Some Topics in Kinetic and Dispersive PDEs

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Abstract of “Some Topics in Kinetic and Dispersive PDEs” by Zhimeng Ouyang, Ph.D., Brown University, May 2021

This dissertation is devoted to the study of some topics in the theory of kinetic equations and nonlinear dispersive systems. It is divided into two parts.

In the first part, we consider the Vlasov-Poisson-Landau system, a classical model for a dilute collisional plasma interacting through Coulombic collisions and with its self-consistent electrostatic field. We establish global stability and well-posedness near the Maxwellian equilibrium state with decay in time and some regularity results for small initial perturbations, in any general bounded domain (including a torus as in a tokamak device), in the presence of specular-reflection boundary condition. We provide a new improved $L^2 \to L^\infty$ framework: L^2 energy estimate combines only with S^p estimate for ultra-parabolic equations.

The second part concerns the long-time behavior of a Wave-Klein-Gordon coupled system in 3D, which is a simplified model for the global nonlinear stability of the Minkowski space-time for self-gravitating massive fields. We study the global behavior of solutions to such systems, and prove modified wave operators for small and smooth data with mild decay at infinity. The key novelty comes from a crucial observation that the asymptotic dynamics is dictated by the resonant interactions. As a consequence, our main results include the derivation of a resonant system with good error bounds, and a detailed description of the asymptotic dynamics of such quasilinear evolution system of hyperbolic and dispersive type.

Contents

Chapter One

Introduction

In this thesis, we study a variety of nonlinear partial differential equations which arise in physical contexts, including the Vlasov-Poisson-Landau system in kinetic theory, as well as nonlinear dispersive equations such as the Wave-Klein-Gordon system in general relativity. The results we obtain involve global existence and uniqueness of solutions, stability of equilibrium, and large-time asymptotic behavior.

1.1 Kinetic Equations in Bounded Domains

Background. Kinetic theory describes the motion of a large number ($\sim 10^{23}$) of particles. When the particles interact through Coulombic collisions, the density function $F(t,x,v)$ satisfies the Landau equation

$$\partial_t F + v \cdot \nabla_x F = \mathcal{Q}[F,F]\,, \tag{1.1.1}$$

with the collision term $\mathcal{Q}$ being a bilinear Fokker-Planck-type operator. Here $F(t,x,v)$ denotes the probability a particle is present in the position $x \in \Omega$ with velocity $v \in \mathbb{R}^3$ at time $t \in \mathbb{R}_+$. For charged particles in plasmas, if we take the self-consistent electrostatic field into account, the density functions $F_\pm(t,x,v)$ (for ions and electrons) satisfy the Vlasov-Poisson-Landau system

$$\begin{aligned} \partial_t F_\pm + v \cdot \nabla_x F_\pm \pm \mathbf{E} \cdot \nabla_v F_\pm \;&=\; \mathcal{Q}\left[F_+, F_\pm\right] + \mathcal{Q}\left[F_-, F_\pm\right], \\ \mathbf{E}(t,x) \;&=\; \nabla_x\, \Delta_x^{-1} \int_{\mathbb{R}^3} \big(F_+ - F_-\big) \mathrm{d}v\,. \end{aligned} \tag{1.1.2}$$

In bounded domains, we prescribe the specular-reflection boundary condition

$$F(t,x,v) = F(t,x,R_x v), \tag{1.1.3}$$

where $R_x v = v - 2(n_x \cdot v)n_x$ and n_x is the unit outward normal vector at the boundary.

For the analysis of kinetic equations in bounded domains, the fundamental difficulty lies in the singularity near the grazing set. This poses a major challenge in establishing the global well-posedness (existence and uniqueness) and time decay of solutions.

Main Result. In Chapter 2, we establish the global well-posedness and asymptotic stability (almost exponential decay) for small initial perturbation near the Maxwellian equilibrium state μ. The analysis is based on low-regularity Sobolev-type and Hölder norms.

Our results in [30] and [12] mark the first successful approach in developing global well-posedness of kinetic equations with Landau collisions in general bounded domains including non-convex domains (e.g. in a "donut-shape" torus, which resembles a tokamak device). It not only presents systematic mechanism to tackle the singularity coming from the grazing set, but also provides insight into more physically-relevant settings.

Methodology. The proof relies on two key ingredients: energy estimate and S^p theory. The L^2 energy structure provides the existence of the solution and bounds of $f = \mu^{-\frac{1}{2}}(F - \mu)$. However, since $\mathcal{Q}$ contains nonlinear diffusion in the velocity space, L^∞ is not enough to guarantee the uniqueness as in the Boltzmann equation [27]; some control of $\nabla_v f$ is needed. We apply S^p estimates for the ultra-parabolic equations (counterpart of $W^{2,p}$ estimates for the parabolic equations) to bound the velocity derivatives and justify the uniqueness.

In order to implement such a strategy, we introduce several new techniques:

- We develop a “two-tier energy method”, which is based on a bootstrap argument, and the delicate interaction of boundedness and decay.
- In order to utilize the S^p theory, which is applicable only in the whole space domain, we design a new flattening-reflection extension beyond the specular boundary.
- We connect the weighted L^2 estimates and the S^p theory through the $L^2 - L^\infty$ framework and a refined De Giorgi–Nash–Moser theory [30]. In our recent paper [12], we develop a more elegant approach based on the S^p embedding theorem to simplify this argument.

Eventually, we obtain the weighted L^2 and L^∞ estimates and decay, Hölder continuity and desired regularity results.

Future Plan. Inspired by my completed projects above, I plan to explore other more involved models and settings. Some ongoing and potential future projects include the Vlasov-Maxwell-Landau system, the Landau equation in bounded domains with the diffusive-reflection boundary condition.

1.2 Asymptotic Dynamics of Nonlinear Dispersive Systems

The main theme of Chapter 3 is the study of the long-time behavior of some nonlinear hyperbolic and dispersive systems in cases where linear scattering does not hold and nonlinear effects have a strong impact on the qualitative behavior of solutions.

Background. We consider the Wave–Klein-Gordon (W-KG) system in three dimensions:

$$\begin{aligned} (\partial_t^2 - \Delta)u \;&=\; |\nabla_{t,x} v|^2 + v^2\,, \\ (\partial_t^2 - \Delta + 1)v \;&=\; u\Delta v\,. \end{aligned} \tag{1.2.1}$$

This coupled system consists of a semi-linear wave equation for u and a quasi-linear Klein-Gordon equation for v.

The system was derived in [56] as a simplified model for the Einstein–Klein-Gordon (E-KG) system, which describes the coupled evolution of the Lorentzian metric and a self-gravitating massive scalar field. Physically, to study the global stability [45, 46] and asymptotic dynamics of E-KG system near the Minkowski space-time is one of the central topics in general relativity. The W-KG system keeps the same linear structure as the E-KG system in harmonic gauge, but only preserves quadratic interactions that involve the massive scalar field.

Main Result. This research focuses on the long-time asymptotics of the small-data solutions to the W-KG system. In details, we prove the modified wave operators.

The key novelty lies in deriving improved "effective" dynamics that control the asymptotic behavior. More specifically, for any solution (u_∞, v_∞) of the resonant system, there exists a solution (u, v) to the W-KG system such that as $t \to \infty$,

$$\left\|\nabla_{t,x}\left[u(t) - u_\infty(t)\right]\right\|_{L^2} + \left\|\nabla_{t,x}\left[v(t) - v_\infty(t)\right]\right\|_{L^2} + \left\|v(t) - v_\infty(t)\right\|_{L^2} \to 0. \quad (1.2.2)$$

Here (u_∞, v_∞) captures the resonant interactions. Roughly speaking,

$$(\partial_t - \mathrm{i}\Lambda_{wa})u_\infty(t) \sim \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\Big[(\partial_t - \mathrm{i}\Lambda_{wa})u_\infty(0) + H(t)\Big], \quad (1.2.3)$$

$$(\partial_t - \mathrm{i}\Lambda_{kg})v_\infty(t) \sim \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathrm{e}^{\mathrm{i}\Theta(t)}(\partial_t - \mathrm{i}\Lambda_{kg})v_\infty(0), \quad (1.2.4)$$

where $\Lambda_{wa} = |\nabla|$ and $\Lambda_{kg} = \langle\nabla\rangle = \sqrt{|\nabla|^2 + 1}$. Also, $H(t)$ is a low-frequency approximation of $\int_0^t |v(s)|^2\,\mathrm{d}s$ and $\Theta(t) \sim \int_0^t u_{\mathrm{low}}(s)\,\mathrm{d}s$ where u_{low} is a low-frequency part of u_∞ with proper multipliers.

Our analysis provides a complete characterization of the global behavior and scattering for the W-KG system. In particular, our results demonstrate that the wave operator problem actually captures the more precise scattering information, including the critical integrable terms (which decays like $\langle t\rangle^{-1}$).

Methodology. One of the main difficulties comes from the fact that we have a genuine system in the sense that the linear evolution of wave and Klein-Gordon equations admit different modes of propagation, combining to create new dynamics that cannot simply be predicted by looking at each component of the system in isolation.

On a more technical level, it turns out that the main feature of the system above is the slow decay of the low frequencies of wave component u in the interior of the light cone and in particular along the characteristics associated to the Klein-Gordon operator. This nonlinear effect ultimately leads to modified scattering for the Klein-Gordon component.

Our proof is based on a combination of the energy method and the dispersive analysis. We utilize both the weak formulation (energy structure) and mild formulation (Duhamel's formula). Energy estimate provides basic control of quantities under Sobolev-type norms. Dispersive estimate offers sufficiently fast time decay in L^∞ norm to handle the time integral of nonlinear terms.

In particular, for the wave operators, we focus on controlling the perturbation globally using a combination of tools from harmonic analysis (Fourier transform, dyadic decomposition), oscillatory integrals (non-stationary/stationary phase analysis), bilinear estimates (pseudo-products operators with singular multipliers), ODE and geometric methods (normal forms, vector fields), and delicately designed function spaces and decay-rate. We carefully analyze the nonlinear term in the frequency space and bound the resonant and non-resonant interactions in an intricate manner.

Future Plan. In the future, we plan to study the full nonlinear E-KG system. We intend to work on the wave-coordinates, which makes the system quasi-linear. However, the wave component is no longer asymptotically free and requires a phase-correction to the asymptotic behavior; its slow decay also makes the energy estimates significantly more complicated. This can be compensated by introducing proper bilinear estimates and a more careful analysis of the quadratic interactions in both semi-linear and quasi-linear levels.

Chapter Two

The Vlasov-Poisson-Landau System with the Specular-Reflection Boundary Condition

2.1 Introduction and Setup

2.1.1 The Vlasov-Poisson-Landau System

In this chapter we are interested in the global well-posedness and stability of the Vlasov-Poisson-Landau (VPL) system in three dimensions, which is considered as a fundamental collisional plasma model. In the absence of magnetic effects, the dynamics of dilute charged particles (e.g. electrons and ions) can be described by the Vlasov-Landau equations:

$$\begin{aligned}
&\partial_t F_+ + v\cdot\nabla_x F_+ + \frac{e_+}{m_+}\mathbf{E}\cdot\nabla_v F_+ \;=\; \mathcal{Q}\left[F_+,F_+\right] + \mathcal{Q}\left[F_-,F_+\right],\\
&\partial_t F_- + v\cdot\nabla_x F_- - \frac{e_-}{m_-}\mathbf{E}\cdot\nabla_v F_- \;=\; \mathcal{Q}\left[F_+,F_-\right] + \mathcal{Q}\left[F_-,F_-\right],\\
&F_\pm(0,x,v) = F_{0,\pm}(x,v)\,.
\end{aligned} \tag{2.1.1}$$

Here the unknowns $F_\pm(t,x,v)\geq 0$ are the (real-valued) density distribution functions for ions $(+)$ and electrons $(-)$, respectively, at time $t\geq 0$, near the position $x=(x_1,x_2,x_3)\in\Omega\subset\mathbb{R}^3_x$, and having the velocity $v=(v_1,v_2,v_3)\in\mathbb{R}^3_v$, where Ω denotes a bounded domain in $\mathbb{R}^3$. The function $\mathbf{E}(t,x)$ is the self-consistent electrostatic field created by all plasma particles, which depends in a complex way on the distribution functions $F_\pm(t,x,v)$. The constants $e_\pm$, $m_\pm$ represent the magnitude of the particles' electric charges and masses.

The classical VPL system models the dynamics of a collisional plasma interacting with its own electrostatic field as well as its grazing collisions. Such grazing collisions are given by the famous Landau (Fokker-Planck) collision operator, proposed by

Landau in 1936 [1]:

$$\mathcal{Q}\,[F_1,F_2](v) := \frac{c_{12}}{m_1}\,\nabla_v\cdot\left\{\int_{\mathbb{R}^3}\Phi(v-v')\left[\frac{1}{m_1}F_1(v')\nabla_v F_2(v)-\frac{1}{m_2}F_2(v)\nabla_v F_1(v')\right]\mathrm{d}v'\right\},$$

where the (generalized) Landau collision kernel Φ is defined as

$$\Phi(v) := \left\{I_3-\frac{v}{|v|}\otimes\frac{v}{|v|}\right\}\cdot|v|^{\gamma+2}\,,\quad \gamma\in[-d,1\,]\,. \tag{2.1.2}$$

This is a non-negative symmetric 3×3 matrix, and we will focus on the original physical case for Coulomb interaction corresponding to $\gamma=-d=-3$.

Since the Vlasov-Poisson equations are an approximation of the Vlasov-Maxwell equations in the non-relativistic zero-magnetic field limit, the self-consistent electric field $\mathbf{E}(t,x)$ is determined by a scalar electric potential $\phi(t,x)$:

$$\mathbf{E}=-\nabla_x\phi\,, \tag{2.1.3}$$

and the electric potential ϕ satisfies the Poisson equation for electrostatics, which is

$$-\,\Delta_x\phi = 4\pi\rho := 4\pi\int_{\mathbb{R}^3}\left[e_+F_+-e_-F_-\right]\mathrm{d}v\,, \tag{2.1.4}$$

with a Dirichlet or Neumann boundary condition (and some additional conditions to guarantee the solvability). Here $\rho(t,x)$ stands for the electric charge density.

The coupled system (2.1.1) with (2.1.3) and (2.1.4) is called a Vlasov-Poisson-Landau (VPL) system. For notational simplicity, from now on we will consider a model system (for single-species particles, e.g. ions (+)) with all constants normalized

[1] The constant $c_{12}=2\pi e_1^2e_2^2\ln\Lambda$, $\ln\Lambda=\ln\left(\frac{\lambda_D}{b_0}\right)$, where $\lambda_D=\left(\frac{T}{4\pi n_e e^2}\right)^{1/2}$ is the Debye shielding distance and $b_0=\frac{e^2}{3T}$ a typical "distance of closest approach" for a thermal particle.

to be one (The analysis of the full two-species case is similar):

$$\begin{aligned} &\partial_t F + v \cdot \nabla_x F + \mathbf{E} \cdot \nabla_v F = \mathcal{Q}\,[F,F]\,, \\ &F(0,x,v) = F_0(x,v)\,, \end{aligned} \tag{2.1.5}$$

where the distribution function $F(t,x,v) \geq 0$ is defined for $(t,x,v) \in [\,0,\infty) \times \Omega \times \mathbb{R}^3$, and the Landau collision operator (for Coulomb interaction) takes the form

$$\begin{aligned} \mathcal{Q}\,[G,F](v) &:= \nabla_v \cdot \left\{ \int_{\mathbb{R}^3} \Phi(v-v')\,[G(v')\nabla_v F(v) - F(v)\nabla_v G(v')]\,\mathrm{d}v' \right\} \\ &= \partial_i \int_{\mathbb{R}^3} \Phi^{ij}(v-v')\,[G(v')\partial_j F(v) - F(v)\partial_j G(v')]\,\mathrm{d}v'\,, \end{aligned}$$

with the Landau collision kernel ($\gamma = -3$)

$$\Phi^{ij}(v) := \left\{ \delta_{ij} - \frac{v_i v_j}{|v|^2} \right\} \cdot |v|^{-1}\,. \tag{2.1.6}$$

The electrostatic field $\mathbf{E}(t,x) = -\nabla_x \phi(t,x)$, and the potential $\phi(t,x)$ satisfies the Poisson equation:

$$-\Delta_x \phi(t,x) = \rho(t,x) - \rho_0 := \int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v - \rho_0\,, \tag{2.1.7}$$

where the background density ρ_0 is a constant number (to be specified later). The prescribed boundary condition for the Poisson equation is either of Dirichlet type (if we fix values of the potential at the boundary), (for example,)

$$\phi(t,x) = 0 \quad \text{for } x \in \partial\Omega \text{ and } t \geq 0\,,$$

or of Neumann type (if we are given fixed normal component of the electric field across the surface)

$$\frac{\partial \phi(t,x)}{\partial n} = \nabla_x \phi(t,x) \cdot n_x = 0 \quad \text{for } x \in \partial\Omega \text{ and } t \geq 0\,,$$

which means that the electric field at the surface (of a perfect magnetic conductor) can only have a tangential component:

$$\mathbf{E}_\perp = \mathbf{E}(t,x) \cdot n_x = 0 \quad \text{for } x \in \partial\Omega \text{ and } t \geq 0\,.$$

For the Dirichlet boundary condition, no extra restriction is needed; while solving Poisson equation for the potential ϕ with a Neumann boundary condition requires the neutral condition to ensure the existence:

$$\int_\Omega \left[\int_{\mathbb{R}^3} F(t,x,v)\,\mathrm{d}v - \rho_0 \right] \mathrm{d}x = 0 \quad \text{for all } t \geq 0\,,$$

which follows by setting $\rho_0 := \frac{1}{|\Omega|} \iint_{\Omega\times\mathbb{R}^3} F_0(x,v)\,\mathrm{d}v\mathrm{d}x$ and the conservation law of mass (2.1.22). Also a zero-mean condition

$$\int_\Omega \phi(t,x)\,\mathrm{d}x = 0 \quad \text{for all } t \geq 0$$

guarantees the uniqueness of solutions.

Maxwellian and Equations for the Perturbation

Our goal in this article is to construct unique global solutions for the Vlasov-Poisson-Landau system (2.1.5) and (2.1.7) near equilibrium state – the (normalized) global Maxwellian:

$$\mu(v) = e^{-|v|^2}\,.$$

Accordingly, we define the standard perturbation $f(t,x,v)$ to μ by

$$F(t,x,v) = \mu(v) + \sqrt{\mu(v)}\, f(t,x,v)\,.$$

Letting $f(t,x,v)$ be our new unknown, the Vlasov-Poisson-Landau system for the perturbation now reads as follows (see the definition of L and Γ in (2.1.10), (2.1.11), (2.1.12), and (2.1.13)):

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f + \mathbf{E}_f\cdot\nabla_v f + Lf = \Gamma[f,f] + \big\{\mathbf{E}_f\cdot v\big\} f + 2\big\{\mathbf{E}_f\cdot v\big\}\sqrt{\mu}\,, \\ &f(0,x,v) = f_0(x,v)\,, \end{aligned} \tag{2.1.8}$$

where $\mathbf{E}_f = -\nabla_x\phi_f$ is determined by

$$-\Delta_x\phi_f = \int_{\mathbb{R}^3}\sqrt{\mu}\, f\,\mathrm{d}v \tag{2.1.9}$$

with the Dirichlet BC

$$\phi_f = 0 \quad \text{on } \partial\Omega$$

or the Neumann BC

$$\frac{\partial\phi_f}{\partial n} = 0 \quad \text{on } \partial\Omega$$

with $\int_\Omega \phi_f\,\mathrm{d}x = 0\,,\ \forall\, t\geq 0\,,$ and the neutral condition for solvability now becomes

$$\iint_{\Omega\times\mathbb{R}^3} f(t,x,v)\sqrt{\mu(v)}\,\mathrm{d}v\mathrm{d}x = 0 \quad \text{for all } t\geq 0\,,$$

which is automatically satisfied from the conservation of mass (2.1.24) under the assumption that the initial data F_0 has the same mass as the Maxwellian μ, i.e.

$$\int_{\mathbb{R}^3}\mu\,\mathrm{d}v = \frac{1}{|\Omega|}\iint_{\Omega\times\mathbb{R}^3} F_0(x,v)\,\mathrm{d}v\mathrm{d}x = \rho_0\,.$$

By expanding the bilinear collision operator $\mathcal{Q}[G,F] = \mathcal{Q}[\,\mu+\sqrt{\mu}g, \mu+\sqrt{\mu}f\,]$ and noting that $\mathcal{Q}[\mu,\mu]=0$, we can decompose it into a linear part and a nonlinear part:

$$\begin{aligned}\mathcal{Q}[\,\mu+\sqrt{\mu}g, \mu+\sqrt{\mu}f\,] \;&=\; \mathcal{Q}[\mu,\mu] + \mathcal{Q}[\,\mu, \sqrt{\mu}f\,] + \mathcal{Q}[\sqrt{\mu}g, \mu] + \mathcal{Q}[\sqrt{\mu}g, \sqrt{\mu}f\,] \\ &=:\; \mu^{1/2}\big\{Af + Kg + \Gamma[g,f]\big\}\,.\end{aligned}$$

We define the linear operator

$$L := -A - K \tag{2.1.10}$$

with the expressions

$$\begin{aligned}Af := \mu^{-1/2}\mathcal{Q}[\,\mu, \mu^{1/2}f\,] &= \mu^{-1/2}\,\partial_i\big\{\mu^{1/2}\sigma^{ij}[\partial_j f + v_j f\,]\big\} \\ &= \partial_i\left[\sigma^{ij}\partial_j f\right] - \sigma^{ij}v_i v_j f + \partial_i\sigma^i f \\ &= \nabla_v\cdot(\sigma\nabla_v f) + (\nabla_v - v)\cdot(\sigma v)f\,,\end{aligned} \tag{2.1.11}$$

$$Kf := \mu^{-1/2}\mathcal{Q}[\,\mu^{1/2}f, \mu\,] = -\mu^{-1/2}\,\partial_i\Big\{\mu\Big[\Phi^{ij} * \big\{\mu^{1/2}\left[\partial_j f + v_j f\right]\big\}\Big]\Big\}\,, \tag{2.1.12}$$

and the nonlinear operator

$$\begin{aligned}\Gamma[g,f] := \mu^{-1/2}\mathcal{Q}[\,\mu^{1/2}g, \mu^{1/2}f\,] = \partial_i\left[\big\{\Phi^{ij} * \left[\mu^{1/2}g\right]\big\}\,\partial_j f\right] - \big\{\Phi^{ij} * \left[v_i\mu^{1/2}g\right]\big\}\,\partial_j f \\ - \partial_i\left[\big\{\Phi^{ij} * \left[\mu^{1/2}\partial_j g\right]\big\}\,f\,\right] + \big\{\Phi^{ij} * \left[v_i\mu^{1/2}\partial_j g\right]\big\}\,f\,.\end{aligned} \tag{2.1.13}$$

Here we introduce notation for the diffusion matrix (collision frequency) that captures the dissipation of the Landau collision kernel:

$$\sigma_u^{ij}(v) := \big[\Phi^{ij} * u\big](v) = \int_{\mathbb{R}^3} \Phi^{ij}(v-v')\, u(v')\, \mathrm{d}v' \,, \tag{2.1.14}$$

$$\sigma_u(v) := \big[\Phi * u\big](v)\,, \qquad \sigma := \sigma_\mu = \Phi * \mu \,, \tag{2.1.15}$$

$$\sigma^i := \sigma^{ij} v_j = \big[\Phi^{ij} * \mu\big]\, v_j = \Phi^{ij} * [v_j \mu]\,. \tag{2.1.16}$$

The last equality is due to the property that for any fixed i,

$$\sum_j \Phi^{ij}(w)\, w_j \equiv 0\,, \quad w := v - v' \tag{2.1.17}$$

using the specific structure (2.1.6) of the Landau kernel Φ.

Domain and Boundary Condition

Throughout this chapter, our domain $\Omega := \{x \in \mathbb{R}^3 : \zeta(x) < 0\}$ is connected and bounded with $\zeta(x)$ being a smooth function. We also assume that $\nabla\zeta(x) \neq 0$ on the boundary $\partial\Omega = \{x : \zeta(x) = 0\}$. The outward unit normal vector n_x at $x \in \partial\Omega$ is given by

$$n_x := \frac{\nabla\zeta(x)}{|\nabla\zeta(x)|}\,,$$

and it can be extended smoothly near the boundary $\partial\Omega$. Additionally, we say that Ω has a rotational symmetry if there exists a point x_0 and a vector ω such that

$$\big[(x - x_0) \times \omega\big] \cdot n_x = 0 \tag{2.1.18}$$

for all $x \in \partial\Omega$. It is worth noting that our results and methods also apply to a more general class of "non-convex" and "non-simply connected" domains.

We denote also by $\gamma := \partial\Omega \times \mathbb{R}^3$ the phase boundary, and then we split γ into an outgoing boundary γ_+, an incoming boundary γ_-, and the singular boundary γ_0 (i.e., the "grazing set") for grazing velocities, respectively defined as

$$\gamma_+ := \{(x,v) \in \gamma :\, n_x \cdot v > 0\},$$
$$\gamma_- := \{(x,v) \in \gamma :\, n_x \cdot v < 0\},$$
$$\gamma_0 := \{(x,v) \in \gamma :\, n_x \cdot v = 0\}.$$

Our problem concerns the *specular-reflection boundary condition*, which in terms of the density distribution function F, can be formulated as

$$F(t,x,v)|_{\gamma_-} = F(t,x,\mathcal{R}_x v)|_{\gamma_+} \tag{2.1.19}$$

for all $t \geq 0$, where for $(x,v) \in \gamma$,

$$\mathcal{R}_x v := v - 2(n_x \cdot v) n_x\,.$$

This is equivalent to a specular-reflection boundary condition satisfied by the perturbation f:

$$f(t,x,v)|_{\gamma_-} = f(t,x,\mathcal{R}_x v)|_{\gamma_+}\,, \quad \forall\, t \geq 0\,. \tag{2.1.20}$$

Conservation Laws and H-Theorem

The conservation laws will play crucial role in the energy estimates.

The conservation of electric charge is described by the continuity equation

$$\partial_t \rho \, + \, \nabla_x \cdot \mathbf{j} \, = \, 0 \,, \tag{2.1.21}$$

where $\rho(t,x) \, := \, \int_{\mathbb{R}^3} F \, \mathrm{d}v$ is the charge density, and $\mathbf{j}(t,x) \, := \, \int_{\mathbb{R}^3} v \, F \, \mathrm{d}v$ is called the current density. In terms of the perturbation f, we may take

$$\begin{aligned} \rho[f](t,x) \, &:= \int_{\mathbb{R}^3} \sqrt{\mu} \, f \, \mathrm{d}v \,, \\ \mathbf{j}[f](t,x) \, &:= \int_{\mathbb{R}^3} v \sqrt{\mu} \, f \, \mathrm{d}v \,. \end{aligned}$$

Under the specular-reflection boundary condition (2.1.19), It is well-known that both total mass and total energy are conserved for the Vlasov-Poisson-Landau system (2.1.5):

$$\frac{\mathrm{d}}{\mathrm{d}t} \iint_{\Omega \times \mathbb{R}^3} F(t) \, \mathrm{d}v \mathrm{d}x \, \equiv 0 \,, \tag{2.1.22}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \left\{ \iint_{\Omega \times \mathbb{R}^3} |v|^2 F(t) \mathrm{d}v \mathrm{d}x + \int_\Omega |\mathbf{E}(t)|^2 \mathrm{d}x \right\} \equiv 0 \,. \tag{2.1.23}$$

Assuming (without loss of generality, by some rescaling) that initially F_0 has the same mass and total energy as the Maxwellian μ, we can then rewrite the mass-energy conservation laws for the system (2.1.8)(2.1.9) with (2.1.20), in terms of the perturbation f:

$$\iint_{\Omega \times \mathbb{R}^3} f(t) \sqrt{\mu} \, \mathrm{d}v \mathrm{d}x \, \equiv 0 \,, \tag{2.1.24}$$

$$\iint_{\Omega \times \mathbb{R}^3} |v|^2 f(t) \sqrt{\mu} \, \mathrm{d}v \mathrm{d}x \, \equiv - \int_\Omega \left| \mathbf{E}_f(t) \right|^2 \mathrm{d}x \,. \tag{2.1.25}$$

Through the coupled Poisson equation we may further deduce

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\mathbf{E}_f(t)\cdot n\,\mathrm{d}S \;&=\; -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\nabla\phi_f(t)\cdot n\,\mathrm{d}S \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\frac{\partial\phi_f}{\partial n}(t)\,\mathrm{d}S\\ &=\; -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\Delta\phi_f(t)\,\mathrm{d}x \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_{\Omega}\left(\int_{\mathbb{R}^3}\sqrt{\mu}\,f(t)\,\mathrm{d}v\right)\mathrm{d}x\\ &=\; \frac{\mathrm{d}}{\mathrm{d}t}\iint_{\Omega\times\mathbb{R}^3}\sqrt{\mu}\,f(t)\,\mathrm{d}v\mathrm{d}x \;\equiv\; 0\,,\end{aligned}\tag{2.1.26}$$

which means that the flux of the self-consistent electric field is conserved as well. This property will be used in the energy estimate in Section 2.4 . Under the same assumption we also have

$$\int_{\partial\Omega}\mathbf{E}_f(t)\cdot n\,\mathrm{d}S \;\equiv\; 0\,.\tag{2.1.27}$$

Moreover, if the domain Ω has a rotational symmetry (2.1.18), then we will assume additionally that the corresponding conservation of angular momentum is valid for all $t \geq 0$:

$$\iint_{\Omega\times\mathbb{R}^3}\big[(x-x_0)\times\omega\big]\cdot v\,f(t,x,v)\sqrt{\mu(v)}\,\mathrm{d}v\mathrm{d}x = 0\,.\tag{2.1.28}$$

We also recall the celebrated H-Theorem of Boltzmann – the "entropy"

$$\mathcal{H}(t) := \iint_{\Omega\times\mathbb{R}^3}F(t)\ln F(t)\,\mathrm{d}v\mathrm{d}x$$

for the system (2.1.5)(2.1.7) is non-increasing as time passes:

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathcal{H}(t) \leq 0\,, \quad \forall\, t \geq 0\,.$$

This suggests the asymptotic stability of the Maxwellian μ, for which the entropy is minimized. (cf. [67])

2.1.2 Linearization and Reformulation

We linearize the equation (2.1.8) by replacing the nonlinear dependence on $f(t,x,v)$ with a given $g(t,x,v)$:

$$\begin{aligned} &\partial_t f + v\cdot\nabla_x f + \mathbf{E}_g\cdot\nabla_v f + Lf = \Gamma[g,f] + \big\{\mathbf{E}_g\cdot v\big\} f + 2\big\{\mathbf{E}_f\cdot v\big\}\sqrt{\mu}\,, \\ &f(0,x,v) = f_0(x,v)\,, \end{aligned} \tag{2.1.29}$$

where $\mathbf{E}_g = -\nabla_x\phi_g$ and ϕ_g is solved from

$$-\Delta_x\phi_g = \int_{\mathbb{R}^3}\sqrt{\mu}\,g\,\mathrm{d}v \tag{2.1.30}$$

with either Dirichlet BC or Neumann BC with $\int_\Omega \phi_g\,\mathrm{d}x = 0\,,\ \forall\, t\ge 0\,.$

Next we rearrange the terms $-Lf = (A+K)f$ and $\Gamma[g,f]$ as

$$-Lf + \Gamma[g,f] \;=\; \overline{A}_g f + \overline{K}_g f \tag{2.1.31}$$

with

$$\begin{aligned} \overline{A}_g f &:= \partial_i\left[\big\{\Phi^{ij}*\big[\mu+\mu^{1/2}g\big]\big\}\,\partial_j f\right] \\ &\qquad - \big\{\Phi^{ij}*\big[v_j\mu^{1/2}g\big]\big\}\,\partial_i f - \big\{\Phi^{ij}*\big[\mu^{1/2}\partial_j g\big]\big\}\,\partial_i f \\ &=: \nabla_v\cdot\big(\sigma_G\nabla_v f\big) + a_g\cdot\nabla_v f \end{aligned} \tag{2.1.32}$$

and

$$\begin{aligned} \overline{K}_g f &:= Kf + \partial_i\sigma^i f - \sigma^{ij}v_iv_j f \\ &\qquad - \partial_i\big\{\Phi^{ij}*\big[\mu^{1/2}\partial_j g\big]\big\}\,f + \big\{\Phi^{ij}*\big[v_i\mu^{1/2}\partial_j g\big]\big\}\,f\,, \end{aligned} \tag{2.1.33}$$

so that all the terms of $\overline{A}_g f$ contain at least one v-derivatives of f, while $\overline{K}_g f$ has no v-derivatives of f. Notice that the v-derivatives $\partial_j f$ in Kf can always be moved to $\mu^{1/2}$ via integration by parts and then outside the convolution by the property (2.1.17) of the Landau kernel.

Combining terms of the same order in v, we reformulate the linearized equation (2.1.29) as

$$\partial_t f+v\cdot\nabla_x f = \nabla_v\cdot\big(\sigma_G\nabla_v f\big)+\big\{a_g-\mathbf{E}_g\big\}\cdot\nabla_v f+\Big\{\overline{K_g}f+\big(v\cdot\mathbf{E}_g\big)f+2\sqrt{\mu}\,v\cdot\mathbf{E}_f\Big\}\,, \tag{2.1.34}$$

where

$$\sigma_G := \Phi * G = \Phi * \left[\mu + \mu^{1/2}g\right], \tag{2.1.35}$$

$$a_g^i := -\,\Phi^{ij} * \left[v_j\mu^{1/2}g + \mu^{1/2}\partial_j g\right], \tag{2.1.36}$$

$$\mathbf{E}_g := \nabla_x\,\Delta_x^{-1}\int_{\mathbb{R}^3}\sqrt{\mu}\,g\,\mathrm{d}v\,, \tag{2.1.37}$$

and the operator $\overline{K_g}$ is defined in (2.1.33). This equation can be viewed as a kinetic Fokker-Planck equation of the form (cf. [24])

$$\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot\big(\mathbf{A}\,\nabla_v f\big) + \mathbf{B}\cdot\nabla_v f + \mathbf{C}f\,, \tag{2.1.38}$$

where g appears only in the coefficients of the linearized operator for f. We can also fit it into a more general class of hypoelliptic/ultraparabolic operators of Kolmogorov type (with rough coefficients), which allows us to apply some known estimates and results regarding this kind of operators (see Section 2.6).

When the size of g (e.g. $\|g\|_{\infty,\vartheta}\sim\varepsilon$) is sufficiently small, the coefficients in

(2.1.38) have the following basic properties (see Section 2.3 for more details):

$$\begin{aligned}\mathbf{A}(t,x,v) :=&\ \sigma_G(t,x,v)\\ &= \left\{\Phi * \left[\mu + \mu^{1/2} g(t,x,v)\right]\right\}(v)\end{aligned}$$

is a 3×3 non-negative matrix, retaining the same analytical properties (e.g. eigenvalues/spectrum) as σ: (cf. Lemma 3 in [26] and Lemma 2.4 in [53])

$$0 < (1+|v|)^{-3}\,\mathbf{I} \lesssim \mathbf{A}(v) \lesssim (1+|v|)^{-1}\,\mathbf{I}\,.$$

This makes the second-order v-derivatives $\nabla_v \cdot \big(\mathbf{A}\,\nabla_v \circ\big)$ an elliptic operator (but *not uniformly* elliptic in v). Moreover,

$$\mathbf{B}(t,x,v) := a_g(t,x,v) - \mathbf{E}_g(t,x)$$

is a uniformly bounded 3-dimensional vector with (cf. Appendix A in [24])

$$\|\mathbf{B}[g]\|_\infty \le \|a_g\|_\infty + \|\mathbf{E}_g\|_\infty \lesssim \|g\|_\infty < \varepsilon\,.$$

Finally,

$$\mathbf{C}f := \overline{K}_g f + \big(v\cdot \mathbf{E}_g\big)f + 2\sqrt{\mu}\,v\cdot \mathbf{E}_f$$

is an operator bounded in some weighted L^p space. (cf. Lemma 2.9 and Lemma 7.1 in [53])

2.1.3 Definition and Notation

Throughout the chapter, C will generally denote an universal constant that may vary from line to line. The notation $A \lesssim B$ means that $A \le CB$ for some universal constant $C > 0$; we will use $\gtrsim$ and $\simeq$ in a similar standard way. Below is a collection of definitions and notations:

Weighted Norms

$$w(v) := \langle v \rangle = \sqrt{1 + |v|^2}\ ,$$

$$|f|_{p,\vartheta} := \left| \langle v \rangle^{\vartheta} f \right|_{L^p(\mathbb{R}^3)} = \left(\int_{\mathbb{R}^3} \langle v \rangle^{p\vartheta} |f(v)|^p \,\mathrm{d}v \right)^{\frac{1}{p}}\ ,$$
$$\|f\|_{p,\vartheta} := \left\| \langle v \rangle^{\vartheta} f \right\|_{L^p(\Omega\times\mathbb{R}^3)} = \left(\iint_{\Omega\times\mathbb{R}^3} \langle v \rangle^{p\vartheta} |f(x,v)|^p \,\mathrm{d}v\mathrm{d}x \right)^{\frac{1}{p}}\ ,$$

$$|f|_{\infty,\vartheta} := \left| \langle v \rangle^{\vartheta} f \right|_{L^\infty(\mathbb{R}^3)} = \sup_{\mathbb{R}^3} \langle v \rangle^{\vartheta} |f(v)|\,,$$
$$\|f\|_{\infty,\vartheta} := \left\| \langle v \rangle^{\vartheta} f \right\|_{L^\infty(\Omega\times\mathbb{R}^3)} = \sup_{\Omega\times\mathbb{R}^3} \langle v \rangle^{\vartheta} |f(x,v)|\,.$$

Dissipation and Energy

$$\langle f, g \rangle := \int_{\mathbb{R}^3} fg \,\mathrm{d}v\,,$$
$$(f, g) := \iint_{\Omega\times\mathbb{R}^3} fg \,\mathrm{d}v\mathrm{d}x\,,$$

$$\langle f, g \rangle_\sigma := \int_{\mathbb{R}^3} \left[\sigma^{ij} \partial_i f \partial_j g + \sigma^{ij} v_i v_j fg \right] \mathrm{d}v\,,$$
$$(f, g)_\sigma := \iint_{\Omega\times\mathbb{R}^3} \left[\sigma^{ij} \partial_i f \partial_j g + \sigma^{ij} v_i v_j fg \right] \mathrm{d}v\mathrm{d}x\,,$$

$$|f|^2_{\sigma,\vartheta} = \langle f, f\rangle_{\sigma,\vartheta} := \int_{\mathbb{R}^3} \langle v\rangle^{2\vartheta} \left[\sigma^{ij}\partial_i f \partial_j f + \sigma^{ij} v_i v_j f^2\right] \mathrm{d}v \,,$$
$$\|f\|^2_{\sigma,\vartheta} = (f, f)_{\sigma,\vartheta} := \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta} \left[\sigma^{ij}\partial_i f \partial_j f + \sigma^{ij} v_i v_j f^2\right] \mathrm{d}v\mathrm{d}x \,,$$

- Instant Energy:

$$\mathcal{I}_\vartheta[f(t)] := \left\|f(t)\right\|^2_{2,\vartheta} + \left\|\mathbf{E}_f(t)\right\|^2_{L^2_x} \,,$$

- Dissipation Rate:

$$\mathcal{D}_\vartheta[f(t)] := \left\|f(t)\right\|^2_{\sigma,\vartheta} + \left\|\mathbf{E}_f(t)\right\|^2_{L^2_x} \,,$$

- Total Energy (Instant + Accumulative):

$$\mathcal{E}_\vartheta[f(t)] := \mathcal{I}_\vartheta[f(t)] + \int_0^t \mathcal{D}_\vartheta[f(\tau)] \,\mathrm{d}\tau \,.$$

Trace and Integral over Phase-Boundary

$$\gamma f := f|_\gamma \,, \qquad \gamma_\pm f := f|_{\gamma_\pm} \,,$$
$$\|\gamma_\pm f\|_{L^p(\gamma_\pm)} := \left(\iint_{\gamma_\pm} \left|\gamma_\pm f(x,v)\right|^p |v\cdot n_x| \mathrm{d}v\mathrm{d}S_x\right)^{\frac{1}{p}} \,.$$

2.2 Main Results and Ideas

2.2.1 Background and Motivation

Despite the important role kinetic theory plays for describing a plasma, we are not aware any global well-posed theory for classical kinetic models in a “donut shape” torus, which resembles a tokamak device. The fundamental difficulty lies in the singularity for kinetic equations near the grazing set γ_0. Even for the linear free streaming operator $\partial_t + v \cdot \nabla_x$, the grazing set γ_0 is always characteristic but not uniformly characteristic, the most challenging case in the regularity study of hyperbolic PDE in a bounded domain. In general, singularity and discontinuity is expected to develop from the grazing set γ_0, and propagate inside for a non-convex domain [25, 52, 32], and the solutions are of BV at the best. On the other hand, regularity is expected to deteriorate near the grazing set γ_0, but could be confined and localized in a convex domain [40, 33].

