{coHmg}_n\mathcal{V}$. We define

$$\mathbf{R}_n \overset{\text{def}}{=} \mathrm{cofib}\,(\mathbf{Q}_{n-1} \to \mathbf{Q}_n) : \mathsf{FI}\mathcal{V} \to \mathrm{coHmg}_n\mathcal{V}$$

Proposition 20. *When restricted to* $\mathrm{Hmg}_n\mathcal{V}$ *and* $\mathrm{coHmg}_n\mathcal{V}$ *respectively, the functors* $\mathbf{R}_n$ *and* $\mathbf{D}_n$ *are inverses. In particular,* n*-homogeneous* FI*-objects are classified by* $\mathfrak{S}_n$*-objects.*

Proof. Given $E \in \mathrm{Hmg}_n\mathcal{V}$, we consider the following commutative diagram.

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{D}_n\mathbf{R}_n E \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] \arrow[d] & E \arrow[r] \arrow[d] & \mathbf{R}_n E \arrow[d] \\
\mathbf{Q}_{n-1}E \arrow[r] & 0 \arrow[r] & \mathbf{P}_{n-1}\mathbf{R}_n E
\end{tikzcd}$$

We begin by considering the middle row. This is a fiber sequence. The bottom row is $\mathbf{P}_{n-1}$ applied to the middle row and therefore also a fiber sequence. The top row is the fiber of the natural transformation from the middle row to the bottom row and is therefore also a fiber sequence. This proves that

$$\mathrm{id}_{\mathrm{Hmg}_n\mathcal{V}} \cong \mathbf{D}_n\mathbf{R}_n$$

The other direction follows from a similar argument. □

Definition 21. Given $E \in \mathsf{FI}\mathcal{V}$ and $n \in \mathsf{FI}$, we define

$$\mathbf{C}_n E \stackrel{\text{def}}{=} \mathbf{R}_n\mathbf{D}_n E(n) \cong \underset{T\subseteq n}{\mathrm{tocofib}}\, \mathbf{D}_n E(T)$$

This is a $\mathfrak{S}_n$-object, and we call $\mathbf{C}_n E$ the *nth Taylor coefficient* of E.

3.2 The aggregate coefficient functor

The $\mathbf{C}_n$ are left adjoint functors and so controlled by their restrictions to representable FI-objects. We therefore wish to calculate $\mathbf{C}_n F_{S,X}$, and the first step toward that goal is calculating $\mathbf{P}_n F_{S,X}$.

Proposition 22. *For $S \in \mathsf{FI}$ and $X \in \mathcal{V}$,*

$$\mathbf{P}_n F_{S,X} \cong \lim_{\substack{T \subseteq S \\ |T| \leq n}} F_{T,X}$$

Proof. For just this proof, let us denote

$$F_{S,X}^{(n)} \overset{\text{def}}{=} \lim_{\substack{T \subseteq S \\ |T| \leq n}} F_{T,X}$$

Because $F_{X,S}^{(n)}$ is a limit of n-excisive FI-objects and therefore n-excisive, it is enough to show that

$$\mathbf{P}_n F_{S,X} \cong \mathbf{P}_n F_{S,X}^{(n)}$$

By the Yoneda lemma, it is enough to show that for all $E \in \mathrm{Exc}_n \mathcal{V}$,

$$\mathsf{FI}\mathcal{V}(F_{S,X}, E) \cong \mathsf{FI}\mathcal{V}\Big(F_{S,X}^{(n)}, E\Big)$$

We have

$$\begin{aligned}
\mathsf{FI}\mathcal{V}(F_{S,X}, E) &\cong \mathcal{V}(X, E(S)) \\
&\cong \mathcal{V}\left(X, \operatorname*{colim}_{\substack{T \subseteq S \\ T \leq n}} E(T)\right) \\
&\cong \operatorname*{colim}_{\substack{T \subseteq S \\ T \leq n}} \mathsf{FI}\mathcal{V}(F_{T,X}, E) \\
&\cong \mathsf{FI}\mathcal{V}\Big(F_{S,X}^{(n)}, E\Big)
\end{aligned}$$

□

Definition 23. In what follows, we denote $G_{n,X} \overset{\text{def}}{=} \mathbf{D}_n F_{n,X}$ and $G_n \overset{\text{def}}{=} \mathbf{D}_n F_n$.

Corollary 24.

$$G_{n,X} \cong \operatorname*{tofib}_{S \subseteq n} F_{S,X}$$

Corollary 25.

$$\mathbf{D}_n F_{m,X} \cong \prod_{\substack{S\subseteq m\\|S|=n}} G_{S,X}$$

Proof. Using Proposition 22,

$$\begin{aligned}
\mathbf{D}_n F_{m,X} &\cong \operatorname{fib} \mathbf{P}_n F_{m,X} \to \mathbf{P}_{n-1} F_{m,X}\\
&\cong \operatorname{fib} \lim_{\substack{S\subseteq m\\|S|\le n}} F_{S,X} \to \mathbf{P}_{n-1} \lim_{\substack{S\subseteq m\\|S|\le n}} F_{S,X}\\
&\cong \lim_{\substack{S\subseteq m\\|S|\le n}} \operatorname{fib} F_{S,X} \to \mathbf{P}_{n-1} F_{S,X}\\
&\cong \operatorname{Ran}_{\{S\subset m:|S|=n\}}^{\{S\subset m:|S|\le n\}} G_{S,X}\\
&\cong \prod_{\substack{S\subseteq m\\|S|=n}} G_{S,X}
\end{aligned}$$

□

Proposition 26. *The Taylor coefficients of representable* FI*-objects are given by*

$$\mathbf{C}_n F_{m,X} \cong \mathsf{FI}(n,m) \pitchfork X$$

Proof. Using the preceding corollaries, we have

$$\begin{aligned}
\mathbf{C}_n F_{m,X} &= \underset{T\subseteq n}{\operatorname{tocofib}} \prod_{\substack{U\subseteq m\\|U|=n}} \underset{S\subseteq U}{\operatorname{tofib}} F_{S,X}(T)\\
&\cong \prod_{\substack{U\subseteq m\\|U|=n}} \underset{S\subseteq U}{\operatorname{tofib}}\ \underset{T\subseteq n}{\operatorname{tocofib}} F_{S,X}(T)\\
&\cong \prod_{\substack{U\subseteq m\\|U|=n}} \underset{T\subseteq n}{\operatorname{tocofib}} F_{U,X}(T) \qquad (3.1)\\
&\cong \prod_{\substack{U\subseteq m\\|U|=n}} F_{U,X}(n) \qquad (3.2)\\
&\cong \prod_{\substack{U\subseteq m\\|U|=n}} \mathsf{FI}(U,n) \otimes X
\end{aligned}$$

$$\cong \prod_{\substack{U\subseteq m\\ |U|=n}} \mathsf{FI}(n,U) \pitchfork X \tag{3.3}$$

$$\cong \mathsf{FI}(n,m) \pitchfork X$$

where eq. (3.1) uses that when $|S| < n$, $F_{S,X}$ is $n-1$-excisive so that

$$\operatorname*{tocofib}_{T\subseteq n} F_{S,X}(T) \cong 0$$

and hence

$$\operatorname*{tofib}_{S\subseteq U} \operatorname*{tocofib}_{T\subseteq n} F_{S,X}(T) \cong \mathrm{fib}\left(\operatorname*{tocofib}_{T\subseteq n} F_{U,X}(T) \to 0\right) \cong \operatorname*{tocofib}_{T\subseteq n} F_{U,X}(T)$$

and eq. (3.2) uses that when $|T| < |U|$,

$$F_{U,X}(T) = \mathsf{FI}(U,T) \otimes X = \emptyset \otimes X \cong 0$$

so that

$$\operatorname*{tocofib}_{T\subseteq n} F_{U,X}(T) \cong \mathrm{cofib}\left(0 \to F_{U,X}(n)\right) \cong F_{U,X}(n) \qquad \square$$

Corollary 27. *For $E \in \mathsf{FI}\mathcal{V}$, $\mathbf{C}_n E$ is functorial in $n \in \mathsf{FI}$, so that we obtain an aggregate Taylor coefficient functor*

$$\mathbf{C} : \mathsf{FI}\mathcal{V} \to \mathsf{FI}\mathcal{V}$$

and we extend the construction to $E \in \mathrm{ExSeq}\mathcal{V}$ with the formula

$$\mathbf{C} : E \mapsto \lim_{n\in\mathbb{N}^{\mathrm{op}}} \mathbf{CP}_n E$$

Definition 28. Denote by $\mathrm{Supp}_n\mathcal{V}$ the image of $\mathrm{Ran}^{\mathsf{FI}}_{\mathsf{FI}_{\leq n}}$. Denote the reflection functor by

$$\mathbf{U}_n : \mathsf{FI}\mathcal{V} \to \mathrm{Supp}_n\mathcal{V}$$

Observation 29. For $E \in \mathrm{ExSeq}\mathcal{V}$,

$$\mathbf{CP}_n E \cong \mathbf{U}_n \mathbf{C} E$$

3.3 Derivatives

Let us give a more direct description of $\mathbf{C}E$ in terms of the FI-object E.

Notation 30. For $E \in \mathsf{FI}\mathcal{V}$ and $n \in \mathsf{FI}$, define a new FI-object

$$\boldsymbol{\Delta}^n E \overset{\text{def}}{=} \underset{S \subseteq n}{\mathrm{tocofib}}\, E(S \sqcup -)$$

Given a map $f : n \to n'$ in FI, abbreviate

$$n' \setminus f \overset{\text{def}}{=} n' \setminus \mathrm{img}\, f$$

Further, given $k \in \mathsf{FI}$, define

$$j^\dagger :: k \to n' \sqcup k$$

by

$$j^\dagger(a) \overset{\text{def}}{=} \begin{cases} j^{-1}(a) & a \in j(n' \setminus f) \\ a & a \notin j(n' \setminus f) \end{cases}$$

and define

$$g_{f,k} \overset{\text{def}}{=} \sum_{j : n' \setminus f \to k} E\left(f \sqcup j^\dagger\right) : E(n \sqcup k) \to E(n' \sqcup k)$$

Theorem 31. *The Taylor coefficients of E are given by*

$$\mathbf{C}E(n) \cong \underset{k \in \mathsf{FI}}{\mathrm{colim}}\, \boldsymbol{\Delta}^n E(k)$$

The morphism $\mathbf{C}E(f)$ is determined by the maps $g_{f,k}$.

Proof. Since our constructions preserve colimits in E, we need only verify that the theorem holds for representable FI-objects, and this in turn allows us to reduce further to the case $\mathcal{V} = \mathcal{S}p$ and thence to just the F_n. To verify the formula for objects, observe that

$$\operatorname*{colim}_{k \in \mathsf{FI}} F_m(n+k)$$

can be identified with the suspension spectrum of the set of partial bijections from m to n. The construction $\mathbf{\Delta}^n$ commutes with suspension, and by the fact we used in the proof of Proposition 9, $\mathbf{\Delta}^n$ kills off exactly those partially defined injections which do not cover n. A partially defined injection $m \to n$ which covers n is the same data as an injection $n \to m$, and since this set is finite, its suspension spectrum is isomorphic to its dual. For the remainder of the proof, we make use of the isomorphism $\mathbf{C}F_m(n) \cong \Sigma^\infty \mathsf{FI}(n,m)_+$. We that

$$\mathbf{C}F_m(f) : \mathbf{C}F_m(n) \to \mathbf{C}F_m(n')$$

is determined by adjointness by the map

$$\mathsf{FI}(n,m) \mapsto \Omega^\infty \Sigma^\infty \mathsf{FI}(n',m)_+$$

given by

$$(i : n \to m) \mapsto \sum_{\substack{i' : n' \to m \\ i' = fi}} \eta(i')$$

where η is the unit of the $\Sigma^\infty_+ \dashv \Omega^\infty$ adjunction and the sum is taken with respect to the E_∞-structure of the infinite loop space. To verify the formula for morphisms, we must verify the commutativity of the following square:

$$\begin{tikzcd} F_m(n+k) \arrow[r] \arrow[d, "g_{f,k}"] & \mathbf{C}F_m(n) \arrow[d, "\mathbf{C}F_m(f)"] \\ F_m(n'+k) \arrow[r] & \mathbf{C}F_m(n') \end{tikzcd}$$

By adjointness, the canonical map $F_m(n+k) \to \mathbf{C}F_m(n)$ is determined by the map

$$\mathsf{FI}(m, n+k) \to \Omega^\infty \mathbf{C}F_m(n)$$

which sends an injection $i : m \to n+k$ to 0 if $n \not\subseteq i(m)$ and to $\eta(i^*)$ otherwise, where

$$i^* \stackrel{\text{def}}{=} \Big(a \mapsto i^{-1}(a)\Big) : n \to m$$

We can conclude by observing that given an injection $i : m \to n+k$ representing an injection $i^* : n \to m$, each injection $n' \to m$ which restricts to i^* is represented once by a summand of $g_{f,k}(i)$. □

Corollary 32. *By Theorem 31 and Recollection 4,*

$$\mathbf{C}(\mathbf{\Delta}^n E)(-) \cong \mathbf{C}(E)(- \sqcup n)$$

Definition 33. Call an FI-object E *finitely supported* if there exists $n \in \mathbb{N}$ such that $E \in \mathrm{Supp}_n \mathcal{V}$. Denote by $\mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$ – the ∞-category of *torsion* FI-objects – the coreflective sub-∞-category of $\mathsf{FI}\mathcal{V}$ generated by the finitely supported FI-objects.