Since the regularity plays an important role in establishing uniqueness in the kinetic theory, such a loss of regularity creates fundamental difficulties in the PDE study. Many techniques with Sobolev norms developed for the free space are not suitable, and completely new mathematical frameworks need to be developed. Furthermore, such a loss of regularity can even alter our basic understanding of classical boundary behavior for kinetic theory [71].

Thanks to the possibility of regularity in a convex domain for kinetic theory, there have been important advances in various kinetic models for a plasma in convex domains. In [40, 41], global well-posedness is established for the collisionless Vlasov-Poisson system. In recent work of [4, 3], global well-posedness is established for the collisional Vlasov-Poisson-Boltzmann system near Maxwellian distributions.

An $L^2 \to L^\infty$ framework for the Boltzmann equation is developed in [27] for study of well-posedness in bounded domains, which is based on the basic L^2 energy estimate and a bootstrap to L^∞ bound thanks to an interaction between free streaming characteristics and averaging in velocity in the collision operator. See subsequent developments along this direction in [18, 17, 2]. In particular, a quantitative estimate of macroscopic part $\mathcal{P}f$ in terms of the microscopic part $(I-\mathcal{P})f$ for the L^2-positivity estimate is established in [16].

Our work continues the recent program to study an $L^2 \to L^\infty$ framework for studying the Landau equation [53, 30]. It should be noted that due to nonlinear diffusion in the velocity space, L^∞ is not quite enough to ensure uniqueness as in the Boltzmann equation; some control of $\nabla_v f$ is needed. The framework relies on L^2 energy estimate, and a recent novel De Giorgi–Nash–Moser theory for the Landau equation, which is developed in [24]. Finally, De Giorgi–Nash–Moser theory leads to application of $S^p_{\mathcal{L}}$ estimates (counterpart of $W^{2,p}$ estimates for parabolic PDEs) to control $\nabla_v f$ for the uniqueness. The main novelty of our work is to give a more direct and simplified approach with only L^2 energy and $S^p_{\mathcal{L}}$ estimates in the perturbative regime, bypassing the recent De Giorgi–Nash–Moser theory for the Landau equation.

2.2.2 Statement of the Main Theorem

We are now ready to state our main theorem, which is a low-regularity global well-posedness and stability result for the model Vlasov-Poisson-Landau system (2.1.5)(2.1.7).

Theorem 2.2.1 (Main Theorem)**.** *Assume that for some sufficiently large velocity-weight exponent $\vartheta_0 \in \mathbb{R}$, the initial data $f_0(x,v) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the smallness assumption*

$$\|f_0\|_{\infty,\vartheta_0} + \|\nabla_x f_0\|_{\infty,\vartheta_0} + \|\nabla_v f_0\|_{\infty,\vartheta_0} + \|\nabla_v^2 f_0\|_{\infty,\vartheta_0} \leq \varepsilon_0 \tag{2.2.1}$$

for a sufficiently small constant $\varepsilon_0 \ll 1$. Let $F_0(x,v) = \mu + \sqrt{\mu}\, f_0(x,v) \geq 0$ and has the same mass as the Maxwellian μ. Then we have the following global a priori estimates:

- (Energy estimates & L^2 decay). *Moreover, the solution* $f(t,x,v)$ *satisfies the uniform weighted energy bounds, for* $\vartheta < \vartheta_0 - \frac{3}{2}$,

$$\sup_{t\in[0,\infty)} \mathcal{E}_\vartheta[f(t)] \,\le\, C_\vartheta\, \mathcal{E}_\vartheta[f(0)] \,\lesssim\, \varepsilon_0^2\,, \tag{2.2.2}$$

and the (almost exponential) time-decay, if $\vartheta + k < \vartheta_0 - \frac{3}{2}$,

$$\|f(t)\|_{2,\vartheta} + \left\|\mathbf{E}_f(t)\right\|_{H^1_x} \lesssim \varepsilon_0 \left(1 + \frac{t}{2k}\right)^{-k} \tag{2.2.3}$$

for any $t \ge 0$.

- (L^∞ bounds & pointwise decay). *For any* $k \ge 0$ *there is an increasing function* $\vartheta' = \vartheta'(k)$ *such that when* $\vartheta + \vartheta' \le \vartheta_0$ *, the weighted pointwise decay bounds hold:*

$$\|f(t)\|_{\infty,\vartheta} + \|\mathbf{E}_f(t)\|_\infty \lesssim \varepsilon_0\,(1+t)^{-k} \tag{2.2.4}$$

for all $t \ge 0$.

- (Regularity results). *In addition, it holds that*

$$\|f\|_{C^{0,\alpha}((0,\infty)\times\Omega\times\mathbb{R}^3)} + \|\mathbf{E}_f\|_{C^{1,\alpha}((0,\infty)\times\Omega)} \lesssim \varepsilon_0 \tag{2.2.5}$$

for some $\alpha \in (0,1]$ *, and*

$$\|\nabla_v f\|_{L^\infty((0,\infty)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_0\,. \tag{2.2.6}$$

- $\partial_t f$ *also satisfies all the estimates above.*

Remark 2.2.1. (1) Based on our a priori estimates, it is rather standard to establish global well-posedness. We refer to [13] for the construction of strong solutions to the specular problem.

(2) There exists $\overline{\vartheta}_0$ such that the system is globally well-posed for initial data satisfying $\vartheta_0 \geq \overline{\vartheta}_0$.

(3) For fixed ϑ_0, this theorem guarantees that the solution decays at least in the rate of $k(\vartheta_0)$. If $\vartheta_0 \to \infty$, then $k \to \infty$. Hence, if we desire a faster decaying solution, we should require that the initial data belongs proper weighted L^∞ space. As [35] states, this is called the almost exponential decay. Better data, faster decay.

2.2.3 Bootstrap Proposition

We implement the following bootstrap scheme:

Proposition 2.2.2 (Bootstrap Estimates)**.** *Let $f(t,x,v)$ be a solution to the VPL-specular problem (2.1.8)(2.1.9)(2.1.20) (or (2.1.29)(2.1.30)(2.1.20)) on some time interval $[0,T]$, $T \geq 1$, with initial data f_0 satisfying the assumption (2.2.1).*

Assume also that, for any $t \in [0,T]$, the solution f (and g) satisfies the bootstrap hypothesis:

$$\langle t\rangle^{k_1}\big\|f(t)\big\|_{2,\vartheta_1} + \langle t\rangle^{k_1}\big\|\partial_t f(t)\big\|_{2,\vartheta_1} \lesssim \varepsilon_1 , \tag{2.2.7}$$

$$\langle t\rangle^{k_2}\big\|f(t)\big\|_{\infty,\vartheta_2} + \langle t\rangle^{k_2}\big\|\partial_t f(t)\big\|_{\infty,\vartheta_2} \lesssim \varepsilon_1 , \tag{2.2.8}$$

and

$$\|f\|_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} + \|\partial_t f\|_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1 , \tag{2.2.9}$$

$$\|D_v f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} + \|D_v \partial_t f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1 , \tag{2.2.10}$$

where $\langle t\rangle := \sqrt{1+t^2}$, $k_i \geq 0$ $(i=1,2)$, $p > 14$, $\varepsilon_0 = \varepsilon_1^q \ll 1$ for some $q > 1$, and see (2.6.3) for the definition of the $S^p_{\mathcal{L}}$ norm.

Then the following improved bounds hold for any $t \in [0,T]$ *:*

$$\langle t\rangle^{k_1}\big\|f(t)\big\|_{2,\vartheta_1} + \langle t\rangle^{k_1}\big\|\partial_t f(t)\big\|_{2,\vartheta_1} \lesssim \varepsilon_1^{\,r}\,, \tag{2.2.11}$$

$$\langle t\rangle^{k_2}\big\|f(t)\big\|_{\infty,\vartheta_2} + \langle t\rangle^{k_2}\big\|\partial_t f(t)\big\|_{\infty,\vartheta_2} \lesssim \varepsilon_1^{\,r}\,, \tag{2.2.12}$$

and

$$\|f\|_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} + \|\partial_t f\|_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^{\,s}\,, \tag{2.2.13}$$

$$\|D_v f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} + \|D_v\partial_t f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^{\,s}\,, \tag{2.2.14}$$

for some $r, s > 1$ *.*

Remark 2.2.2. Given that the bootstrap proposition above holds true, Theorem 2.2.1 follows from a local existence result combined with a continuity argument (see Section 2.9 for the proof). The local existence is presented in Theorem 2.9.1 and it can be obtained by using the well-posedness for the linearized equation (cf. [30, Section 2 − 5]) with a standard iteration argument under the initial assumption (2.2.1). The majority of the rest of this chapter is devoted to the proof of Proposition 2.2.2.

The bounds (2.2.7)(2.2.8) and (2.2.11)(2.2.12) are our (low-order) weighted energy and pointwise decay estimates (which guarantees existence). They are essentially Sobolev-type estimates, which can be obtained through the so-called "two-tier" energy method. It is presented in Section 2.4.

The bounds (2.2.9)(2.2.10) and (2.2.13)(2.2.14) provide uniform control on the velocity-derivatives, (which ensures the uniqueness). Also, it provides further regularity results. The proof is based on the ultra-parabolic equations and the S^p theory.

2.2.4 Overview and Idea of the Proof

We now discuss the strategy for our proofs and some of the main features of our arguments.

Upshots of the Proof:

- Bootstrap scheme: The control of various norms in our work is tangled together: (weighted)-L^2 decay $\to S^p \to C^\alpha$ & D_v regularity $\to$ (weighted)-L^∞ decay $\to$ (weighted)-L^2 decay. In fact, each step requires almost all the other estimates. In particular, the justification of uniqueness relies on (weighted)-L^2, D_v and (weighted)-L^∞ bounds.

- Energy estimate: We delicately implement the celebrated "macro-micro decomposition method" (kernel estimate by contradiction argument) to the Vlasov-Poisson-Landau system to obtain the weighted L^2 decay:

  - In order to handle the problematic nonlinear term containing the electric field, we need to multiply e^ϕ to combine it with $v \cdot \nabla_x f$. This further requires the estimates of $\partial_t \phi \sim \partial_t f$.

  - Since σ-norm is not strong enough to control the L^2 norm, we need to introduce weights and use interpolation, leading to almost exponential decay. Also, we require decay of $\partial_t f$ be sufficiently fast.

  - "Two-tier energy method": decay $\to$ energy bound $\to$ decay

  - $\partial_t f$ estimate: combine the counterpart of f

- S^p estimate: We greatly simplify the $L^2 - L^\infty - C^\alpha$ framework in [53] and [30]. Instead of the DeGiorgi-Nash-Moser method, we use S^p embedding theorem to obtain Hölder continuity.

- Specular boundary problem: The S^p estimate is based on an extension of the domain Ω beyond the boundary, which is achievable for the specular case, using the flattening-reflection technique in [30].

2.2.5 Organization

This chapter is organized as follows. In Section 2.3, we present some preliminary lemmas about the Landau operators. In Section 2.4, we justify the estimates (2.2.11)(2.2.12). In Section 2.5 – 2.8, we develop the S^p theory to justify the uniqueness. In particular, in Section 2.5 and 2.6, we adapt the general S^p theory to our equations. In Section 2.7, we prove the S^p bounds (2.2.13) and $\|D_v f\|_{L^\infty}$ bound (2.2.14). In Section 2.8, we justify the $\|f\|_{L^\infty}$ estimates, which is also necessary for the uniqueness proof, with some byproducts (Hölder regularity and pointwise decay). Finally, in Section 2.9, we prove the main theorem.

2.3 Preliminaries: Basic Properties and Estimates

For convenience, we collect some basic facts and estimates in this subsection, including the structure of the Landau collision kernel, from which we may characterize the σ-norm of dissipation and hereby estimate the Landau operators in terms of this norm. We will omit some of those proofs but instead refer the reader to [26, Section 2, 3], [53, Section 2], and [5, Section 2] for more details. Moreover, we prepare some preliminary L^p and elliptic estimates for our proof of the energy and $S^p_{\mathcal{L}}$ bounds in later sections.

Throughout this subsection, if not otherwise stated, f, g and g_i represent general functions of certain variables.

Lemma 2.3.1 (Embedding of Weighted Spaces)**.** *Let f be a function defined for $(x,v)\in\Omega\times\mathbb{R}^3$, and $p\in(1,\infty)$. For any $\vartheta\in\mathbb{R}$ and $l>\frac{3}{p}$, there is a uniform constant $C_p>0$ (depending only on the domain Ω and l) such that*

$$\|f\|_{p,\vartheta}\ \le\ C_p\|f\|_{\infty,\vartheta+l}\,. \tag{2.3.1}$$

Remark 2.3.1. As a corollary when $p=2$, we see that the weighted L^∞ spaces can be embedded into weighted L^2 spaces at the cost of (at least) a finite $\left(\frac{3}{2}\right)^+$ order of velocity-weight loss. In particular, for initial data, our assumption on weighted L^∞ norm guarantees contol of L^2 norm.

Proof. We estimate

$$\begin{aligned}
\|f\|^p_{p,\vartheta} &= \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{p\vartheta}|f(x,v)|^p\,\mathrm{d}v\mathrm{d}x\\
&= \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{-pl}\left|\langle v\rangle^{\vartheta+l}f(x,v)\right|^p\mathrm{d}v\mathrm{d}x\\
&\le \|f\|^p_{\infty,\vartheta+l}\cdot\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{-pl}\,\mathrm{d}v\mathrm{d}x\\
&= C_p^p\,\|f\|^p_{\infty,\vartheta+l}\,,
\end{aligned}$$

where the constant

$$C_p=C_p(\Omega,l):=\ |\Omega|^{1/p}\left(\int_{\mathbb{R}^3}\langle v\rangle^{-pl}\,\mathrm{d}v\right)^{\frac{1}{p}}<\infty$$

for $l>\frac{3}{p}$, and $C_p\sim|\Omega|^{1/p}\left(\frac{1}{pl-3}\right)^{1/p}$ is uniformly bounded as $l\to+\infty$. □

Next is a model estimate to be applied repeatedly in the following lemmas about the properties related to the Landau kernel (cf. [5, Proposition 2.2 − 2.4], [26, Lemma 2, 3] and [53, Lemma 2.2 − 2.5]).

Lemma 2.3.2. *Let* $\vartheta > -3$, $k(v) \in C^\infty\big(\mathbb{R}^3\backslash\{0\}\big)$ *and* $m(v) \in C^\infty\big(\mathbb{R}^3\big)$. *Assume that for any multi-index* $\beta \geq 0$,

$$
\begin{aligned}
\big|D^\beta k(v)\big| \;&\leq\; C'_\beta\, |v|^{\vartheta-|\beta|}\,, \\
\big|D^\beta m(v)\big| \;&\leq\; C'_\beta\, e^{-\tau_\beta |v|^2}\,,
\end{aligned}
\tag{2.3.2}
$$

for some $C'_\beta > 0$ *and* $\tau_\beta > 0$. *Then there is* $C_\beta > 0$ *such that*

$$
\big|D^\beta \big[k * m\big](v)\big| \;\leq\; C_\beta\, \langle v\rangle^{\vartheta-|\beta|}\,. \tag{2.3.3}
$$

Corollary 2.3.2.1 (Pointwise Estimate of σ_G^{ij} and a_g^i)**.** *Let the (generalized) Landau kernel* $\Phi(v)$ *be given by* (2.1.2) *for* $-3 \leq \gamma \leq 1$. *Then* $\sigma^{ij}(v)$ *and* $\sigma^i(v)$ *defined in* (2.1.14) – (2.1.16) *are smooth functions of* v *such that*

$$
\big|D_v^\beta \sigma^{ij}(v)\big| \;+\; \big|D_v^\beta \sigma^{i}(v)\big| \;\leq\; C_\beta\, \langle v\rangle^{\gamma+2-|\beta|} \tag{2.3.4}
$$

for some $C_\beta > 0$ *with* $|\beta| \in \mathbb{N}$.

In our case of the Coulomb interaction (i.e. $\gamma = -3$ *), if* $G = \mu + \sqrt{\mu} g$ *with* $\sup_t \|g\|_\infty \leq \varepsilon \ll 1$, *then*

$$
\big|D_v^\beta \sigma_G^{ij}(t,x,v)\big| \;\lesssim\; \langle v\rangle^{-1-|\beta|} \;\leq\; 1 \tag{2.3.5}
$$

for $|\beta| = 0, 1$, *and*

$$
\big|a_g^i(t,x,v)\big| \;\lesssim\; \sup_t \|g\|_\infty \cdot \langle v\rangle^{-1} \;\leq\; \varepsilon\,, \tag{2.3.6}
$$

where σ_G^{ij} *is given in* (2.1.35) *and* a_g^i *in* (2.1.36).

Proof. Applying Lemma 2.3.2 to $k = \Phi^{ij}$ with

$$\left|D^\beta \Phi^{ij}(v)\right| \;\le\; C'_\beta\, |v|^{\gamma+2-|\beta|}$$

and $m = \mu\,, v_j\mu$ satisfying (2.3.2), the bound (2.3.4) follows immediately from (2.3.3).

For (2.3.5) and (2.3.6), from the definitions we clearly have

$$\left|D_v^\beta \sigma_G^{ij}(t,x,v)\right| \;\le\; \left|D_v^\beta \sigma^{ij}(v)\right| \;+\; \left|D_v^\beta \sigma_{\sqrt{\mu}g}^{ij}(t,x,v)\right| \;\le\; C_\beta\big(1 + \sup_t \|g\|_\infty\big)\cdot \langle v\rangle^{-1-|\beta|}\ ,$$

by following the proof of Lemma 2.3.2 with slight modification, and

$$\left|a_g^i(t,x,v)\right| \;\le\; \left|\Phi^{ij} * \big[v_j \mu^{1/2} g\big]\right| + \left|\Phi^{ij} * \big[\mu^{1/2}\partial_j g\big]\right| \;\lesssim\; \sup_t \|g\|_\infty \cdot \langle v\rangle^{-1}\ ,$$

where (for the second term) we transfer the derivatives on g using integration by parts. □

Lemma 2.3.3 (Structure of the Diffusion Matrix)**.** *Let* $\sigma = \big[\sigma^{ij}\big]_{1\le i,\,j\le 3}$ *be the diffusion matrix in* (2.1.15)*. For any* $v \in \mathbb{R}^3$*,* $\sigma(v)$ *is a* $3{\times}3$ *real symmetric matrix. Thus it can be diagonalized by an orthogonal matrix consisting of associated eigenvectors.*

The spectrum of $\sigma(v)$ *consists of a simple eigenvalue* $\lambda_1(v) > 0$ *associated with the eigenvector* v*, and a double eigenvalue* $\lambda_2(v) > 0$ *associated with the eigenspace* $v^\perp$*. Furthermore, the eigenvalues can be written explicitly as*

$$\lambda_1(v) \;=\; \int_{\mathbb{R}^3} \Big\{1 - |\widehat{v}\cdot\widehat{w}|^2\Big\}\, \mu(v-w)\, |w|^{\gamma+2}\, \mathrm{d}w\ , \tag{2.3.7}$$

$$\lambda_2(v) \;=\; \int_{\mathbb{R}^3} \Big\{1 - \frac{1}{2}\, |\widehat{v}\times\widehat{w}|^2\Big\}\, \mu(v-w)\, |w|^{\gamma+2}\, \mathrm{d}w\ , \tag{2.3.8}$$

where $\widehat{v} := \frac{v}{|v|}$, $\widehat{w} := \frac{w}{|w|}$. *Asymptotically, as* $|v| \to \infty$,

$$\lambda_1(v) \sim c_1 \langle v \rangle^{\gamma} , \quad \lambda_2(v) \sim c_2 \langle v \rangle^{\gamma+2}$$

for some constants $c_1, c_2 > 0$.

Consequently, for $\gamma = -3$, *and* $G = \mu + \sqrt{\mu} g$ *with* $\sup_t \|g\|_\infty \leq \varepsilon \ll 1$, *the matrix* $\sigma_G(t,x,v)$ *has three positive eigenvalues satisfying that for all* $t \geq 0$, $x \in \Omega$, *and for any* $v \in \mathbb{R}^3$,

$$\frac{1}{C} \langle v \rangle^{-3} \leq \lambda_G(t,x,v) \leq C \langle v \rangle^{-1} \tag{2.3.9}$$

for some constant $C > 0$, *meaning that* σ_G *is positive-definite/elliptic (but not uniformly elliptic in* $v \in \mathbb{R}^3$, *i.e. does not have a strictly positive lower bound independent of* v *).*

Corollary 2.3.3.1 (Lower Bound for the σ-norm)**.** *For any* $3d$ *- vector field* $\mathbf{g}(v) = \langle g_1, g_2, g_3 \rangle$, *let* $\mathsf{q}_\sigma[\mathbf{g}](v) := \mathbf{g}^T \sigma \mathbf{g} = \sum_{i,j=1}^3 \sigma^{ij} g_i\, g_j$ *denote the quadratic form associated with the diffusion matrix* $\sigma(v)$. *Denote also by* $P_v \mathbf{g} := (\mathbf{g} \cdot \widehat{v})\, \widehat{v}$ *the projection of* $\mathbf{g}(v)$ *onto the subspace spanned by the vector* v . *Then*

$$\mathsf{q}_\sigma[\mathbf{g}](v) = \lambda_1(v) \left| P_v \mathbf{g} \right|^2 + \lambda_2(v) \left| [I - P_v] \mathbf{g} \right|^2 ,$$

where $\lambda_1(v), \lambda_2(v)$ *are eigenvalues of* $\sigma(v)$ *given by* (2.3.7) *and* (2.3.8).

Let $|\cdot|_{\sigma,\vartheta}$ *and* $\|\cdot\|_{\sigma,\vartheta}$ *be the weighted* σ*-norms defined in Section* 2.1.3. *Then there exists* $C_\vartheta > 0$ *such that*

$$\begin{aligned} |g|^2_{\sigma,\vartheta} &\geq C_\vartheta \left\{ \left| \langle v \rangle^{\vartheta - \frac{3}{2}} \left| P_v(\nabla_v g) \right| \right|_2^2 + \left| \langle v \rangle^{\vartheta - \frac{1}{2}} \left| [I - P_v](\nabla_v g) \right| \right|_2^2 + \left| \langle v \rangle^{\vartheta - \frac{1}{2}} g \right|_2^2 \right\} \\ &\gtrsim |g|^2_{2,\vartheta - \frac{1}{2}} , \end{aligned} \tag{2.3.10}$$

and therefore,

$$\|g\|_{\sigma,\vartheta} \gtrsim \|g\|_{2,\vartheta-\frac{1}{2}} \,. \tag{2.3.11}$$

Remark 2.3.2. Although the σ-norm contains first-order velocity derivatives, it is still not strong enough to control the L^2 norm. However, we manage to bound the L^2 by this σ-norm at the cost of a minimal half power of weight loss.

The next two lemmas on the Landau operators are useful in the energy estimates (Section 2.4 and Section 2.9). We first record a basic estimate about the linear collision operator L (cf. the proof of [26, Lemma 6] and [53, Lemma 2.7]).

Lemma 2.3.4 (Linear Estimate for L)**.** *Let $L = -A - K$ be given by (2.1.10) – (2.1.12), and let $\vartheta \in \mathbb{R}$. For any small $\delta > 0$, there exists $C_{\vartheta,\delta} = C_\vartheta(\delta) > 0$ such that*

$$(1-\delta)|g|^2_{\sigma,\vartheta} - C_{\vartheta,\delta}\,|\mu g|_2^2 \le -\left\langle w^{2\vartheta} A g, g\right\rangle \le (1+\delta)|g|^2_{\sigma,\vartheta} + C_{\vartheta,\delta}\,|\mu g|_2^2 \,, \tag{2.3.12}$$

and

$$\left|\left\langle w^{2\vartheta} K g_1, g_2\right\rangle\right| \le \left(\delta|g_1|_{\sigma,\vartheta} + C_{\vartheta,\delta}|\mu g_1|_2\right)|g_2|_{\sigma,\vartheta} \,. \tag{2.3.13}$$

Here the Maxwellian μ is a convenient choice of a mollified characteristic function of a ball, which can be replaced by any function of v with sufficiently fast decay at infinity. As a consequence (by taking $\delta \le \frac{1}{4}$), we have

$$\frac{1}{2}|g|^2_{\sigma,\vartheta} - C_\vartheta|g|^2_\sigma \le \left\langle w^{2\vartheta} L g, g\right\rangle \le \frac{3}{2}|g|^2_{\sigma,\vartheta} + C_\vartheta|g|^2_\sigma \,, \tag{2.3.14}$$

and

$$\frac{1}{2}|g|^2_{\sigma,\vartheta} - C_\vartheta|g|^2_{2,\vartheta} \le \left\langle w^{2\vartheta} L g, g\right\rangle \le \frac{3}{2}|g|^2_{\sigma,\vartheta} + C_\vartheta|g|^2_{2,\vartheta} \,. \tag{2.3.15}$$

For the nonlinear collision operator Γ, we estimate $\big(w^{2\vartheta}\Gamma[g_1, g_2], g_3\big)$ in terms of $\|\cdot\|_\infty$, $\|\cdot\|_{2,\vartheta}$ and $\|\cdot\|_{\sigma,\vartheta}$ without higher-order regularity. Note that $\Gamma[g_1, g_2]$ is non-symmetric, so we need to estimate this nonlinear term in two different ways, with the energy norm on two variables respectively. (cf. [26, Theorem 3] and [53, Theorem 2.8])

Lemma 2.3.5 (Nonlinear Estimate for Γ)**.** *Let* Γ *be defined as in* (2.1.13)*, then for every* $\vartheta \in \mathbb{R}$*, there exists* $C_\vartheta > 0$ *such that*

$$\left|\big(w^{2\vartheta}\Gamma[g_1, g_2], g_3\big)\right| \;\leq\; C_\vartheta \, \|g_1\|_\infty \|g_2\|_{\sigma,\vartheta} \|g_3\|_{\sigma,\vartheta} \,. \tag{2.3.16}$$

Moreover, for any $\vartheta \leq -2$*, we have*

$$\left|\big(w^{2\vartheta}\Gamma[g_1, g_2], g_3\big)\right| \;\leq\; C_\vartheta \big(\|g_2\|_\infty + \|D_v g_2\|_\infty\big) \cdot \min\left\{\|g_1\|_{2,\vartheta}\,, \|g_1\|_{\sigma,\vartheta}\right\} \|g_3\|_{\sigma,\vartheta} \,. \tag{2.3.17}$$

The following estimates will be used in the proof of the $S^p_{\mathcal{L}}$ bound in Section 2.7.1.

Lemma 2.3.6 (L^p Estimate of $\overline{K}_g f$)**.** *Let* $\overline{K}_g f$ *be defined as in* (2.1.33) *with* g *satisfying the assumption* (2.2.8)*, then for every* $\vartheta \geq 0$ *and* $1 \leq p \leq \infty$*, it holds that*

$$\begin{aligned} \big\|\overline{K}_g f\big\|_{L^p_{t,x,v}(|v|\sim n)} &\lesssim n^{-\vartheta} \left(\|f\|_{L^p_{t,x,v}} + \big\|D_v f\big\|_{L^p_{t,x,v}} + \big\|\langle v\rangle^\vartheta f\big\|_{L^p_{t,x,v}(|v|\sim n)}\right) \\ &\lesssim n^{-\vartheta} \left(\big\|D_v f\big\|_{L^p_{t,x,v}} + \big\|\langle v\rangle^\vartheta f\big\|_{L^p_{t,x,v}}\right) \end{aligned} \tag{2.3.18}$$

for any $n \in \mathbb{N}$.

Proof. We split the operator as $\overline{K_g} = K + J_g$, where

$$
\begin{aligned}
Kf &:= -\,\mu^{-1/2}\,\partial_i\Big\{\mu\Big[\Phi^{ij}*\big\{\mu^{1/2}\big[\partial_j f + v_j f\big]\big\}\Big]\Big\} \\
&= 2v_i\mu^{1/2}\Big[\Phi^{ij}*\big\{\mu^{1/2}\big[\partial_j f + v_j f\big]\big\}\Big] - \mu^{1/2}\Big[\partial_i\Phi^{ij}*\big\{\mu^{1/2}\big[\partial_j f + v_j f\big]\big\}\Big] \\
J_g f &:= \partial_i\sigma^i f - \sigma^{ij}v_iv_j f - \partial_i\left\{\Phi^{ij}*\big[\mu^{1/2}\partial_j g\big]\right\} f + \left\{\Phi^{ij}*\big[v_i\mu^{1/2}\partial_j g\big]\right\} f\,.
\end{aligned}
$$

We will first estimate Kf as

$$
\big\|Kf\big\|_{L^p(|v|\sim n)} \lesssim n^{-\vartheta}\left(\|f\|_{L^p} + \big\|D_v f\big\|_{L^p}\right)\,. \tag{2.3.19}
$$

Notice that for $\alpha \in (0,1)$, $\vartheta \geq 0$, and $k = 0,1$,

$$
\langle v\rangle^k\mu^\alpha\big|_{|v|\sim n} \lesssim n^{-\vartheta}\langle v\rangle^{\vartheta+k}\mu^\alpha \lesssim n^{-\vartheta}\,.
$$

Also, in light of the proof of Lemma 2.3.3 which provides spectrum of the Landau kernel $\Phi(v)$, it suffices to show that

$$
\big\|\mu^\alpha\cdot\left\{|v|^\lambda * \left[\mu^\gamma h\right]\right\}(v)\big\|_{L^p_v} \lesssim \|h\|_{L^p_v}\,,
$$

where $\alpha, \gamma \in (0,1)$, $\lambda \in (-3,-1]$, and $h(t,x,v)$ represents either f or $\partial_{v_i}f$. Using the Hölder inequality, the p-th power of LHS above is bounded by

$$
\begin{aligned}
&\int_{\mathbb{R}^3}\mu^{\alpha p}(v)\left|\int_{\mathbb{R}^3}\big|v-v'\big|^\lambda\mu^\gamma(v')h(v')\,\mathrm{d}v'\right|^p\mathrm{d}v \\
\leq &\int_{\mathbb{R}^3}\mu^{\alpha p}(v)\left(\int_{\mathbb{R}^3}\big|v-v'\big|^{\lambda p'}\mu^{\gamma p'}(v')\mathrm{d}v'\right)^{\frac{p}{p'}}\left(\int_{\mathbb{R}^3}\big|h(v')\big|^p\mathrm{d}v'\right)\mathrm{d}v \\
\lesssim &\,\|h\|^p_{L^p_v}\cdot\int_{\mathbb{R}^3}\mu^{\alpha p}(v)\langle v\rangle^{\lambda p}\,\mathrm{d}v \\
\lesssim &\,\|h\|^p_{L^p_v}\,.
\end{aligned}
$$

The second inequality above follows in a similar manner as the proof of Lemma 2.3.2 (cf. [26, Lemma 2]) with obvious modifications.

Now for the other part, we clearly have

$$\big\|J_g f\big\|_{L^p(|v|\sim n)} \leq \big(C + \sup_t \|g\|_\infty\big) \, \|f\|_{L^p(|v|\sim n)} \lesssim n^{-\vartheta} \big\|\langle v\rangle^\vartheta f\big\|_{L^p(|v|\sim n)} \,. \tag{2.3.20}$$

Finally, combining (2.3.19) and (2.3.20) gives us the first inequality of (2.3.18), which immediately yields the second one. $\square$

The last estimate for the electric field will be used repeatedly across the chapter.

Lemma 2.3.7 (L^p Estimate of $\mathbf{E}_f$)**.** *Let* $\mathbf{E}_f := -\nabla_x \phi_f = \nabla_x \Delta_x^{-1} \int_{\mathbb{R}^3} \sqrt{\mu}\, f \,\mathrm{d}v$ *be the electric field with potential* ϕ_f *solved from the Poisson equation*

$$-\Delta_x \phi_f(t,x) = \int_{\mathbb{R}^3} \sqrt{\mu}\, f \,\mathrm{d}v \; =: \rho[f](t,x)$$

with either zero-Dirichlet or zero-Neumann BC, then for $1 < p < \infty$ *, it holds that for any* $t \geq 0$ *,*

$$\big\|\mathbf{E}_f(t)\big\|_{W^{1,p}_x} \lesssim \big\|f(t)\big\|_{L^p_{x,v}} \,. \tag{2.3.21}$$

Moreover, when $p = \infty$ *, we have*

$$\|\mathbf{E}_f\|_{L^\infty_{t,x}} \lesssim \|f\|_{L^\infty_{t,x,v}} \,. \tag{2.3.22}$$

Proof. For $1 < p < \infty$, based on the L^p estimate for elliptic equations, we have

$$\begin{aligned}
\|\mathbf{E}_f(t)\|_{W_x^{1,p}} \lesssim \|\phi_f(t)\|_{W_x^{2,p}} \lesssim \|\rho[f](t)\|_{L_x^p} &= \left\| \int_{\mathbb{R}^3} \sqrt{\mu(v)}\, f(t,\cdot,v)\, \mathrm{d}v \right\|_{L_x^p} \\
&\le \int_{\mathbb{R}^3} \left\| \sqrt{\mu(v)}\, f(t,\cdot,v) \right\|_{L_x^p} \mathrm{d}v = \int_{\mathbb{R}^3} \sqrt{\mu(v)}\, \|f(t,\cdot,v)\|_{L_x^p} \mathrm{d}v \\
&\le \|f(t)\|_{L_{x,v}^p} \cdot |\sqrt{\mu}\,|_{L_v^{p'}} \lesssim \|f(t)\|_{L_{x,v}^p} .
\end{aligned}$$

The third and fourth inequalities are due to Minkowski's integral inequality and Hölder's inequality, respectively.

Additionally, in view of the continuous embedding $W_x^{1,p}(\Omega) \hookrightarrow L_x^\infty(\Omega)$ for $p > 3$, we also have

$$\begin{aligned}
\|\mathbf{E}_f(t)\|_{L_x^\infty} \lesssim \|\mathbf{E}_f(t)\|_{W_x^{1,p}} \lesssim \|\rho[f](t)\|_{L_x^p} &= \left\| \int_{\mathbb{R}^3} \sqrt{\mu(v)}\, f(t,\cdot,v)\, \mathrm{d}v \right\|_{L_x^p} \\
&\le \|f(t)\|_\infty \cdot |\Omega|^{\frac{1}{p}} \int_{\mathbb{R}^3} \sqrt{\mu}\, \mathrm{d}v \\
&\lesssim \|f(t)\|_\infty \le \sup_{t\ge 0} \|f(t)\|_\infty
\end{aligned}$$

for any $t \ge 0$, then (2.3.22) follows. □

2.4 Energy Estimates: Weighted L^2 Decay

In this section we establish uniform weighted energy estimates and L^2 time-decay for the VPL-specular problem (2.1.8)(2.1.9)(2.1.20) assuming small weighted L^∞ norms and decay of the solutions f and its time-derivative $\partial_t f$.

Our strategy is to adapt arguments in [3], [30], and [53]. In addition, we will utilize techniques in [3, Section 7] to control the extra terms related to the electric field.