Corollary 34. *For any $E \in \mathsf{FI}\mathcal{V}$, $\mathbf{C}E \in \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$.*

Proof. This follows from the facts that $\mathbf{C}E$ is finitely supported when E is representable, that $\mathbf{C}$ is a left adjoint functor, and that every FI-object is an iterated colimit of representable FI-objects. □

We formalize the notion that an analytic FI-object is determined by its "germ at infinity."

Corollary 35. *For* $E \in \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$, $\mathbf{P}E \cong 0$.

Proof. It is sufficient to prove that $\mathbf{C}E \cong 0$ for E finitely supported. We begin by showing that $\mathbf{C}E(0) \cong \operatorname{colim} E \cong 0$ when $E|_{\mathsf{FI}_{\geq n}} \cong 0$. By [Lur09, Proposition 4.1.3.1] – Quillen's Theorem A for quasicategories, originally due to Joyal – it suffices to show that for all $m \in \mathsf{FI}$,

$$B(m \downarrow \mathsf{FI}_{\geq n}) \cong *$$

where $B : \mathcal{C}at_\infty \to \mathcal{S}$ is the classifying space functor, since this implies that

$$0 = \operatorname*{colim}_{\mathsf{FI}_{\geq n}} 0 \cong \operatorname*{colim}_{\mathsf{FI}_{\geq n}} E|_{\mathsf{FI}_{\geq n}} \cong \operatorname*{colim}_{\mathsf{FI}} E$$

Note that we have an equivalence

$$(k, f : m \to k) \mapsto k \setminus f : m \downarrow \mathsf{FI}_{\geq n} \cong \mathsf{FI}_{\geq n-m}$$

so it will be sufficient to establish that $B\mathsf{FI}_{\geq n} \cong *$ for all $n \in \mathbb{N}$. Let

$$\iota_{\geq n} : \mathsf{FI}_{\geq n} \to \mathsf{FI}$$

denote the inclusion functor and write

$$\kappa_{\geq n} \stackrel{\text{def}}{=} S \mapsto S \sqcup n : \mathsf{FI} \to \mathsf{FI}_{\geq n}$$

Then we have natural transformations

$$\mathrm{id}_{\mathsf{FI}} \to \iota_{\geq n}\kappa_{\geq n}$$

and

$$\mathrm{id}_{\mathsf{FI}_{\geq n}} \to \kappa_{\geq n}\iota_{\geq n}$$

each given by the canonical inclusion $S \to S \sqcup n$. Upon taking classifying spaces, these natural transformations become homotopies, so that $B\iota_{\geq n}$ and $B\kappa_{\geq n}$ are inverse homotopy equivalences between $B\mathsf{FI}$ and $B\mathsf{FI}_{\geq n}$. But $B\mathsf{FI} \cong *$ because FI has an initial object. For any $S \in \mathsf{FI}$, we have that if E vanishes on $\mathsf{FI}_{\geq n}$, then so does $\mathbf{\Delta}^S E$, so for E finitely supported,

$$\mathbf{C}E(S) \cong \mathbf{C}(\mathbf{\Delta}^S E)(0) \cong 0 \qquad \square$$

Corollary 36. *For $E \in \mathsf{FI}\mathcal{V}^{\mathrm{Anly}}$ and any $n \in \mathbb{N}$,*

$$E \cong \mathrm{Ran}_{\mathsf{FI}_{\geq n}}^{\mathsf{FI}} \mathrm{Res}_{\mathsf{FI}}^{\mathsf{FI}_n} E$$

Conjecture 37. Corollary 35 implies that $\mathsf{FI}\mathcal{V}^{\mathrm{Anly}}$ is a full sub-∞-category of the right orthogonal complement of $\mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$. We conjecture the converse: that $\mathsf{FI}\mathcal{V}^{\mathrm{Anly}}$ in fact is the right orthogonal complement of $\mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$.

3.4 Recovering excision sequences from coefficients

Notation 38. Denote by $\mathbf{Z}$ the right adjoint to $\mathbf{C}$. We have used $\mathbf{C}$ to refer to functors with various domains including $\mathsf{FI}\mathcal{V}$, $\mathrm{ExSeq}\mathcal{V}$, and $\mathrm{Exc}_n\mathcal{V}$, and we shall similarly use $\mathbf{Z}$ to refer to functors with these various codomains.

Observation 39. For $E \in \mathsf{FI}\mathcal{V}$,

$$\begin{aligned}
\mathbf{Z}E &\cong \int_{m\in\mathsf{FI}} \mathbf{C}F_m \pitchfork E(m) \\
&\cong \int_{m\in\mathsf{FI}} \mathcal{S}p(\mathsf{FI}(m,-),\mathbb{S}) \pitchfork E(m) \\
&\cong \int_{m\in\mathsf{FI}} \mathsf{FI}(m,-) \otimes E(m)
\end{aligned}$$

Lemma 40. *The unit*

$$\eta_{F_{X,n}} : F_{X,n} \to \mathbf{Z}\mathbf{C}F_{X,n}$$

is an isomorphism.

Proof.

$$\begin{aligned}\mathbf{ZC}F_{X,n} &\cong \int_{m\in\mathsf{FI}} \mathsf{FI}(m,-)\otimes \mathsf{FI}(m,n) \pitchfork X\\ &\cong \int_{m\in\mathsf{FI}} \mathsf{FI}(m,n) \pitchfork \mathsf{FI}(m,-)\otimes X\\ &\cong \mathsf{FI}(n,-)\otimes X\\ &\cong F_{X,n}\end{aligned}$$

□

Definition 41. We call a $\mathfrak{S}_n$-object A *tame* if the norm map

$$\mathsf{FI}(-,n) \pitchfork_{\mathfrak{S}_n} A \to \mathsf{FI}(-,n) \pitchfork^{\mathfrak{S}_n} A$$

is a natural isomorphism. In this case we also call the cohomogeneous FI-object $\mathrm{Lan}_{\mathfrak{S}_n}^{\mathsf{FI}} A$ tame. We also call any m-excisive FI-object that is a finite colimit of tame cohomogeneous FI-objects tame, any excision sequence of tame excisive FI-objects tame, and any analytic FI-object with a tame Taylor tower tame. We denote the ∞-categories of such $\mathfrak{S}_n\mathcal{V}^{\mathrm{Tame}}$, $\mathrm{coHmg}_n\mathcal{V}^{\mathrm{Tame}}$, $\mathrm{Exc}_m\mathcal{V}^{\mathrm{Tame}}$, $\mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}}$, and $\mathsf{FI}\mathcal{V}^{\mathrm{Anly,Tame}}$ respectively.