To start with, we rearrange the Vlasov-Landau equation (2.1.8) as

$$\partial_t f + v\cdot\nabla_x f + Lf - 2\sqrt{\mu}\, v\cdot\mathbf{E}_f \;=\; \Gamma[f,f] - \mathbf{E}_f\cdot\nabla_v f + \big(v\cdot\mathbf{E}_f\big) f\,, \tag{2.4.1}$$

so that LHS has purely linear terms, leaving all the nonlinear terms on RHS. Throughout this section, we will focus on the a priori estimates of the solution.

Theorem 2.4.1 (Energy Estimates and L^2 Time-Decay). *Let $f(t,x,v)$ be a solution to the VPL-specular problem (2.1.8)(2.1.9)(2.1.20) (or (2.1.29)(2.1.30)(2.1.20) with $g=f$) and additional conservation law of angular momentum (2.1.28) if Ω has a rotational symmetry on some time interval $[0,T]$, $T\geq 1$, with initial data f_0 satisfying (2.2.1). Suppose also the bootstrap decay bounds (2.2.7)(2.2.8), and in particular that*

$$\sup_{t\in[0,T]}\|f(t)\|_{\infty,\overline{\vartheta}} + \sup_{t\in[0,T]}\|\partial_t f(t)\|_{\infty,\overline{\vartheta}} + \sup_{t\in[0,T]}\big\|\nabla_v\partial_t f(t)\big\|_{\infty} \lesssim \varepsilon_1\,, \tag{2.4.2}$$

for some $\overline{\vartheta}\geq 3$, and where $\varepsilon_1\ll 1$ is small enough.

Then the uniform weighted energy bounds holds

$$\begin{aligned}\sup_{t\in[0,T]}\mathcal{E}_\vartheta[f(t)] &:= \sup_{t\in[0,T]}\left(\big\|f(t)\big\|_{2,\vartheta}^2 + \big\|\mathbf{E}_f(t)\big\|_{L^2_x}^2 + \int_0^t\big\|f(\tau)\big\|_{\sigma,\vartheta}^2\,\mathrm{d}\tau + \int_0^t\big\|\mathbf{E}_f(\tau)\big\|_{L^2_x}^2\,\mathrm{d}\tau\right)\\ &\lesssim \varepsilon_0^2\,.\end{aligned} \tag{2.4.3}$$

Furthermore, the solution decays almost exponentially with respect to time:

$$\|f(t)\|_{2,\vartheta} + \big\|\mathbf{E}_f(t)\big\|_{H^1_x} \lesssim \varepsilon_0\langle t\rangle^{-k} \tag{2.4.4}$$

for any $t \in [0,T]$ *. Moreover, we also have*

$$\mathcal{E}_{\vartheta}[\partial_t f(t)] \lesssim \varepsilon_0^2, \tag{2.4.5}$$

and

$$\|\partial_t f(t)\|_{2,\vartheta} + \left\|\partial_t \mathbf{E}_f(t)\right\|_{H_x^1} \lesssim \varepsilon_0 \langle t\rangle^{-k} . \tag{2.4.6}$$

Remark 2.4.1. The time-decay result (2.4.4) is called almost-exponential decay, in that the solution decays with any polynomial rate in time. This decay rate is optimal for the Landau-Poisson system in view of our current setting as well as method, and the reason is twofold:

(1) On one hand, the instant energy functional at each time is stronger than the dissipation rate due to the particular structure of Landau operator, so we have to perform interpolation between hierarchies of weighted energy norms such that dissipation can bound a power of energy, which yields only algebraic decay.

(2) On the other hand, in comparison to periodic domains where more regular initial assumption grants faster decay (for example, exponential decay, cf. [69]), we can only resort to low-regularity techniques in the presence of specular-reflection boundary, which at best yield algebraic decay.

(3) When the domain is rotational invariant, the conservation of angular momentum is necessary to ensure the validity of this theorem. This is mainly owing to the proof of Lemma 2.4.3, which leads to the positivity of L in Corollary 2.4.3.1.

It is known that the linear Landau operator L given by (2.1.10) is a self-adjoint nonnegative operator on $L_v^2(\mathbb{R}^3)$. Its null space (kernel) is a five-dimensional subspace of $L^2_v(\mathbb{R}^3)$ spanned by $\left\{\sqrt{\mu}\,,\, v\sqrt{\mu}\,,\, |v|^2\sqrt{\mu}\,\right\}$, which are called collision invariants. We introduce the following notations:

Definition 2.4.1 (Projection onto the null space of L)**.** Let $\mathbf{N}(L) := \left\{\, h \in L^2_v(\mathbb{R}^3) : Lh = 0\,\right\}$ denote the null space of the linear operator L with basis $e_0 := \sqrt{\mu}$, $e_i := v_i\sqrt{\mu}$ $(i = 1, 2, 3)$, $e_4 := |v|^2\sqrt{\mu}$, and write

$$\mathbf{N}(L) \;=\; \operatorname{span}\left\{\sqrt{\mu}\,,\, v_i\sqrt{\mu}\ \ (i = 1, 2, 3)\,,\, |v|^2\sqrt{\mu}\,\right\}\,. \tag{2.4.7}$$

For the function $f(t, x, v)$ with fixed (t, x), we define the projection of f in $L^2_v(\mathbb{R}^3)$ onto $\mathbf{N}(L)$ as

$$\mathcal{P}f\,(t, x, v) \;:=\; \sum_{i=0}^{4} \big\langle f(t, x, \cdot\,), e_i \big\rangle\, e_i \;=:\; \big\{\mathfrak{a}_f(t, x) + v \cdot \mathfrak{b}_f(t, x) + |v|^2\, \mathfrak{c}_f(t, x)\big\}\,\sqrt{\mu}\,,$$

where

$$\begin{aligned}
\mathfrak{a}_f(t, x) \;&:=\; \big\langle f(t, x, \cdot\,), e_0\big\rangle \;=\; \int_{\mathbb{R}^3} f(t, x, \cdot\,)\sqrt{\mu}\,\mathrm{d}v\,,\\
\mathfrak{b}^i_f(t, x) \;&:=\; \big\langle f(t, x, \cdot\,), e_i\big\rangle \;=\; \int_{\mathbb{R}^3} f(t, x, \cdot\,)v_i\sqrt{\mu}\,\mathrm{d}v\,,\\
\mathfrak{c}_f(t, x) \;&:=\; \big\langle f(t, x, \cdot\,), e_4\big\rangle \;=\; \int_{\mathbb{R}^3} f(t, x, \cdot\,)|v|^2\sqrt{\mu}\,\mathrm{d}v\,.
\end{aligned}$$

We then recall a basic property of L (see [26, Lemma 5] for the proof, or as a byproduct in showing the structure (2.4.7) of L).

Lemma 2.4.2 (Semi-Positivity of L)**.** *There exists $\delta > 0$ such that*

$$\big\langle Lh, h\big\rangle \;\geq\; \delta\,\big|(I - \mathcal{P})h\big|^2_\sigma \tag{2.4.8}$$

for a general function $h(v)$.

2.4.1 Positivity of L

A crucial step toward proving the L^2 decay is to estimate $\mathcal{P}f$ in terms of $(I-\mathcal{P})f$. The following lemma is an adaptation of [27, Proposition 1] and [29, Proposition 2]. Since most of the proof is identical to that of [29], we will skip the details and only provide the key ideas and steps. The new terms related to the fields can be controlled in an obvious way.

Lemma 2.4.3. *Under the same assumptions as in Theorem 2.4.1, we have*

$$\int_s^t \left\|\mathcal{P}f(\tau)\right\|_\sigma^2 \mathrm{d}\tau + \int_s^t \left\|\mathbf{E}_f(\tau)\right\|_{L^2_x}^2 \mathrm{d}\tau \lesssim \int_s^t \left\|(I-\mathcal{P})f(\tau)\right\|_\sigma^2 \mathrm{d}\tau \tag{2.4.9}$$

for all $0 \le s < t$ *with* $|t-s| \in \mathbb{Z}^+$.

As an immediate consequence of the lemma above, we obtain the positivity of L with Landau dissipation, a key ingredient in the proof of Theorem 2.4.1.

Corollary 2.4.3.1 (Coercivity Estimate on L)**.** *With the same assumptions above, we have a sufficiently small constant* $\delta' > 0$ *such that*

$$\int_s^t \big(Lf(\tau), f(\tau)\big)\,\mathrm{d}\tau \ \ge\ \delta' \left\{\int_s^t \left\|f(\tau)\right\|_\sigma^2 \mathrm{d}\tau + \int_s^t \left\|\mathbf{E}_f(\tau)\right\|_{L^2_x}^2 \mathrm{d}\tau\right\} \tag{2.4.10}$$

for all $0 \le s < t$ *with* $|t-s| \in \mathbb{Z}^+$.

Sketch of Proof. Recalling that L is semi-positive by Lemma 2.4.2, we clearly have

$$\big(Lf, f\big) \ \ge\ \delta \left\|(I-\mathcal{P})f\right\|_\sigma^2 \tag{2.4.11}$$

for some $\delta > 0$. Then it suffices to bound the remaining part on $\mathcal{P}f$ by the counterpart of $(I-\mathcal{P})f$ using Lemma 2.4.3. □

Proof of Lemma 2.4.3. This proof is an adaptation of the proof of [29, Proposition 2], so we only point out the main difference.

We consider the equation (2.4.1). To handle the most problematic nonlinear term $(v \cdot \mathbf{E})f$, we combine it with the linear streaming term $v \cdot \nabla_x f$ (which also contains an extra v factor) by multiplying both sides of (2.4.1) by e^{ϕ}, so we can rewrite the equation as

$$\partial_t\big(e^{\phi} f\big) + v \cdot \nabla_x\big(e^{\phi} f\big) + L\big(e^{\phi} f\big) - 2e^{\phi}\sqrt{\mu}\, v \cdot \mathbf{E} \;=\; e^{\phi}\,\Gamma[f,f] - e^{\phi}\,\mathbf{E} \cdot \nabla_v f + e^{\phi} f\, \partial_t \phi\,. \tag{2.4.12}$$

We first justify the lemma for $s=0$, $t=1$. If the estimate (2.4.9) is not true no matter how small ε_1 is, then there exist a sequence of solutions f_n to (2.4.1) with $f = f_n$ such that for any n

$$\int_0^1 \|(I-\mathcal{P})f_n(\tau)\|_\sigma^2 \,\mathrm{d}\tau \le \frac{1}{n}\int_0^1 \|\mathcal{P} f_n(\tau)\|_\sigma^2 \,\mathrm{d}\tau, \tag{2.4.13}$$

and $\|f_n\|_{\infty,\vartheta} < \frac{1}{n}$ for given $\vartheta > \frac{3}{2}$.

We first prove the weak compactness of f_n. For any fixed $\vartheta < 0$, we multiply (2.4.12) for $f = f_n$ by $\mathrm{e}^{\phi_n}\langle v\rangle^{2\vartheta} f_n$ and integrate both sides of the resulting equation. Similar to the argument in [29, Proposition 2] and using the fact that $\mathrm{e}^{\phi_n} \sim 1$ by the bootstrap assumption, we obtain

$$\|\langle v\rangle^{\vartheta} f_n(t)\|_{L^2}^2 + \int_{t_0}^{t} \|f_n(\tau)\|_{\sigma,\vartheta}^2 \,\mathrm{d}\tau \le C e^{t-t_0} \|\langle v\rangle^{\vartheta} f_n(t_0)\|_{L^2}^2, \tag{2.4.14}$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{t_0}^{t}\|f_n(\tau)\|_\sigma^2\,\mathrm{d}\tau=\|f_n(t)\|_\sigma^2\geq C\left\|\langle v\rangle^{-\frac{1}{2}}f_n(t_0)\right\|_{L^2}^2-C'\int_{t_0}^{t}\|f(\tau)\|_\sigma^2\,\mathrm{d}\tau, \tag{2.4.15}$$

for some $C, C' > 0$. By the bootstrap assumption (2.2.8) and the Grönwall inequality, we obtain that

$$\int_{t_0}^{t}\|f_n(\tau)\|_\sigma^2\,\mathrm{d}\tau\geq C(1-\mathrm{e}^{-C'(t-t_0)})\|\langle v\rangle^{-\frac{1}{2}}f_n(t_0)\|_{L^2}^2. \tag{2.4.16}$$

Now we define the normalized term Z_n of f_n as

$$Z_n:=\frac{f_n}{C_n}\,,$$

with

$$C_n:=\sqrt{\int_0^1\|\mathcal{P}f_n(\tau)\|_\sigma^2\,\mathrm{d}\tau}\,.$$

Then Z_n satisfies the equation

$$\begin{aligned}&\partial_t\big(e^{C_n\phi_n}Z_n\big)+v\cdot\nabla_x\big(e^{C_n\phi_n}Z_n\big)+L\big(e^{C_n\phi_n}Z_n\big)-2e^{C_n\phi_n}\sqrt{\mu}\,v\cdot\mathbf{E}_n\\&=e^{C_n\phi_n}\,\Gamma[f_n,Z_n]-C_ne^{C_n\phi_n}\,\mathbf{E}_n\cdot\nabla_vZ_n+C_ne^{C_n\phi_n}Z_n\,\partial_t\phi_n\,,\end{aligned} \tag{2.4.17}$$

where C_n are uniformly bounded by a small constant, and $\mathbf{E}_n := -\nabla_x\phi_n$ with ϕ_n determined by the Poisson equation

$$-\Delta_x\phi_n=\int_{\mathbb{R}^3}\sqrt{\mu}\,Z_n\,\mathrm{d}v\;=:\rho[Z_n]\,. \tag{2.4.18}$$

Thus, following similar argument in [29, Proposition 2], we obtain the uniform bound

$$\sup_{0\leq\tau\leq1}\|\langle v\rangle^{-\frac{1}{2}}Z_n(\tau)\|_{L^2}^2\leq C$$

for some $C > 0$. Also, by the normalization we already have $\int_0^1 \|Z_n(\tau)\|_\sigma^2 \,\mathrm{d}\tau = 1$. Since the eigenvalues $\lambda(v)$ of the matrix $\sigma(v)$ satisfies the bound (2.3.9), the normed vector space with the norm $\|\cdot\|_\sigma$ can be understood as a weighted L^2 Sobolev space and is reflexive. Therefore, there exists the weak limit Z of Z_n in $\int_0^1 \|\cdot\|_\sigma^2 \,\mathrm{d}\tau$. Also, by (2.4.13), we have

$$\int_0^1 \|(I-\mathcal{P})Z_n(\tau)\|_\sigma^2 \,\mathrm{d}\tau \le \frac{1}{n} \to 0\,. \tag{2.4.19}$$

By the triangle inequality, we also have that $\int_0^1 \|\mathcal{P}Z_n(\tau)\|_\sigma^2 \,\mathrm{d}\tau$ is uniformly bounded from above. In addition, the norm $\|\cdot\|_\sigma$ is an anisotropic Sobolev norm with respect to direction of the velocity v by definition. Then by Alaoglu's theorem and Eberlein-Shmulian's theorem, $\mathcal{P}Z_n$ converges weakly to $\mathcal{P}Z$ in $\int_0^1 \|\cdot\|_\sigma^2 \,\mathrm{d}\tau$ up to a subsequence. Thus, we conclude that $(I-\mathcal{P})Z = 0$ and $Z = \mathcal{P}Z$.

Also, by taking the limit $n \to \infty$, we note that the limit Z satisfies

$$\partial_t Z + v \cdot \nabla_x Z \;-\; 2\sqrt{\mu} v \cdot \mathbf{E}_Z = 0 \tag{2.4.20}$$

in the sense of distribution.

Now our main strategy has two steps:

- Step 1: Show that the convergence $Z_n \to Z$ is actually strong in $\int_0^1 \|\cdot\|_\sigma^2 \,\mathrm{d}\tau$ by proving the compactness. This is almost the same as the argument in [29, Proposition 2], so we omit the details. The basic idea is to show the compactness in the interior and rule out the possibility of concentration near $t = 0$, $t = 1$ or $x \in \partial\Omega$.
- Step 2: We use the equation (2.4.20) and conservation laws to show that Z is

actually zero. This will be our focus below.

Eventually, we confirm that this leads to a contradiction and thus Lemma 2.4.3 must holds in $[0,1]$. Finally, we extend the domain to arbitrary $[s,t]$ with $|t-s|\geq 1$.

Z **is indeed zero.** The limit equation for Z is (since all nonlinear terms in the equation for f vanish after normalization)

$$\partial_t Z + v\cdot\nabla_x Z + 2\sqrt{\mu}v\cdot\mathbf{E} = 0. \tag{2.4.21}$$

Here assume that

$$Z = \sqrt{\mu}\Big(a + b\cdot v + c\left|v\right|^2\Big), \tag{2.4.22}$$

and

$$\mathbf{E} = \nabla_x\Delta_x^{-1}\int_{\mathbb{R}^3}\sqrt{\mu}Z\mathrm{d}v := -\nabla_x\phi, \tag{2.4.23}$$

with either DBC or NBC on ϕ.

The conservation laws for mass and energy are as follows:

$$\partial_t\int_{\Omega\times\mathbb{R}^3}\sqrt{\mu}Z = 0, \quad \partial_t\int_{\Omega\times\mathbb{R}^3}\left|v\right|^2\sqrt{\mu}Z = 0. \tag{2.4.24}$$

Also, combined with the normalized initial condition, we have

$$\int_{\Omega\times\mathbb{R}^3}\sqrt{\mu}Z(t) = \int_{\Omega\times\mathbb{R}^3}\sqrt{\mu}Z(0) = 0, \tag{2.4.25}$$

$$\int_{\Omega\times\mathbb{R}^3}\left|v\right|^2\sqrt{\mu}Z(t) = \int_{\Omega\times\mathbb{R}^3}\left|v\right|^2\sqrt{\mu}Z(0) = -\int_{\Omega}\left|\mathbf{E}\right|^2(0). \tag{2.4.26}$$

If the domain Ω is rotational invariant

$$[(x-x_0)\times\omega]\cdot n=0, \tag{2.4.27}$$

then the angular momentum is conserved

$$\int_{\Omega\times\mathbb{R}^3}[(x-x_0)\times\omega]\cdot v\sqrt{\mu}Z=0. \tag{2.4.28}$$

This is essentially $\int_{\Omega\times\mathbb{R}^3}[(x-x_0)\times v\sqrt{\mu}Z]\cdot\omega=0$, which is consistent with the classical definition of angular momentum. Note that all conservation laws can be directly derived from the equation (2.4.21) itself, with the normalization (2.4.25) on initial data.

We can analyze Z as follows:

Step 1: We insert (2.4.22) into (2.4.21) and compare the coefficient of v for each order. It is easy to check that $\partial_t a=0$ (at lowest order) and $\nabla_x c=0$ (at highest order). Hence, we have

$$a=a(x),\quad c=c(t). \tag{2.4.29}$$

Using conservation of mass (2.4.24), we know $\partial_t c=0$, which means c is a constant.

Based on (2.4.25), we know

$$\int_\Omega a=0. \tag{2.4.30}$$

Step 2: Inserting (2.4.22) into (2.4.21) and checking the term of order v_i, we get

$$\partial_t b_i + \partial_{x_i}(a - \phi) = 0. \tag{2.4.31}$$

Inserting (2.4.22) into (2.4.21) and checking the term of order v_i^2 and $v_i v_j$, we get

$$\partial_{x_i} b_i = 0, \quad \partial_{x_j} b_i + \partial_{x_i} b_j = 0 \ \text{ for } \ i \neq j. \tag{2.4.32}$$

Hence, $\nabla_x b$ is a skew-symmetric matrix. Hence, using the Helmholtz decomposition (or direct computation), b must take the form

$$b = \bar{b}(t) + \varpi(t) \times x \tag{2.4.33}$$

for some $\bar{b}(t)$ and $\varpi(t)$.

From (2.4.31), we know

$$\partial_t(\partial_{x_j} b_i) = \partial_t(\partial_{x_i} b_j). \tag{2.4.34}$$

Hence, $\partial_t \nabla_x b$ is a symmetric matrix. From the last step, we know $\partial_t \nabla_x b$ is a skew-symmetric matrix. These two requirements imply that $\partial_t \nabla_x b$ is a zero matrix. Hence, $\nabla_x b$ is a time-independent. Therefore, $\varpi = \text{const}$. We know

$$b = \bar{b}(t) + \varpi \times x. \tag{2.4.35}$$

Considering (2.4.31), we know that $\partial_t b$ is independent of time. Hence, $\bar{b} = b_1 + b_0 t$. At the boundary, due to the specular reflection boundary, we always have $b \cdot n = 0$. Hence, by taking a boundary point x at which n_x is parallel to b_0, we get $b_0 = 0$. In summary, we have

$$b = b_1 + \overline{\omega} \times x. \tag{2.4.36}$$

Step 3: Since b is independent of time, in (2.4.31), we have

$$\partial_{x_i}(a - \phi) = 0, \tag{2.4.37}$$

which implies

$$a - \phi = \text{const.} \tag{2.4.38}$$

Considering the definition

$$\Delta_x \phi = a, \tag{2.4.39}$$

we know

$$\Delta_x \phi = \phi + \text{const.} \tag{2.4.40}$$

- For NBC case, using the normalization condition $\displaystyle\int_\Omega \phi = 0$, directly energy estimates yields

$$\int_\Omega |\nabla_x \phi|^2 + \int_\Omega |\phi|^2 = 0. \tag{2.4.41}$$

Hence, we know $\phi = 0$ and thus a is a constant. Based on (2.4.30), such a constant must be zero. Hence, $a = 0$.

- For DBC, taking Laplacian in (2.4.38), we obtain

$$\Delta_x a - \Delta_x \phi = 0. \tag{2.4.42}$$

Considering (2.4.39), we have

$$\Delta_x a - a = 0. \tag{2.4.43}$$

The DBC implies $\phi = 0$ on $\partial\Omega$, which implies $a = \text{const}$ on $\partial\Omega$. Due to the maximum principle of the elliptic equation, we get a does not change sign in Ω. Then using the average property (2.4.30), we know the constant must be zero. Hence, $a = 0$ and $\phi = \text{const}$.

All in all, in both cases, we confirm that ϕ is a constant and $a = 0$. This further implies $\mathbf{E} = 0$ and thus $c = 0$ from (2.4.26).

Step 4: So far, we have confirmed that $a = c = 0$ and $b = b_1 + \overline{\omega} \times x$. Based on [27, Page 748-749] and [29], and using the conservation of angular momentum, we obtain $b = 0$. (In detail, we may deduce that $Z = \overline{\omega} \times (x - x_0) v\sqrt{\mu}$ in Ω and $[\overline{\omega} \times (x - x_0)] \cdot n = 0$ on $\partial\Omega$ by decomposing b_1. If the domain is not rotational invariant, then there is no nonzero $\overline{\omega}$ to make the above true. On the other hand, if the domain is rotational invariant, we need additional conservation of angular momentum to justify $\overline{\omega} = 0$.)

In summary, we conclude that $Z = 0$ and this leads to a contradiction.

Generalizing to arbitrary $0 \le s < t$ with $|t-s| \in \mathbb{Z}^+$. Let $t - s = N$ for some positive integer N. We split

$$[s,t] = \left(\bigcup_{j=0}^{N-1} [s+j, s+j+1] \right). \tag{2.4.44}$$

On each interval $[s+j, s+j+1]$ for $j = 0, 1, ..., N-1$, we define $f^j(r,x,v) \stackrel{\text{def}}{=} f(r+s+j, x, v)$. Then clearly $f^j(r,x,v)$ is a solution on the time interval $r \in [0,1]$ with the new initial condition $f^j(0,x,v) = f(s+j,x,v)$. Note that due to the bootstrap assumption, $\mathcal{E}_\vartheta(f^j(t))$ is uniformly (in j) bounded from above (which is used to show the boundedness of sequence Z_n). Hence, using the previous argument, we obtain the positivity estimates.

Hence, by assembling all these domains, the positivity estimate actually holds for the full interval $[s,t]$ with $|t-s| = N$. □

2.4.2 Weighted L^2 Bound and Decay

We are now ready to prove the main theorem of this section:

Proof of Theorem 2.4.1. The proof is a modification of [53, Theorem 1.4], whose idea is brought from [68, Theorem 5.1]. Our main observation based on the relationship

$$\|h\|_{\sigma,\vartheta} \gtrsim \|h\|_{2,\vartheta-\frac{1}{2}} \tag{2.4.45}$$

is that even though $\|\cdot\|_{\sigma,\vartheta}$ is not stronger than $\|\cdot\|_{2,\vartheta}$ (but instead dominates the energy norm at the cost of losing some weight), it is still possible to bound the instant energy by a fractional power of the dissipation rate via interpolations with stronger energy norms of higher weight powers. This leads to the result of almost exponential decay.

It is also worth pointing out that our coupled electric field poses considerable difficulty in the energy-decay estimate. To overcome this, we combine various delicate techniques and tricks in [28, Section 3 & 4] and [3, Section 7] to handle the extra terms related to the electric field.

If $t \in [0,1)$, then by the local well-posedness result, the estimates naturally follow (since the initial data is sufficiently small), so we focus on the $t \geq 1$ case.

Step 1. *Energy Estimate without weight ($\vartheta = 0$).* Throughout this proof we consider at a rearrangement of the Vlasov-Landau equation (2.1.8) with all linear terms on the left and nonlinearities on the right:

$$\partial_t f + v \cdot \nabla_x f + Lf - 2\sqrt{\mu}\, v \cdot \mathbf{E} \;=\; \Gamma[f,f] - \mathbf{E} \cdot \nabla_v f + \big(v \cdot \mathbf{E}\big) f\,, \tag{2.4.46}$$

where $\mathbf{E} := \mathbf{E}_f = -\nabla_x \phi_f$ with $\phi := \phi_f$ determined by the Poisson equation

$$-\Delta_x \phi = \int_{\mathbb{R}^3} \sqrt{\mu}\, f \,\mathrm{d}v \;=:\; \rho[f]\,. \tag{2.4.47}$$

To handle the most problematic nonlinear term $(v \cdot \mathbf{E})f$, we combine it with the linear streaming term $v \cdot \nabla_x f$ (which also contains an extra v factor) by multiplying both sides of (2.4.46) by e^{ϕ}, so we can rewrite the equation as

$$\partial_t\big(e^{\phi} f\big) + v \cdot \nabla_x\big(e^{\phi} f\big) + L\big(e^{\phi} f\big) - 2e^{\phi}\sqrt{\mu}\, v \cdot \mathbf{E} \;=\; e^{\phi}\,\Gamma[f,f] - e^{\phi}\,\mathbf{E} \cdot \nabla_v f + e^{\phi} f\, \partial_t \phi\,. \tag{2.4.48}$$

Multiplying $e^{\phi}f$ on both sides of (2.4.48) and integrating in $(x,v)\in\Omega\times\mathbb{R}^3$, we get

$$\begin{aligned}
&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left\|e^{\phi}f\right\|_2^2+\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\nabla_x\cdot\left\{v\left(e^{\phi}f\right)^2\right\}\\
&\quad+\iint_{\Omega\times\mathbb{R}^3}\left(e^{\phi}f\right)L\left(e^{\phi}f\right)-\iint_{\Omega\times\mathbb{R}^3}2\left(e^{2\phi}f\right)\sqrt{\mu}\,v\cdot\mathbf{E}\\
&=\iint_{\Omega\times\mathbb{R}^3}\left(e^{2\phi}f\right)\Gamma[f,f]-\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}e^{2\phi}\,\mathbf{E}\cdot\nabla_v\left(f^2\right)+\iint_{\Omega\times\mathbb{R}^3}\left(e^{\phi}f\right)^2\partial_t\phi\,.
\end{aligned}\tag{2.4.49}$$

We need to estimate every term of (2.4.49) in terms of the energy norm $\|\cdot\|_2$ or dissipation rate $\|\cdot\|_\sigma$, for which we will use repeatedly $e^{\phi}\sim 1$ with $\|\phi\|_\infty\lesssim\|f\|_\infty\ll 1$ (see the proof of Lemma 2.3.7).

First we observe that the following two terms vanish by using integration by parts (divergence theorem):

$$\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}e^{2\phi}\,\mathbf{E}\cdot\nabla_v\left(f^2\right)\mathrm{d}v\mathrm{d}x\;=\;0\,,\tag{2.4.50}$$

$$\begin{aligned}
\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\nabla_x\cdot\left\{v\left(e^{\phi}f\right)^2\right\}\mathrm{d}v\mathrm{d}x\;&=\;\frac{1}{2}\iint_{\partial\Omega\times\mathbb{R}^3}\left(e^{\phi}f\right)^2(v\cdot n_x)\,\mathrm{d}v\mathrm{d}S_x\\
&=\;\frac{1}{2}\int_{\partial\Omega}e^{2\phi}\left[\left(\int_{\{v\cdot n_x>0\}}-\int_{\{v\cdot n_x<0\}}\right)f^2|v\cdot n_x|\,\mathrm{d}v\right]\mathrm{d}S_x\\
&=\;0\,.
\end{aligned}\tag{2.4.51}$$

Here the boundary terms cancel with each other in view of the specular-reflection boundary condition (2.1.20).

For the other nonlinear terms, we have

$$\iint_{\Omega\times\mathbb{R}^3}\left(e^{2\phi}f\right)\Gamma[f,f]\;\lesssim\;\|f\|_\infty\left\|e^{\phi}f\right\|_\sigma^2\;\lesssim\;\|f\|_\infty\|f\|_\sigma^2\tag{2.4.52}$$

by (2.3.16) in Lemma 2.3.5, and

$$\iint_{\Omega\times\mathbb{R}^3} \left(e^{\phi} f\right)^2 \partial_t \phi \lesssim \|\partial_t\phi\|_\infty \left\|e^{\phi} f\right\|_2^2 \lesssim \|\partial_t f\|_2 \|f\|_2^2\,, \tag{2.4.53}$$

where for the last inequality we modify the proof of Lemma 2.3.7. Note that we cannot bound this term in terms of $\|f\|_\sigma^2$, which ultimately leads to an extra term on the RHS in (2.4.69) not being absorbed, because the σ-norm is not strong enough to control the L^2 norm.

The last term on the LHS of (2.4.49) is more troublesome to deal with. We first manipulate this term as follows:

$$\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3} 2\left(e^{2\phi} f\right)\sqrt{\mu}\, v\cdot \mathbf{E} &= \iint_{\Omega\times\mathbb{R}^3} 2\left(e^{2\phi} f\right)\sqrt{\mu}\, v\cdot \nabla_x\phi = \iint_{\Omega\times\mathbb{R}^3} f\sqrt{\mu}\, v\cdot\nabla_x\left(e^{2\phi}\right)\\
&= \iint_{\Omega\times\mathbb{R}^3} \nabla_x\cdot\left\{v\sqrt{\mu}\left(e^{2\phi} f\right)\right\} - e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f\\
&= \iint_{\partial\Omega\times\mathbb{R}^3} \left(e^{2\phi} f\right)\sqrt{\mu}\,(v\cdot n_x)\,\mathrm{d}v\mathrm{d}S_x - \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f\\
&= -\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f\;,
\end{aligned} \tag{2.4.54}$$

where again, we use integration by parts (divergence theorem), and the boundary term on the third line vanishes by the specular boundary condition. Then we want to get control of $-\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f$ from the original equation itself: this time multiplying both sides of (2.4.46) by the factor $e^{2\phi}\sqrt{\mu}$ and integrating over $\Omega\times\mathbb{R}^3$, we obtain

$$\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\partial_t f + \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f + \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, Lf - \iint_{\Omega\times\mathbb{R}^3} 2\, e^{2\phi}\mu\, v\cdot\mathbf{E}\\
= &\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\Gamma[f,f] - \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\mathbf{E}\cdot\nabla_v f + \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\left(v\cdot\mathbf{E}\right) f\;.
\end{aligned} \tag{2.4.55}$$

Obviously, the last term on the LHS vanishes due to symmetry/oddness, and by the orthogonality to the Landau operators, we see that

$$\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, Lf \;=\; \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\Gamma[f,f] \;=\; 0\,. \tag{2.4.56}$$

Also, integrating by parts gives

$$-\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\mathbf{E}\cdot\nabla_v f \;+\; \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\big(v\cdot\mathbf{E}\big)f \;=\; 0\,. \tag{2.4.57}$$

To sum up, we may equate the following two terms from (2.4.55) and further deduce that

$$\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\, v\cdot\nabla_x f \;&=\; \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\sqrt{\mu}\,\partial_t f \\
&=\; \int_\Omega e^{2\phi}\,\partial_t\Big(\int_{\mathbb{R}^3}\sqrt{\mu}\,f\,\mathrm{d}v\Big)\,\mathrm{d}x \;=\; -\int_\Omega e^{2\phi}\partial_t\Delta_x\phi\,\mathrm{d}x \\
&=\; \int_\Omega 2\,e^{2\phi}(\nabla_x\phi)\cdot(\nabla_x\partial_t\phi)\,\mathrm{d}x \;-\; \int_{\partial\Omega} e^{2\phi}\,\partial_t\Big(\frac{\partial_x\phi}{\partial n_x}\Big)\,\mathrm{d}S_x \\
&=\; \int_\Omega e^{2\phi}\partial_t|\nabla_x\phi|^2\,\mathrm{d}x \;=\; \int_\Omega \partial_t\big(e^{2\phi}|\nabla_x\phi|^2\big) - \int_\Omega 2e^{2\phi}|\nabla_x\phi|^2\partial_t\phi \\
&=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{2\phi}|\mathbf{E}|^2 - \int_\Omega 2e^{2\phi}|\mathbf{E}|^2\partial_t\phi\,,
\end{aligned} \tag{2.4.58}$$

by using the Poisson equation (2.4.47) on the second line, followed by integration by parts (divergence theorem). Note also that the boundary term on the third line vanishes under either the zero-Neumann BC ($\frac{\partial\phi}{\partial n}\big|_{\partial\Omega}\equiv 0$) or the zero-Dirichlet BC ($\phi|_{\partial\Omega}\equiv 0$), in which case this term reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\frac{\partial\phi}{\partial n}(t)\,\mathrm{d}S \;=\; -\frac{\mathrm{d}}{\mathrm{d}t}\int_{\partial\Omega}\mathbf{E}(t)\cdot n\,\mathrm{d}S \;\equiv\; 0\,,$$

thanks to the conservation of flux for the self-consistent electric field (see (2.1.26)).

Hence, combining (2.4.54) and (2.4.58) reveals that

$$-\iint_{\Omega\times\mathbb{R}^3} 2\left(e^{2\phi}f\right)\sqrt{\mu}\, v\cdot\mathbf{E} \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega e^{2\phi}|\mathbf{E}|^2 \,-\, \int_\Omega 2e^{2\phi}|\mathbf{E}|^2\partial_t\phi\,. \tag{2.4.59}$$

We will combine the first term above with the first term of (2.4.49). After moving the second to the other side, we bound it as

$$\int_\Omega 2e^{2\phi}|\mathbf{E}|^2\partial_t\phi \;\lesssim\; \|\partial_t\phi\|_\infty\left\|e^\phi\mathbf{E}\right\|_2^2 \;\lesssim\; \|\partial_t f\|_2\|f\|_\sigma^2\,. \tag{2.4.60}$$

Here we justify the last inequality by modifying the proof of Lemma 2.3.7:

$$\begin{aligned}
\left\|e^\phi\mathbf{E}\right\|_2 \;\lesssim\; \|\mathbf{E}\|_2 \;\lesssim\; \left\|\rho[f]\right\|_2 &= \left\|\int_{\mathbb{R}^3}\sqrt{\mu}\,f\,\mathrm{d}v\right\|_{L^2_x} \\
&\le \int_{\mathbb{R}^3}\left\|\sqrt{\mu}\,f\right\|_{L^2_x}\mathrm{d}v \;=\; \int_{\mathbb{R}^3}\langle v\rangle^{\frac12}\sqrt{\mu}\,\left\|\langle v\rangle^{-\frac12}f\right\|_{L^2_x}\mathrm{d}v \\
&\le \left\|\langle v\rangle^{-\frac12}f\right\|_{L^2_{x,v}}\cdot\left|\langle v\rangle^{\frac12}\sqrt{\mu}\,\right|_{L^2_v} \\
&\lesssim \|f\|_{2,-\frac12} \;\lesssim\; \|f\|_\sigma\,.
\end{aligned} \tag{2.4.61}$$

Now in (2.4.49) we are left with the third term on the LHS, which at first can be bounded below as

$$\iint_{\Omega\times\mathbb{R}^3}\left(e^\phi f\right)L\left(e^\phi f\right) \;=\; \left(L\left(e^\phi f\right),\, e^\phi f\right) \;\ge\; \delta\left\|e^\phi(I-\mathcal{P})f\right\|_\sigma^2 \;\gtrsim\; \left\|e^\phi(I-\mathcal{P})f\right\|_\sigma^2\,, \tag{2.4.62}$$

due to the semi-positivity of L (see Lemma 2.4.2).

Summarizing all above, inserting (2.4.50) (2.4.51) (2.4.52) (2.4.53) (2.4.59) (2.4.60) (2.4.62) into (2.4.49), we arrive at

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|e^{\phi}f\right\|_2^2+\left\|e^{\phi}\mathbf{E}\right\|_2^2\right)+\left\|e^{\phi}(I-\mathcal{P})f\right\|_\sigma^2 \lesssim \left(\|f\|_\infty+\|\partial_t f\|_2\right)\|f\|_\sigma^2+\|\partial_t f\|_2\|f\|_2^2\,, \tag{2.4.63}$$

and it naturally follows that

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{1}{2}\left\|e^{\phi}f\right\|_2^2+\left\|e^{\phi}\mathbf{E}\right\|_2^2\right)+\left\|e^{\phi}(I-\mathcal{P})f\right\|_\sigma^2\\ &\lesssim \left(\|f\|_\infty+\|\partial_t f\|_2\right)\|f\|_\sigma^2+\|\partial_t f\|_2\left(\frac{1}{2}\left\|e^{\phi}f\right\|_2^2+\left\|e^{\phi}\mathbf{E}\right\|_2^2\right)\,.\end{aligned} \tag{2.4.64}$$

Define

$$\psi(t) := -\int_0^t \left\|\partial_t f(\tau)\right\|_2 \mathrm{d}\tau\,. \tag{2.4.65}$$

From the bootstrap assumption (2.2.7), we know $|\psi(t)| \ll 1$, and thus $e^{\psi}\sim 1$. Then multiplying both sides of (2.4.64) above by the integrating factor e^{ψ}, we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\left[e^{\psi}\left(\frac{1}{2}\left\|e^{\phi}f\right\|_2^2+\left\|e^{\phi}\mathbf{E}\right\|_2^2\right)\right]+e^{\psi}\left\|e^{\phi}(I-\mathcal{P})f\right\|_\sigma^2 \lesssim \left(\|f\|_\infty+\|\partial_t f\|_2\right)\|f\|_\sigma^2\,. \tag{2.4.66}$$

In order to get the coercivity bound (2.4.10), we need to first integrate over time (for any $0\le s<t$) and obtain

$$\begin{aligned}&\left(\frac{1}{2}\left\|e^{\psi(t)}e^{\phi(t)}f(t)\right\|_2^2+\left\|e^{\psi(t)}e^{\phi(t)}\mathbf{E}(t)\right\|_2^2\right)-\left(\frac{1}{2}\left\|e^{\psi(s)}e^{\phi(s)}f(s)\right\|_2^2+\left\|e^{\psi(s)}e^{\phi(s)}\mathbf{E}(s)\right\|_2^2\right)\\ &+\int_s^t e^{\phi(\tau)}e^{\phi(\tau)}\left\|(I-\mathcal{P})f(\tau)\right\|_\sigma^2\mathrm{d}\tau\\ &\lesssim\left(\sup_{\tau\ge 0}\|f\|_\infty+\sup_{\tau\ge 0}\|\partial_t f\|_2\right)\cdot\int_s^t\left\|f(\tau)\right\|_\sigma^2\mathrm{d}\tau\,.\end{aligned} \tag{2.4.67}$$

Since $1\lesssim e^{\phi(t)}\lesssim 1$ and $1\lesssim e^{\psi(t)}\lesssim 1$ uniformly for any t, applying Lemma 2.4.3

(following the proof of Corollary 2.4.3.1) yields

$$\Big(\big\| e^{\psi(t)} e^{\phi(t)} f(t) \big\|_2^2 + \big\| e^{\psi(t)} e^{\phi(t)} \mathbf{E}(t) \big\|_2^2 \Big) \;+\; \delta' \left\{ \int_s^t \big\| f(\tau) \big\|_\sigma^2 \, \mathrm{d}\tau \;+\; \int_s^t \big\| \mathbf{E}(\tau) \big\|_2^2 \, \mathrm{d}\tau \right\}$$
$$\lesssim \; \Big(\big\| e^{\psi(s)} e^{\phi(s)} f(s) \big\|_2^2 + \big\| e^{\psi(s)} e^{\phi(s)} \mathbf{E}(s) \big\|_2^2 \Big) + \Big(\sup_{\tau \geq 0} \|f\|_\infty + \sup_{\tau \geq 0} \|\partial_t f\|_2 \Big) \cdot \int_s^t \big\| f(\tau) \big\|_\sigma^2 \, \mathrm{d}\tau \tag{2.4.68}$$

for all $0 \leq s < t$ with $|t - s| \in \mathbb{Z}^+$.

Finally, under the a priori assumption

$$\sup_t \|f\|_\infty + \sup_t \|\partial_t f\|_2 \;\ll\; 1\,,$$

this corresponding term can be absorbed into the LHS, and therefore we end up with

$$\Big(\big\| e^{\psi(t)} e^{\phi(t)} f(t) \big\|_2^2 + \big\| e^{\psi(t)} e^{\phi(t)} \mathbf{E}(t) \big\|_2^2 \Big) \;+\; \delta' \left\{ \int_s^t \big\| f(\tau) \big\|_\sigma^2 \, \mathrm{d}\tau \;+\; \int_s^t \big\| \mathbf{E}(\tau) \big\|_2^2 \, \mathrm{d}\tau \right\} \tag{2.4.69}$$
$$\leq \; \big\| e^{\psi(s)} e^{\phi(s)} f(s) \big\|_2^2 + \big\| e^{\psi(s)} e^{\phi(s)} \mathbf{E}(s) \big\|_2^2 \;.$$

Step 2. Weighted Energy Estimate ($\vartheta > 0$). Multiplying both sides of (2.4.48) by $\langle v \rangle^{2\vartheta} e^{\phi} f$ for $\vartheta > 0$ and integrating over $\Omega \times \mathbb{R}^3$, we get

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big\| \langle v \rangle^{\vartheta} e^{\phi} f \big\|_2^2 \;+\; \frac{1}{2} \iint_{\Omega \times \mathbb{R}^3} \nabla_x \cdot \Big\{ v \langle v \rangle^{2\vartheta} \big(e^{\phi} f \big)^2 \Big\}$$
$$+ \iint_{\Omega \times \mathbb{R}^3} \langle v \rangle^{2\vartheta} \big(e^{\phi} f \big) \, L\big(e^{\phi} f \big) \;-\; \iint_{\Omega \times \mathbb{R}^3} 2 \, \langle v \rangle^{2\vartheta} \big(e^{2\phi} f \big) \sqrt{\mu} \, v \cdot \mathbf{E}$$
$$= \iint_{\Omega \times \mathbb{R}^3} \langle v \rangle^{2\vartheta} \big(e^{2\phi} f \big) \, \Gamma[f, f] \;-\; \frac{1}{2} \iint_{\Omega \times \mathbb{R}^3} \langle v \rangle^{2\vartheta} e^{2\phi} \, \mathbf{E} \cdot \nabla_v \big(f^2 \big) \;+\; \iint_{\Omega \times \mathbb{R}^3} \langle v \rangle^{2\vartheta} \big(e^{\phi} f \big)^2 \, \partial_t \phi \,. \tag{2.4.70}$$

This time we aim to estimate every term of (2.4.70) in terms of the weighted energy norm $\|\cdot\|_{2,\vartheta}$ or (weighted) dissipation rate $\|\cdot\|_{\sigma,\vartheta}$.

For the LHS, using the divergence theorem, the second term vanishes since

$$\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\nabla_x\cdot\Big\{v\langle v\rangle^{2\vartheta}\big(e^{\phi}f\big)^2\Big\} \;=\; \frac{1}{2}\bigg(\iint_{\gamma_+}-\iint_{\gamma_-}\bigg)\big(e^{\phi}f\big)^2\langle v\rangle^{2\vartheta}|v\cdot n_x|\,\mathrm{d}v\mathrm{d}S_x \;=\; 0\,, \tag{2.4.71}$$

again by symmetry/oddness with the specular boundary condition on f . As for the third term, we instead use (2.3.14) in Lemma 2.3.4 to bound it below as

$$\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi}f\big)\,L\big(e^{\phi}f\big) \;=\; \big(\langle v\rangle^{2\vartheta}L\big(e^{\phi}f\big)\,,\,e^{\phi}f\,\big) \;\gtrsim\; \big\|e^{\phi}f\big\|_{\sigma,\vartheta}^2-C_\vartheta\big\|e^{\phi}f\big\|_{\sigma}^2\,. \tag{2.4.72}$$

Also, for the fourth term, the trick in $\vartheta=0$ case does not apply, so we move this term to the RHS and bound it directly using Hölder's inequality as well as Cauchy-Schwarz inequality with a small parameter $\varepsilon>0$:

$$\begin{aligned}\iint_{\Omega\times\mathbb{R}^3}2\,\langle v\rangle^{2\vartheta}\big(e^{2\phi}f\big)\sqrt{\mu}\,v\cdot\mathbf{E} \;&\lesssim\; \big|\langle v\rangle^{\vartheta+\frac{3}{2}}\sqrt{\mu}\,\big|_\infty\big\|e^{\phi}f\big\|_{2,\vartheta-\frac{1}{2}}\big\|e^{\phi}\mathbf{E}\big\|_2\\ &\le\; C_\vartheta\big\|e^{\phi}f\big\|_{\sigma,\vartheta}\|f\|_\sigma\\ &\le\; \varepsilon\,\|f\|_{\sigma,\vartheta}^2\;+\;C_{\vartheta,\varepsilon}\,\|f\|_\sigma^2\,.\end{aligned} \tag{2.4.73}$$

Here the second inequality holds because of the relation $\|\cdot\|_{2,\vartheta-1/2}\lesssim\|\cdot\|_{\sigma,\vartheta}$, and also due to the same reason as in (2.4.61) .

For the RHS, we estimate (similarly to the case without weight)

$$\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{2\phi}f\big)\,\Gamma[f,f] \;\lesssim\; \|f\|_\infty\big\|e^{\phi}f\big\|_{\sigma,\vartheta}^2 \;\lesssim\; \|f\|_\infty\|f\|_{\sigma,\vartheta}^2 \tag{2.4.74}$$

by (2.3.16), and

$$-\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}e^{2\phi}\,\mathbf{E}\cdot\nabla_v\big(f^2\big) \;=\; \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\mathbf{E}\cdot\nabla_v\Big(\langle v\rangle^{2\vartheta}\Big)\,e^{2\phi}f^2$$
$$\lesssim\; \|\mathbf{E}\|_\infty\big\|e^{\phi}f\big\|^2_{2,\vartheta-\frac{1}{2}} \;\lesssim\; \|f\|_\infty\|f\|^2_{\sigma,\vartheta} \tag{2.4.75}$$

via integration by parts in v, and

$$\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi}f\big)^2\,\partial_t\phi \;\lesssim\; \|\partial_t\phi\|_\infty\big\|e^{\phi}f\big\|^2_{2,\vartheta} \;\lesssim\; \|\partial_t f\|_2\|f\|^2_{2,\vartheta}\,, \tag{2.4.76}$$

where in the last inequality we modify the proof of Lemma 2.3.7.

Collecting all the estimates in (2.4.71)–(2.4.76) above and inserting them into (2.4.70), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|e^{\phi}f\big\|^2_{2,\vartheta}+\big\|e^{\phi}f\big\|^2_{\sigma,\vartheta} \;\lesssim\; \big(\|f\|_\infty+\varepsilon\big)\|f\|^2_{\sigma,\vartheta}+C_\vartheta\big\|e^{\phi}f\big\|^2_{\sigma}+C_{\vartheta,\varepsilon}\,\|f\|^2_{\sigma}+\|\partial_t f\|_2\|f\|^2_{2,\vartheta}\,, \tag{2.4.77}$$

which further yields

$$\frac{\mathrm{d}}{\mathrm{d}t}\big\|e^{\phi}f\big\|^2_{2,\vartheta}\;+\;\|f\|^2_{\sigma,\vartheta}\;\lesssim\; C_\vartheta\|f\|^2_{\sigma}\;+\;\|\partial_t f\|_2\|f\|^2_{2,\vartheta}\,, \tag{2.4.78}$$

by choosing $\varepsilon>0$ small enough and assuming $\sup_t\|f\|_\infty \ll 1$ so the corresponding term can be absorbed into the LHS. Hence, multiplying e^{ψ} on both sides (to kill the last term on RHS), then integrating over time (for $0\le s<t$), we finally get

$$\big\|e^{\psi(t)}e^{\phi(t)}f(t)\big\|^2_{2,\vartheta}\;+\;\int_s^t\big\|f(\tau)\big\|^2_{\sigma,\vartheta}\,\mathrm{d}\tau\;\lesssim\;\big\|e^{\psi(s)}e^{\phi(s)}f(s)\big\|^2_{2,\vartheta}+C_\vartheta\int_s^t\big\|f(\tau)\big\|^2_{\sigma}\,\mathrm{d}\tau\,. \tag{2.4.79}$$

Step 3. Uniform Weighted Energy Bounds. The idea of the following two steps is inspired by the so-called "Two-tier" energy method in [36]. The uniform energy bound (2.4.3) and time-decay result (2.4.4) follow from weighted energy estimates with arbitrarily strong velocity-weight and sufficiently fast decay-rate. To be specific, we need to obtain a weighted energy inequality ($\vartheta', \vartheta \in \mathbb{N}$)

$$\begin{aligned}&\left(\sum_{\vartheta'=0}^{2\vartheta}\left\|e^{\psi(t)}e^{\phi(t)}f(t)\right\|_{2,\vartheta'/2}^2+\left\|e^{\psi(t)}e^{\phi(t)}\mathbf{E}(t)\right\|_2^2\right)\\&\quad+\delta'\left\{\sum_{\vartheta'=0}^{2\vartheta}\int_s^t\left\|f(\tau)\right\|_{\sigma,\vartheta'/2}^2\mathrm{d}\tau+\int_s^t\left\|\mathbf{E}(\tau)\right\|_2^2\mathrm{d}\tau\right\}\\&\lesssim\sum_{\vartheta'=0}^{2\vartheta}\left\|e^{\psi(s)}e^{\phi(s)}f(s)\right\|_{2,\vartheta'/2}^2+\left\|e^{\psi(s)}e^{\phi(s)}\mathbf{E}(s)\right\|_2^2\end{aligned}\tag{2.4.80}$$

by summing up (2.4.69) and (2.4.79) from previous two steps over $\vartheta' \in [0, 2\vartheta] \cap \mathbb{N}$ (for any given $\vartheta \in \mathbb{N}$) with proper weight (summation factor), such that the term $\sum_{\vartheta'} C_{\vartheta'}\int_s^t\left\|f(\tau)\right\|_\sigma^2\mathrm{d}\tau$ can be absorbed to LHS. Alternatively, we may derive this energy inequality in a more rigorous way via induction on the weight-power $\vartheta \in \mathbb{N}$ (in a similar fashion to the proof of [53, Theorem 1.4]). Note that we cannot absorb RHS here into LHS because σ-norm cannot control the L^2 norm for fixed ϑ (at least for the largest ϑ).

The estimate (2.4.80) only holds when $t-s \in \mathbb{Z}^+$ due to Lemma 2.4.3. For general interval (s,t), we write

$$t-s=N+r,$$

where $N \in \mathbb{N}$ and $r \in [0,1)$. Then for the interval $[s+r,t]$, using the similar argument as (2.4.80), and consider $\mathrm{e}^{\phi} \sim 1$ and $\mathrm{e}^{\psi} \sim 1$, we have

$$\begin{aligned}&\left(\sum_{\vartheta'=0}^{2\vartheta}\left\|f(t)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(t)\right\|_{2}^{2}\right)\\&\quad+\delta'\left\{\sum_{\vartheta'=0}^{2\vartheta}\int_{s+r}^{t}\left\|f(\tau)\right\|_{\sigma,\vartheta'/2}^{2}\mathrm{d}\tau+\int_{s+r}^{t}\left\|\mathbf{E}(\tau)\right\|_{2}^{2}\mathrm{d}\tau\right\}\\&\lesssim\sum_{\vartheta'=0}^{2\vartheta}\left\|f(s+r)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(s+r)\right\|_{2}^{2}.\end{aligned}\tag{2.4.81}$$

Based on the local well-posedness result (see Theorem 2.9.1), we obtain the bounds in the interval $[s, s+r]$

$$\begin{aligned}&\left(\sum_{\vartheta'=0}^{2\vartheta}\left\|f(s+r)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(s+r)\right\|_{2}^{2}\right)\\&\quad+\left\{\sum_{\vartheta'=0}^{2\vartheta}\int_{s}^{s+r}\left\|f(\tau)\right\|_{\sigma,\vartheta'/2}^{2}\mathrm{d}\tau+\int_{s}^{s+r}\left\|\mathbf{E}(\tau)\right\|_{2}^{2}\mathrm{d}\tau\right\}\\&\lesssim\sum_{\vartheta'=0}^{2\vartheta}\left\|f(s)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(s)\right\|_{2}^{2}.\end{aligned}\tag{2.4.82}$$

Note that the initial condition (2.9.1) and the bootstrap assumptions ensure a universal time extension for all $s \geq 0$. Therefore, it is valid to apply Theorem 2.9.1 as long as we have the smallness of $\|f(s)\|_{\infty,\vartheta}$ for $s > 0$.

Summing up (2.4.81) and (2.4.82) and absorbing $\sum_{\vartheta'=0}^{2\vartheta}\left\|f(s+r)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(s+r)\right\|_{2}^{2}$ to the LHS, we obtain the bounds in the interval $[s, t]$

$$\begin{aligned}&\left(\sum_{\vartheta'=0}^{2\vartheta}\left\|f(t)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(t)\right\|_{2}^{2}\right)\\&\quad+\delta'\left\{\sum_{\vartheta'=0}^{2\vartheta}\int_{s}^{t}\left\|f(\tau)\right\|_{\sigma,\vartheta'/2}^{2}\mathrm{d}\tau+\int_{s}^{t}\left\|\mathbf{E}(\tau)\right\|_{2}^{2}\mathrm{d}\tau\right\}\\&\lesssim\sum_{\vartheta'=0}^{2\vartheta}\left\|f(s)\right\|_{2,\vartheta'/2}^{2}+\left\|\mathbf{E}(s)\right\|_{2}^{2}.\end{aligned}\tag{2.4.83}$$

Define an instantaneous weighted energy functional

$$\begin{aligned}\mathcal{W}_\vartheta(t) &:= \sum_{\vartheta'=0}^{2\vartheta} \big\|f(t)\big\|_{2,\vartheta'/2}^2 + \big\|\mathbf{E}(t)\big\|_2^2 \\ &\simeq \big\|f(t)\big\|_{2,\vartheta}^2 + \big\|\mathbf{E}(t)\big\|_2^2 ,\end{aligned}$$

and the weighted dissipation rate

$$\begin{aligned}\mathcal{V}_\vartheta(t) &:= \sum_{\vartheta'=0}^{2\vartheta} \big\|f(t)\big\|_{\sigma,\vartheta'/2}^2 + \big\|\mathbf{E}(t)\big\|_2^2 \\ &\simeq \big\|f(t)\big\|_{\sigma,\vartheta}^2 + \big\|\mathbf{E}(t)\big\|_2^2 .\end{aligned}$$

Then the estimate (2.4.83) above is actually

$$\mathcal{W}_\vartheta(t) + C_\vartheta \int_s^t \mathcal{V}_\vartheta(\tau)\,\mathrm{d}\tau \le \mathcal{W}_\vartheta(s) . \tag{2.4.84}$$

As the first portion of our theorem, we show that the weighted energy (defined in Section 2.1.3)

$$\mathcal{E}_\vartheta[f(t)] := \mathcal{I}_\vartheta[f(t)] + \int_0^t \mathcal{D}_\vartheta[f(\tau)]\,\mathrm{d}\tau \simeq \mathcal{W}_\vartheta(t) + \int_0^t \mathcal{V}_\vartheta(\tau)\,\mathrm{d}\tau$$

is uniformly bounded for arbitrarily large ϑ (under the assumption of integrable decay for $\|\partial_t f\|_2$): Taking $s = 0$ in (2.4.84) results in

$$\mathcal{E}_\vartheta[f(t)] \simeq \mathcal{W}_\vartheta(t) + \int_0^t \mathcal{V}_\vartheta(\tau)\,\mathrm{d}\tau \lesssim_\vartheta \mathcal{W}_\vartheta(0) \simeq \mathcal{E}_\vartheta[f(0)] \lesssim \varepsilon_0^2 , \tag{2.4.85}$$

which concludes the uniform weighted energy bound (2.4.3).

<u>*Step 4*</u>. *L^2 Time-Decay.* Having got the boundedness, we now show that the weighted L^2 norms decay at any algebraic rate.

Based on the observation that for fixed ϑ, the dissipation rate $\|\cdot\|_{\sigma,\vartheta}$ is not stronger than the instant energy $\|\cdot\|_{2,\vartheta}$ by (2.4.45), we shall perform interpolation for $\langle v\rangle^{2\vartheta}$ between the weight functions $\langle v\rangle^{2\vartheta-1}$ and $\langle v\rangle^{2\vartheta+k}$ for any given $k\in\mathbb{N}$, then bound the stronger norm by using the result (2.4.85) of last step. This yields

$$\big\|f(t)\big\|_{2,\vartheta} \;\leq\; \big\|f(t)\big\|_{2,\vartheta+\frac{k}{2}}^{\frac{1}{k+1}}\big\|f(t)\big\|_{2,\vartheta-\frac{1}{2}}^{\frac{k}{k+1}} \;\leq\; \Big(C_{\vartheta,k}\,\|f_0\|_{2,\vartheta+\frac{k}{2}}\Big)^{\frac{1}{k+1}}\big\|f(t)\big\|_{2,\vartheta-\frac{1}{2}}^{\frac{k}{k+1}} \;\lesssim\; \varepsilon_0^{\frac{1}{k+1}}\big\|f(t)\big\|_{\sigma,\vartheta}^{\frac{k}{k+1}}\;,$$

which further implies the dissipation rate

$$\mathcal{V}_\vartheta(t) \;\geq\; C_{\vartheta,k}\,\varepsilon_0^{-\frac{2}{k}}\,\mathcal{W}_\vartheta(t)^{\frac{k+1}{k}}\;.$$

Let

$$\mathcal{Z}_\vartheta(s) := C_{\vartheta,k}\,\varepsilon_0^{-\frac{2}{k}}\int_s^\infty \mathcal{W}_\vartheta(\tau)^{\frac{k+1}{k}}\,\mathrm{d}\tau\,. \tag{2.4.86}$$

Then we get

$$\mathcal{Z}_\vartheta'(s) = -\,C_{\vartheta,k}\,\varepsilon_0^{-\frac{2}{k}}\,\mathcal{W}_\vartheta(s)^{\frac{k+1}{k}}, \tag{2.4.87}$$

From (2.4.84) with $t=\infty$, we have

$$\mathcal{Z}_\vartheta(s) \;\lesssim\; \mathcal{W}_\vartheta(s)\,. \tag{2.4.88}$$

Combining (2.4.87) and (2.4.88) yields

$$\mathcal{Z}_\vartheta'(s) + \overline{C}_{\vartheta,k}\,\varepsilon_0^{-\frac{2}{k}}\,\mathcal{Z}_\vartheta(s)^{\frac{k+1}{k}} \leq 0\,. \tag{2.4.89}$$

This is a Bernoulli-type differential inequality for $\mathcal{Z}_\vartheta(s)$. We solve it over $[0,s]$ by standard ODE method with integrating factor $-\frac{1}{k}\mathcal{Z}_\vartheta^{-\frac{k+1}{k}}$ to obtain

$$\mathcal{Z}_\vartheta(s) \lesssim_{\vartheta,k} \varepsilon_0^2 \left(1+\frac{s}{k}\right)^{-k}. \tag{2.4.90}$$

On the other hand, raising both sides of (2.4.84) to the power $\frac{k+1}{k}$, we have that for any $s \le t$ with $|t-s| \ge 1$,

$$\mathcal{W}_\vartheta(t)^{\frac{k+1}{k}} \le \mathcal{W}_\vartheta(s)^{\frac{k+1}{k}}. \tag{2.4.91}$$

Integrating the above inequality over $s \in \left[\frac{t}{2}, t-1\right]$ (for $t>2$), we obtain

$$\left(\frac{t}{2}-1\right)\mathcal{W}_\vartheta(t)^{\frac{k+1}{k}} \lesssim_{\vartheta,k} \varepsilon_0^{\frac{2}{k}}\, \mathcal{Z}_\vartheta\left(\frac{t}{2}\right). \tag{2.4.92}$$

Therefore, combining (2.4.92) with (2.4.90), we have (for t large)

$$\mathcal{W}_\vartheta(t)^{\frac{k+1}{k}} \lesssim_{\vartheta,k} \varepsilon_0^{2+\frac{2}{k}} \langle t\rangle^{-1}\left(1+\frac{t}{k}\right)^{-k}, \tag{2.4.93}$$

which further yields

$$\mathcal{W}_\vartheta(t) \lesssim_{\vartheta,k} \varepsilon_0^2 \left(1+\frac{t}{k}\right)^{-k}. \tag{2.4.94}$$

Finally, with $\left\|\mathbf{E}(t)\right\|_{H^1} \lesssim \left\|f(t)\right\|_2$ by Lemma 2.3.7, we conclude (2.4.4) in our theorem.

Step 5. $\partial_t f$ Estimates. Lastly, to close the a priori estimates, we still need to prove the similar L^2 decay bound for $\partial_t f$. We follow the same procedure as everything before in this section.

Let $\dot{f} := \partial_t f$ and $\dot{\mathbf{E}} := \partial_t \mathbf{E}$. Taking ∂_t derivative of (2.4.46), we get the equation

$$\begin{aligned} &\partial_t \dot{f} + v \cdot \nabla_x \dot{f} + L\dot{f} - 2\sqrt{\mu}\, v \cdot \dot{\mathbf{E}} \\ &= \Big\{ \Gamma[\dot{f}, f] + \Gamma[f, \dot{f}] \Big\} - \Big\{ \dot{\mathbf{E}} \cdot \nabla_v f + \mathbf{E} \cdot \nabla_v \dot{f} \Big\} + \Big\{ \big(v \cdot \dot{\mathbf{E}}\big) f + \big(v \cdot \mathbf{E}\big) \dot{f} \Big\} , \end{aligned} \tag{2.4.95}$$

where $\dot{\mathbf{E}} := \partial_t \mathbf{E}_f = \mathbf{E}_{\dot{f}} = -\nabla_x \phi_{\dot{f}}$ with $\dot{\phi} := \partial_t \phi_f = \phi_{\dot{f}}$ satisfying

$$- \Delta_x \dot{\phi} = \int_{\mathbb{R}^3} \sqrt{\mu}\, \dot{f} \,\mathrm{d}v \; =: \rho[\dot{f}] \, . \tag{2.4.96}$$

First of all, we check that the kernel estimate and coercivity result hold by repeating exactly the arguments of Lemma 2.4.3 and Corollary 2.4.3.1 for the new equations with harmless temporal derivatives. The argument is almost identical, so we omit the details.

Next, similar to the (weighted) energy estimate for f, upon multiplying (2.4.95) by the factor e^{ϕ}, the last term $(v \cdot \mathbf{E}) \dot{f}$ is cancelled. We first multiply $e^{\phi} \dot{f}$ (resp. $\langle v \rangle^{2\vartheta} e^{\phi} \dot{f}$ for the weighted case) on both sides and integrate over $\Omega \times \mathbb{R}^3$ to get

$$\begin{aligned} &\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \big\| e^{\phi} \dot{f} \big\|_2^2 + \frac{1}{2} \iint_{\Omega \times \mathbb{R}^3} \nabla_x \cdot \Big\{ v \big(e^{\phi} \dot{f} \big)^2 \Big\} + \iint_{\Omega \times \mathbb{R}^3} \big(e^{\phi} \dot{f} \big) L \big(e^{\phi} \dot{f} \big) - \iint_{\Omega \times \mathbb{R}^3} 2 \big(e^{2\phi} \dot{f} \big) \sqrt{\mu}\, v \cdot \dot{\mathbf{E}} \\ &= \iint_{\Omega \times \mathbb{R}^3} \big(e^{2\phi} \dot{f} \big) \Gamma[\dot{f}, f] + \iint_{\Omega \times \mathbb{R}^3} \big(e^{2\phi} \dot{f} \big) \Gamma[f, \dot{f}] - \iint_{\Omega \times \mathbb{R}^3} e^{2\phi} \dot{f}\, \dot{\mathbf{E}} \cdot \nabla_v f - \frac{1}{2} \iint_{\Omega \times \mathbb{R}^3} e^{2\phi}\, \mathbf{E} \cdot \nabla_v \big(\dot{f}^2 \big) \\ &\quad + \iint_{\Omega \times \mathbb{R}^3} e^{2\phi} \big(v \cdot \dot{\mathbf{E}} \big) f \dot{f} + \iint_{\Omega \times \mathbb{R}^3} \big(e^{\phi} \dot{f} \big)^2 \dot{\phi} \, , \end{aligned}$$

and then we examine this resulting equation term by term. The estimates from Step 1 to Step 3 of similar form are valid for $\dot{f}$ with only a few changes in the proof which we will point out below:

In the case without weight ($\vartheta = 0$): For the LHS's fourth term, in order to avoid producing higher-order derivatives terms (which we cannot control) we modify our previous argument in a more direct way. Extracting the main contribution of this term, we deduce that

$$\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3} 2\,\dot f\sqrt{\mu}\,v\cdot\dot{\mathbf{E}} \;&=\; \iint_{\Omega\times\mathbb{R}^3} 2\,\dot f\sqrt{\mu}\,v\cdot\nabla_x\dot\phi \;=\; -\iint_{\Omega\times\mathbb{R}^3} 2\,\dot\phi\sqrt{\mu}\,v\cdot\nabla_x\dot f \\
&=\; -\int_\Omega 2\,\dot\phi\,\nabla_x\cdot\Big(\int_{\mathbb{R}^3} v\sqrt{\mu}\,\dot f\,\mathrm{d}v\Big)\,\mathrm{d}x \;=\; \int_\Omega 2\,\dot\phi\,\partial_t\Big(\int_{\mathbb{R}^3}\sqrt{\mu}\,\dot f\,\mathrm{d}v\Big)\,\mathrm{d}x \\
&=\; -\int_\Omega 2\,\dot\phi\,\partial_t\Delta_x\dot\phi\,\mathrm{d}x \;=\; \int_\Omega 2\,(\nabla_x\dot\phi)\cdot(\nabla_x\partial_t\dot\phi)\,\mathrm{d}x \;=\; \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega|\dot{\mathbf{E}}|^2\,.
\end{aligned}$$

Here the four equality is due to the continuity equation (2.1.21) of conservation laws (which is essentially the same kind of conserved quantity derived in a similar way from the equation (2.4.46) in Step 1). Then the remainder can be easily controlled by

$$\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3} 2\,(e^{2\phi}-1)\,\dot f\sqrt{\mu}\,v\cdot\dot{\mathbf{E}} \;&\lesssim\; \big\|e^{2\phi}-1\big\|_\infty\big\|v\sqrt{\mu}\,\dot f\big\|_2\,\|\dot{\mathbf{E}}\|_2 \\
&\lesssim\; \|\phi\|_\infty\|\dot f\|_\sigma^2\,,
\end{aligned}$$

and thus be absorbed as $\|\phi\|_\infty \lesssim \|f\|_\infty \ll 1$.

Another difficulty is that, the nonlinearity generates two more terms containing $\|f\|_2^2$ and $\|f\|_\sigma^2$ related to f rather than $\dot f$ (which cannot be estimated directly, but requires some special treatment): to be specific,

$$\iint_{\Omega\times\mathbb{R}^3}\big(e^{2\phi}\dot f\,\big)\,\Gamma[\dot f, f] \;\lesssim\; \|\dot f\|_\infty\|f\|_\sigma\|\dot f\|_\sigma \;\lesssim\; \|\dot f\|_\infty\big(\|f\|_\sigma^2+\|\dot f\|_\sigma^2\big)\,,$$

and

$$\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\dot f\,\dot{\mathbf{E}}\cdot\nabla_v f \;&=\; \iint_{\Omega\times\mathbb{R}^3} e^{2\phi} f\,\dot{\mathbf{E}}\cdot\nabla_v\dot f \;\lesssim\; \|\nabla_v\dot f\|_\infty\|f\|_2\|\dot{\mathbf{E}}\|_2 \\
&\lesssim\; \|\nabla_v\dot f\|_\infty\|f\|_2\|\dot f\|_\sigma \;\lesssim\; \|\nabla_v\dot f\|_\infty\big(\|f\|_2^2+\|\dot f\|_\sigma^2\big)\,,
\end{aligned}$$

which requires the additional assumption of smallness of $\|\nabla_v \dot{f}\|_\infty$ Also, the third extra term

$$\iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\big(v\cdot\dot{\mathbf{E}}\big)f\dot{f} \;\lesssim\; \big\|\langle v\rangle^{\frac{1}{2}}v\,f\big\|_\infty\|\dot{f}\|_\sigma\|\dot{\mathbf{E}}\|_2 \;\lesssim\; \|f\|_{\infty,\frac{3}{2}}\|\dot{f}\|_\sigma^2$$

needs the smallness of weighted L^∞ norm.

In summary, under the assumption that

$$\sup_t\|f\|_{\infty,\overline{\vartheta}} + \sup_t\|\dot{f}\|_{\infty,\overline{\vartheta}} + \sup_t\|\nabla_v\dot{f}\|_\infty \ll 1 \tag{2.4.97}$$

for some $\overline{\vartheta}\geq 3$ and using the similar technique to multiply e^ψ on both sides, we have

$$\begin{aligned}
&\Big(\big\|e^{\psi(t)}e^{\phi(t)}\dot{f}(t)\big\|_2^2 + \big\|e^{\psi(t)}e^{\phi(t)}\dot{\mathbf{E}}(t)\big\|_2^2\Big) + \delta'\left\{\int_s^t\big\|\dot{f}(\tau)\big\|_\sigma^2\,\mathrm{d}\tau + \int_s^t\big\|\dot{\mathbf{E}}(\tau)\big\|_2^2\,\mathrm{d}\tau\right\}\\
\lesssim\;&\Big(\big\|e^{\psi(s)}e^{\phi(s)}\dot{f}(s)\big\|_2^2 + \big\|e^{\psi(s)}e^{\phi(s)}\dot{\mathbf{E}}(s)\big\|_2^2\Big)\\
&+\int_s^t\big\|\dot{f}(\tau)\big\|_\infty\big\|f(\tau)\big\|_\sigma^2\,\mathrm{d}\tau + \int_s^t\big\|\nabla_v\dot{f}(\tau)\big\|_\infty\big\|f(\tau)\big\|_2^2\,\mathrm{d}\tau\\
\leq&\Big(\big\|e^{\psi(s)}e^{\phi(s)}\dot{f}(s)\big\|_2^2 + \big\|e^{\psi(s)}e^{\phi(s)}\dot{\mathbf{E}}(s)\big\|_2^2\Big)\\
&+\sup_{\tau\geq 0}\|\dot{f}\|_\infty\cdot\int_s^t\big\|f(\tau)\big\|_\sigma^2\,\mathrm{d}\tau + \sup_{\tau\geq 0}\|\nabla_v\dot{f}\|_\infty\cdot\int_s^t\big\|f(\tau)\big\|_2^2\,\mathrm{d}\tau
\end{aligned} \tag{2.4.98}$$

for any $0\leq s<t$ with $|t-s|\in\mathbb{Z}^+$.

For the weighted case ($\vartheta>0$), most of the estimates are similar to those of weighted version for f in Step 2 simply replacing certain norms on f by those on

$\dot{f}$, except for the three additional terms from nonlinearity. In particular,

$$
\begin{aligned}
-\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}e^{2\phi}\dot{f}\,\dot{\mathbf{E}}\cdot\nabla_v f \;&=\; \iint_{\Omega\times\mathbb{R}^3} e^{2\phi}\,\dot{\mathbf{E}}\cdot\nabla_v\Big(\langle v\rangle^{2\vartheta}\Big)\,f\dot{f} \;+\; \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}e^{2\phi}f\,\dot{\mathbf{E}}\cdot\nabla_v\dot{f} \\
&\lesssim\; \|\dot{\mathbf{E}}\|_\infty\|f\|_{\sigma,\vartheta}\|\dot{f}\|_{\sigma,\vartheta} \;+\; \|\nabla_v\dot{f}\|_\infty\|f\|_{2,2\vartheta}\|\dot{\mathbf{E}}\|_2 \\
&\lesssim\; \|f\|_{\sigma,\vartheta}\|\dot{f}\|^2_{\sigma,\vartheta} \;+\; \|\nabla_v\dot{f}\|_\infty\big(\|f\|^2_{2,2\vartheta}+\|\dot{f}\|^2_{\sigma,\vartheta}\big)\;.
\end{aligned}
$$

The rest of this substep is a simple modification of the $\vartheta=0$ case. Eventually, under the assumption that

$$
\sup_t\|f\|_{\sigma,\vartheta} \;+\; \sup_t\|f\|_{\infty,\vartheta+\frac{3}{2}} \;+\; \sup_t\|\dot{f}\|_\infty \;+\; \sup_t\|\nabla_v\dot{f}\|_\infty \;\ll\; 1 \tag{2.4.99}
$$

and using the techniques of multiplying e^{ψ} on both sides, we obtain the weighted energy estimate

$$
\begin{aligned}
&\big\|e^{\psi(t)}e^{\phi(t)}\dot{f}(t)\big\|^2_{2,\vartheta} \;+\; \int_s^t\big\|\dot{f}(\tau)\big\|^2_{\sigma,\vartheta}\,\mathrm{d}\tau \\
\lesssim\;& \big\|e^{\psi(s)}e^{\phi(s)}\dot{f}(s)\big\|^2_{2,\vartheta}+C_\vartheta\int_s^t\big\|\dot{f}(\tau)\big\|^2_{\sigma}\,\mathrm{d}\tau \\
&+\sup_{\tau\geq0}\|\dot{f}\|_\infty\cdot\int_s^t\big\|f(\tau)\big\|^2_{\sigma,\vartheta}\,\mathrm{d}\tau \;+\; \sup_{\tau\geq0}\|\nabla_v\dot{f}\|_\infty\cdot\int_s^t\big\|f(\tau)\big\|^2_{2,2\vartheta}\,\mathrm{d}\tau\;,
\end{aligned}
\tag{2.4.100}
$$

Now we use these two inequalities (2.4.98) and (2.4.100) to conclude

$$
\begin{aligned}
&\left(\sum_{\vartheta'=0}^{\vartheta} \left\| e^{\psi(t)} e^{\phi(t)} \dot{f}(t) \right\|_{2,\vartheta'/2}^2 + \left\| e^{\psi(t)} e^{\phi(t)} \dot{\mathbf{E}}(t) \right\|_2^2 \right) \\
&\quad + \delta' \left\{ \sum_{\vartheta'=0}^{\vartheta} \int_s^t \left\| \dot{f}(\tau) \right\|_{\sigma,\vartheta'/2}^2 \mathrm{d}\tau + \int_s^t \left\| \dot{\mathbf{E}}(\tau) \right\|_2^2 \mathrm{d}\tau \right\} \\
&\lesssim \left(\sum_{\vartheta'=0}^{\vartheta} \left\| e^{\psi(s)} e^{\phi(s)} \dot{f}(s) \right\|_{2,\vartheta'/2}^2 + \left\| e^{\psi(s)} e^{\phi(s)} \dot{\mathbf{E}}(s) \right\|_2^2 \right) \\
&\quad + \sup_{\tau \geq 0} \| \dot{f} \|_\infty \cdot \sum_{\vartheta'=0}^{\vartheta} \int_s^t \left\| f(\tau) \right\|_{\sigma,\vartheta'/2}^2 \mathrm{d}\tau + \sup_{\tau \geq 0} \| \nabla_v \dot{f} \|_\infty \cdot \sum_{\vartheta'=0}^{\vartheta} \int_s^t \left\| f(\tau) \right\|_{2,\vartheta'}^2 \mathrm{d}\tau,
\end{aligned}
\tag{2.4.101}
$$

by summing up over $\vartheta' \in [0, \vartheta] \cap \mathbb{N}$ for any given $\vartheta \in \mathbb{N}$ with proper weight.