We call an FI-object *cotame* if it lies in the image of $\mathrm{Res}_{\mathrm{ExSeq}\mathcal{V}}^{\mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}}}\mathbf{C}$. We denote the ∞-category of cotame FI-objects $\mathsf{FI}\mathcal{V}^{\mathrm{coTame}}$. We denote by $\mathrm{Supp}_n\mathcal{V}^{\mathrm{coTame}}$ the full sub-∞-category of $\mathrm{Supp}_n\mathcal{V}$ spanned by cotame objects.

Theorem 42. *Let $A \in \mathfrak{S}_n\mathcal{V}^{\mathrm{Tame}}$ and denote*

$$E \overset{\mathrm{def}}{=} \mathsf{FI}(n,-)\otimes A$$

and endow E with the diagonal $\mathfrak{S}_n$-action so that

$$E_{\mathfrak{S}_n} \cong \mathrm{Lan}_{\mathfrak{S}_n}^{\mathsf{FI}} \in \mathrm{coHmg}_n \mathcal{V}^{\mathrm{Tame}}$$

Then

$$\eta : E_{\mathfrak{S}_n} \to \mathbf{ZC} E_{\mathfrak{S}_n}$$

is an isomorphism.

Proof. Fixing total orders on n and k determines an isomorphism of $\mathfrak{S}_n$-spaces

$$\mathsf{FI}(n,k) \cong \mathrm{Lan}_*^{\mathfrak{S}_n} \{S \subseteq k : |S| = n\}$$

exhibiting $\mathsf{FI}(n,k)$ as a free $\mathfrak{S}_n$-space. Tensoring an $\mathfrak{S}_n$-object with a free $\mathfrak{S}_n$-space yields a free $\mathfrak{S}_n$-object, so $E(k)$ is a free $\mathfrak{S}_n$-object for each $k \in \mathsf{FI}$. Then by [Lur17, Example 6.1.6.26], the norm maps

$$Nm\,(E(k)) : E(k)_{\mathfrak{S}_n} \to E(k)^{\mathfrak{S}_n}$$

are isomorphisms and hence the norm map

$$Nm\,(E) : E_{\mathfrak{S}_n} \to E^{\mathfrak{S}_n}$$

is an isomorphism.

Because left adjoint functors of stable ∞-categories are exact, we have a commutative square

$$\begin{tikzcd}
\mathbf{C}(E_{\mathfrak{S}_n}) \arrow[r, "\mathbf{C}(Nm\,(E))"] \arrow[d] & \mathbf{C}\left(E^{\mathfrak{S}_n}\right) \arrow[d, "f"] \\
(\mathbf{C}E)_{\mathfrak{S}_n} \arrow[r, "Nm\,(\mathbf{C}E)"] & (\mathbf{C}E)^{\mathfrak{S}_n}
\end{tikzcd}$$

where the left arrow is an isomorphism because $\mathbf{C}$ is a left adjoint, the top arrow is

$\mathbf{C}$ applied to the norm and hence an isomorphism by the preceding argument, and the bottom arrow is an isomorphism by our assumption on A, so the right arrow f must also be an isomorphism.

The adjunction $\mathbf{C} \dashv \mathbf{Z}$ now gives us a new commutative square

$$\begin{tikzcd} E^{\mathfrak{S}_n} \arrow[r, "(\eta_E)^{\mathfrak{S}_n}"] \arrow[d, "\eta_{E^{\mathfrak{S}_n}}"] & (\mathbf{ZC}E)^{\mathfrak{S}_n} \arrow[d] \\ \mathbf{ZC}\left(E^{\mathfrak{S}_n}\right) \arrow[r, "\mathbf{Z}f"] & \mathbf{Z}(\mathbf{C}E)^{\mathfrak{S}_n} \end{tikzcd}$$

We know that all the morphisms in this square except the left one are isomorphisms, so that one must be as well.

We play our game one last time. The naturality of the unit gives us the commutative square

$$\begin{tikzcd} E_{\mathfrak{S}_n} \arrow[r, "Nm\,(E)"] \arrow[d, "\eta_{E_{\mathfrak{S}_n}}"] & E^{\mathfrak{S}_n} \arrow[d] \\ \mathbf{ZC}(E_{\mathfrak{S}_n}) \arrow[r, "\mathbf{ZC}(Nm\,(E))"] & \mathbf{ZC}\left(E^{\mathfrak{S}_n}\right) \end{tikzcd}$$

and we conclude that $\eta_{E_{\mathfrak{S}_n}}$ must be an isomorphism since all the other morphisms in the square are isomorphisms. □

Remark 43. The bulk of the foregoing proof can be encapsulated by the claim that the following square commutes:

$$\begin{tikzcd} E_{\mathfrak{S}_n} \arrow[r] \arrow[d] & E^{\mathfrak{S}_n} \arrow[d] \\ \mathbf{ZC}(E_{\mathfrak{S}_n}) \arrow[r] & (\mathbf{ZC}E)^{\mathfrak{S}_n} \end{tikzcd}$$

Lemma 44. *Suppose that $\mathcal{C}$ and $\mathcal{D}$ are stable ∞-catories, that $\mathcal{C}_0 \subseteq \mathcal{C}$ and $\mathcal{D}_0 \subseteq \mathcal{D}$ are full sub-∞-categories, that each object of $\mathcal{C}$ and $\mathcal{D}$ can be expressed as a colimit* *of objects in* $\mathcal{C}_0$ *and* $\mathcal{D}_0$ *respectively, and that there exists an exact functor*

$$L : \mathcal{C} \to \mathcal{D}$$

which restricts to an equivalence

$$\operatorname{Res}_{\mathcal{C}}^{\mathcal{C}_0} L : \mathcal{C}_0 \simeq \mathcal{D}_0$$

Then L *is an equivalence*

$$L : \mathcal{C} \simeq \mathcal{D}$$

Proof. We show that L is surjective and fully faithful. Denote by

$$R : \mathcal{D}_0 \to \mathcal{C}_0$$

the inverse of $\operatorname{Res}_{\mathcal{C}}^{\mathcal{C}_0} L : \mathcal{C}_0$. For establish surjectivity, we have that for $a \in \mathcal{D}$ there exists some finite ∞-category $\mathcal{I}$ and diagram

$$A : \mathcal{I} \to \mathcal{D}_0$$

such that

$$\begin{aligned} a &\cong \operatorname*{colim}_{i \in \mathcal{I}} A(i) \\ &\cong \operatorname*{colim}_{i \in \mathcal{I}} LRA(i) \\ &\cong L\Big(\operatorname*{colim}_{i \in \mathcal{I}} RA(i)\Big) \end{aligned}$$

Next, for $b, c \in \mathcal{C}$, there exist finite ∞-categories $\mathcal{J}$ and $\mathcal{K}$ and functors

$$B : \mathcal{J} \to \mathcal{C}_{\prime}$$

$$C : \mathcal{K} \to \mathcal{C}_{\prime}$$

such that $b \cong \operatorname{colim} B$ and $c \cong \operatorname{colim} C$ so that we have

$$\begin{aligned}
\mathcal{D}(Lb, Lc) &\cong \mathcal{D}\Big(L(\operatorname{colim} j \in \mathcal{J} B(j)), L\Big(\operatorname*{colim}_{k\in\mathcal{K}} C(k)\Big)\Big) \\
&\cong \mathcal{D}\Big(\operatorname*{colim}_{j\in\mathcal{J}} LB(j), \operatorname*{colim}_{k\in\mathcal{K}} LC(k)\Big) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{k\in\mathcal{K}} \mathcal{D}_0(LB, LC) \\
&\cong \lim_{j\in\mathcal{J}} \operatorname*{colim}_{k\in\mathcal{K}} \mathcal{C}_0(B, C) \\
&\cong \mathcal{C}\Big(\operatorname*{colim}_{j\in\mathcal{J}} B(j), \operatorname*{colim}_{k\in\mathcal{K}} C(k)\Big) \\
&\cong \mathcal{C}(b, c)
\end{aligned}$$

establishing full faithfulness. □

Corollary 45. *We have an equivalence*

$$\mathbf{C} : \mathrm{Exc}_n\mathcal{V}^{\mathrm{Tame}} \simeq \mathrm{Supp}_n\mathcal{V}^{\mathrm{coTame}}$$

Corollary 46. *We have an equivalence*

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{coTame}}$$

Corollary 47. *The equivalence of Corollary 46 restricts to an equivalence*

$$\mathbf{C} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly,Tame}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{Tors,coTame}}$$

Proof. This follows from Corollary 34 and Theorem 17. □

Example 48. When $\mathcal{V}$ is $\mathbb{Q}$-linear, all $\mathfrak{S}_n$-objects are tame, so we have equivalences

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

$$\mathbf{C} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$$

Example 49. When $\mathcal{V}$ is the ∞-category of $K(m)$-local spectra for some $m \in \mathbb{N}$ and prime p, all $\mathfrak{S}_n$-objects are tame, so we have equivalences

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V} \simeq \mathsf{FI}\mathcal{V}$$

$$\mathbf{C} : \mathsf{FI}\mathcal{V}^{\mathrm{Anly}} \simeq \mathsf{FI}\mathcal{V}^{\mathrm{Tors}}$$

Definition 50. For $E \in \mathrm{ExSeq}\mathcal{V}$, we say that E is *self-tame* if for all pairs $n, m \in \mathbb{N}$, the map

$$\mathsf{FI}\mathcal{V}(\mathbf{Q}_m E, \mathbf{Q}_n E) \to \mathsf{FI}\mathcal{V}(\mathbf{Q}_m E, \mathbf{ZCQ}_n E)$$

induced by the unit (i.e. the norm map) is an isomorphism. We denote the full sub-∞-category of such $\mathrm{ExSeq}\mathcal{V}^{\mathrm{selfTame}}$. We say that an FI-object is *self-cotame* if it lies in the image of

$$\mathbf{C} : \mathrm{ExSeq}\mathcal{V}^{\mathrm{selfTame}} \to \mathsf{FI}\mathcal{V}$$

and we denote the full sub-∞-category of such $\mathsf{FI}\mathcal{V}^{\mathrm{selfcoTame}}$.

Notation 51. Recall that the *core* of an ∞-category $\mathcal{C}$, denote $\mathrm{core}\,\mathcal{C}$, is its coreflection into $\mathcal{S}$; in other words, $\mathrm{core}\,\mathcal{C}$ is the maximal sub-∞-groupoid of $\mathcal{C}$.

Corollary 52. *We have an equivalence*

$$\mathrm{core}\,\mathbf{C} : \mathrm{core}\,\mathrm{ExSeq}\mathcal{V}^{\mathrm{Tame}} \simeq \mathrm{core}\,\mathsf{FI}\mathcal{V}^{\mathrm{selfcoTame}}$$

CHAPTER 4

Representation Stability

In this section, we explore the connections between FI-calculus and representation stability. For the most part, we will specialize to the case $\mathcal{V} = \mathcal{S}p^{\mathbb{Q}}$, but we begin with an important result that applies more broadly, when $\mathcal{V} = \mathcal{S}p$.

4.1 A bouquet

Theorem 53. *For $k \geq 2n-1$, $G_n(k)$ is a wedge of copies of $\mathbb{S}$.*

Proof. Recall that by Proposition 22, G_n is the total fiber of the n-cube given by $F_S(k)$ as S ranges over the subsets of n. This is equivalent to the n-fold desuspension of the total cofiber of the same n-cube, so we could equivalently show that this total cofiber is a wedge of copies of $\mathbb{S}^n$. For this it would suffice to show that the total cofiber $L(n,k)$ of the n-cube $\mathsf{FI}(S,k)_+$ is a wedge of copies of S^n, where we form the colimit in the ∞-category of pointed spaces rather than of spectra.

As a first step, we show that

$$\operatorname{cofib}(L(n,k) \to L(n, k \sqcup 1)) \cong \bigvee_{x \in n} \Sigma L(n \setminus \{x\}, k) \tag{4.1}$$

Note that

$$\operatorname{cofib}(\mathsf{FI}(S,k)_+ \to \mathsf{FI}(S, k \sqcup 1)_+) \cong \bigvee_{x \in S} \mathsf{FI}(S \setminus \{x\}, k)_+ \tag{4.2}$$

for $S \subseteq n$. For $x \in n$ and $S \subseteq n$, define the n-cube

$$A_x(S) \overset{\text{def}}{=} \begin{cases} \mathsf{FI}(S \setminus \{x\}, k)_+ & x \in S \\ * & x \notin S \end{cases}$$

and note that

$$\operatorname*{tocofib}_{S \subseteq n} A_x(S) \cong \operatorname{cofib}(L(n \setminus \{x\}, k) \to *) \cong \Sigma L(n \setminus \{x\}, k)$$

We can rewrite morphism 4.2 as

$$\operatorname{cofib}(\mathsf{FI}(S,k)_+ \to \mathsf{FI}(S, k \sqcup 1)_+) \cong \bigvee_{x \in n} A_x(S)$$

and taking total cofibers over $S \subseteq n$ yields morphism 4.1.