Due to Lemma 2.4.3, the above result only holds for $t - s \in \mathbb{Z}^+$. Using a similar argument as in Step 3, with the help of local well-posedness result, we can extend the estimate to arbitrary $t - s \geq 1$.

Notice that there is an extra term of f with the highest-power weight (serving as a "source term"), which forces us to combine the two weighted inequalities for f and $\dot{f}$ with different range (summation limits) so that this term can be absorbed. Defining

$$
\begin{aligned}
\widetilde{\mathcal{W}}_\vartheta(t) &:= \sum_{\vartheta'=0}^{2\vartheta+1} \left\| f(t) \right\|_{2,\vartheta'/2}^2 + \sum_{\vartheta'=0}^{\vartheta} \left\| \dot{f}(t) \right\|_{2,\vartheta'/2}^2 + \left\| \mathbf{E}(t) \right\|_2^2 + \left\| \dot{\mathbf{E}}(t) \right\|_2^2 \\
&\simeq \left\| f(t) \right\|_{2,\vartheta+\frac{1}{2}}^2 + \left\| \dot{f}(t) \right\|_{2,\frac{\vartheta}{2}}^2 + \left\| \mathbf{E}(t) \right\|_2^2 + \left\| \dot{\mathbf{E}}(t) \right\|_2^2 ,
\end{aligned}
$$

and

$$
\begin{aligned}
\widetilde{\mathcal{V}}_\vartheta(t) \,:=&\, \sum_{\vartheta'=0}^{2\vartheta+1} \big\|f(t)\big\|^2_{\sigma,\vartheta'/2} + \sum_{\vartheta'=0}^{\vartheta} \big\|\dot{f}(t)\big\|^2_{\sigma,\vartheta'/2} + \big\|\mathbf{E}(t)\big\|_2^2 + \big\|\dot{\mathbf{E}}(t)\big\|_2^2 \\
\simeq&\, \big\|f(t)\big\|^2_{\sigma,\vartheta+\frac{1}{2}} + \big\|\dot{f}(t)\big\|^2_{\sigma,\frac{\vartheta}{2}} + \big\|\mathbf{E}(t)\big\|_2^2 + \big\|\dot{\mathbf{E}}(t)\big\|_2^2 \,.
\end{aligned}
$$

(2.4.101) together with (2.4.83) leads to

$$
\widetilde{\mathcal{W}}_\vartheta(t) + \int_s^t \widetilde{\mathcal{V}}_\vartheta(\tau)\,\mathrm{d}\tau \lesssim \widetilde{\mathcal{W}}_\vartheta(s)\,. \tag{2.4.102}
$$

Finally, mimicking the derivation of (2.4.3) and (2.4.4), the desired bounds for $\partial_t f$ follow. Therefore we complete the proof of the theorem. □

2.5 Extension Across the Specular-Reflection Boundary

For the later use of S^p theory, we need to extend the domain Ω to the whole space.

2.5.1 Extension of Solutions to the Whole Space

In this subsection, we will show step by step the way of extending our equation satisfied on a bounded domain with specular-reflection BC to a whole space problem.

“Boundary-Flattening” Transformation

Let

$$\begin{aligned} \vec{\Psi}: \quad \overline{\Omega}\times\mathbb{R}^3 \quad &\rightarrow \quad \overline{\mathbb{H}}_-\times\mathbb{R}^3 \\ (x,v) \quad &\mapsto \quad (y,w) \stackrel{\text{def}}{=} \big(\vec{\psi}(x), Av\big) \end{aligned} \tag{2.5.1}$$

be the (local) transformation that flattens the boundary, where

$$A \stackrel{\text{def}}{=} \left[\frac{\partial y}{\partial x}\right] = D\vec{\psi}$$

is a non-degenerate 3×3 Jacobian matrix, and the explicit definition of $y=\vec{\psi}(x)$ will be given below. Let

$$\tilde{f}(t,y,w) \stackrel{\text{def}}{=} f\big(t,\vec{\Psi}^{-1}(y,w)\big) = f\big(t,\vec{\psi}^{-1}(y),A^{-1}w\big) = f(t,x,v). \tag{2.5.2}$$

denote the solution under the new coordinates.

Remark 2.5.1. It is crucial that we define our transformation $\vec{\Psi}$ for both (x,v) variables in this certain form so that it preserves the characteristics and the transport operator as explained (see also the subsection of 2.5.1 below for more details).

Suppose the boundary $\partial\Omega$ is (locally) given by the graph $x_3=\rho(x_1,x_2)$, and $\big\{(x_1,x_2,x_3)\in\mathbb{R}^3 : x_3<\rho(x_1,x_2)\big\}\subseteq\Omega$. Inspired by Lemma 15 in [33][2], we define $y=\vec{\psi}(x)$ explicitly as follows:

[2] The authors used spherical-type coordinates to make the map almost globally defined; here we just prefer the standard coordinates for simplicity.

$$\vec{\psi}^{-1}: \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix} \mapsto \vec{\eta}(y_1, y_2) + y_3 \cdot \vec{n}(y_1, y_2)$$

$$= \begin{pmatrix} y_1 \\ y_2 \\ \rho(y_1, y_2) \end{pmatrix} + y_3 \cdot \begin{pmatrix} -\rho_1 \\ -\rho_2 \\ 1 \end{pmatrix}$$

$$= \begin{pmatrix} y_1 - y_3 \cdot \rho_1 \\ y_2 - y_3 \cdot \rho_2 \\ \rho + y_3 \end{pmatrix} =: \begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix},$$

where we denote by $\rho = \rho(y_1, y_2)$, $\rho_i = \partial_i \rho(y_1, y_2), i=1, 2$, and

$$\vec{\eta}(y_1, y_2) \stackrel{\text{def}}{=} \big(y_1, y_2, \rho(y_1, y_2)\big) \in \partial\Omega$$

$$\partial_1 \vec{\eta} \stackrel{\text{def}}{=} \frac{\partial \vec{\eta}}{\partial y_1} = \langle 1, 0, \rho_1 \rangle$$

$$\partial_2 \vec{\eta} \stackrel{\text{def}}{=} \frac{\partial \vec{\eta}}{\partial y_2} = \langle 0, 1, \rho_2 \rangle,$$

then the (outward) normal vector at the point $\vec{\eta}(y_1, y_2) \in \partial\Omega$ is chosen to be

$$\vec{n}(y_1, y_2) \stackrel{\text{def}}{=} \partial_1 \vec{\eta} \times \partial_2 \vec{\eta} = \langle -\rho_1, -\rho_2, 1 \rangle.$$

From the above definition we can see that the transformation $\vec{\psi}$ is “boundary-flattening” because it maps the points on the boundary $\{x_3 = \rho(x_1, x_2)\}$ to the plane $\{y_3 = 0\}$. We also remark that the map is locally well-defined and is a smooth homeomorphism in a tubular neighborhood of the boundary (see Lemma 15 of [33] for the rigorous proof).

Directly we compute the Jacobian matrix

$$A^{-1} = D\vec{\psi}^{-1} = \left[\frac{\partial x}{\partial y}\right] = \left[\partial_1\vec{\eta} + y_3\cdot\partial_1\vec{n}; \partial_2\vec{\eta} + y_3\cdot\partial_2\vec{n}; \vec{n}\right]$$
$$= \begin{pmatrix} 1-y_3\cdot\rho_{11} & -y_3\cdot\rho_{12} & -\rho_1 \\ -y_3\cdot\rho_{12} & 1-y_3\cdot\rho_{22} & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}$$
$$\xrightarrow{\text{on } \partial\Omega:\ y_3=0} \left[\partial_1\vec{\eta}; \partial_2\vec{\eta}; \vec{n}\right]$$
$$= \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix}.$$

So we can write out $\vec{\Psi}^{-1}$ as

$$\vec{\Psi}^{-1}:\quad (y, w) \ \mapsto\ (x, v) \stackrel{\text{def}}{=} \left(\vec{\psi}^{-1}(y), A^{-1}w\right) \tag{2.5.3}$$

$$\begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} \mapsto \begin{pmatrix} 1-y_3\cdot\rho_{11} & -y_3\cdot\rho_{12} & -\rho_1 \\ -y_3\cdot\rho_{12} & 1-y_3\cdot\rho_{22} & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix}$$
$$= \begin{pmatrix} (1-y_3\rho_{11})\cdot w_1 - y_3\rho_{12}\cdot w_2 - \rho_1\cdot w_3 \\ -y_3\rho_{12}\cdot w_1 + (1-y_3\rho_{22})\cdot w_2 - \rho_2\cdot w_3 \\ \rho_1\cdot w_1 + \rho_2\cdot w_2 + w_3 \end{pmatrix} =: \begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix}.$$

Restricted on the boundary $\partial\Omega$ i.e.,$\{y_3=0\}$, the map becomes

$$\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = w_1 \cdot \partial_1\vec{\eta} + w_2 \cdot \partial_2\vec{\eta} + w_3 \cdot \vec{n}$$

$$= \begin{pmatrix} 1 & 0 & -\rho_1 \\ 0 & 1 & -\rho_2 \\ \rho_1 & \rho_2 & 1 \end{pmatrix} \begin{pmatrix} w_1 \\ w_2 \\ w_3 \end{pmatrix} = \begin{pmatrix} w_1 - \rho_1 \cdot w_3 \\ w_2 - \rho_2 \cdot w_3 \\ \rho_1 \cdot w_1 + \rho_2 \cdot w_2 + w_3 \end{pmatrix}.$$

Now we are ready to show the key feature of the transformation $\vec{\Psi}$ – preserving the "specular symmetry" on the boundary: it sends any two points $(x, v), (x, R_x v)$ on the phase boundary $\gamma = \partial\Omega \times \mathbb{R}^3$ with specular-reflection relation to two points on $\{y_3=0\} \times \mathbb{R}^3$ which are also specular-symmetric to each other.

In other words, we have the following commutative diagram (when $x \in \partial\Omega$ i.e., $y_3=0$):

$$\begin{array}{ccc} (y, w) & \xrightarrow{\vec{\Psi}^{-1}} & (x, v) \\ {\scriptstyle R_y}\big\downarrow & & \big\downarrow{\scriptstyle R_x} \\ (y, R_y w) & \xrightarrow{\vec{\Psi}^{-1}} & (x, R_x v) \end{array}$$

from which we can equivalently write

$$A^{-1}\big(R_y w\big) = R_x\big(A^{-1}w\big), \quad \text{if } y_3=0.$$

This can be verified either geometrically by noticing that $\vec{n} = \partial_1\vec{\eta} \times \partial_2\vec{\eta}$, so

$$\begin{aligned} A^{-1}\big(R_y w\big) &= A^{-1}\langle w_1, w_2, -w_3\rangle = w_1 \cdot \partial_1\vec{\eta} + w_2 \cdot \partial_2\vec{\eta} - w_3 \cdot \vec{n} \\ &= R_x\big(w_1 \cdot \partial_1\vec{\eta} + w_2 \cdot \partial_2\vec{\eta} + w_3 \cdot \vec{n}\big) = R_x\big(A^{-1}w\big); \end{aligned}$$

or arithmetically using the definition $R_x v = v - 2(n_x \cdot v) n_x$.

Having this property, the specular reflection boundary condition on the solutions is also preserved:

$$\tilde{f}(t, y, w) = \tilde{f}(t, y, Rw), \quad \text{on } \{y_3 = 0\},$$

where $R \stackrel{\text{def}}{=} \operatorname{diag}\{1, 1, -1\}$, which allows us to construct the mirror extension (as in the next subsection) that is consistent with this restriction and thus is automatically satisfied.

To conclude this part, we carry out more computations for later use:

$$D\vec{\Psi}^{-1} = \left[\frac{\partial(x, v)}{\partial(y, w)}\right] = \left(\begin{array}{c|c} \frac{\partial x}{\partial y} & \frac{\partial x}{\partial w} \\ \hline \frac{\partial v}{\partial y} & \frac{\partial v}{\partial w} \end{array}\right) = \left(\begin{array}{c|c} A^{-1} & 0_{3\times 3} \\ \hline B & A^{-1} \end{array}\right), \tag{2.5.4}$$

$$\begin{aligned} B &\stackrel{\text{def}}{=} \left[\frac{\partial v}{\partial y}\right] \\ &= \begin{pmatrix} \begin{array}{c} -y_3\rho_{111}\cdot w_1 - y_3\rho_{112}\cdot w_2 \\ -\rho_{11}\cdot w_3 \end{array} & \begin{array}{c} -y_3\rho_{112}\cdot w_1 - y_3\rho_{122}\cdot w_2 \\ -\rho_{12}\cdot w_3 \end{array} & -\rho_{11}\cdot w_1 - \rho_{12}\cdot w_2 \\ \begin{array}{c} -y_3\rho_{112}\cdot w_1 - y_3\rho_{122}\cdot w_2 \\ -\rho_{12}\cdot w_3 \end{array} & \begin{array}{c} -y_3\rho_{122}\cdot w_1 - y_3\rho_{222}\cdot w_2 \\ -\rho_{22}\cdot w_3 \end{array} & -\rho_{12}\cdot w_1 - \rho_{22}\cdot w_2 \\ \rho_{11}\cdot w_1 + \rho_{12}\cdot w_2 & \rho_{12}\cdot w_1 + \rho_{22}\cdot w_2 & 0 \end{pmatrix}, \end{aligned}$$

$$
\begin{aligned}
A &= \frac{1}{\det(A^{-1})} \cdot \left(A^{-1}\right)^* \\
&= \left[y_3^2 \cdot \left(\rho_{11}\rho_{22} - \rho_{12}^2\right) + y_3 \cdot \left(2\rho_1\rho_2\rho_{12} - \rho_2^2\rho_{11} - \rho_1^2\rho_{22} - \rho_{11} - \rho_{22}\right) + \left(\rho_1^2 + \rho_2^2 + 1\right)\right]^{-1} \\
&\quad \cdot \begin{pmatrix} (1+\rho_2^2) - y_3 \cdot \rho_{22} & -\rho_1\rho_2 + y_3 \cdot \rho_{12} & \rho_1 + y_3 \cdot (\rho_2\rho_{12} - \rho_1\rho_{22}) \\ -\rho_1\rho_2 + y_3 \cdot \rho_{12} & (1+\rho_1^2) - y_3 \cdot \rho_{11} & \rho_2 + y_3 \cdot (\rho_1\rho_{12} - \rho_2\rho_{11}) \\ -\rho_1 + y_3 \cdot (\rho_1\rho_{22} - \rho_2\rho_{12}) & -\rho_2 + y_3 \cdot (\rho_2\rho_{11} - \rho_1\rho_{12}) & \begin{array}{c} 1 - y_3 \cdot (\rho_{11} + \rho_{22}) \\ + y_3^2 \cdot (\rho_{11}\rho_{22} - \rho_{12}^2) \end{array} \end{pmatrix},
\end{aligned}
$$

$$
\begin{aligned}
C &\stackrel{\text{def}}{=} A^{-T}A^{-1} \\
&= \begin{pmatrix} \begin{array}{c} y_3^2 \cdot (\rho_{11}^2 + \rho_{12}^2) \\ -y_3 \cdot 2\rho_{11} + (\rho_1^2 + 1) \end{array} & \begin{array}{c} y_3^2 \cdot \rho_{12}(\rho_{11} + \rho_{22}) \\ -y_3 \cdot 2\rho_{12} + \rho_1\rho_2 \end{array} & y_3 \cdot (\rho_1\rho_{11} + \rho_2\rho_{12}) \\ \begin{array}{c} y_3^2 \cdot \rho_{12}(\rho_{11} + \rho_{22}) \\ -y_3 \cdot 2\rho_{12} + \rho_1\rho_2 \end{array} & \begin{array}{c} y_3^2 \cdot (\rho_{12}^2 + \rho_{22}^2) \\ -y_3 \cdot 2\rho_{22} + (\rho_2^2 + 1) \end{array} & y_3 \cdot (\rho_1\rho_{12} + \rho_2\rho_{22}) \\ y_3 \cdot (\rho_1\rho_{11} + \rho_2\rho_{12}) & y_3 \cdot (\rho_1\rho_{12} + \rho_2\rho_{22}) & \rho_1^2 + \rho_2^2 + 1 \end{pmatrix},
\end{aligned}
$$

$$
C^{-1} = AA^T = \frac{1}{\det(C)} \cdot C^*.
$$

Remark 2.5.2. Here we just assume ρ is (locally) smooth enough and its derivatives remain uniformly bounded, so that all the coefficients of transformed equations where the above matrices appear will keep roughly the same size as the original ones.

Mirror Extension Across the Specular-Reflection Boundary

After flattening the boundary, we then "flip over" $\tilde{f}$ to the upper half space by setting

$$\overline{f}(t,y',w') \overset{\text{def}}{=} \begin{cases} \tilde{f}(t,y',w'), & \text{if } y' \in \overline{\mathbb{H}}_- \\ \tilde{f}(t,Ry',Rw'), & \text{if } y' \in \overline{\mathbb{H}}_+ \end{cases}, \tag{2.5.5}$$

where $R \overset{\text{def}}{=} \text{diag}\{1,1,-1\}$. The similar notation also applies to other variables. Combined with the corresponding partition of unity, we are able to define our solutions in the whole space (locally).

Remark 2.5.3. The above construction of extension coincides with the specular reflection boundary condition, which in turn makes it a well-defined and continuous extension across the boundary. This observation suggests that, unfortunately, we cannot apply the same kind of extension to other boundary condition cases.

Also, it is worth pointing out the necessity of "continuity of $\overline{f}$ across the boundary" lies in that, on one hand, it ensures $\overline{f}$ is indeed a solution (at least) in the weak sense in the whole space (see Section 2.5.2); on the other hand, g will appear in the coefficients of the ultra-parabolic form, and we require some kind of continuity of the second-order coefficient for the S^p estimate.

Transformed Equations

By using the chain rule with our definitions (2.5.1) and (2.5.2) of the transformation $\vec{\Psi}$, we first compute the transformed equation satisfied by $\tilde{f}$ in the lower half space:[3]

$$\partial_t f = \partial_t \tilde{f},$$

$$\begin{aligned} v \cdot \nabla_x f &= \left(A^{-1}w\right)^T \left\{ A^T \nabla_y \tilde{f} + \left[\tfrac{\partial w}{\partial x}\right]^T \nabla_w \tilde{f} \right\} \\ &= w^T \left(A^{-T}A^T\right) \nabla_y \tilde{f} + \left(A^{-1}w\right)^T \left(A\left[\tfrac{\partial v}{\partial y}\right]A\right)^T \nabla_w \tilde{f} \\ &= w \cdot \nabla_y \tilde{f} + \left(ABw\right) \cdot \nabla_w \tilde{f}, \end{aligned}$$

$$a_g \cdot \nabla_v f = \widetilde{a_g} \cdot \left(A^T \nabla_w \tilde{f}\right) = \left(A\widetilde{a_g}\right) \cdot \nabla_w \tilde{f},$$

$$\mathbf{E}_g \cdot \nabla_v f = \widetilde{\mathbf{E}_g} \cdot \left(A^T \nabla_w \tilde{f}\right) = \left(A\widetilde{\mathbf{E}_g}\right) \cdot \nabla_w \tilde{f},$$

and

$$\nabla_v \cdot \left(\sigma_G \nabla_v f\right) = \nabla_w \cdot \left(\left[A\widetilde{\sigma}_G A^T\right] \nabla_w \tilde{f}\right).$$

See (2.5.4) for explicit definition of A, B, and

$$\begin{aligned} \widetilde{a_g}(t,y,w) &\stackrel{\text{def}}{=} a_g\left(t, \vec{\Psi}^{-1}(y,w)\right) = a_g(t,x,v), \\ \widetilde{\mathbf{E}_g}(t,y) &\stackrel{\text{def}}{=} \mathbf{E}_g\left(t, \vec{\psi}^{-1}(y)\right) = \mathbf{E}_g(t,x), \\ \widetilde{\sigma}_G(t,y,w) &\stackrel{\text{def}}{=} \sigma_G\left(t, \vec{\Psi}^{-1}(y,w)\right) = \sigma_G(t,x,v). \end{aligned}$$

Based on our construction of the extension (2.5.5), we then go on deriving the

[3] We use the column vector convention in the following matrix operation expressions.

equation satisfied by $\overline{f}$ for the upper half space:

$$\partial_t \tilde{f} = \partial_t \overline{f},$$

$$w \cdot \nabla_y \tilde{f} = \big(Rw'\big)^T R^T \nabla_{y'} \overline{f} = w'^T \big(R^T R^T\big) \nabla_{y'} \overline{f} = w' \cdot \nabla_{y'} \overline{f},$$

$$\big(ABw\big) \cdot \nabla_w \tilde{f} = \big(\overline{A}\overline{B} Rw'\big) \cdot R^T \nabla_{w'} \overline{f} = \big(R\overline{A}\overline{B} Rw'\big) \cdot \nabla_{w'} \overline{f},$$

$$\big(A\widetilde{a_g}\big) \cdot \nabla_w \tilde{f} = \big(\overline{A}\overline{a}_g\big) \cdot R^T \nabla_{w'} \overline{f} = \big(R\overline{A}\overline{a}_g\big) \cdot \nabla_{w'} \overline{f},$$

$$\big(A\widetilde{\mathbf{E}_g}\big) \cdot \nabla_w \tilde{f} = \big(\overline{A}\overline{\mathbf{E}}_g\big) \cdot R^T \nabla_{w'} \overline{f} = \big(R\overline{A}\overline{\mathbf{E}}_g\big) \cdot \nabla_{w'} \overline{f},$$

$$\nabla_w \cdot \Big(\big[A\widetilde{\sigma}_G A^T\big] \nabla_w \tilde{f}\Big) = \nabla_{w'} \cdot \Big(\big[R\overline{A}\overline{\sigma}_G \overline{A}^T R\big] \nabla_{w'} \overline{f}\Big),$$

where

$$\overline{A}(y') \stackrel{\text{def}}{=} A(Ry') = A(y),$$
$$\overline{B}(y', w') \stackrel{\text{def}}{=} B(Ry', Rw') = B(y, w),$$

and $\overline{a}_g$, $\overline{\mathbf{E}}_g$, $\overline{\sigma}_G$ are a_g, $\mathbf{E}_g$, σ_G defined with (t, y', w'), respectively.

Summing up the above computations, we now obtain that $\overline{f}$ satisfies (pointwisely) the following equation(s) in the lower and upper space, respectively:

$$\partial_t \overline{f} + w' \cdot \nabla_{y'} \overline{f} = \nabla_{w'} \cdot \big(\mathbb{A} \nabla_{w'} \overline{f}\big) + \mathbb{B} \cdot \nabla_{w'} \overline{f} + \mathbb{C} \overline{f}, \tag{2.5.6}$$

where the coefficients $\mathbb{A}$, $\mathbb{B}$, and $\mathbb{C}$ are piecewise-defined:

$$\mathbb{A}(t, y', w') \stackrel{\text{def}}{=} \begin{cases} \widetilde{\mathbb{A}} \stackrel{\text{def}}{=} A\widetilde{\sigma}_G A^T, & \text{if } y' \in \mathbb{H}_- \\ \overline{\mathbb{A}} \stackrel{\text{def}}{=} R\overline{A}\overline{\sigma}_G \overline{A}^T R, & \text{if } y' \in \mathbb{H}_+ \end{cases}, \tag{2.5.7}$$

$$\mathbb{B}(t,y',w') \stackrel{\text{def}}{=} \begin{cases} \widetilde{\mathbb{B}} \stackrel{\text{def}}{=} ABw' + A\widetilde{a}_g - A\widetilde{\mathbf{E}_g}, & \text{if } y' \in \mathbb{H}_- \\ \overline{\mathbb{B}} \stackrel{\text{def}}{=} R\overline{A}\overline{B}Rw' + R\overline{A}\overline{a}_g - R\overline{A}\overline{\mathbf{E}}_g, & \text{if } y' \in \mathbb{H}_+ \end{cases}, \tag{2.5.8}$$

$$\mathbb{C}(t,y',w') \stackrel{\text{def}}{=} \begin{cases} \widetilde{\mathbb{C}} \stackrel{\text{def}}{=} \widetilde{\overline{K}_g f} + (A^{-1}w')^T\widetilde{\mathbf{E}}_g + 2\sqrt{\mu(w')}(A^{-1}w')^T\widetilde{\mathbf{E}}_g, & \text{if } y' \in \mathbb{H}_- \\ \overline{\mathbb{C}} \stackrel{\text{def}}{=} \overline{\overline{K}_g f} + (\overline{A}^{-1}Rw')^T\overline{\mathbf{E}}_g + 2\sqrt{\mu(w')}(\overline{A}^{-1}Rw')^T\overline{\mathbf{E}}_g, & \text{if } y' \in \mathbb{H}_+ \end{cases}. \tag{2.5.9}$$

Remark 2.5.4. Thanks to our design of the form of transformation (2.5.1) and extension (2.5.5), the transport operator of the equation remains *invariant* after change of variables, which is vital for our future analysis.

It is also worth noting that the new second-order coefficient $\mathbb{A}$ preserves the positivity of σ_G, and thus the hypo-ellipticity of the equation, since A and R are non-degenerate. In fact, the second-order term would also be invariant if $A = D\vec{\psi}$ was an orthogonal matrix; i.e.,$A^TA = AA^T = I$, but that is not necessary in general and actually unrealizable.

2.5.2 Weak Formulation of Extended Equations

After doing the extension, it is important to make sure that across the boundary the equation(s) (2.5.6) are satisfied by $\overline{f}$ in some proper sense (at least in the weak sense). That means $\overline{f}$ should satisfy the following weak formation of equation (2.5.6) in the whole space:

$$\begin{aligned}
&\iint_{\mathbb{R}^3\times\mathbb{R}^3} \left[(\overline{f}\varphi)(t)-(\overline{f}\varphi)(0)\right] dy'dw' \\
&= \int_0^t \iint_{\mathbb{R}^3\times\mathbb{R}^3} \Big\{\overline{f}\left[(\partial_s + w'\cdot\nabla_{y'})\varphi - \mathbb{B}\cdot\nabla_{w'}\varphi\right] + \mathbb{C}f\varphi - \nabla_{w'}\overline{f}\cdot(\mathbb{A}\nabla_{w'}\varphi)\Big\} dy'dw'ds.
\end{aligned}$$

Normally a weak formation is obtained by multiplying the equation by some suitable test function φ and then integrating by parts over the domain where the equation(s) are defined i.e.,$(0,t)\times(\mathbb{H}_-\cup\mathbb{H}_+)\times\mathbb{R}^3$. This process yields

$$\begin{aligned}
&\iint_{\widetilde{\Omega}\times\mathbb{R}^3} \left[(\overline{f}\varphi)(t)-(\overline{f}\varphi)(0)\right] dy'dw' \\
&= \int_0^t \iint_{\widetilde{\Omega}\times\mathbb{R}^3} \Big\{\overline{f}\left[(\partial_s + w'\cdot\nabla_{y'})\varphi - \mathbb{B}\cdot\nabla_{w'}\varphi\right] + \mathbb{C}f\varphi - \nabla_{w'}\overline{f}\cdot(\mathbb{A}\nabla_{w'}\varphi)\Big\} dy'dw'ds \\
&\quad - \int_0^t\int_{\tilde{\gamma}} \overline{f}\varphi d\tilde{\gamma}ds
\end{aligned}$$

Here $\widetilde{\Omega} \overset{\text{def}}{=} \mathbb{H}_-\cup\mathbb{H}_+$, $\tilde{\gamma} \overset{\text{def}}{=} \partial\widetilde{\Omega}\times\mathbb{R}^3 = (\partial\mathbb{H}_-\cup\partial\mathbb{H}_+)\times\mathbb{R}^3$, and $d\tilde{\gamma} \overset{\text{def}}{=} (w'\cdot n_{y'})dS_{y'}dw'$.

Remark 2.5.5. The only boundary-integral term $I_{\tilde{\gamma}} \overset{\text{def}}{=} \int_0^t\int_{\tilde{\gamma}} \overline{f}\varphi d\tilde{\gamma}ds$ above comes from integration by parts in y'. Note that integration by parts in w' does not produce any boundary terms.

Compared with the above definition, this is equivalent to saying that we have to be sure the boundary term vanishes:

$$\int_{\tilde{\gamma}} \overline{f}\varphi d\tilde{\gamma} = \left(\iint_{\partial\mathbb{H}_-\times\mathbb{R}^3} + \iint_{\partial\mathbb{H}_+\times\mathbb{R}^3}\right)\overline{f}\varphi(w'\cdot n_{y'})dS_{y'}dw' = 0,$$

which is indeed true since

$$\overline{f}(t;y_1',y_2',0-;w') = \overline{f}(t;y_1',y_2',0+;w')$$

due to continuity of $\overline{f}$ across the boundary, while the normal vectors at same point of outer and inner boundary are of opposite directions

$$n_{y'}(y'_3\!=\!0-)|_{\partial\mathbb{H}_-} = -n_{y'}(y'_3\!=\!0+)|_{\partial\mathbb{H}_+},$$

plus the coincidence of y'-derivative term (transport operator) on two sides.

Therefore, we can now conclude that $\overline{f}$ is a (weak) solution to the equation (2.5.6) in the whole space.

2.5.3 Continuity of the Coefficients Across the Boundary

The S^p estimate are based on a reformulation of the equation, which is of the form of a class of kinetic Fokker-Planck equations (also called hypoelliptic or ultraparabolic of Kolmogorov type) with rough coefficients:

$$\partial_t f + v\cdot\nabla_x f = \nabla_v\cdot(\mathbf{A}\nabla_v f) + \mathbf{B}\cdot\nabla_v f + \mathbf{C}f.$$

The properties of the coefficients used in [53] for the estimates to hold are as follows: if $\|g\|_\infty$ is sufficiently small,

$\mathbf{A}(t,x,v) \overset{\text{def}}{=} \sigma_G$: 3×3 non-negative matrix, but *not uniformly* elliptic, $0 < (1+|v|)^{-3}I \lesssim \mathbf{A}(v) \lesssim (1+|v|)^{-1}I$ (Lemma 2.4 in [53]); continuous.

$\mathbf{B}(t,x,v) \overset{\text{def}}{=} a_g - \mathbf{E}_g$: essentially bounded $3d$-vector, $\|\mathbf{B}[g]\|_\infty \lesssim \|g\|_\infty \ll 1$.

The ellipticity and boundedness of the new coefficients after extension are easy to check (look back the transformed equations in Section 2.5.1).

Thus we are left with one main task – checking the VMO continuity of the second-order coefficient $\mathbb{A}$ across the boundary, which is necessary only for the S^p estimate (see Theorem 7.2 and Lemma 7.5 in [53]). A direct computation on (2.5.7) gives

$$
\begin{aligned}
\widetilde{\sigma}_G &= \left|\det\left(A^{-1}\right)\right| \cdot \widetilde{\psi} * \left(\tilde{\mu}+\tilde{\mu}^{1/2}\tilde{g}\right), \\
\psi(v) &= |v|^{-1}\cdot I - |v|^{-3}\cdot\left(vv^T\right), \\
\widetilde{\psi}(y,w) &= (w^TA^{-T}A^{-1}w)^{-1/2}\cdot I - (w^TA^{-T}A^{-1}w)^{-3/2}\cdot\left[A^{-1}ww^TA^{-T}\right], \\
&= (w^TCw)^{-1/2}\cdot I - (w^TCw)^{-3/2}\cdot\left[A^{-1}ww^TA^{-T}\right], \text{ and} \\
\tilde{\mu}(y,w) &= e^{-w^TCw}, \quad \overline{\mu}(y',w') = e^{-w'^TR\overline{C}Rw'}.
\end{aligned}
$$

Then we have

$$
\begin{aligned}
\widetilde{\mathbb{A}} &= A\widetilde{\sigma}_GA^T \\
&= \left|\det\left(A^{-1}\right)\right|\cdot\left\{(w^TCw)^{-1/2}\cdot\left[AA^T\right] - (w^TCw)^{-3/2}\cdot\left[ww^T\right]\right\} * \left(\tilde{\mu}+\tilde{\mu}^{1/2}\tilde{g}\right), \\
\overline{\mathbb{A}} &= R\overline{A}\overline{\sigma}_G\overline{A}^TR \\
&= \left|\det\left(\overline{A}^{-1}\right)\right|\cdot\left\{(w'^TR\overline{C}Rw')^{-1/2}\cdot\left[R\overline{A}\overline{A}^TR\right] - (w'^TR\overline{C}Rw')^{-3/2}\cdot\left[w'w'^T\right]\right\} \\
&\qquad * \left(\overline{\mu}+\overline{\mu}^{1/2}\overline{g}\right).
\end{aligned}
$$

Based on our definition, since we have continuity, small L^∞ bound and decay of G (i.e. vanishing at infinity), the fact that $\overline{\mathbb{A}} \in$ VMO naturally follows.

2.5.4 Conclusion

By designing a suitable "boundary-flattening" transformation, we are able to extend our solutions to the whole space for the specular reflection boundary condition case while preserving the form of transformed equation to the largest extent.

Remark 2.5.6. Essentially, this is a "fake-whole-space" problem and we can still utilize the bounded domain techniques we developed here. Moreover, we do not need to worry about the boundary condition any more.

Remark 2.5.7. For problem in bounded domains, if we do not perform domain extension, then the S^p estimates we obtain are merely interior and are possibly not uniform over the whole domain. Extension helps us to strengthen the estimate uniformity, which can help close the proof.

In order to utilize S^p theory, we only need to make sure the transformed/extended equation of

$$\partial_t \overline{f} + w' \cdot \nabla_{y'} \overline{f} - \overline{\sigma}_G^{ij} \partial_{w_i' w_j'} \overline{f} = \overline{S}, \tag{2.5.10}$$

where

$$\overline{S} = \partial_{w_i'} \overline{\sigma}_G^{ij} \partial_{w_j'} \overline{f} + \mathbb{B} \cdot \nabla_{w'} \overline{f} + \mathbb{C} \overline{f} \tag{2.5.11}$$

for $\mathbb{B}$ and $\mathbb{C}$ defined in (2.5.8) and (2.5.9), also satisfies

- (H.1): Ellipticity (eigenvalue bounds) of $\mathbb{A} = \sigma_G$: remain equivalent.
- (H.2): Structure of transport operator: the structure of operator is invariant.
- (H.3): $\mathbb{A} = [\overline{\sigma}_G^{ij}] \in \text{VMO}$, especially across the boundary.

This makes it available to implement S^p techniques in this extended domain. Since the electric field does not change the structure/property of the ultra-parabolic operator, and the construction of flattening/reflection-transformation only depends on the geometry of domain Ω, our extension trick should preserve these properties above.

Remark 2.5.8. For the S^p estimates, we need to estimate the L^p norm of (2.5.11) term by term (in the transformed variable (t, y', w')). Compared to using the original variable (t, x, v), for most terms there is no essential difference (almost the same). The only exception is that we need to raise one more power of weight in estimating $\mathbb{B} \cdot \nabla_{w'} \overline{f}$ due to the extra w' factor coming from the transformation.

Remark 2.5.9. The extension technique relies on two ingredients: transport operator and specular reflection boundary. The fact that an external field does not spoil our extension trick in [30] shows the robustness of our method.

2.6 Some Lemmas for the Ultraparabolic Equations

2.6.1 Structure of the Ultraparabolic Operator

In order to prove the S^p estimates and further regularity results, we rewrite the linearized equation (2.1.34) so that it takes the form of an ultraparabolic equation (cf. [1][64]):

$$\partial_t f + v \cdot \nabla_x f - \sigma_G^{ij} \partial_{v_i v_j} f \ = \mathcal{S} \ , \tag{2.6.1}$$

where the source term

$$\mathcal{S}(t,x,v) \;:=\; \partial_{v_i}\sigma_G^{ij}\partial_{v_j} f + \big\{a_g - \mathbf{E}_g\big\}\cdot\nabla_v f + \Big\{\overline{K_g} f + \big(v\cdot\mathbf{E}_g\big) f + 2\sqrt{\mu}\, v\cdot\mathbf{E}_f\Big\}\,.$$

Our equation (2.6.1) corresponds to a special case of the class of ultraparabolic operators

$$\mathcal{L} := \sum_{i,\,j=1}^{m_0} a_{ij}(\mathbf{z})\,\partial_{x_i x_j} + \sum_{i,\,j=1}^{N} b_{ij}\, x_i\partial_{x_j} - \partial_t \tag{2.6.2}$$

with $m_0 = 3\,,\ N = 6\,,\ \mathbf{z} := (\mathbf{x}, t) \in \mathbb{R}^{6+1}$, $\mathbf{x} := (x_i)_{1\le i\le 6} = (v,x) \in \mathbb{R}^3_v \times \mathbb{R}^3_x$, and

$$a_{ij}(\mathbf{z}) = \sigma_G^{ij}(t,x,v) = \big\{\Phi^{ij} * \big[\mu + \mu^{1/2} g(t,x,v)\big]\big\}\,(v)\,,$$

$b_{14} = b_{25} = b_{36} = -1$ and the rest of b_{ij} are zeros, so that

$$\mathcal{Y} := \sum_{i,\,j=1}^{N} b_{ij}\, x_i\partial_{x_j} - \partial_t = -\big(\partial_t + v\cdot\nabla_x\big)\,.$$

We now verify that the hypothesis made on the coefficients of $\mathcal{L}$ are satisfied in our case:

(H.1) The matrix of the second-order coefficients

$$\mathsf{A}(\mathbf{z}) := \big[a_{ij}(\mathbf{z})\big]_{1\le i,\,j\le 3} = \sigma_G(t,x,v)$$

is symmetric, and if $\|g\|_\infty < \varepsilon$ is sufficiently small, there are constants $c_1, c_2 > 0$ such that

$$c_1\Lambda_1(v)\,|\eta|^2 \;\le\; \eta^T A(\mathbf{z})\,\eta \;\le\; c_2\Lambda_2(v)\,|\eta|^2$$

for every $\mathbf{z} = (t,x,v) \in \mathbb{R}^{6+1}$ and $\eta \in \mathbb{R}^3$. Here $\Lambda_1(v), \Lambda_2(v) > 0$ are equivalent to the eigenvalues of σ_G, satisfying

$$\Lambda_1(v) \simeq (1+|v|)^{-3}\,, \quad \Lambda_2(v) \simeq (1+|v|)^{-1}\,.$$

Therefore, the principal part of $\mathcal{L}$ is an elliptic operator on $\mathbb{R}^3_v$, also it is uniformly elliptic restricted on a bounded subset of $v \in \mathbb{R}^3$. (see Lemma 2.3.3 in Section 2.3)

(H.2) The constant matrix

$$\mathsf{B} := \big[b_{ij}\big]_{1\le i,\,j\le 6} = \begin{bmatrix} 0 & \mathsf{B}_1 \\ 0 & 0 \end{bmatrix}, \qquad \mathsf{B}_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}$$

is upper triangular with $r=1$, $m_0 = m_1 = 3$, and B_1 is a 3×3 block matrix of rank 3.