Next, observe that $\mathsf{FI}(\emptyset, k) \cong *$ and that $\operatorname{cofib}(X \to *)_+ \cong \Sigma X$ for any (unbased) space X (since X is unbased, we regard ΣX as the unreduced suspension of X with the tip of one cone chosen as a basepoint). We therefore see that

$$L(n,k) \cong \Sigma \operatorname*{colim}_{\emptyset \neq S \subseteq n} \mathsf{FI}(S,k) \tag{4.3}$$

where the colimit is taken in the ∞-category of unbased spaces and again we choose a basepoint when we suspend. Let us consider the category of elements of the functor $\mathsf{FI}(-,k) : \mathsf{FI}_{/n,>0} \to \mathcal{S}$. This is the partially ordered set $P(n,k)$ of tuples (S,T,ϕ) where $\emptyset \neq S \subseteq n$, $\emptyset \neq T \subseteq k$, and $\phi : S \cong T$ with order given by $(S,T,\phi) \leq (S',T',\phi')$

if $S \subseteq S'$, $T \subseteq T'$, and $\phi'|_S = \phi$. There is an evident isomorphism $P(n,k) \cong P(k,n)$, and because $P(n,k)$ is the category of elements of $\mathsf{FI}(-,k) : \mathsf{FI}_{/n,>0} \to \mathcal{S}$, we have $NP(n,k) \cong \operatorname{colim}_{\emptyset \neq S \subseteq n} \mathsf{FI}(S,k)$ where N denotes the nerve of the poset.

The symmetry $P(n,k) \cong P(k,n)$ reveals the symmetry $L(n,k) \cong L(k,n)$ by eq. (4.3). Combining this with eq. (4.1), we have

$$\operatorname{cofib}\left(L(n,k) \to L(n \sqcup 1, k)\right) \cong \bigvee_{x \in k} \Sigma L(n, k \setminus \{x\}) \tag{4.4}$$

We make the inductive hypothesis that for some n there exists $C(n)$ such that for all $k \geq C(n)$, $L(n,k)$ is a wedge of copies of S^n. Let $k \geq C(n)$ and consider the long exact sequence in homology induced by eq. (4.4) (but replace k with $k \sqcup 1$). By our inductive hypothesis, for $i \neq n$,

$$H_{i+1}\left(\bigvee_{x \in k \sqcup 1} \Sigma L(n, k \sqcup 1 \setminus \{x\})\right) \cong 0 \cong H_i(L(n, k \sqcup 1))$$

so $H_i(L(n \sqcup 1, k \sqcup 1)) \cong 0$ whenever $i \notin \{n, n+1\}$ and the morphism

$$H_n(L(n, k \sqcup 1)) \twoheadrightarrow H_n(L(n \sqcup 1, k \sqcup 1)) \tag{4.5}$$

is a surjection. Similarly, using eq. (4.1) (and replacing n and k with $n \sqcup 1$ and $k \sqcup 1$ respectively), we have that

$$H_n(L(n \sqcup 1, k \sqcup 1)) \twoheadrightarrow H_n(L(n \sqcup 1, k \sqcup 2)) \tag{4.6}$$

is surjective. Composing morphisms 4.5 and 4.6, we have a surjection

$$H_n(L(n, k \sqcup 1)) \twoheadrightarrow H_n(L(n \sqcup 1, k \sqcup 2)) \tag{4.7}$$

We now show that the map

$$L(n, k \sqcup 1) \longrightarrow L(n \sqcup 1, k \sqcup 2)$$

is nullhomotopic, so that morphism 4.7 and therefore also $H_n(L(n \sqcup 1, k \sqcup 2))$ must be trivial. This is easiest to see in terms of the posets $P(n, k \sqcup 1)$ and $P(n \sqcup 1, k \sqcup 2)$. Let us establish the notation $1 = \{a\}$ and $2 = \{a, b\}$. Consider the following diagram:

$$\begin{tikzcd}
P(n, k \sqcup 1) \arrow[r, "\mathrm{id}"] \arrow[d] \arrow[dr, "f"] & P(n, k \sqcup 1) \arrow[d, "g"] \\
* \arrow[r, "\{a\} \cong \{b\}"] & P(n \sqcup 1, k \sqcup 2)
\end{tikzcd}$$

We let g be the inclusion and

$$f : (S, T, \phi) \mapsto (S \sqcup \{a\}, T \sqcup \{b\}, \phi \sqcup (\{a\} \cong \{b\}))$$

Then there are natural transformations $g \Rightarrow f$ and $(\{a\} \cong \{b\}) \circ * \Rightarrow f$. After taking the nerve, these natural transformations become homotopies, and composing these homotopies yields a null-homotopy of g. But g is the morphism which induces morphism 4.7.

We have proven the following: if there exists $C(n)$ such that for all $k \geq C(n)$, $L(n, k)$ is a wedge of n-spheres, then for all $k \geq C(n)$, the homology of $L(n \sqcup 1, k \sqcup 2)$ is free (since by the long exact sequence from eq. (4.4) it must be a subgroup of the homology of a wedge of spheres) and is concentrated in degree $n + 1$. Since $P(\emptyset, k) = \emptyset$ for all k, $L(\emptyset, k) = S^0$ for all k. For all $k \geq 1$, $L(1, k)$ is the suspension of a non-empty discrete space and therefore a wedge of circles. We will show that for $n \geq 2$ and $k \geq 2n - 1$, $L(n, k)$ is the suspension of a connected space and therefore simply-connected. By induction, $L(n, k)$ is a Moore space of type $M(G, n)$ for G free and is therefore a wedge of n-spheres.

Let us show that when $n \geq 2$ and $k \geq 3$, $NP(n, k)$ is connected. Each simplex is attached to a vertex, and each vertex is connected by a 1-simplex to a vertex of the form $(\{x\}, \{y\}, \phi)$. We will denote vertices $(\{x_1, \ldots, x_j\}, \{y_1, \ldots, y_j\}, \phi)$ by

$$\begin{pmatrix} x_1 & \phi(x_1) \\ \vdots & \vdots \\ x_j & \phi(x_j) \end{pmatrix}$$

Let $x, x' \in n$ be distinct and $y, y', y'' \in k$ be distinct. We must show that $\begin{pmatrix} x & y \end{pmatrix}$ is connected to $\begin{pmatrix} x' & y' \end{pmatrix}$, to $\begin{pmatrix} x' & y \end{pmatrix}$, and to $\begin{pmatrix} x & y' \end{pmatrix}$. We have

$$\begin{pmatrix} x & y \end{pmatrix} < \begin{pmatrix} x & y \\ x' & y' \end{pmatrix} > \begin{pmatrix} x' & y' \end{pmatrix} \tag{4.8}$$

$$\begin{pmatrix} x & y \end{pmatrix} < \begin{pmatrix} x & y \\ x' & y'' \end{pmatrix} > \begin{pmatrix} x' & y'' \end{pmatrix} < \begin{pmatrix} x & y' \\ x' & y'' \end{pmatrix} > \begin{pmatrix} x & y' \end{pmatrix} \tag{4.9}$$

$$\begin{pmatrix} x & y' \end{pmatrix} < \begin{pmatrix} x & y' \\ x' & y \end{pmatrix} > \begin{pmatrix} x' & y \end{pmatrix} \tag{4.10}$$

where we compose sequences 4.9 and 4.10 to obtain a path from $\begin{pmatrix} x & y \end{pmatrix}$ to $\begin{pmatrix} x' & y \end{pmatrix}$.