It is known that under the conditions (H.1) and (H.2), the operator $\mathcal{L}_{\mathbf{z_0}}$ obtained by freezing the coefficients a_{ij} at any fixed point $\mathbf{z_0} \in \mathbb{R}^{N+1}$ is called “hypoelliptic”. In our case, the equation (2.6.1) is parabolic only in the velocity variable, while the (Liouville) transport operator $\mathcal{Y}$ has a mixing effect in the position-velocity phase space. We then identify two algebraic-geometric structures in the space of $(t,x,v) \in \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$ associated to $\mathcal{L}$, or more precisely, induced by the transport operator $\mathcal{Y}$. Note that the “frozen” operators $\mathcal{L}_{\mathbf{z_0}}$ with a constant matrix $\mathsf{A} = \big[a_{ij}\big]_{1\le i,\,j\le 3}$ are *invariant* with respect to the following group actions.

Definition 2.6.1 (Group of Translations)**.** For every $\mathbf{z} := (t,x,v)$, $\mathbf{w} := (\tau,\xi,\nu) \in \mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$, set the binary operation to be

$$\mathbf{z}\circ\mathbf{w} := (t,x,v)\circ(\tau,\xi,\nu) = \big(\,t+\tau,\; x+\xi+\tau v,\; v+\nu\,\big)\,.$$

Then $\big(\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v\,,\,\circ\big)$ is a (non-commutative) Lie group with the identity element $\mathbf{0}:=(0,0,0)$, and the inverse of an element is

$$\mathbf{z}^{-1}:=(t,x,v)^{-1}=\big(-t,\,-x+tv,\,-v\big)\,.$$

Also, the left-translation defined by the mapping $\mathcal{T}_{\mathbf{w}}:\mathbf{z}\mapsto\mathbf{w}\circ\mathbf{z}$ gives a group action on itself, and so we call this the "left-translation group associated to $\mathcal{L}$".

Definition 2.6.2 (Group of Dilations)**.** The group of positive real numbers under multiplication, denoted by $(\mathbb{R}^+,\,\cdot\,)$, defines a group action on $\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v$: for any $\lambda>0$, it acts on $\mathbf{z}:=(t,x,v)$ as

$$\lambda*\mathbf{z}\;:=\mathsf{D}(\lambda)\,(t,x,v)=\big(\,\lambda^2 t,\,\lambda^3 x,\,\lambda v\,\big)\,,$$

where we can write

$$\mathsf{D}(\lambda):=\mathrm{diag}\big(\lambda^2,\lambda^3\mathrm{I}_3,\lambda\mathrm{I}_3\big)\,,$$

and we call $\big(\mathsf{D}(\lambda)\big)_{\lambda>0}$ the "dilation group associated to $\mathcal{L}$".

Remark 2.6.1. (1) The operator $\mathcal{L}_{\mathbf{z_0}}$ is *invariant* with respect to both group actions above. For the left-translations, we have

$$\mathcal{L}_{\mathbf{z_0}}\big[f(\mathbf{w}\circ\mathbf{z})\big]=\big(\mathcal{L}_{\mathbf{z_0}}f\big)(\mathbf{w}\circ\mathbf{z})\,;$$

with respect to the dilation group $\big(\mathsf{D}(\lambda)\big)_{\lambda>0}$, in particular, $\mathcal{L}_{\mathbf{z_0}}$ is homogeneous of degree 2:

$$\mathcal{L}_{\mathbf{z_0}}\big[f(\lambda*\mathbf{z})\big]=\lambda^2\cdot\big(\mathcal{L}_{\mathbf{z_0}}f\big)(\lambda*\mathbf{z})\,.$$

(2) The translation group preserves the Lebesgue measure in $\mathbb{R}^{6+1}$, because the

Jacobian determinant of the transformation $\mathcal{T}_{\mathbf{w}}$

$$\det \mathbf{J}_{\mathcal{T}_{\mathbf{w}}} = \left| \frac{\partial(\mathbf{w} \circ \mathbf{z})}{\partial \mathbf{z}} \right| \equiv 1 \,.$$

On the other hand, we will denote by the integer

$$Q + 2 \,:= 1 \times 3 + 3 \times 3 + 2 = 14$$

the "homogeneous dimension" of $\mathbb{R}^{6+1}$ with respect to $\big(\mathsf{D}(\lambda)\big)_{\lambda>0}$, where $Q = 12$ is called the corresponding "spatial homogeneous dimension", in view that

$$\det \mathsf{D}(\lambda) = \lambda^{Q+2} \,.$$

We introduce now a quasi-norm and a quasi-distance in the space $\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$, related to the groups of translations and dilations defined above. In terms of the topology generated by this quasi-distance, we will add a compatible Hölder condition on the variable coefficients for the second-order term.

Definition 2.6.3 (Quasi-Norm)**.** For any $\mathbf{z} := (t, x, v) \in \mathbb{R}^{6+1} \backslash \{\mathbf{0}\}$, define the "quasi-norm" $\|\mathbf{z}\| := \rho$ to be the unique positive solution to the equation

$$\frac{t^2}{\rho^4} + \frac{|x|^2}{\rho^6} + \frac{|v|^2}{\rho^2} = 1 \,;$$

while if $\mathbf{z} = \mathbf{0}$, we set $\|\mathbf{z}\| = 0$.

Remark 2.6.2. We call $\| \cdot \|$ a "quasi-norm" because it does not exactly satisfy the three requirements to qualify as a usual norm, but instead has the following properties:

(a) Positive-definiteness: $\|\mathbf{z}\| \geq 0$; $\|\mathbf{z}\| = 0 \Leftrightarrow \mathbf{z} = \mathbf{0}$;

(b) $\mathsf{D}(\lambda)$-homogeneity of degree 1 (scalability): $\|\lambda * \mathbf{z}\| = \lambda \|\mathbf{z}\|\,,\ \forall\, \lambda > 0\,;$

(c) Generalized triangle inequality (subadditivity): $\|\mathbf{z} \circ \mathbf{w}\| \leq C_{\mathcal{Y}}\big(\|\mathbf{z}\| + \|\mathbf{w}\|\big)$ for some constant $C_{\mathcal{Y}} \geq 1$ depending only on the operator $\mathcal{Y}$;

(d) Bounded inverse (stability): $\|\mathbf{z}^{-1}\| \leq C_{\mathcal{Y}} \|\mathbf{z}\|$ for some positive constant $C_{\mathcal{Y}}$.

With this quasi-norm we associate the "quasi-distance":

Definition 2.6.4 (Quasi-Distance)**.** For every $\mathbf{z} := (t, x, v)$, $\mathbf{w} := (\tau, \xi, \nu) \in \mathbb{R}^{6+1}$, define

$$\mathsf{d}(\mathbf{z}, \mathbf{w}) := \left\|\mathbf{w}^{-1} \circ \mathbf{z}\right\|,$$

where

$$\mathbf{w}^{-1} \circ \mathbf{z} := (\tau, \xi, \nu)^{-1} \circ (t, x, v) = \big(t - \tau,\ x - \xi - (t - \tau)\nu,\ v - \nu\big)\,.$$

In addition, we define also the d-ball centered at $\mathbf{z}$ with radius r as

$$\mathfrak{B}_r(\mathbf{z}) := \left\{\mathbf{w} \in \mathbb{R}^{6+1} : \mathsf{d}(\mathbf{z}, \mathbf{w}) < r\right\}\,.$$

Remark 2.6.3*.* From Remark 2.6.2 it follows that d is a "quasi-distance" satisfying the relaxed axioms: there exists some positive constant $C_{\mathcal{Y}}$ such that

(a) Positive-definiteness: $\mathsf{d}(\mathbf{z}, \mathbf{w}) \geq 0\,;\ \mathsf{d}(\mathbf{z}, \mathbf{w}) = 0 \Leftrightarrow \mathbf{z} = \mathbf{w}\,;$

(b) Quasi-symmetry: $\frac{1}{C_{\mathcal{Y}}}\, \mathsf{d}(\mathbf{w}, \mathbf{z}) \leq \mathsf{d}(\mathbf{z}, \mathbf{w}) \leq C_{\mathcal{Y}}\, \mathsf{d}(\mathbf{w}, \mathbf{z})\,;$

(c) Generalized triangle inequality: $\mathsf{d}(\mathbf{z}, \mathbf{w}) \leq C_{\mathcal{Y}}\big[\mathsf{d}(\mathbf{z}, \mathbf{y}) + \mathsf{d}(\mathbf{y}, \mathbf{w})\big]\,.$

Moreover, note that $\mathfrak{B}_r(\mathbf{0}) = \mathsf{D}(r)\, \mathfrak{B}_1(\mathbf{0})$ due to the $\mathsf{D}(\lambda)$-homogeneity of degree 1 of $\|\cdot\|$ (see Remark 2.6.2). Together with the invariance of Lebesgue measure with respect to the translation group, and the notion of homogeneous dimension from $\det \mathsf{D}(\lambda) = \lambda^{Q+2}$ (see Remark 2.6.1), we have

$$\mathfrak{m}\big(\mathfrak{B}_r(\mathbf{z})\big) = r^{Q+2}\,\mathfrak{m}\big(\mathfrak{B}_1(\mathbf{0})\big)\,,$$

where $\mathfrak{m}(\cdot)$ denotes the Lebesgue measure and so $\mathfrak{m}\big(\mathfrak{B}_1(\mathbf{0})\big)$ has the same value as the volume of the Euclidean unit ball in $\mathbb{R}^{6+1}$. Therefore it implies that $(\,\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v,\ \mathrm{d}\mathbf{z},\ \mathsf{d}\,)$ is a homogeneous metric-measure space.

The quasi-distance d induces a topology on $\mathbb{R}_t\times\mathbb{R}^3_x\times\mathbb{R}^3_v$ in a natural way, which allows us to define the Hölder space $C^{k,\alpha}_{\mathcal{L}}$ and the space $\mathrm{VMO}_{\mathcal{L}}$ accordingly:

Definition 2.6.5 ($C^{k,\alpha}_{\mathcal{L}}$ Space)**.** Let Π be an open set in $\mathbb{R}^{N+1}$. We say a real-valued function f on Π satisfies the $\mathcal{L}$-Hölder condition, or has (uniform) $\mathcal{L}$-Hölder continuity, if there exist constants $\alpha\in(0,1]$ and $M>0$ such that

$$\big|f(\mathbf{z}) - f(\mathbf{w})\big| \;\le\; M\big\|\mathbf{w}^{-1}\circ\mathbf{z}\big\|^{\alpha}$$

for every $\mathbf{z},\mathbf{w}\in\Pi$. In this case, f can be equipped with a semi-norm

$$|f|_{C^{0,\alpha}_{\mathcal{L}}(\Pi)} := \sup_{\mathbf{z}\neq\mathbf{w}\in\Pi}\frac{|f(\mathbf{z})-f(\mathbf{w})|}{\|\mathbf{w}^{-1}\circ\mathbf{z}\|^{\alpha}}\,.$$

The $\mathcal{L}$-Hölder space $C^{k,\alpha}_{\mathcal{L}}(\Pi)$ consists of functions on Π having continuous derivatives up to order k and such that the k-th partial derivatives are (uniformly) $\mathcal{L}$-Hölder continuous with exponent α. Additionally, if the function f and its derivatives up to order k are bounded on the closure of (bounded) Π, then the $\mathcal{L}$-Hölder space $C^{k,\alpha}_{\mathcal{L}}(\overline{\Pi})$ is a Banach space with respect to the norm

$$\|f\|_{C^{k,\alpha}_{\mathcal{L}}} := \|f\|_{C^k} + \max_{|\beta|=k}\big|D^{\beta}f\big|_{C^{0,\alpha}_{\mathcal{L}}}\,,$$

where β ranges over multi-indices and

$$\|f\|_{C^k(\Pi)} := \max_{|\beta|\le k} \sup_{\mathbf{z}\in\Pi} \left|D^\beta f(\mathbf{z})\right| .$$

Definition 2.6.6 (VMO$_{\mathcal{L}}$ Space)**.** For every function $f \in L^1_{\mathrm{loc}}\big(\mathbb{R}^{N+1}\big)$, we set

$$\begin{aligned} \|f\|_* &:= \sup_{\mathfrak{B}} \fint_{\mathfrak{B}} \big|f(\mathbf{z}) - f_{\mathfrak{B}}\big| \mathrm{d}\mathbf{z} \,, \\ \eta_f(r) &:= \sup_{\rho\le r} \fint_{\mathfrak{B}_\rho} \big|f(\mathbf{z}) - f_{\mathfrak{B}_\rho}\big| \mathrm{d}\mathbf{z} \,. \end{aligned}$$

Then we define

$$\begin{aligned} \mathrm{BMO}_{\mathcal{L}}\big(\mathbb{R}^{N+1}\big) &:= \Big\{ f \in L^1_{\mathrm{loc}}\big(\mathbb{R}^{N+1}\big) : \|f\|_* < +\infty \Big\} \,, \\ \mathrm{VMO}_{\mathcal{L}}\big(\mathbb{R}^{N+1}\big) &:= \Big\{ f \in \mathrm{BMO}_{\mathcal{L}}\big(\mathbb{R}^{N+1}\big) : \lim_{r\to 0} \eta_f(r) = 0 \Big\} \,. \end{aligned}$$

Te get the desired a priori estimates, we need to make an additional regularity assumption on the second-order coefficients of $\mathcal{L}$:

(H.3) The coefficients $a_{ij}(\mathbf{z}) := \sigma_G^{ij}(t,x,v)$ belong to the space VMO$_{\mathcal{L}}$ (the space of "vanishing mean oscillation" functions with respect to $\mathcal{L}$), i.e.

$$a_{ij}(\mathbf{z}) \in \mathrm{VMO}_{\mathcal{L}}\big(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v\big) \,.$$

To have this, it suffices to assume that there are constants $\alpha \in (0,1]$ and $M' > 0$ such that

$$|g|_{C^{0,\alpha}_{\mathcal{L}}} \le M' \,.$$

2.6.2 Sobolev Estimates for the Ultraparabolic Equation

Before presenting our main lemmas, we define the following Sobolev-type spaces, that we will denote by $S^p_{\mathcal{L}}(\Pi)$, in order to emphasize their dependence on the operator $\mathcal{L}$.

Definition 2.6.7 ($S^p_{\mathcal{L}}$ Space)**.** Let $\Pi \subset \mathbb{R}^{N+1}$ be an open set. For $p \in (1,\infty)$, we define the $S^p_{\mathcal{L}}$ space

$$S^p_{\mathcal{L}}(\Pi) := \Big\{ f \in L^p(\Pi) : \partial_{v_i} f \,,\, \partial_{v_i v_j} f \,,\, \mathcal{Y} f \in L^p(\Pi) \,,\ i,j = 1,2,3 \Big\},$$

and assign the norm

$$\|f\|_{S^p_{\mathcal{L}}(\Pi)} := \Big(\big\|f\big\|^p_{L^p(\Pi)} + \big\|D_v f\big\|^p_{L^p(\Pi)} + \big\|D^2_{vv} f\big\|^p_{L^p(\Pi)} + \big\|\mathcal{Y} f\big\|^p_{L^p(\Pi)} \Big)^{\frac{1}{p}}, \tag{2.6.3}$$

where $\mathcal{Y} = -\big(\partial_t + v \cdot \nabla_x\big)$, and we use the simplified notations

$$\big\|D_v f\big\|^p_{L^p(\Pi)} := \sum_{i=1}^{3} \big\|\partial_{v_i} f\big\|^p_{L^p(\Pi)},$$
$$\big\|D^2_{vv} f\big\|^p_{L^p(\Pi)} := \sum_{i,j=1}^{3} \big\|\partial_{v_i v_j} f\big\|^p_{L^p(\Pi)}.$$

Remark 2.6.4. The definition of the $S^p_{\mathcal{L}}$ space is related to the structure of the operator $\mathcal{L}$ especially the transport operator $\mathcal{Y}$, capturing precisely its anisotropy, while the variable coefficients $a_{ij}(\mathbf{z})$ do not play any role.

Now we provide a key local a priori estimate for the derivatives $\partial_{v_i v_j} f$ and $\mathcal{Y} f$ of solutions to the ultraparabolic equation $-\mathcal{L} f = \mathcal{S}$, if the conditions (H.1) – (H.3) are fulfilled. The proofs and further references about the results listed below can be found in [1][64].

Lemma 2.6.1 (Local $S^p_{\mathcal{L}}$ Estimates)**.** *Let* Π *be a bounded open set in* $\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v$, *and* f *be a solution in* Π *to the equation* (2.6.1) *of the form* $-\mathcal{L}f = \mathcal{S}$ *, where the ultraparabolic operator* $\mathcal{L}$ *is given by* (2.6.2)*. Suppose that the conditions (H.1)–(H.3) are verified in* Π *, and that* $f \in L^p(\Pi)$ *,* $\mathcal{S} \in L^p(\Pi)$ *with* $p \in (1, \infty)$*.*

Then $\mathcal{Y}f \in L^p_{\text{loc}}(\Pi)$ *,* $\partial_{v_i v_j} f \in L^p_{\text{loc}}(\Pi)$ $(i, j = 1, 2, 3)$ *. More precisely, for every open subset* $\Pi' \subset\subset \Pi$ *, there exists a positive constant* C_1 *depending on* p, $\text{dist}(\Pi', \Pi)$ *, eigenvalues (elliptic constant) of* $\mathsf{A}(\mathbf{z})$ *, and* α *,* M *related to the Hölder semi-norm of* $a_{ij}(\mathbf{z})$ *, such that*

$$\left\|\mathcal{Y}f\right\|_{L^p(\Pi')} \leq C_1 \left(\|\mathcal{S}\|_{L^p(\Pi)} + \|f\|_{L^p(\Pi)} \right) , \tag{2.6.4}$$

$$\left\|D^2_{vv}f\right\|_{L^p(\Pi')} \leq C_1 \left(\|\mathcal{S}\|_{L^p(\Pi)} + \|f\|_{L^p(\Pi)} \right) . \tag{2.6.5}$$

Remark 2.6.5. The proof of this lemma is based on representation formulas derived from the fundamental solution of $\mathcal{L}$, as well as some singular integral estimates (cf. [1, Section 2]). Those arguments in the original proof reveals the way how the constant C_1 depends on parameters:

$$C_1 = C_2(\Lambda_1, \Lambda_2) \cdot C_3\big(\text{dist}(\Pi', \Pi),\, p, \alpha, M\big) ,$$

where Λ_1, Λ_2 are the smallest and the largest eigenvalue of $\mathsf{A}(\mathbf{z})$, respectively. To be specific, there exists $\alpha' > 0$ such that

$$C_2(\Lambda_1, \Lambda_2) \leq C\big(\Lambda_1^{-1} + \Lambda_2\big)^{\alpha'} .$$

2.6.3 The Embedding Theorem for $S^p_{\mathcal{L}}$ Spaces

For the Sobolev-type space $S^p_{\mathcal{L}}$ defined above, we recall some embedding results of Sobolev and Morrey types, which will play an important role in the proofs in Section 2.7. The proofs and further references about the results listed below can be found in [1][64].

Lemma 2.6.2 (Sobolev-Morrey Embedding)**.** *Let* $u \in S^p_{\mathcal{L}}(\mathbb{R}^{N+1})$ *with* $p \in (1,\infty)$.

1) If $2p < Q+2$ *, then* $u \in L^{q_1}(\mathbb{R}^{N+1})$ *, where* $\frac{1}{q_1} = \frac{1}{p} - \frac{2}{Q+2}$ *;*

2) if $2p > Q+2$ *and* $p < Q+2$ *, then* $u \in C^{0,\alpha_1}_{\mathcal{L}}(\mathbb{R}^{N+1})$ *, where* $\alpha_1 = \frac{2p-(Q+2)}{p}$ *;*

3) if $p < Q+2$ *, then* $\partial_{v_i} u \in L^{q_2}(\mathbb{R}^{N+1})$ *for* $i = 1,2,3$ *, where* $\frac{1}{q_2} = \frac{1}{p} - \frac{1}{Q+2}$ *;*

4) if $p > Q+2$ *, then* $\partial_{v_i} u \in C^{0,\alpha_2}_{\mathcal{L}}(\mathbb{R}^{N+1})$ *for* $i = 1,2,3$ *, where* $\alpha_2 = \frac{p-(Q+2)}{p}$.

Moreover, the above inclusions are continuous embeddings, (i.e. $S^p_{\mathcal{L}}(\mathbb{R}^{N+1}) \hookrightarrow L^q(\mathbb{R}^{N+1})$ *or* $S^p_{\mathcal{L}}(\mathbb{R}^{N+1}) \hookrightarrow C^{0,\alpha}_{\mathcal{L}}(\mathbb{R}^{N+1})$ *for the first two cases).*

Corollary 2.6.2.1. *Letting* $N+1 = 7$ *and* $Q+2 = 14$ *for our problem, we have the following continuous embeddings into Hölder spaces: for any* $u \in S^p_{\mathcal{L}}(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ *with* $p \in (1,\infty)$ *,*

1) if $7 < p < 14$ *, then* $u \in C^{0,\alpha_1}_{\mathcal{L}}(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ *with* $\alpha_1 = 2 - \frac{14}{p} \in (0,1)$ *, such that*

$$|u|_{C^{0,\alpha_1}_{\mathcal{L}}} \lesssim \|u\|_{S^p_{\mathcal{L}}} \ ;$$

2) if $p > 14$ *, then* $\partial_{v_i} u \in C^{0,\alpha_2}_{\mathcal{L}}(\mathbb{R}_t \times \mathbb{R}^3_x \times \mathbb{R}^3_v)$ *for* $i = 1,2,3$ *, with* $\alpha_2 = 1 - \frac{14}{p} \in (0,1)$ *, such that*

$$|D_v u|_{C^{0,\alpha_2}_{\mathcal{L}}} \lesssim \|u\|_{S^p_{\mathcal{L}}} \ .$$

Remark 2.6.6. The continuous embeddings above are actually a direct consequence of the Sobolev-Morrey inequalities.

Finally, we conclude this section by a Gagliardo-Nirenberg type interpolation inequality, which gives the bound on the intermediate derivatives $D_v u$ from the control of the highest derivatives $D^2_{vv} u$ and additional information on the function u itself.

Lemma 2.6.3 (Interpolation I). *Let $u \in S^p_{\mathcal{L}}(\Pi)$ with $p \in (1, \infty)$, and Π being an open set in $\mathbb{R}^{N+1}$. For any $0 < \varepsilon < 1$, there exists a constant $C_\varepsilon = \frac{C}{\varepsilon} > 0$ (depending only on p) such that*

$$\left\| D_v u \right\|_{L^p(\Pi)} \leq \varepsilon \left\| D^2_{vv} u \right\|_{L^p(\Pi)} + C_\varepsilon \left\| u \right\|_{L^p(\Pi)} . \tag{2.6.6}$$

2.7 $S^p_{\mathcal{L}}$ Estimates and Hölder Continuity

2.7.1 $S^p_{\mathcal{L}}$ Bound

In this section we prove the $S^p_{\mathcal{L}}$ bound in (2.2.13).

Proposition 2.7.1. *With the notations and hypothesis in Proposition 2.2.2, we have*

$$\|f\|_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^s \tag{2.7.1}$$

for some $s > 1$.

Proof. With the assumption (2.2.9), (in the interior) we have $|g|_{C^{0,\alpha}_{\mathcal{L}}} \lesssim \varepsilon_1$ for some

$\alpha \in (0,1]$ thanks to Lemma 2.6.2 (and particularly Corollary 2.6.2.1) of the Sobolev-Morrey embedding, Also, based on the discussion in Section 2.5, (after extension) $\mathsf{A}(\mathbf{z}) = \sigma_G(t,x,v)$ belongs to $\mathrm{VMO}_{\mathcal{L}}$. Hence, it satisfies the continuity condition (H.3).

Thus we may apply Lemma 2.6.1 with

$$\Pi' = \Pi'_n := (0,T) \times \Omega \times \left\{ v \in \mathbb{R}^3 : n - 1/2 < |v| < n + 1/2 \right\} ,$$
$$\Pi = \Pi_n := (0,T{+}1) \times \widetilde{\Omega} \times \left\{ v \in \mathbb{R}^3 : n - 1 < |v| < n + 1 \right\} ,$$

for all $n \in \mathbb{N}$, where $\widetilde{\Omega}$ denotes an extension of the domain Ω beyond the boundary (constructed in Section 2.5). We also need a time extension for $-1 \le t < 0$ using the initial data. (see [53, (7.1)] for the construction). In particular, this requires the initial condition (2.2.1).

Then the local $S^p_{\mathcal{L}}$ estimates give

$$\left\| \mathcal{Y} f \right\|^p_{L^p(\Pi'_n)} \le C^p_n \left(\|\mathcal{S}\|^p_{L^p(\Pi_n)} + \|f\|^p_{L^p(\Pi_n)} \right) , \tag{2.7.2}$$
$$\left\| D^2_{vv} f \right\|^p_{L^p(\Pi'_n)} \le C^p_n \left(\|\mathcal{S}\|^p_{L^p(\Pi_n)} + \|f\|^p_{L^p(\Pi_n)} \right) , \tag{2.7.3}$$

where

$$\mathcal{S} = \partial_{v_i} \sigma^{ij}_G \partial_{v_j} f + \left\{ a_g - \mathbf{E}_g \right\} \cdot \nabla_v f + \left\{ \overline{K}_g f + \left(v \cdot \mathbf{E}_g \right) f + 2\sqrt{\mu}\, v \cdot \mathbf{E}_f \right\} .$$

From Remark 2.6.5 we can write the constant as

$$C_n = C(\varepsilon_1)\, n^a \sim O(n^a)$$

for some $a > 0$, and $C(\varepsilon_1)$ denotes a general positive constant depending on

$|g|_{C^{0,\alpha}_{\mathcal{L}}} \lesssim \varepsilon_1\,.$

Also, we have a standard interpolation (see Lemma 2.6.3)

$$\left\|D_v f\right\|^p_{L^p(\widetilde{\Pi})} \leq \varepsilon \left\|D^2_{vv} f\right\|^p_{L^p(\widetilde{\Pi})} + \frac{C}{\varepsilon}\left\|f\right\|^p_{L^p(\widetilde{\Pi})}\,, \tag{2.7.4}$$

where $\widetilde{\Pi}$ can be Π'_n for $n = 1, 2, \cdots$ (and ε to be determined later). Therefore, by (2.7.2)(2.7.3) and (2.7.4),

$$\begin{aligned}
\|f\|^p_{S^p_{\mathcal{L}}(\Pi'_n)} &= \|f\|^p_{L^p(\Pi'_n)} + \left\|D_v f\right\|^p_{L^p(\Pi'_n)} + \left\|D^2_{vv} f\right\|^p_{L^p(\Pi'_n)} + \left\|\mathcal{Y} f\right\|^p_{L^p(\Pi'_n)} \\
&\leq C(\varepsilon_1)\, n^{ap} \cdot \Big\{ \|\mathcal{S}\|^p_{L^p(\Pi_n)} + \|f\|^p_{L^p(\Pi_n)} \Big\} \\
&\leq C(\varepsilon_1)\, n^{ap} \cdot \Big\{ \left\|\partial_{v_i}\sigma^{ij}_G \partial_{v_j} f\right\|^p_{L^p(\Pi_n)} + \left\|\big(a_g - \mathbf{E}_g\big)\cdot\nabla_v f\right\|^p_{L^p(\Pi_n)} \\
&\qquad + \left\|\overline{K}_g f\right\|^p_{L^p(\Pi_n)} + \left\|\big(v\cdot\mathbf{E}_g\big) f\right\|^p_{L^p(\Pi_n)} + \left\|2\sqrt{\mu}\, v\cdot\mathbf{E}_f\right\|^p_{L^p(\Pi_n)} + \|f\|^p_{L^p(\Pi_n)} \Big\}\,.
\end{aligned} \tag{2.7.5}$$

Summing up over $|v| \sim n \in \mathbb{N}$ the inequality above, we estimate the RHS term by term:

$$\begin{aligned}
\sum_{n\in\mathbb{N}} C(\varepsilon_1)\, n^{ap}\cdot\|f\|^p_{L^p(\Pi_n)} &\lesssim \sum_{n\in\mathbb{N}} n^{ap}\cdot n^{-p\vartheta}\left\|\langle v\rangle^{\vartheta} f\right\|^p_{L^p(\Pi_n)} \\
&\lesssim \left\|\langle v\rangle^{\vartheta} f\right\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\cdot\sum_{n\in\mathbb{N}} n^{p(a-\vartheta)} \\
&\lesssim \left\|\langle v\rangle^{\vartheta} f\right\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,.
\end{aligned} \tag{2.7.6}$$

Here we make the infinite series convergent by adding some sufficiently large velocity-weight with exponent $\vartheta > a + \frac{1}{p}$ (so that $p(a-\vartheta) < -1$).

Since $\left\|\partial_{v_i}\sigma^{ij}_G\right\|_\infty \lesssim 1$, $\|a_g\|_\infty \lesssim \|g\|_\infty$ (Corollary 2.3.2.1), and $\|\mathbf{E}_g\|_\infty \lesssim \|g\|_\infty$

(Lemma 2.3.7), we have

$$
\begin{aligned}
&\sum_{n\in\mathbb{N}} C(\varepsilon_1)\, n^{ap}\cdot\Big\{\big\|\partial_{v_i}\sigma_G^{ij}\partial_{v_j}f\big\|^p_{L^p(\Pi_n)}+\big\|\big(a_g-\mathbf{E}_g\big)\cdot\nabla_v f\big\|^p_{L^p(\Pi_n)}\Big\}\\
&\lesssim \sum_{n\in\mathbb{N}} n^{ap}\,\big\|D_v f\big\|^p_{L^p(\Pi_n)}\\
&\lesssim \sum_{n\in\mathbb{N}} n^{ap}\cdot\Big(\varepsilon_n\,\big\|D^2_{vv}f\big\|^p_{L^p(\Pi_n)}+\,\varepsilon_n^{-1}\,\|f\|^p_{L^p(\Pi_n)}\Big)\\
&\le \sum_{n\in\mathbb{N}} n^{p(a-\vartheta)}\varepsilon'\big\|D^2_{vv}f\big\|^p_{L^p(\Pi_n)}\,+\sum_{n\in\mathbb{N}} n^{ap}\cdot n^{\vartheta p}\varepsilon'^{-1}\|f\|^p_{L^p(\Pi_n)}\\
&\lesssim \varepsilon'\big\|D^2_{vv}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\cdot\sum_{n\in\mathbb{N}} n^{p(a-\vartheta)}\,+\,\varepsilon'^{-1}\big\|\langle v\rangle^{2\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\cdot\sum_{n\in\mathbb{N}} n^{p(a-\vartheta)}\\
&\lesssim \varepsilon'\varepsilon_1^p\,+\,\varepsilon'^{-1}\big\|\langle v\rangle^{2\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,.
\end{aligned}
\tag{2.7.7}
$$

In the second inequality we apply (2.7.4) to each $n\in\mathbb{N}$ with $\varepsilon_n=\varepsilon' n^{-p\vartheta}$, where ε' can be chosen as a suitable small parameter (to be determined later).

Similarly, by Lemma 2.3.6 as well as (2.7.4),

$$
\begin{aligned}
&\sum_{n\in\mathbb{N}} C(\varepsilon_1)\, n^{ap}\cdot\big\|\overline{K}_g f\big\|^p_{L^p(\Pi_n)}\\
&\lesssim \sum_{n\in\mathbb{N}} n^{ap}\cdot n^{-p\vartheta}\Big(\|f\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}+\big\|D_v f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}+\big\|\langle v\rangle^{\vartheta}f\big\|^p_{L^p(\Pi_n)}\Big)\\
&\lesssim \Big(\big\|D_v f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}+\big\|\langle v\rangle^{\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\Big)\cdot\sum_{n\in\mathbb{N}} n^{p(a-\vartheta)}\\
&\lesssim \varepsilon'\big\|D^2_{vv}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,+\,\varepsilon'^{-1}\|f\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,+\,\big\|\langle v\rangle^{\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\\
&\lesssim \varepsilon'\varepsilon_1^p\,+\,\varepsilon'^{-1}\big\|\langle v\rangle^{\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,.
\end{aligned}
\tag{2.7.8}
$$

Using $\|\mathbf{E}_g\|_\infty \lesssim \|g\|_\infty$ and $\|\mathbf{E}_f\|_{L^p} \lesssim \|f\|_{L^p}$ (see Lemma 2.3.7), we also have

$$\begin{aligned}\sum_{n\in\mathbb{N}} C(\varepsilon_1)\, n^{ap} \cdot \big\|\big(v\cdot\mathbf{E}_g\big)f\big\|^p_{L^p(\Pi_n)} &\lesssim \sum_{n\in\mathbb{N}} n^{ap}\cdot n^{-p\vartheta}\|g\|^p_\infty\big\|\langle v\rangle^{\vartheta+1}f\big\|^p_{L^p(\Pi_n)} \\ &\lesssim \varepsilon_1^p\,\big\|\langle v\rangle^{\vartheta+1}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\cdot\sum_{n\in\mathbb{N}} n^{p(a-\vartheta)} \\ &\lesssim \varepsilon_1^p\,\big\|\langle v\rangle^{\vartheta+1}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,,\end{aligned}\tag{2.7.9}$$

with $\vartheta > a + \frac{1}{p}$, and

$$\begin{aligned}\sum_{n\in\mathbb{N}} C(\varepsilon_1)\, n^{ap}\cdot\big\|2\sqrt{\mu}\,v\cdot\mathbf{E}_f\big\|^p_{L^p(\Pi_n)} &\lesssim \sum_{n\in\mathbb{N}} n^{ap}\cdot n^{-p\vartheta}\big\|2\langle v\rangle^{\vartheta}\sqrt{\mu}\,v\cdot\mathbf{E}_f\big\|^p_{L^p(\Pi_n)} \\ &\lesssim \big\|2\langle v\rangle^{\vartheta}\sqrt{\mu}\,v\cdot\mathbf{E}_f(t,x)\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)} \\ &\lesssim \big\|\mathbf{E}_f\big\|^p_{L^p((0,T)\times\Omega)}\cdot\big|\,2\langle v\rangle^{\vartheta+1}\sqrt{\mu}\,\big|^p_{L^p(\mathbb{R}^3)} \\ &\lesssim \|f\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)}\,.\end{aligned}\tag{2.7.10}$$

Combining (2.7.5) – (2.7.10), we obtain (for some $\vartheta^* > 1 + 2a + \frac{2}{p}$)

$$\begin{aligned}\|f\|^p_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} &\lesssim \varepsilon'\varepsilon_1^p + \varepsilon'^{-1}\big\|\langle v\rangle^{2\vartheta}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)} + \varepsilon_1^p\,\big\|\langle v\rangle^{\vartheta+1}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)} \\ &\lesssim \varepsilon'^{-1}\big\|\langle v\rangle^{\vartheta^*}f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)} + o\big(\varepsilon_1^p\big)\,.\end{aligned}\tag{2.7.11}$$

Finally, it remains to bound the weighted L^p norm: using the standard interpolation (for $2 < p < \infty$, by Hölder inequality), followed by the strong (almost-exponential) L^2 decay (Theorem 2.4.1, which allows us to utilize the "stronger" initial condition (2.2.1) with sufficiently large weight, as well as the bootstrap assumption

(2.2.8)), we see that

$$\begin{aligned}\big\|\langle v\rangle^{\vartheta^*} f\big\|^p_{L^p((0,T)\times\Omega\times\mathbb{R}^3)} = \int_0^T \big\|f(t)\big\|^p_{p,\vartheta^*}\,\mathrm{d}t &\le \int_0^T \big\|f(t)\big\|^2_{2,\vartheta^*}\big\|f(t)\big\|^{p-2}_{\infty,\vartheta^*}\,\mathrm{d}t \\ &\lesssim \varepsilon_1^{p-2}\int_0^T \langle t\rangle^{-2k}\|f_0\|^2_{2,\vartheta^*+k}\,\mathrm{d}t \\ &\lesssim \varepsilon_1^{p-2}\varepsilon_0^2\int_0^T \langle t\rangle^{-2k}\,\mathrm{d}t \\ &\lesssim \varepsilon_1^{p-2+2q} = \varepsilon_1^{p\left(1+\frac{2q-2}{p}\right)}\,.\end{aligned} \tag{2.7.12}$$

Note that for any $k > \frac{1}{2}$, the time integral above is uniformly bounded with respect to T. Hence, upon letting $s' = 1 + \frac{2q-2}{p} > 1$ and $\varepsilon' = \varepsilon_1^{p\left(\frac{s'-1}{2}\right)} = \varepsilon_1^{q-1}$, we deduce from (2.7.11) and (2.7.12) that $\|f\|^p_{S^p_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^{sp}$ with $s = \frac{s'+1}{2} = 1+\frac{q-1}{p} > 1$, which completes the proof. $\square$

2.7.2 Hölder Continuity

As a direct corollary of the $S^p_{\mathcal{L}}$ bound (Proportion 2.7.1) and the embeddings for $S^p_{\mathcal{L}}$ spaces (Corollary 2.6.2.1), we deduce the following uniform Hölder Continuity results.