□

4.2 A dictionary

Throughout the remainder of this section, we will at times treat rational vector spaces as rational spectra concentrated in dimension 0. For X a spectrum, we denote by $X^{\mathbb{Q}}$ its rationalization. We call functors

$$\mathsf{FI} \to \mathbb{Q}\mathbf{Vect}$$

FI-*modules.*

Theorem 54. *For some $n \in \mathbb{N}$, let E be an n-excisive* FI*-object taking values in rational spectra with finitely generated homology groups. Then the* FI*-modules $H_i(E)$ are representation stable.*

Proof. When $n = 0$, E must be constant. An objectwise-finite FI-module is representation stable if and only if it is finitely generated by [CEF15, Theorem 1.13]. Moreover, the category of $\mathbf{Fin}\mathbb{Q}\mathbf{Vect}$-FI-objects is Noetherian by [CEF15, Theorem 1.3]. The homology of an m-cohomogeneous FI-object is generated entirely in degree m, so the cofiber sequence $\mathbf{Q}_{n-1}E \to E \to \mathbf{R}_n E$ gives us our desired result, since $H_i(E)$ is the extension of a sub-FI-module of $H_i \mathbf{R}_n E$ by a quotient FI-module of $H_i(\mathbf{Q}_{n-1}E)$, both of which are finitely generated. □

We know from Theorem 53 that the homology of $G_n(k)$ is concentrated in dimension 0 for $k \geq 2n - 1$. Let us get to know $H_0(G_n, \mathbb{Q})$ better. First we recall some facts about the representation theory of the symmetric groups.

Recollection 55. Given an irreducible rational representation V of $\mathfrak{S}_n$, its complexification $V \otimes_{\mathbb{Q}} \mathbb{C}$ is an irreducible complex representation of $\mathfrak{S}_n$, so the representation theory of $\mathfrak{S}_n$ is the same over any characteristic 0 field, regardless of algebraic closure. This is a well-known fact, but can be seen for instance from the facts that the so-called Specht modules can be defined over the integers [Sag01, Section 2.3] and account for all irreducible complex representations [Sag01, Section 2.4].

We call a finite, weakly decreasing sequence of natural numbers $\lambda = (\lambda_1, \ldots, \lambda_j)$ a numerical partition.[1] We write

$$|\lambda| \stackrel{\text{def}}{=} \sum_i \lambda_i$$

[1]These are usually called "partitions" of n in the combinatorics literature, but we wish to distinguish them from sets of disjoint sets whose union is n.

and say that λ is a numerical partition of $|\lambda|$. The notation $\lambda \vdash n$ is synonymous with $|\lambda| = n$.

Recall e.g. from [Sag01, Section 2.3] that the Specht modules of n are in bijection with numerical partitions of n. We denote by $V(\lambda)$ the Specht module corresponding to a numerical partition λ. Given a numerical partition λ and $k \geq \lambda_1 + |\lambda|$, we define $\lambda[k]$ to be the numerical partition $(k - |\lambda|, \lambda)$. We define $w(\lambda) \overset{\text{def}}{=} |\lambda| - \lambda_1$ and we say that $w(\lambda)$ is the weight of λ. Observe that $w(\lambda[k]) = |\lambda|$ when $\lambda[k]$ is defined.

For $\mu \vdash n$, we have a $\mathfrak{S}_n$-representation M^μ called a Young permutation representation and defined in [Sag01, Section 2.1]. By [Sag01, Section 2.10], for $\lambda \vdash n$, the dimension of $\mathbb{Q}\mathbf{Vect}_{\mathfrak{S}_n}(V(\lambda), M^\mu) = K_{\lambda,\mu}$, where $K_{\lambda,\mu}$ is a Kostka number: the number of semistandard Young tableaux of shape λ and content μ. This means the following. For $\lambda = (\lambda_1, \ldots, \lambda_j)$, we consider n boxes arranged in j rows with λ_i boxes in row i. A semistandard tableau of shape λ and content μ is a way of filling these boxes with natural numbers such that the number i occurs μ_i times, the columns of our tableau are strictly increasing from top to bottom, and the rows of our tableau are weakly increasing from left to right. A standard λ tableau simply means a semistandard λ tableau with content (1^n).

Finally, recall some notation from [CEF15]. Given a rational $\mathfrak{S}_n$-representation $V : \mathfrak{S}_n \to \mathbb{Q}\mathbf{Vect}$, we define $M(V)_\bullet \overset{\text{def}}{=} \mathrm{Lan}_{\mathfrak{S}_n}^{\mathsf{FI}} V$, the left Kan extension of V to FI. We also recall the FI-object $V(\lambda)_\bullet$ defined in [CEF15, Proposition 3.1.4], which is representation stable and satisfies $V(\lambda)_k \cong V(\lambda[k])$ when $k \geq \lambda_1$ and $V(\lambda)_k \cong 0$ otherwise.

Observe that $G_n^{\mathbb{Q}}$ has a natural action of $\mathfrak{S}_n$ because F_n does. In the following theorem, we characterize $G_n^{\mathbb{Q}}$ along with its $\mathfrak{S}_n$-action.

Theorem 56. *We have an isomorphism in the category* $Fun(\mathfrak{S}_n \times \mathsf{FI}_{\geq 2n}, \mathcal{S}p)$

$$G_n^{\mathbb{Q}} \cong \bigoplus_{\lambda \vdash n} V(\lambda) \boxtimes V(\lambda)_{\bullet}$$

Proof. Given a rational $\mathfrak{S}_k$-spectrum X and $\lambda \vdash k$, define $\chi_\lambda(X)$ to be the Euler characteristic of the spectrum $\mathcal{S}p_{\mathfrak{S}_k}(V(\lambda), X)$ if it exists. Then

$$\chi_\lambda\left(G_n^{\mathbb{Q}}(k)\right) = \sum_{0 \leq i \leq n} (-1)^{n-i} \binom{n}{i} \chi_\lambda\left(F_i^{\mathbb{Q}}(k)\right) \tag{4.11}$$

We observe that $F_i^{\mathbb{Q}}(k) \cong M^{\left(k-i,1^i\right)}$ so that

$$\chi_\lambda\left(F_i^{\mathbb{Q}}(k)\right) = K_{\lambda,(k-i,1^i)}$$

In a semistandard tableau of shape λ and content $(k-i, 1^i)$, the $k-i$ 1s must be in left-most boxes of the first row of the tableau, so we need only consider $\lambda \vdash k$ such that $\lambda_1 \geq k-n$, and since $k \geq 2n$, we have $k - \lambda_1 \leq n \leq k-i$. This means that there are no boxes directly below any box after the $k-i$th box in the first row of our tableau, so for any subset of $\{2, \ldots, i+1\}$, there is exactly one way to fill out the rest of the first row of our tableau. Define $\lambda' \stackrel{\text{def}}{=} (\lambda_2, \ldots, \lambda_j)$. Once we have chosen how to fill the top row of our semistandard λ-tableau under construction, any standard λ' tableau determines a unique semistandard λ tableau with the given first row (since all of the relevant numbers in the first row are 1s and all of our remaining numbers after the first row are unique and greater than 1). Note that $\lambda_1 = k - w(\lambda)$, so we have shown that

$$K_{\lambda,(k-i,1^i)} = \binom{i}{\lambda_1 - k + i} K_{\lambda',\lambda'} = \binom{i}{w(\lambda)} K_{\lambda',\lambda'}$$