Corollary 2.7.1.1. *With the notations and hypothesis in Proposition 2.2.2, we have that*

1) if $p > 7$, then with $\alpha_1 = \min\left\{1\,,\,2-\frac{14}{p}\right\} \in (0,1]$,

$$|f|_{C^{0,\alpha_1}_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^s\,; \tag{2.7.13}$$

2) if $p > 14$, then with $\alpha_2 = 1-\frac{14}{p} \in (0,1)$,

$$|D_v f|_{C^{0,\alpha_2}_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^s\,, \tag{2.7.14}$$

where $s > 1$ *can be chosen the same as in Proportion 2.7.1.*

2.7.3 L^∞ Bound on $\nabla_v f$

We now point out that Corollary 2.7.1.1 easily contains the bound for $\|\nabla_v f\|_{L^\infty}$ in (2.2.14), a crucial element to ensure the uniqueness (see Section 2.9 for the proof).

Corollary 2.7.1.2. *With the notations and hypothesis in Proposition 2.2.2 (for* $p > 14$ *), we have*

$$\|D_v f\|_{L^\infty((0,T)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_1^s \tag{2.7.15}$$

for some $s > 1$ *.*

Proof. Note also that taking $t = \tau$, $x = \xi$, $v_j = \nu_j$ $(j \neq i)$ gives $\left\|\mathbf{w}^{-1} \circ \mathbf{z}\right\| = |v_i - \nu_i|$, and so

$$\left|\partial_{v_i} f\right| = \left|\lim_{\nu_i \to v_i} \frac{f(t,x,v) - f(\cdots, \nu_i, \cdots)}{v_i - \nu_i}\right| \leq \sup_{\mathbf{z} \neq \mathbf{w}} \frac{|f(\mathbf{z}) - f(\mathbf{w})|}{\|\mathbf{w}^{-1} \circ \mathbf{z}\|} = |f|_{C^{0,1}_{\mathcal{L}}},$$

provided that the limit exists (almost everywhere) i.e. the derivatives are well-defined in some (weak) sense.

If $p > 14$, then this result can be deduced immediately from (2.7.14). ∎

2.8 Pointwise Estimates and Regularity Results

2.8.1 Weighted L^∞ Decay

So far we have gained control of the energy (L^2 norm) of our solutions (Theorem 2.4.1) and have known that they are Hölder continuous (Corollary 2.7.1.1). Remembering a prototype of Gagliardo-Nirenberg interpolation below allows us to conclude that the solution f actually belongs to some other "intermediate spaces", and in particular, is uniformly bounded. (cf. [58, Proposition 12.84])

Lemma 2.8.1 (Interpolation II). *Let $u : \mathbb{R}^n \to \mathbb{R}$ be Hölder continuous with exponent $0 < \alpha < 1$ and such that $u \in L^q(\mathbb{R}^n)$ for some $1 \le q \le \infty$. Then u belongs to $L^r(\mathbb{R}^n)$ for all $q \le r \le \infty$ and to $C^{0,\beta}(\mathbb{R}^n)$ for all $0 < \beta \le \alpha$, with*

$$\begin{aligned} \|u\|_{L^r(\mathbb{R}^n)} &\le c\,\|u\|_{L^q(\mathbb{R}^n)}^{\theta_1} |u|_{C^{0,\alpha}(\mathbb{R}^n)}^{1-\theta_1}\,, \\ |u|_{C^{0,\beta}(\mathbb{R}^n)} &\le c\,\|u\|_{L^q(\mathbb{R}^n)}^{\theta_2} |u|_{C^{0,\alpha}(\mathbb{R}^n)}^{1-\theta_2}\,, \end{aligned}$$

for some constant $c = c(n,\alpha,q) > 0$, where $\theta_1 = \frac{\alpha+n/r}{\alpha+n/q}$ and $\theta_2 = \frac{\alpha-\beta}{\alpha+n/q}$.

Motivated by these interpolation inequalities above, we further prove the weighted L^∞ decay bound in (2.2.12) based on our known weighted L^2 decay and Hölder continuity results. Intuitively, the heuristics is that a continuous function with small L^2 norm should have magnitude of comparable size. In fact, it suffices to show the following:

Proposition 2.8.2. *For any $\delta > 0$, there exists $\varepsilon = \varepsilon(\delta) > 0$ such that if $\langle t\rangle^{k_1}\|f(t)\|_{2,\vartheta_1} < \varepsilon$ for some $k_1, \vartheta_1 \ge 0$, and $|f|_{C^{0,\alpha}_{\mathcal{L}}} \le M$ with some $\alpha \in (0,1)$ and $M > 0$, then $\langle t\rangle^{k_2}\|f(t)\|_{\infty,\vartheta_2} < \delta$, where $\left(1+\frac{6}{\alpha}\right) k_2 \le k_1$ and $\left(1+\frac{6}{\alpha}\right)\vartheta_2 \le \vartheta_1$.*

Proof. We use the method of contradiction. Assuming the contrary, there exists $\delta_0 > 0$ such that no matter how small $\varepsilon > 0$ is, we always have $\langle t\rangle^{k_2}\|f(t)\|_{\infty,\vartheta_2} \geq \delta_0$. In other words, without loss of generality, there is a point $\mathbf{z} = (t, x, v)$ where $\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2} f(t, x, v) \geq \delta_0$, or, equivalently,

$$f(t, x, v) \geq \frac{\delta_0}{\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\,. \tag{2.8.1}$$

From $|f|_{C^{0,\alpha}_{\mathcal{L}}} \leq M$ we have for any point $\mathbf{w} = (\tau, \xi, \nu)$,

$$\big|f(\mathbf{z}) - f(\mathbf{w})\big| \leq M\big\|\mathbf{w}^{-1}\circ\mathbf{z}\big\|^{\alpha}\,.$$

Let $\widehat{\mathsf{d}}\,(\widehat{\mathbf{z}}, \widehat{\mathbf{w}})$ denote the restriction of the quasi-distance d on the phase-space $\mathbb{R}^3_x\times\mathbb{R}^3_v$, where $\widehat{\mathbf{z}} := (x, v)$, $\widehat{\mathbf{w}} := (\xi, \nu)$. Then $\big(\mathbb{R}^3_x\times\mathbb{R}^3_v\,,\, d\widehat{\mathbf{z}}\,,\, \widehat{\mathsf{d}}\,\big)$ is a homogeneous metric-measure space with "spatial homogeneous dimension" $Q = 12$. For fixed t, consider a $\widehat{\mathsf{d}}$-disk

$$\mathfrak{D}^t_r(\widehat{\mathbf{z}}) := \Big\{\widehat{\mathbf{w}} \in \mathbb{R}^3_x \times \mathbb{R}^3_v : \widehat{\mathsf{d}}\,(\widehat{\mathbf{z}}, \widehat{\mathbf{w}}) < r\Big\}$$

in the phase-space centered at $\widehat{\mathbf{z}} = (x, v)$ with radius $r = \Big(\frac{\delta_0}{2M\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\Big)^{\frac{1}{\alpha}}$. Then for every point (t, ξ, ν) with $\widehat{\mathbf{w}} = (\xi, \nu) \in \mathfrak{D}^t_r(\widehat{\mathbf{z}})$, we have

$$\big|f(t, x, v) - f(t, \xi, \nu)\big| \leq \frac{\delta_0}{2\,\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\,,$$

which, together with (2.8.1), implies

$$f(t, \xi, \nu) \geq \frac{\delta_0}{2\,\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\,. \tag{2.8.2}$$

Also, it is easy to check that $\langle v\rangle \sim \langle \nu\rangle$ in $\mathfrak{D}^t_r(\widehat{\mathbf{z}})$ up to an error of $O(\delta_0)$.

Now we estimate the weighted L^2 norm with (given) time-growth:

$$\begin{aligned}
\langle t\rangle^{k_1}\|f(t)\|_{2,\vartheta_1} &\geq \langle t\rangle^{k_1}\left(\iint_{\mathfrak{D}_r^t(\widehat{\mathbf{z}})}\langle\nu\rangle^{2\vartheta_1}\big|f(t,\xi,\nu)\big|^2\mathrm{d}\nu\mathrm{d}\xi\right)^{\frac{1}{2}}\\
&\gtrsim \langle t\rangle^{k_1}\langle v\rangle^{\vartheta_1}\frac{\delta_0}{\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\cdot\mathfrak{m}\big(\mathfrak{D}_r^t(\widehat{\mathbf{z}})\big)^{\frac{1}{2}}\\
&\simeq \langle t\rangle^{k_1}\langle v\rangle^{\vartheta_1}\frac{\delta_0}{\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\left(\frac{\delta_0}{\langle t\rangle^{k_2}\langle v\rangle^{\vartheta_2}}\right)^{\frac{6}{\alpha}}\\
&= \delta_0^{1+\frac{6}{\alpha}}\,\langle t\rangle^{k_1-\left(1+\frac{6}{\alpha}\right)k_2}\,\langle v\rangle^{\vartheta_1-\left(1+\frac{6}{\alpha}\right)\vartheta_2}\,.
\end{aligned}$$

Letting $k_1-\left(1+\frac{6}{\alpha}\right)k_2\geq 0$ and $\vartheta_1-\left(1+\frac{6}{\alpha}\right)\vartheta_2\geq 0$, we obtain

$$\langle t\rangle^{k_1}\|f(t)\|_{2,\vartheta_1}\gtrsim\delta_0^{1+\frac{6}{\alpha}}\,. \tag{2.8.3}$$

Note that this lower bound is a fixed positive number. However, on the other hand, our assumption is that $\langle t\rangle^{k_1}\|f(t)\|_{2,\vartheta_1}<\varepsilon$ for arbitrarily small $\varepsilon>0$. We thus get a contradiction. $\square$

2.8.2 Hölder Regularity

As a byproduct, we have the following result:

Corollary 2.8.2.1. *With the notation and hypothesis in Proposition 2.2.2, we have*

$$\|f\|_{C^{0,\alpha}_{\mathcal{L}}((0,T)\times\Omega\times\mathbb{R}^3)}+\|\mathbf{E}_f\|_{C^{1,\alpha}((0,\infty)\times\Omega)}\lesssim\varepsilon_1^s \tag{2.8.4}$$

for some $s>1$, and where α takes the same values as α_1 in (2.7.13).

Remark 2.8.1. This regularity result actually includes the bound of $\|\mathbf{E}_f\|_{W^{1,\infty}_{t,x}}$ for f being a solution to the coupled system, which is an improvement compared to the L^∞ estimate for the Poisson equation in Lemma 2.3.7.

Proof. The Hölder continuity i.e. semi-norm bound (2.7.13) in Corollary 2.7.1.1, together with the L^∞ bound (2.2.12), gives bound for the Hölder norm $\|f\|_{C^{0,\alpha}_{\mathcal{L}}}$.

For the corresponding electric field $\mathbf{E}_f := -\nabla_x \phi_f = \nabla_x \Delta_x^{-1} \int_{\mathbb{R}^3} \sqrt{\mu}\, f \,\mathrm{d}v =: \nabla_x \Delta_x^{-1} \rho[f]$, we apply the standard Schauder's estimates for elliptic equations to get (cf. [54, Section 4.1])

$$\begin{aligned}
\big\|\mathbf{E}_f(t)\big\|_{C^{1,\alpha}_x} \lesssim \big\|\phi_f(t)\big\|_{C^{2,\alpha}_x} \lesssim \big\|\rho[f](t)\big\|_{C^{0,\alpha}_x} &= \left\|\int_{\mathbb{R}^3} \sqrt{\mu(v)}\, f(t,\cdot,v)\,\mathrm{d}v\right\|_{C^{0,\alpha}_x} \\
&\le \int_{\mathbb{R}^3} \sqrt{\mu(v)}\, \big\|f(t,\cdot,v)\big\|_{C^{0,\alpha}_x} \mathrm{d}v \\
&\le \|f\|_{C^{0,\alpha}_{\mathcal{L}}} \cdot \int_{\mathbb{R}^3} \sqrt{\mu(v)}\,\mathrm{d}v \lesssim \|f\|_{C^{0,\alpha}_{\mathcal{L}}}
\end{aligned}$$

for any $t \ge 0$. Also, by the definition of Hölder norms (see Definition 2.6.5),

$$\begin{aligned}
\big\|\mathbf{E}_f\big\|_{C^{1,\alpha}_t} &\le \big\|\mathbf{E}_f\big\|_{L^\infty} + \big\|\partial_t \mathbf{E}_f\big\|_{L^\infty} + \big|\partial_t \mathbf{E}_f\big|_{C^{0,\alpha}_t} \\
&\lesssim \|f\|_{L^\infty} + \|\partial_t f\|_{L^\infty} + |\partial_t f|_{C^{0,\alpha}_{\mathcal{L}}}.
\end{aligned}$$

In summary, we have

$$\big\|\mathbf{E}_f\big\|_{C^{1,\alpha}_{t,x}} \lesssim \|f\|_{C^{0,\alpha}_{\mathcal{L}}} + \|\partial_t f\|_{C^{0,\alpha}_{\mathcal{L}}}.$$

□

Remark 2.8.2. In a similar manner as Step 5 in the proof of Theorem 2.4.1 and using the same notation, we take ∂_t derivative of the equation (2.6.1) (treated as nonlinear equation), and arrange the resulting equation in the desired ultraparabolic form for $\dot{f}$.

Then we may redo everything for $\dot{f}$ with apparent modifications. Eventually, we arrive at the bounds (2.2.12),(2.2.13), and (2.2.14) for $\dot{f}$.

2.9 Proof of the Main Theorem: Global Well-posedness

In this final section we will prove the global well-posedness of solutions stated in Theorem 2.2.1, including global existence, uniqueness, and the bounds (2.2.2) – (2.2.6) global in time. Additionally, by a similar argument as [53] and [30], we can show the non-negativity of the density-distribution function F.

2.9.1 Global Regularity

We start with the construction of global solutions to the Vlasov-Poisson-Landau system (2.1.8) (2.1.9).

Step 1. Local Existence. The proof of this result is an obvious modification of the argument in [30, Section 2 − 5] combined with the standard approximation technique. Since this is not the major emphasis of this chapter, we will simply record the result.

Theorem 2.9.1 (Local Well-posedness). *There exists a sufficiently small constant $\varepsilon_0 \ll 1$ such that for some large velocity-weight exponent $\vartheta_0 \in \mathbb{R}$, the initial data $f_0(x,v) : \Omega \times \mathbb{R}^3 \to \mathbb{R}$ satisfies the smallness assumption*

$$\|f_0\|_{\infty,\vartheta_0} + \|\nabla_x f_0\|_{\infty,\vartheta_0} + \|\nabla_v f_0\|_{\infty,\vartheta_0} + \|\nabla_v^2 f_0\|_{\infty,\vartheta_0} \leq \varepsilon_0 \,. \tag{2.9.1}$$

Let $F_0(x,v) = \mu + \sqrt{\mu}\, f_0(x,v) \geq 0$ and has the same mass as the Maxwellian μ. Then we have the following conclusions:

- (Existence & Uniqueness). *There exists a unique solution $f(t,x,v)$ on $[0,1] \times \Omega \times \mathbb{R}^3$ to the Vlasov-Poisson-Landau system (2.1.8)(2.1.9) for perturbation* *with the specular-reflection boundary condition* (2.1.20) *and the conservation laws* (2.1.24)(2.1.25). *Also,* $F(t,x,v) = \mu + \sqrt{\mu}\, f(t,x,v) \geq 0$ *satisfies the system* (2.1.5)(2.1.7) *with* (2.1.19).

- (Energy estimates). *Moreover, the solution* $f(t,x,v)$ *satisfies the uniform weighted energy bounds, for* $\vartheta < \vartheta_0 - \frac{3}{2}$ *,*

$$\sup_{t\in[0,1]} \mathcal{E}_\vartheta[f(t)] \leq C_\vartheta\, \mathcal{E}_\vartheta[f(0)] \lesssim \varepsilon_0^2\,. \tag{2.9.2}$$

- (L^∞ bounds). *There is* $\vartheta' > 0$ *such that when* $\vartheta + \vartheta' \leq \vartheta_0$ *, the weighted pointwise bounds hold:*

$$\|f(t)\|_{\infty,\vartheta} + \|\mathbf{E}_f(t)\|_\infty \lesssim \varepsilon_0 \tag{2.9.3}$$

for all $t \in [0,1]$ *.*

- ($S^p_{\mathcal{L}}$ bounds). *For the* $S^p_{\mathcal{L}}$ *norm defined in* (2.6.3)*, we have*

$$\|f\|_{S^p_{\mathcal{L}}((0,1)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_0\,. \tag{2.9.4}$$

- (Regularity results). *In addition, it holds that*

$$\|f\|_{C^{0,\alpha}((0,1)\times\Omega\times\mathbb{R}^3)} + \|\mathbf{E}_f\|_{C^{1,\alpha}((0,1)\times\Omega)} \lesssim \varepsilon_0 \tag{2.9.5}$$

for some $\alpha \in (0,1]$ *, and*

$$\|\nabla_v f\|_{L^\infty((0,1)\times\Omega\times\mathbb{R}^3)} \lesssim \varepsilon_0\,. \tag{2.9.6}$$

- *$\partial_t f$ also satisfies all the estimates above.*

Step 2. Bootstrapping. The global regularity part of the theorem is a consequence of Proposition 2.2.2 and the local existence result, via a standard bootstrap-continuity argument.

Define

$$T_*(\varepsilon) = \sup \Big\{ T > 0 : \text{for every choice of initial data satisfying } \mathcal{E}_\vartheta[f(0)] < \varepsilon, \text{ there exists a solution on } [0,T] \text{ achieving the data and satisfying } \mathcal{E}_\vartheta[f(t)] \leq \varepsilon_1 \text{ for } 0 < t \leq T \Big\}.$$

Local well-posedness theorem justifies that such $T_* > 0$ is well-defined. Also, T_* is non-increasing with respect to ε. Now we claim that there exists $0 < \varepsilon_0 < \varepsilon$ such that $T_*(\varepsilon_0) = \infty$. If this is not true, then $0 < T_*(\varepsilon_0) < \infty$. We will show the contradiction that actually the solution can be extended pass $T_*(\varepsilon_0)$.

For every $T_1 \in (0, T_*)$, we know there exists a solution achieving the initial data $\mathcal{E}_\vartheta[f(0] < \varepsilon_0 < \varepsilon$ and satisfying $\mathcal{E}_\vartheta[f(t)] < \varepsilon_1$ for any $0 < t \leq T_1$. The boostrapping theorem justifies that now we actually have the improved estimate $\mathcal{E}_\vartheta[f(T_1)] < \varepsilon_1^s$.

Let T_1 be the new initial time. Local well-posedness theorem implies that now we have extended solution to $t \in [T_1, T_1 + \tilde{T}]$. Since we can always take T_1 sufficiently close to T_*, such that $T_1 + \tilde{T} > T_*$.

Then it remains the verify that for $t \in [T_1, T_1 + \tilde{T}]$, we have $\mathcal{E}_\vartheta[f(t)] < \varepsilon_1$. Local well-posedness theorem tells us now in the extended interval, we have $\mathcal{E}_\vartheta[f(t)] \leq C_\vartheta \mathcal{E}_\vartheta[f(T_1)] \leq C_\vartheta \varepsilon_1^s$. Since $s > 1$, for sufficiently small ε_1, we always have $C_\vartheta \varepsilon_1^s \leq \varepsilon_1$.

Hence, the estimate also holds.

All in all, we extend the solution beyond T_* which achieves the initial data and satisfies the estimate $\mathcal{E}_\vartheta[f(t)] < \varepsilon_1$. Then this is clearly a contradiction to T_* definition. Therefore, the claim is proved and the solution exists globally.

2.9.2 Uniqueness

We now prove that the solution is unique.

Proof. Suppose that $(f_1\,, \mathbf{E}_1)$ and $(f_2\,, \mathbf{E}_2)$ are two global solutions to the VPL-specular problem (2.1.8)(2.1.9)(2.1.20) with bounds (2.2.2)–(2.2.6) holding true globally. Then for $i = 1, 2\,,$ $(f_i\,, \mathbf{E}_i)$ satisfies

$$\partial_t f_i + v \cdot \nabla_x f_i + L f_i - 2\sqrt{\mu}\, v \cdot \mathbf{E}_i \;=\; \Gamma[f_i, f_i] - \mathbf{E}_i \cdot \nabla_v f_i + \big(v \cdot \mathbf{E}_i\big) f_i\,, \tag{2.9.7}$$

where $\mathbf{E}_i := \mathbf{E}_{f_i} = -\nabla_x \phi_{f_i}$ with $\phi_i := \phi_{f_i}$ solved from the Poisson equation

$$-\,\Delta_x \phi_i = \int_{\mathbb{R}^3} \sqrt{\mu}\, f_i \,\mathrm{d}v \;=: \rho[f_i]\,. \tag{2.9.8}$$

Note that (2.9.7) has all the linear terms on its LHS, while terms on RHS are the nonlinearities. Let $\tilde{f} := f_1 - f_2$ be the difference of two solutions and so $\widetilde{\mathbf{E}} := \mathbf{E}_1 - \mathbf{E}_2 = \mathbf{E}_{\tilde{f}}$. Then from (2.9.7) we get the equation

$$\begin{aligned} &\partial_t \tilde{f} + v \cdot \nabla_x \tilde{f} + L\tilde{f} - 2\sqrt{\mu}\, v \cdot \widetilde{\mathbf{E}} \\ &= \Big\{ \Gamma[\tilde{f}, f_1] + \Gamma[f_2, \tilde{f}\,] \Big\} - \Big\{ \widetilde{\mathbf{E}} \cdot \nabla_v f_1 + \mathbf{E}_2 \cdot \nabla_v \tilde{f} \Big\} + \Big\{ \big(v \cdot \widetilde{\mathbf{E}}\big) f_1 + \big(v \cdot \mathbf{E}_2\big) \tilde{f} \Big\}\,. \end{aligned} \tag{2.9.9}$$

For a similar reason as in the proof of Theorem 2.4.1 (in the energy estimate), we multiply e^{ϕ_2} on both sides of the equation above, so that the last term on RHS can be merged into the second term on LHS. This way the equation becomes

$$\begin{aligned} &\partial_t\big(e^{\phi_2}\tilde{f}\,\big) + v\cdot\nabla_x\big(e^{\phi_2}\tilde{f}\,\big) + L\big(e^{\phi_2}\tilde{f}\,\big) - 2e^{\phi_2}\sqrt{\mu}\,v\cdot\widetilde{\mathbf{E}} \\ = {}& J_1 + J_2 + J_3 + J_4\ , \end{aligned} \tag{2.9.10}$$

where

$$\begin{aligned} J_1 &:= e^{\phi_2}\,\Gamma[\tilde{f}, f_1] \,+\, e^{\phi_2}\,\Gamma[f_2, \tilde{f}\,]\ , \\ J_2 &:= -\Big\{e^{\phi_2}\,\widetilde{\mathbf{E}}\cdot\nabla_v f_1 \,+\, e^{\phi_2}\,\mathbf{E}_2\cdot\nabla_v\tilde{f}\Big\}\ , \\ J_3 &:= e^{\phi_2}\big(v\cdot\widetilde{\mathbf{E}}\big)f_1\ , \\ J_4 &:= e^{\phi_2}\tilde{f}\,\partial_t\phi_2\ . \end{aligned}$$

Multiplying both sides of (2.9.10) by $\langle v\rangle^{2\vartheta}e^{\phi_2}\tilde{f}$ for $\vartheta\le -2$ (the same weight power as in Lemma 2.3.5, in order to apply the estimate (2.3.17) for $\Gamma[\,\cdot\,,\,\cdot\,]$ term) and integrating over $(x,v)\in\Omega\times\mathbb{R}^3$, we get

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\big\|e^{\phi_2}\tilde{f}\,\big\|_{2,\vartheta}^2 \,+\, \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\nabla_x\cdot\Big\{v\big(e^{\phi_2}\tilde{f}\,\big)^2\Big\} \\ &\quad+ \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,L\big(e^{\phi_2}\tilde{f}\,\big) \,-\, \iint_{\Omega\times\mathbb{R}^3}2\,\langle v\rangle^{2\vartheta}\big(e^{2\phi_2}\tilde{f}\,\big)\sqrt{\mu}\,v\cdot\widetilde{\mathbf{E}} \\ = &\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,J_1 \,+\, \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,J_2 \\ &\quad+ \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,J_3 \,+\, \iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,J_4\ . \end{aligned} \tag{2.9.11}$$

Now we estimate term by term from (2.9.11): We first observe that $e^{\phi_2}\sim 1$ since $\|\phi_2\|_\infty \lesssim \|f_2\|_\infty \ll 1$ (see the proof of Lemma 2.3.7).

On the LHS, the second term vanishes since

$$\frac{1}{2}\iint_{\Omega\times\mathbb{R}^3}\nabla_x\cdot\Big\{v\big(e^{\phi_2}\tilde{f}\,\big)^2\Big\} \;=\; \frac{1}{2}\bigg(\iint_{\gamma_+}-\iint_{\gamma_-}\bigg)\big(e^{\phi_2}\tilde{f}\,\big)^2|v\cdot n_x|\,\mathrm{d}v\mathrm{d}S_x \;=\; 0\;, \tag{2.9.12}$$

using the divergence theorem along with the specular boundary condition. By (2.3.15) in Lemma 2.3.4, the third term can be bounded below as

$$\iint_{\Omega\times\mathbb{R}^3}\langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde{f}\,\big)\,L\big(e^{\phi_2}\tilde{f}\,\big) \;=\; \Big(\langle v\rangle^{2\vartheta}L\big(e^{\phi_2}\tilde{f}\,\big)\,,\,e^{\phi_2}\tilde{f}\Big) \;\gtrsim\; \big\|e^{\phi_2}\tilde{f}\,\big\|^2_{\sigma,\vartheta}-C_\vartheta\big\|e^{\phi_2}\tilde{f}\,\big\|^2_{2,\vartheta}\,. \tag{2.9.13}$$

Also, we move the fourth term to the other side and bound it above using Hölder's inequality:

$$\iint_{\Omega\times\mathbb{R}^3}2\,\langle v\rangle^{2\vartheta}\big(e^{2\phi_2}\tilde{f}\,\big)\sqrt{\mu}\,v\cdot\widetilde{\mathbf{E}} \;\lesssim\; \big|\langle v\rangle^{\vartheta+1}\sqrt{\mu}\,\big|_\infty\big\|e^{\phi_2}\tilde{f}\big\|_{2,\vartheta}\big\|e^{\phi_2}\widetilde{\mathbf{E}}\big\|_2 \;\lesssim\; \big\|e^{\phi_2}\tilde{f}\big\|^2_{2,\vartheta}\,. \tag{2.9.14}$$

The last inequality is valid because

$$\begin{aligned}
\big\|e^{\phi_2}\widetilde{\mathbf{E}}\big\|_2 \lesssim \|\widetilde{\mathbf{E}}\|_2 \lesssim \big\|\rho[\tilde{f}\,]\big\|_2 \;&=\; \bigg\|\int_{\mathbb{R}^3}\sqrt{\mu}\,\tilde{f}\,\mathrm{d}v\bigg\|_{L^2_x}\\
&\le\; \int_{\mathbb{R}^3}\big\|\sqrt{\mu}\,\tilde{f}\big\|_{L^2_x}\mathrm{d}v \;=\; \int_{\mathbb{R}^3}\langle v\rangle^{-\vartheta}\sqrt{\mu}\,\big\|\langle v\rangle^{\vartheta}\tilde{f}\big\|_{L^2_x}\mathrm{d}v\\
&\le\; \big\|\langle v\rangle^{\vartheta}\tilde{f}\big\|_{L^2_{x,v}}\cdot\big|\langle v\rangle^{-\vartheta}\sqrt{\mu}\,\big|_{L^2_v}\\
&\lesssim\; \|\tilde{f}\|_{2,\vartheta} \;\lesssim\; \big\|e^{\phi_2}\tilde{f}\big\|_{2,\vartheta}\,,
\end{aligned}$$

by modifying the proof of Lemma 2.3.7 (with $\vartheta\le-2$).

For the RHS, we estimate

$$
\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\, J_1 \;&=\; \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\Big\{e^{\phi_2}\,\Gamma[\tilde f, f_1] \;+\; e^{\phi_2}\,\Gamma[f_2,\tilde f\,]\Big\} \\
&\lesssim\; \big(\|f_1\|_\infty + \|D_v f_1\|_\infty\big)\,\big\|e^{\phi_2}\tilde f\big\|^2_{\sigma,\vartheta} + \|f_2\|_\infty\big\|e^{\phi_2}\tilde f\big\|^2_{\sigma,\vartheta} \\
&\lesssim\; \varepsilon_0\,\big\|e^{\phi_2}\tilde f\big\|^2_{\sigma,\vartheta}\,,
\end{aligned}
\tag{2.9.15}
$$

where we apply (2.3.17) to the first term and (2.3.16) to the second (see Lemma 2.3.5). Also,

$$
\begin{aligned}
&\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\, J_2 \\
&= -\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\Big\{e^{\phi_2}\,\widetilde{\mathbf{E}}\cdot\nabla_v f_1 \;+\; e^{\phi_2}\,\mathbf{E}_2\cdot\nabla_v\tilde f\,\Big\} \\
&= -\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)e^{\phi_2}\,\widetilde{\mathbf{E}}\cdot\nabla_v f_1 \;+\; \frac{1}{2}\iint_{\Omega\times\mathbb{R}^3} \mathbf{E}_2\cdot\nabla_v\Big(\langle v\rangle^{2\vartheta}\Big)\big(e^{\phi_2}\tilde f\,\big)^2 \\
&\lesssim\; \|D_v f_1\|_\infty\|\widetilde{\mathbf{E}}\|_\infty\big\|e^{\phi_2}\tilde f\big\|_{2,\vartheta}\big|\langle v\rangle^{\vartheta}\big|_2 \;+\; \|\mathbf{E}_2\|_\infty\big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta} \\
&\lesssim\; \big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta}\,,
\end{aligned}
\tag{2.9.16}
$$

noting that with $\vartheta \le -2$ we have $\big|\langle v\rangle^{\vartheta}\big|_2 < \infty$. In addition,

$$
\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\, J_3 \;&=\; \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)e^{\phi_2}\big(v\cdot\widetilde{\mathbf{E}}\big)f_1 \\
&\lesssim\; \|f_1\|_{\infty,\vartheta+1}\big\|e^{\phi_2}\tilde f\big\|_{2,\vartheta}\big\|e^{\phi_2}\widetilde{\mathbf{E}}\big\|_2 \\
&\lesssim\; \big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta}\,,
\end{aligned}
\tag{2.9.17}
$$

and

$$
\begin{aligned}
\iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)\, J_4 \;&=\; \iint_{\Omega\times\mathbb{R}^3} \langle v\rangle^{2\vartheta}\big(e^{\phi_2}\tilde f\,\big)e^{\phi_2}\tilde f\,\partial_t\phi_2 \\
&\lesssim\; \|\partial_t\phi_2\|_\infty\big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta} \\
&\lesssim\; \|\partial_t f_2\|_2\big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta} \;\lesssim\; \big\|e^{\phi_2}\tilde f\big\|^2_{2,\vartheta}\,.
\end{aligned}
\tag{2.9.18}
$$

Collecting all the estimates in (2.9.12)−(2.9.18) above and plugging them into (2.9.11), we then absorb $\varepsilon_0 \| e^{\phi_2} \tilde{f} \|^2_{\sigma,\vartheta}$ from (2.9.15) into LHS and obtain

$$\frac{\mathrm{d}}{\mathrm{d}t} \| e^{\phi_2} \tilde{f} \|^2_{2,\vartheta} + \| e^{\phi_2} \tilde{f} \|^2_{\sigma,\vartheta} - C_\vartheta \| e^{\phi_2} \tilde{f} \|^2_{2,\vartheta} \lesssim \| e^{\phi_2} \tilde{f} \|^2_{2,\vartheta} \,,$$

which further yields

$$\frac{\mathrm{d}}{\mathrm{d}t} \| e^{\phi_2} \tilde{f} \|^2_{2,\vartheta} \leq C \| e^{\phi_2} \tilde{f} \|^2_{2,\vartheta} \,. \tag{2.9.19}$$

Hence, by the Gronwall inequality, we have that for every $t \in [0, \infty)$,

$$\| e^{\phi_2} \tilde{f}(t) \|^2_{2,\vartheta} \leq e^{Ct} \| e^{\phi_2} \tilde{f}(0) \|^2_{2,\vartheta} = 0 \,,$$

noticing that $f_1(0) = f_2(0) = f_0$ and thus $\tilde{f}(0) \equiv 0$. This actually implies that $\tilde{f}(t) \equiv 0$ for all $t \geq 0$, since $e^{\phi_2} > 0$ and also due to the continuity of the solutions. Therefore, we conclude that the solution exists uniquely. □

Chapter Three

Modified Wave Operators for the Wave-Klein-Gordon System

3.1 Introduction

3.1.1 The Wave-Klein-Gordon System

In this chapter we are interested in the global behavior of the Wave-Klein-Gordon (W-KG) system in $3+1$ space-time dimensions:

$$\begin{aligned} -\Box u \;&=\; A^{\alpha\beta}\partial_\alpha v \partial_\beta v + Dv^2 \,, \\ (-\Box + 1)v \;&=\; uB^{\alpha\beta}\partial_\alpha\partial_\beta v + Euv \,, \end{aligned} \tag{3.1.1}$$

where $\Box := -\partial_t^2 + \Delta$ is the d'Alembert operator. The unknowns $u, v : \mathbb{R}_t^+ \times \mathbb{R}_x^3 \to \mathbb{R}$ are real-valued functions, and $A^{\alpha\beta}$, $B^{\alpha\beta}$, D and E are real constants. Without loss of generality we may assume that $A^{\alpha\beta} = A^{\beta\alpha}$ and $B^{\alpha\beta} = B^{\beta\alpha}$, $\alpha, \beta \in \{0, 1, 2, 3\}$. For convenience we will also assume that $B^{00} = 0$.[1]

Our main focus is the description of asymptotic behavior for small solutions to a prototype of the above system:

$$\begin{aligned} (\partial_t^2 - \Delta)u \;&=\; |\nabla_{t,x} v|^2 + v^2 \,, \\ (\partial_t^2 - \Delta + 1)v \;&=\; u\Delta v \,. \end{aligned} \tag{3.1.2}$$

This coupled system consists of a semi-linear wave equation for u and a quasi-linear Klein-Gordon equation for v.

---

[1] This can be achieved by adding some higher order terms in the second equation in (3.1.1), which do not change the analysis.

3.1.2 Background and Motivation

The system (3.1.1) was derived by LeFloch-Ma [56] as a simplified model for the Einstein-Klein-Gordon (E-KG) system, which describes the coupled evolution of the Lorentzian metric and a self-gravitating massive scalar field. We refer to LeFloch-Ma [56, 57], Ionescu-Pausader [45, 46, 47] and Wang [70] for more details.

Consider an Einstein field equation for an unknown space-time $(M, \mathbf{g})$

$$G_{\alpha\beta} := R_{\alpha\beta} - \frac{1}{2} R\, \mathbf{g}_{\alpha\beta} = T_{\alpha\beta}, \tag{3.1.3}$$

where $G_{\alpha\beta}$ is the Einstein tensor, $\mathbf{g}$ is a Lorentzian metric, $R_{\alpha\beta}$ is the Ricci tensor and R is the scalar curvature, and $T_{\alpha\beta}$ is the energy-momentum tensor of matter in the space. In the presence of massive field ψ, $T_{\alpha\beta}$ is given by

$$T_{\alpha\beta} := \mathbf{D}_\alpha \psi \mathbf{D}_\beta \psi - \frac{1}{2} \mathbf{g}_{\alpha\beta} \left(\mathbf{D}_\mu \psi \mathbf{D}^\mu \psi + \psi^2 \right), \tag{3.1.4}$$

where $\mathbf{D}$ denotes covariant derivatives. Using Bianchi identities $\mathbf{D}^\alpha G_{\alpha\beta} = 0$, we may derive the evolution of the massive scalar field ψ as

$$\Box_{\mathbf{g}} \psi - \psi = 0, \tag{3.1.5}$$

where $\Box_{\mathbf{g}} := \mathbf{g}^{\alpha\beta} \mathbf{D}_\alpha \mathbf{D}_\beta$. If we choose the wave coordinates satisfying $-\Box_{\mathbf{g}} x^\alpha = 0$ for $\alpha = 0, 1, 2, 3$, then $(\mathbf{g}, \psi)$ satisfies the Einstein-Klein-Gordon system

$$R_{\alpha\beta} - \mathbf{D}_\alpha \psi \mathbf{D}_\beta \psi = \frac{\psi^2}{2} \mathbf{g}_{\alpha\beta}, \tag{3.1.6}$$

$$\Box_{\mathbf{g}} \psi - \psi = 0. \tag{3.1.7}$$

Further under the harmonic gauge, the above system reduces to

$$-\widetilde{\Box}_{\mathbf{g}}\,\mathbf{g}_{\alpha\beta} = 2\partial_\alpha\psi\partial_\beta\psi + \psi^2\mathbf{g}_{\alpha\beta} - F^{\geq 2}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g}) = 0, \tag{3.1.8}$$

$$-\widetilde{\Box}_{\mathbf{g}}\psi + \psi = 0, \tag{3.1.9}$$

where $\widetilde{\Box}_{\mathbf{g}} := \mathbf{g}^{\alpha\beta}\partial_\alpha\partial_\beta$, and $F^{\geq 2}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g})$ contains all higher-order terms.

The global stability and asymptotic behavior of the Minkowski space-time are central topics in general relativity. The small data global well-posedness, regularity and stability of the Einstein-Klein-Gordon system were studied in LeFloch-Ma [57], Ionescu-Pausader [46] and Wang [70].