Combining this with eq. (4.11), we have

$$\chi_\lambda\left(G_n^{\mathbb{Q}}(k)\right) = K_{\lambda',\lambda'} \sum_{0 \leq i \leq n} (-1)^{n-i} \binom{n}{i} \binom{i}{w(\lambda)} \tag{4.12}$$

Note that the quantity $\binom{n}{i}\binom{i}{w(\lambda)}$ is the number of pairs $B \subseteq A \subseteq n$ such that $|B| = w(\lambda)$ and $|A| = i$. We could also count these pairs by choosing $B \subseteq n$ and then choosing $A \setminus B \subseteq n \setminus B$. This observation gives us the identity

$$\binom{n}{i}\binom{i}{w(\lambda)} = \binom{n}{w(\lambda)}\binom{n-w(\lambda)}{i-w(\lambda)}$$

Substituting this into eq. (4.12), we have

$$\begin{aligned}
\chi_\lambda\left(G_n^{\mathbb{Q}}(k)\right) &= K_{\lambda',\lambda'}\binom{n}{w(\lambda)}\sum_{0\le i\le n}(-1)^{n-i}\binom{n-w(\lambda)}{i-w(\lambda)} \\
&= K_{\lambda',\lambda'}\binom{n}{w(\lambda)}\sum_{0\le j\le n-w(\lambda)}(-1)^{n-w(\lambda)-j}\binom{n-w(\lambda)}{j} \\
&= K_{\lambda',\lambda'}\binom{n}{w(\lambda)}(1-1)^{n-w(\lambda)} \\
&= \begin{cases} K_{\lambda',\lambda'} & n = w(\lambda) \\ 0 & n \neq w(\lambda) \end{cases} \qquad (4.13)
\end{aligned}$$

This establishes that for $\lambda \vdash k$, $\chi_\lambda\left(G_n^{\mathbb{Q}}(k)\right) \neq 0$ if and only if $w(\lambda) = n$. This implies that for $k \geq 2n$ and V an irreducible $\mathfrak{S}_k$-representation, $\mathcal{S}p_{\mathfrak{S}_k}\left(V, G_n^{\mathbb{Q}}(k)\right) \neq 0$ if and only if $V \cong V(\lambda')_k$ for some $\lambda' \vdash n$.

Note that by Theorem 53, $H_0\left(G_n^{\mathbb{Q}}\right)$ is a sub-$\mathfrak{S}_n \times \mathsf{FI}_{\geq 2n}$-module of $H_0\left(F_n^{\mathbb{Q}}\right)$. Observe that $F_n^{\mathbb{Q}} = M(\mathbb{Q}[\mathfrak{S}_n])$. By Maschke's Theorem,

$$\begin{aligned}
\mathbb{Q}[\mathfrak{S}_n] &\cong \bigoplus_{\lambda\vdash n}\operatorname{End}(V(\lambda)) \\
&\cong \bigoplus_{\lambda\vdash n} V(\lambda)^* \boxtimes V(\lambda) \\
&\cong \bigoplus_{\lambda\vdash n} V(\lambda) \boxtimes V(\lambda)
\end{aligned}$$

where the last isomorphism holds because in characteristic zero, finite dimensional representations of $\mathfrak{S}_n$ are self-dual. We therefore have that

$$F_n^{\mathbb{Q}} \cong \bigoplus_{\lambda \vdash n} V(\lambda) \boxtimes M\left(V(\lambda)\right)$$

By [CEF15, Lemma 3.2.3 and Proposition 3.4.1], the weight n irreducible sub-representations of the $M(V(\lambda))(k)$ form a sub-FI-object of $M(V(\lambda))$ and indeed are exactly $V(\lambda)_\bullet$. □

Corollary 57. *For E an n-homogeneous rational FI-object with $\mathbf{C}_n E \cong V(\mu)$ for $\mu \vdash n$, $E|_{\mathsf{FI}_{\geq 2n}} \cong V(\mu)_\bullet|_{\mathsf{FI}_{\geq 2n}}$.*

Proof. Restricting to $\mathsf{FI}_{\geq 2n}$, we have

$$\begin{aligned}
E &\cong V(\mu) \wedge_{\mathfrak{S}_n} G_n^{\mathbb{Q}} \\
&\cong \bigoplus_{\lambda \vdash n} \left(V(\mu) \otimes V(\lambda)\right)_{\mathfrak{S}_n} \boxtimes V(\lambda)_\bullet \\
&\cong \bigoplus_{\lambda \vdash n} \left(V(\mu) \otimes V(\lambda)\right)^{\mathfrak{S}_n} \boxtimes V(\lambda)_\bullet \\
&\cong \bigoplus_{\lambda \vdash n} \left(V(\mu)^* \otimes V(\lambda)\right)^{\mathfrak{S}_n} \boxtimes V(\lambda)_\bullet \\
&\cong \bigoplus_{\lambda \vdash n} \mathrm{Hom}_{\mathfrak{S}_n}(V(\mu), V(\lambda)) \boxtimes V(\lambda)_\bullet \\
&\cong V(\mu)_\bullet
\end{aligned}$$

□

Since every rational $\mathfrak{S}_n$-spectrum is a direct sum of ((de)suspensions of) spectra of the form appearing in the hypothesis of Corollary 57, we now have an elementary dictionary allowing us to translate between rational $\mathfrak{S}_n$-spectra and n-homogeneous rational FI-objects – in other words, we have made the equivalence in Proposition 20 explicit (in the rational case). In fact, we have the following additional corollary.

Corollary 58. *For E a rational FI-object and $k \geq 2n$, there is an isomorphism of*

$\mathfrak{S}_k$-spectra

$$\mathbf{P}_n E(k) \cong \bigvee_{i \leq n} D_i E(k)$$

Proof. The result follows from an inductive argument. The case $n = 0$ holds since $\mathbf{P}_0 = \mathbf{D}_0$. For the inductive step, we apply Corollary 57 and Schur's Lemma to the long exact sequence in homology associated to the fiber sequence $\mathbf{D}_n E(k) \to \mathbf{P}_n E(k) \to \mathbf{P}_{n-1} E(k)$. This establishes that the two sides of our equation agree on the level of homology (including the $\mathfrak{S}_k$-action), and $\mathbb{Q}[\mathfrak{S}_k]$ is semisimple, this implies that they agree as $\mathfrak{S}_k$-spectra $\square$

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