Heuristically, in the Einstein-Klein-Gordon system, if we replace the deviation of the Lorentzian metric $\mathbf{g}$ from the Minkowski metric by a scalar function u, replace the massive scalar field ψ by v, and neglect the higher-order interactions involving ψ (i.e. ignoring $F^{\geq 2}_{\alpha\beta}(\mathbf{g}, \partial\mathbf{g})$), we arrive at the Wave-Klein-Gordon system (3.1.1).

The small data global well-posedness, regularity and stability of the Wave-Klein-Gordon system were studied in LeFloch-Ma [56], Ionescu-Pausader [45], Dong-Wyatt [14]. The lower-dimensional results can be found in Ma [60, 59], Ifrim-Stingo [42] and Stingo [66]. Some earlier results on wave and Klein-Gordon type equations can be found in Georgiev [21] and Katayama [50].

Roughly speaking, LeFloch-Ma [56, 57], Wang [70], and Dong-Wyatt [14] used a refined hyperbolic foliation method (see also LeFloch-Ma [55]), which requires restricted initial data (e.g. compactly supported). This restriction can be slightly lifted to treat more general data (see Ma [61] for a model equation).

On the other hand, Ionescu-Pausader [45, 46, 47] relies on the so-called space-time resonance method. It combines the energy method and Fourier analysis to study the global well-posedness of quasi-linear dispersive equations. Generally speaking, energy method combined with vector fields may handle the higher-order regularity terms, and Fourier analysis with normal forms could justify the dispersive decay of lower-regularity terms. This method has been proven to be very effective in many kinds of nonlinear dispersive equations. We refer to Delort-Fang [6], Delort-Fang-Xue [7], Germain-Masmoudi-Shatah [22, 23], Gustafson-Nakanishi-Tsai [37], Deng [8], Deng-Ionescu-Pausader [9], Deng-Ionescu-Pausader-Pusateri [10], Guo-Ionescu-Pausader [31], Guo-Pausader [34], Ionescu-Pausader [43, 44], Deng-Pusateri [11], and Ionescu-Pusateri [48]. In this work, we will follow this path to study the long-time behavior of the Wave-Klein-Gordon system.

3.1.3 Main Result

The Wave-Klein-Gordon system does not have a null structure, but it presents an intricate resonant pattern, which is reflected in the asymptotic behavior. Ionescu-Pausader [45, Remark 1.3] pointed out the modification to linear scattering for the Wave-Klein-Gordon system. We interpret their result as follows.

Theorem 3.1.1 (Modified Scattering). *Let (u, v) be a global solution of the system (3.1.2). Then there exists a scattering profile $\left(V_\infty^{wa}, V_\infty^{kg}\right)$ such that*

$$\lim_{t\to\infty} \left\| \left(\partial_t - \mathrm{i}\,|\nabla|\,\right) u(t) - \mathrm{e}^{-\mathrm{i}t|\nabla|} V_\infty^{wa} \right\|_{L^2} = 0, \tag{3.1.10}$$

and

$$\lim_{t\to\infty} \left\| \left(\partial_t - \mathrm{i}\langle\nabla\rangle\right) v(t) - \mathrm{e}^{-\mathrm{i}t\langle\nabla\rangle + \mathrm{i}\Theta(t,\nabla)} V_\infty^{kg} \right\|_{L^2} = 0, \tag{3.1.11}$$

where $\Theta(t,\nabla)$ is a phase-correction operator depending on u.

Remark 3.1.1. This theorem indicates that the wave component u scatters linearly and the Klein-Gordon component v undergoes nonlinear scattering.

In this chapter, we will focus on the the converse statement – existence of wave operators, namely that every possible asymptotic behavior is achieved. More precisely, we intend to justify that any asymptotic dynamic arises in a unique way as the limit of a solution to the W-KG system.

Theorem 3.1.2 (Main Theorem – Modified Wave Operator, informal version)**.** *Assume the scattering data $\big(V_\infty^{wa}(x), V_\infty^{kg}(x)\big)$ is sufficiently small under suitable weighted Sobolev-type norms. Then there exists a unique global solution (u,v) of the system (3.1.2) such that*

$$\lim_{t\to\infty}\left\|\big(\partial_t - \mathrm{i}\,|\nabla|\,\big)u(t) - \mathrm{e}^{-\mathrm{i}t|\nabla|}\left(V_\infty^{wa} + \check{\mathcal{H}}_\infty(t)\right)\right\|_{L^2} = 0, \tag{3.1.12}$$

and

$$\lim_{t\to\infty}\left\|\big(\partial_t - \mathrm{i}\langle\nabla\rangle\big)v(t) - \mathrm{e}^{-\mathrm{i}t\langle\nabla\rangle+\mathrm{i}\mathcal{D}_\infty(t,\nabla)}V_\infty^{kg}\right\|_{L^2} = 0. \tag{3.1.13}$$

Here $\check{\mathcal{H}}_\infty(t,x)$ is a function depending on V_∞^{kg} and $\mathcal{D}_\infty(t,\nabla)$ is a phase-correction operator that explicitly depends on $(V_\infty^{wa}, V_\infty^{kg})$.

Remark 3.1.2. We would like to make a few remarks:

(1) This theorem is just an informal version and the precise statement of the theorem (Theorem 3.2.1) is in Section 3.2.6. $\check{\mathcal{H}}_\infty$ is the inverse Fourier transform of $\mathcal{H}_\infty$ defined in (3.2.65) and $\mathcal{D}_\infty(t,\nabla)$ is the operator corresponding to a modification $\mathcal{D}_\infty(t,\xi)$ of the free Klein-Gordon dispersion (see (3.2.68) for the definition).

(2) In the Fourier space, $\mathcal{D}_\infty(t,\xi)$ is a real-valued nonlinear function of $\widehat{V_\infty^{kg}}$ which takes into account the low-frequency part of $\widehat{V_\infty^{wa}}$. In particular, the fact that $|\mathcal{D}_\infty(t,\xi)| \to \infty$ as $t \to \infty$ indicates a genuine nonlinear scattering for the KG component.

(3) Compared with Theorem 3.1.1, we may notice that Theorem 3.1.2 has an extra term $\check{\mathcal{H}}_\infty$ (which is $\sim O(1)$ as $t \to \infty$). This term actually provides a more precise description of the asymptotic behavior for the Wave-Klein-Gordon system. In fact, the notation V_∞^{wa} in the two theorems does not refer to the same quantity. Instead, we may write $\widetilde{V_\infty^{wa}} = V_\infty^{wa} + \check{\mathcal{H}}_\infty +$ fast-decaying term, for $\widetilde{V_\infty^{wa}}$ in Theorem 3.1.1 and V_∞^{wa} in Theorem 3.1.2.

(4) Our result also applies to the original W-KG system (3.1.1). For simplicity we will work with the model system (3.1.2), but all our arguments can be carried out for the general case (3.1.1) as well.

3.1.4 Difficulties and Methods

The energy structure for the Wave-Klein-Gordon system can be explained heuristically as follows (see Ionescu-Pausader [45, Section 1.3.1]): The structure of the W-KG system is characterized by the bilinear interactions

$$\begin{aligned} \text{Wave} &\leftarrow \text{KG} \times \text{KG} \\ \text{KG} &\leftarrow \text{Wave} \times \text{KG} \end{aligned}$$

Let $\mathcal{L}$ be a commutative vector field and ∂ be either time or space derivative. Then applying $\mathcal{L}$ to the Wave-Klein-Gordon system (3.1.2) yields

$$\Box\big(\mathcal{L}[u]\big) \simeq \partial v \cdot \mathcal{L}[\partial v], \tag{3.1.14}$$

$$(\Box + 1)\big(\mathcal{L}[v]\big) \simeq \mathcal{L}[u] \cdot \partial^2 v + u \cdot \mathcal{L}[\partial^2 v]. \tag{3.1.15}$$

It is clear that the order of derivatives for the nonlinear terms are imbalanced. Since the standard energy estimate may only control

$$\|\mathcal{L}[\partial u]\|_{L^2} \quad \text{and} \quad \|\mathcal{L}[v]\|_{H^1}\,, \tag{3.1.16}$$

then we have difficulty bounding $\mathcal{L}[u]\cdot\partial^2 v$ (in the sense that $\mathcal{L}[u]$ is not included in the energy and $\partial^2 v$ cannot be handled via integration by parts). Notice that $u \cdot \mathcal{L}[\partial^2 v]$ may be handled using integration by parts in the energy estimate, so we will not discuss it in the following. To overcome the above difficulty, Ionescu-Pausader [45] introduced $|\nabla|^{-\frac{1}{2}}$ term in the energy structure of the wave equation.

Ideally, the wave component $\|u\|_{W^{k,\infty}} \lesssim \langle t\rangle^{-1}$ and the Klein-Gordon component $\|v\|_{W^{k,\infty}} \lesssim \langle t\rangle^{-\frac{3}{2}}$ based on the standard dispersive estimates. For the wave equation, energy estimate implies

$$\begin{aligned}\partial_t \left\| |\nabla|^{-\frac{1}{2}} \mathcal{L}[\partial u] \right\|_{L^2} &\lesssim \left\| |\nabla|^{-\frac{1}{2}} \Big(\partial v \cdot \mathcal{L}[\partial v]\Big) \right\|_{L^2} \\ &\lesssim \langle t\rangle^{\frac{1}{2}} \|\partial v\|_{L^\infty} \|\mathcal{L}[\partial v]\|_{L^2} \lesssim \langle t\rangle^{-1} \|\mathcal{L}[\partial v]\|_{L^2}\,,\end{aligned} \tag{3.1.17}$$

for the frequency $|\xi| \gtrsim \langle t\rangle^{-1}$. On the other hand, for the Klein-Gordon equation, energy estimate implies

$$\begin{aligned}\partial_t \|\mathcal{L}[v]\|_{H^1} &\lesssim \left\|\mathcal{L}[u]\cdot\partial^2 v\right\|_{L^2} \lesssim \left\||\nabla|^{-\frac{1}{2}}\left(|\nabla|^{-\frac{1}{2}}\mathcal{L}[\partial u]\right)\cdot\partial^2 v\right\|_{L^2} \\ &\lesssim \langle t\rangle^{\frac{1}{2}}\left\|\partial^2 v\right\|_{L^\infty}\left\||\nabla|^{-\frac{1}{2}}\mathcal{L}[\partial u]\right\|_{L^2} \lesssim \langle t\rangle^{-1}\left\||\nabla|^{-\frac{1}{2}}\mathcal{L}[\partial u]\right\|_{L^2},\end{aligned} \tag{3.1.18}$$

for the frequency $|\xi| \gtrsim \langle t\rangle^{-1}$. Now if we can control the energy

$$\left\||\nabla|^{-\frac{1}{2}}\mathcal{L}[\partial u]\right\|_{L^2} \quad \text{and} \quad \|\mathcal{L}[v]\|_{H^1}, \tag{3.1.19}$$

we may close the bootstrapping.

In the above analysis, we leave out the part with frequency $|\xi| \lesssim \langle t\rangle^{-1}$. This cannot be handled by the energy structure with dispersive estimate, and this part will not decay to zero as $t \to \infty$. In other words, in the long run, only the part with $|\xi| \lesssim \langle t\rangle^{-1}$ will play a role in the asymptotic behavior. In the Wave-Klein-Gordon system, we will extract this part and call it a resonant system for (u^{RS}, v^{RS}).

For the modified wave operator, we assume that the initial data for the resonant system has been prescribed, i.e. (u^{RS}, v^{RS}) is given. We intend to construct a solution (u, v) to the Wave-Klein-Gordon system (3.1.2) such that $u^{RS} \to u$ and $v^{RS} \to v$ as $t \to \infty$. Denote $g^{wa} = u - u^{RS}$ and $g^{kg} = v - v^{RS}$. We insert it into the Wave-Klein-Gordon system (3.1.2) to get an equation for (g^{wa}, g^{kg}). This equation has similar structure as (3.1.2) with some extra terms:

$$\begin{aligned}\Box g^{wa} &= -\Box u^{RS} + \left|\partial g^{wa} + \partial u^{RS}\right|^2 + \left|g^{wa} + u^{RS}\right|^2, \\ (\Box + 1) g^{kg} &= -(\Box + 1) v^{RS} + (g^{wa} + u^{RS})\Delta(g^{kg} + v^{RS}).\end{aligned} \tag{3.1.20}$$

We will design a fixed-point argument such that (g^{wa}, g^{kg}) is well-posed for $t \in [T, \infty)$

and satisfies

$$\lim_{t\to\infty} g^{wa}(t) = \lim_{t\to\infty} g^{kg}(t) = 0. \tag{3.1.21}$$

The nonlinear terms can be further divided into three categories: quadratic terms (two components are both g^{wa} or g^{kg}), linear terms (one component is g^{wa} or g^{kg} and the other one is u^{RS} or v^{RS}), and forcing terms (two components are both u^{RS} or v^{RS}). In particular, the most difficult estimate lies in the forcing terms, and we have to delicately utilize the cancellation with the resonant system.

Compared with Ionescu-Pausader [45], where they allow the solutions to grow mildly in energy norms as $t \to \infty$ (i.e. nonlinear estimate is like $\langle t\rangle^{-1+\delta}$), our case is more challenging, in the sense that we must prove (g^{wa}, g^{kg}) decays as $t \to \infty$ (i.e. nonlinear estimate is like $\langle t\rangle^{-1-\delta}$). The faster decay rate is extremely crucial for our proof. This is particularly difficult for the forcing terms since the resonant system at most decays at $\langle t\rangle^{-1}$, so we must delicately analyze the difference between the forcing terms and the resonant part $-\Box u^{RS}$ and $-(\Box + 1)v^{RS}$ in (3.1.20), and eliminate the low-frequency contributions.

To summarize, the key to our proof is the energy estimate and analysis of the three types of nonlinear terms. The most difficult terms are forcing terms related to the resonant system. In particular, we focus on controlling the perturbation globally using a combination of tools from Fourier and harmonic analysis (Fourier transform, dyadic decomposition), oscillatory integrals (non-stationary/stationary phase analysis), bilinear estimates (pseudo-products operators with singular multipliers), ODE and geometric methods (normal forms, vector fields), and delicately designed function spaces and decay rate.

3.2 Setup of the Problem

3.2.1 Reformulation of the Equations: Duhamel's Formula

Throughout this chapter the Fourier transform is defined as

$$\widehat{f}(\xi) = \mathscr{F}[f](\xi) = (2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{-\mathrm{i}x\cdot\xi} f(x)\,\mathrm{d}x\,. \tag{3.2.1}$$

Define the operators on $\mathbb{R}^3$

$$\Lambda_{wa} := |\nabla| = \mathscr{F}^{-1}|\xi|\mathscr{F}\,, \qquad \Lambda_{kg} := \langle\nabla\rangle = \sqrt{|\nabla|^2+1} = \mathscr{F}^{-1}\langle\xi\rangle\mathscr{F}\,, \tag{3.2.2}$$

which relate to the dispersion relations for the wave and Klein-Gordon equation, respectively.

We denote also the corresponding multipliers in frequency space by

$$\begin{aligned} &\Lambda_{wa,+}(\xi) = \Lambda_{wa}(\xi) := |\xi|\,, && \Lambda_{kg,+}(\xi) = \Lambda_{kg}(\xi) := \langle\xi\rangle = \sqrt{|\xi|^2+1}\,,\\ &\Lambda_{wa,-}(\xi) = -\Lambda_{wa,+}(\xi)\,, && \Lambda_{kg,-}(\xi) = -\Lambda_{kg,+}(\xi)\,. \end{aligned} \tag{3.2.3}$$

The (complex-valued) *normalized solutions* U^{wa}, U^{kg} of the system (3.1.2) are defined by

$$U^{wa}(t) := [\partial_t - \mathrm{i}\Lambda_{wa}]\,u(t)\,, \qquad U^{kg}(t) := [\partial_t - \mathrm{i}\Lambda_{kg}]\,v(t)\,, \tag{3.2.4}$$

along with

$$
\begin{aligned}
&U^{wa,+} := U^{wa}\,, && U^{kg,+} := U^{kg}\,,\\
&U^{wa,-} := \overline{U^{wa}} = [\partial_t + \mathrm{i}\Lambda_{wa}]\,u(t)\,, && U^{kg,-} := \overline{U^{kg}} = [\partial_t + \mathrm{i}\Lambda_{kg}]\,v(t)\,,
\end{aligned}
\tag{3.2.5}
$$

so that the solutions u, v can be recovered from U^{wa}, U^{kg} by the formulas

$$
\begin{aligned}
&\partial_t u = \tfrac{1}{2}\left(U^{wa} + \overline{U^{wa}}\right) = \mathrm{Re}\left(U^{wa}\right)\,, && \partial_t v = \tfrac{1}{2}\left(U^{kg} + \overline{U^{kg}}\right) = \mathrm{Re}\left(U^{kg}\right)\,,\\
&\Lambda_{wa} u = \tfrac{\mathrm{i}}{2}\left(U^{wa} - \overline{U^{wa}}\right) = -\mathrm{Im}\left(U^{wa}\right)\,, && \Lambda_{kg} v = \tfrac{\mathrm{i}}{2}\left(U^{kg} - \overline{U^{kg}}\right) = -\mathrm{Im}\left(U^{kg}\right)\,,\\
&u = \tfrac{\mathrm{i}}{2\Lambda_{wa}}\left(U^{wa} - \overline{U^{wa}}\right) = -\tfrac{1}{|\nabla|}\,\mathrm{Im}\left(U^{wa}\right)\,, && v = \tfrac{\mathrm{i}}{2\Lambda_{kg}}\left(U^{kg} - \overline{U^{kg}}\right) = -\tfrac{1}{\langle\nabla\rangle}\,\mathrm{Im}\left(U^{kg}\right)\,.
\end{aligned}
\tag{3.2.6}
$$

Conjugating with the linear propagator operator $\mathrm{e}^{\mathrm{i}t\Lambda} := \mathscr{F}^{-1} e^{\mathrm{i}t\Lambda(\xi)} \mathscr{F}$, we define the associated *profiles* V^{wa}, V^{kg} by

$$
V^{wa}(t) := \mathrm{e}^{\mathrm{i}t\Lambda_{wa}} U^{wa}(t)\,, \qquad V^{kg}(t) := \mathrm{e}^{\mathrm{i}t\Lambda_{kg}} U^{kg}(t)\,, \tag{3.2.7}
$$

and denote also

$$
\begin{aligned}
&V^{wa,+} := V^{wa} = \mathrm{e}^{\mathrm{i}t|\nabla|}\, U^{wa}(t)\,, && V^{kg,+} := V^{kg} = \mathrm{e}^{\mathrm{i}t\langle\nabla\rangle}\, U^{kg}(t)\,,\\
&V^{wa,-} := \overline{V^{wa}} = \mathrm{e}^{-\mathrm{i}t|\nabla|}\, \overline{U^{wa}}(t)\,, && V^{kg,-} := \overline{V^{kg}} = \mathrm{e}^{-\mathrm{i}t\langle\nabla\rangle}\, \overline{U^{kg}}(t)\,,
\end{aligned}
\tag{3.2.8}
$$

which under the Fourier transform become

$$
\begin{aligned}
&\widehat{V^{wa,+}}(t,\xi) = \mathrm{e}^{\mathrm{i}t|\xi|}\, \widehat{U^{wa}}(t,\xi)\,, && \widehat{V^{kg,+}}(t,\xi) = \mathrm{e}^{\mathrm{i}t\langle\xi\rangle}\, \widehat{U^{kg}}(t,\xi)\,,\\
&\widehat{V^{wa,-}}(t,\xi) = \mathrm{e}^{-\mathrm{i}t|\xi|}\, \widehat{\overline{U^{wa}}}(t,\xi)\,, && \widehat{V^{kg,-}}(t,\xi) = \mathrm{e}^{-\mathrm{i}t\langle\xi\rangle}\, \widehat{\overline{U^{kg}}}(t,\xi)\,.
\end{aligned}
\tag{3.2.9}
$$

Remark 3.2.1. From the above definition we can see that

$$\widehat{V^{wa,-}}(t,\xi) = \overline{\widehat{V^{wa,+}}}(t,\xi) = \overline{\widehat{V^{wa,+}}(t,-\xi)}, \quad \widehat{V^{kg,-}}(t,\xi) = \overline{\widehat{V^{kg,+}}}(t,\xi) = \overline{\widehat{V^{kg,+}}(t,-\xi)}. \tag{3.2.10}$$

Now we reformulate the system (3.1.2) to derive the Duhamel's formulas for $\widehat{V^{wa}}$ and $\widehat{V^{kg}}$. Observing that both equations in (3.1.2) are of hyperbolic and dispersive type with quadratic nonlinearities, we first write them in terms of the normalized variables U^{wa}, U^{kg} as quadratic dispersive equations of the form

$$\begin{aligned} (\partial_t + \mathrm{i}\Lambda_{wa})\, U^{wa} &= \mathcal{N}^{wa} = \mathcal{Q}^{wa}(v,v) := |\nabla_{t,x} v|^2 + v^2\,, \\ (\partial_t + \mathrm{i}\Lambda_{kg})\, U^{kg} &= \mathcal{N}^{kg} = \mathcal{Q}^{kg}(u,v) := u\Delta v\,, \end{aligned} \tag{3.2.11}$$

where the (quadratic) nonlinearities can be expressed as pseudo-product operators

$$\begin{aligned} \mathcal{N}^{wa} &= \mathcal{T}^{wa}_{a(\xi,\eta)}(v,v) = \sum_{\pm,\pm} \mathcal{T}^{wa}_{a_{\pm\pm}}(U^{kg,\pm}, U^{kg,\pm})\,, \\ \mathcal{N}^{kg} &= \mathcal{T}^{kg}_{b(\xi,\eta)}(u,v) = \sum_{\pm,\pm} \mathcal{T}^{kg}_{b_{\pm\pm}}(U^{kg,\pm}, U^{wa,\pm}) \end{aligned} \tag{3.2.12}$$

of the general form

$$\mathcal{T}_{m(\xi,\eta)}(f,g) := \mathscr{F}^{-1}\widehat{\mathcal{Q}(f,g)} = \mathscr{F}^{-1}\int m(\xi,\eta)\widehat{f}(\xi-\eta)\widehat{g}(\eta)\,\mathrm{d}\eta$$

with the symbol-type multiplier $m(\xi,\eta)$ carrying information of the quadratic interaction (e.g. dispersion/wave-type, derivatives).

Since the normalized solutions U^{wa}, U^{kg} display oscillations itself, and we want to isolate all the oscillations in a unique factor $\mathrm{e}^{\mathrm{i}t\Phi(\xi,\eta)}$, we then need to introduce the profiles V^{wa}, V^{kg} as in (3.2.7) so that (3.2.11) can be rewritten in terms of the new unknowns as

$$\partial_t V^{wa} = \mathrm{e}^{\mathrm{i}t\Lambda_{wa}} \mathcal{N}^{wa} := \mathrm{e}^{\mathrm{i}t\Lambda_{wa}} \sum_{\pm,\pm} \mathcal{T}^{wa}_{a_{\pm\pm}} \big(\mathrm{e}^{\mp\mathrm{i}t\Lambda_{kg}} V^{kg,\pm}, \mathrm{e}^{\mp\mathrm{i}t\Lambda_{kg}} V^{kg,\pm}\big)\,, \tag{3.2.13}$$

$$\partial_t V^{kg} = \mathrm{e}^{\mathrm{i}t\Lambda_{kg}} \mathcal{N}^{kg} := \mathrm{e}^{\mathrm{i}t\Lambda_{kg}} \sum_{\pm,\pm} \mathcal{T}^{kg}_{b_{\pm\pm}} \big(\mathrm{e}^{\mp\mathrm{i}t\Lambda_{kg}} V^{kg,\pm}, \mathrm{e}^{\mp\mathrm{i}t\Lambda_{wa}} V^{wa,\pm}\big)\,, \tag{3.2.14}$$

from which we see that V^{wa}, V^{kg} evolve purely nonlinearly (which is more stable).

Taking the Fourier transform (combined with its differentiation and product-convolution properties) and using the formulas (3.2.6) and identities (3.2.9), we obtain

$$\partial_t \widehat{V^{wa}}(t,\xi) = \mathrm{e}^{\mathrm{i}t\Lambda_{wa}(\xi)} \widehat{\mathcal{N}^{wa}} := \sum_{\iota_1,\iota_2\in\{+,-\}} \mathbf{I}^{\iota_1\iota_2}_{wa}\big[V^{kg,\iota_1}, V^{kg,\iota_2}\big](t,\xi)\,, \tag{3.2.15}$$

$$\partial_t \widehat{V^{kg}}(t,\xi) = \mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{\mathcal{N}^{kg}} := \sum_{\iota_1,\iota_2\in\{+,-\}} \mathbf{I}^{\iota_1\iota_2}_{kg}\big[V^{kg,\iota_1}, V^{wa,\iota_2}\big](t,\xi)\,, \tag{3.2.16}$$

where

$$\begin{aligned}
\mathbf{I}^{\iota_1\iota_2}_{wa}\big[F,G\big](t,\xi) &:= \frac{1}{4}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{\iota_1\iota_2}_{wa}(\xi,\eta)} a_{\iota_1\iota_2}(\xi,\eta) \widehat{F}(t,\xi-\eta)\widehat{G}(t,\eta)\,\mathrm{d}\eta\,, \\
\Phi^{\iota_1\iota_2}_{wa}(\xi,\eta) &:= \Lambda_{wa}(\xi) - \Lambda_{kg,\iota_1}(\xi-\eta) - \Lambda_{kg,\iota_2}(\eta) \\
&= |\xi| - \iota_1\langle\xi-\eta\rangle - \iota_2\langle\eta\rangle\,, \\
a_{\iota_1\iota_2}(\xi,\eta) &:= 1 + \frac{\iota_1\iota_2}{\Lambda_{kg}(\xi-\eta)\Lambda_{kg}(\eta)}\big[(\xi-\eta)\cdot\eta - 1\big] = 1 + \iota_1\iota_2\,\frac{(\xi-\eta)\cdot\eta - 1}{\langle\xi-\eta\rangle\langle\eta\rangle}\,,
\end{aligned} \tag{3.2.17}$$

and

$$
\begin{aligned}
\mathbf{I}^{\iota_1\iota_2}_{kg}\big[F,G\big](t,\xi) &:= \frac{1}{4}(2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{\iota_1\iota_2}_{kg}(\xi,\eta)} b_{\iota_1\iota_2}(\xi,\eta)\widehat{F}(t,\xi-\eta)\widehat{G}(t,\eta)\,\mathrm{d}\eta\,,\\
\Phi^{\iota_1\iota_2}_{kg}(\xi,\eta) &:= \Lambda_{kg}(\xi) - \Lambda_{kg,\iota_1}(\xi-\eta) - \Lambda_{wa,\iota_2}(\eta)\\
&= \langle\xi\rangle - \iota_1\langle\xi-\eta\rangle - \iota_2|\eta|\,,\\
b_{\iota_1\iota_2}(\xi,\eta) &:= \iota_1\iota_2\,\frac{|\xi-\eta|^2}{\Lambda_{kg}(\xi-\eta)\Lambda_{wa}(\eta)} = \iota_1\iota_2\,\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle|\eta|}\,.
\end{aligned}
\tag{3.2.18}
$$

In later proofs we will also use the notation $\widehat{\mathbf{I}}[\widehat{F},\widehat{G}]$ for the same expression as $\mathbf{I}[F,G]$, i.e.

$$
\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\big[\widehat{F},\widehat{G}\big](t,\xi) = \mathbf{I}^{\iota_1\iota_2}_{wa}\big[F,G\big](t,\xi),
$$

$$
\widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}}\big[\widehat{F},\widehat{G}\big](t,\xi) = \mathbf{I}^{\iota_1\iota_2}_{kg}\big[F,G\big](t,\xi).
$$

Our analysis is largely based on the nonlinear phases $\Phi^{\iota_1\iota_2}$, which measure the quadratic interactions between different wave-types.

3.2.2 Vector Fields

The major component of our analysis relies on the energy estimates for the system (3.1.2). These energy estimates involve vector fields, which correspond to the symmetries of the linearized equations.

We denote by $\partial_0 := \partial_t$ and $\partial_i := \partial_{x_i}$ for $i = 1, 2, 3$. Define the Lorentz vector fields Γ_j and the rotation vector fields Ω_{jk},

$$\Gamma_j := x_j\partial_t + t\partial_j, \quad \Omega_{jk} := x_j\partial_k - x_k\partial_j, \tag{3.2.19}$$

for $j,k = 1,2,3$. These vector fields commute with both the wave operator $-\Box$ and the Klein-Gordon operator $-\Box + 1$. For any multi-index $a = (a_1, a_2, a_3) \in \mathbb{N}^3$, $b = (b_1, b_2, b_3) \in \mathbb{N}^3$, with $|a| = \sum_{j=1}^3 a_j$, $|b| = \sum_{j=1}^3 b_j$, we define

$$\Gamma^a := \Gamma_1^{a_1}\Gamma_2^{a_2}\Gamma_3^{a_3}, \quad \Omega^b := \Omega_{23}^{b_1}\Omega_{31}^{b_2}\Omega_{12}^{b_3}. \tag{3.2.20}$$

For any $n \in \mathbb{N}$, define $\mathcal{V}_n$ as the set of differential operators of the form

$$\mathcal{V}_n := \left\{\mathcal{L} = \Gamma^a\Omega^b : |a| + |b| \le n\right\}. \tag{3.2.21}$$

Define $U^{wa}_{\mathcal{L}}$ and $U^{kg}_{\mathcal{L}}$ in a similar manner as U^{wa} and U^{kg}, with $\mathcal{L}[u]$ instead of u and $\mathcal{L}[v]$ instead of v. The same notation applies to all the other variables, like $V^{wa}_{\mathcal{L}}$ and $V^{kg}_{\mathcal{L}}$:

$$\begin{aligned} U^{wa}_{\mathcal{L}}(t) &:= (\partial_t - \mathrm{i}\Lambda_{wa})(\mathcal{L}[u])(t), & U^{kg}_{\mathcal{L}}(t) &:= (\partial_t - \mathrm{i}\Lambda_{kg})(\mathcal{L}[v])(t), \\ V^{wa}_{\mathcal{L}}(t) &:= \mathrm{e}^{\mathrm{i}t\Lambda_{wa}}U^{wa}_{\mathcal{L}}(t), & V^{kg}_{\mathcal{L}}(t) &:= \mathrm{e}^{\mathrm{i}t\Lambda_{kg}}U^{kg}_{\mathcal{L}}(t). \end{aligned} \tag{3.2.22}$$

Similarly, the functions $\mathcal{L}[u], \mathcal{L}[v]$ can be recovered from the normalized variables $U^{wa}_{\mathcal{L}}, U^{kg}_{\mathcal{L}}$ by the formulas

$$\begin{aligned} \partial_t(\mathcal{L}[u]) &= \tfrac{1}{2}\left(U^{wa}_{\mathcal{L}} + \overline{U^{wa}_{\mathcal{L}}}\right) = \mathrm{Re}\left(U^{wa}_{\mathcal{L}}\right), & \partial_t(\mathcal{L}[v]) &= \tfrac{1}{2}\big(U^{kg}_{\mathcal{L}} + \overline{U^{kg}_{\mathcal{L}}}\big) = \mathrm{Re}\big(U^{kg}_{\mathcal{L}}\big), \\ \Lambda_{wa}(\mathcal{L}[u]) &= \tfrac{\mathrm{i}}{2}\left(U^{wa}_{\mathcal{L}} - \overline{U^{wa}_{\mathcal{L}}}\right) = -\mathrm{Im}\left(U^{wa}_{\mathcal{L}}\right), & \Lambda_{kg}(\mathcal{L}[v]) &= \tfrac{\mathrm{i}}{2}\big(U^{kg}_{\mathcal{L}} - \overline{U^{kg}_{\mathcal{L}}}\big) = -\mathrm{Im}\big(U^{kg}_{\mathcal{L}}\big), \\ \mathcal{L}[u] &= \tfrac{\mathrm{i}}{2\Lambda_{wa}}\left(U^{wa}_{\mathcal{L}} - \overline{U^{wa}_{\mathcal{L}}}\right) = -\tfrac{1}{|\nabla|}\,\mathrm{Im}\left(U^{wa}_{\mathcal{L}}\right), & \mathcal{L}[v] &= \tfrac{\mathrm{i}}{2\Lambda_{kg}}\big(U^{kg}_{\mathcal{L}} - \overline{U^{kg}_{\mathcal{L}}}\big) = -\tfrac{1}{\langle\nabla\rangle}\,\mathrm{Im}\big(U^{kg}_{\mathcal{L}}\big). \end{aligned} \tag{3.2.23}$$

With the definitions above, the system (3.2.11) yields, for any $\mathcal{L} \in \mathcal{V}_n$,

$$\begin{aligned}(\partial_t + \mathrm{i}\Lambda_{wa})U^{wa}_{\mathcal{L}} &= \mathcal{N}^{wa}_{\mathcal{L}} := \mathcal{L}\big[|\nabla_{t,x}v|^2 + v^2\big],\\ (\partial_t + \mathrm{i}\Lambda_{kg})U^{kg}_{\mathcal{L}} &= \mathcal{N}^{kg}_{\mathcal{L}} := \mathcal{L}\big[u\Delta v\big].\end{aligned} \tag{3.2.24}$$

For any smooth function $f(x)$, we define $f_{\mathcal{L}}(x)$ when the vector field $\mathcal{L}$ is applied to f. Note that in our formulation, all the variables can be categorized into three levels: (u, v) level; (U^{wa}, U^{kg}) level; (V^{wa}, V^{kg}) level. f is typically a function at (V^{wa}, V^{kg}) level, but $\mathcal{L}$ can only be directly applied at (u, v) level, so $f_{\mathcal{L}} \neq \mathcal{L}[f]$. Instead, we have to go back and forth between the three levels. To be specific, for example,

$$V^{wa}_{\mathcal{L}} = \mathrm{e}^{\mathrm{i}t\Lambda_{wa}}\big(\partial_t - \mathrm{i}\Lambda_{wa}\big)\mathcal{L}\left[\frac{\mathrm{i}}{2\Lambda_{wa}}\Big(\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}V^{wa} - \mathrm{e}^{\mathrm{i}t\Lambda_{wa}}\overline{V^{wa}}\Big)\right], \tag{3.2.25}$$

$$V^{kg}_{\mathcal{L}} = \mathrm{e}^{\mathrm{i}t\Lambda_{kg}}\big(\partial_t - \mathrm{i}\Lambda_{kg}\big)\mathcal{L}\left[\frac{\mathrm{i}}{2\Lambda_{kg}}\Big(\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}V^{kg} - \mathrm{e}^{\mathrm{i}t\Lambda_{kg}}\overline{V^{kg}}\Big)\right]. \tag{3.2.26}$$

3.2.3 Resonant System

We now discuss the main intuition behind the derivation of the "resonant system" which drives the asymptotic behavior of our original system. As explained in Section 3.2.1, we will focus on the evolution of $\widehat{V^{wa}}$ and $\widehat{V^{kg}}$ by looking at the Duhamel's formulas (3.2.15) and (3.2.16).

Quadratic Phases

In (3.2.15) and (3.2.16), the behavior of the oscillatory integral is largely characterized by the phase functions $\Phi_{wa}^{\iota_1\iota_2}$ and $\Phi_{kg}^{\iota_1\iota_2}$. which have the general form

$$\begin{aligned} &\Phi_{\sigma\mu\nu} : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R}, \\ &\Phi_{\sigma\mu\nu}(\xi, \eta) := \Lambda_\sigma(\xi) - \Lambda_\mu(\xi - \eta) - \Lambda_\nu(\eta) \end{aligned} \tag{3.2.27}$$

for

$$\sigma, \mu, \nu \in \mathcal{P} := \{(wa, +), (wa, -), (kg, +), (kg, -)\}. \tag{3.2.28}$$

In fact, our resonant/stationary-phase analysis is based on essentially only one type of quadratic phase functions

$$\begin{aligned} \Phi(\xi_1, \xi_2) &= \Lambda_{kg}(\xi_1) \pm \Lambda_{kg}(\xi_2) \pm \Lambda_{wa}(\xi_1 + \xi_2) \\ &= \langle \xi_1 \rangle \pm \langle \xi_2 \rangle \pm |\xi_1 + \xi_2|, \end{aligned} \tag{3.2.29}$$

which has the lower bound

$$\left|\Phi(\xi_1, \xi_2)\right| \gtrsim \frac{|\xi_1 + \xi_2|}{(1 + |\xi_1| + |\xi_2|)^2}. \tag{3.2.30}$$

Based on (3.2.17) and (3.2.18), if $|\xi|, |\xi - \eta|, |\eta| \in [0, b]$ for $b \geq 1$, then

$$\left|\Phi_{wa}^{\iota_1\iota_2}(\xi, \eta)\right| \geq \frac{|\xi|}{4b^2}, \quad \left|\Phi_{kg}^{\iota_1\iota_2}(\xi, \eta)\right| \geq \frac{|\eta|}{4b^2}. \tag{3.2.31}$$

Therefore, we expect that the interactions where the wave component has very small frequencies, in particular when $t|\xi| \lesssim 1$ or $t|\eta| \lesssim 1$, will play an important role in the analysis.

Analysis of the Resonances

Recall that (3.2.15) and (3.2.16) yield Duhamel's formulas of the form

$$\widehat{V}(t,\xi) \;=\; \widehat{V_\infty}(\xi) - \int_t^\infty \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}s\Phi(\xi,\eta)} m(\xi,\eta) \widehat{V}(s,\xi-\eta) \widehat{V}(s,\eta)\, \mathrm{d}\eta \mathrm{d}s\,. \tag{3.2.32}$$

In order to prove long-range scattering, it is desirable to control the above oscillatory integral uniformly in time. From the viewpoint of "stationary phase", we may want to take advantage of the oscillating factor $\mathrm{e}^{\mathrm{i}s\Phi(\xi,\eta)}$. This can create rapid decay as time becomes large (and thus yields smallness), except on the sets where the phase is stationary in either of the integration variables:

- Time resonant set $\mathcal{T} := \big\{(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \Phi(\xi,\eta) = 0\big\}$ – stationary over time s.
- Space resonant set $\mathcal{S} := \big\{(\xi,\eta) \in \mathbb{R}^3 \times \mathbb{R}^3 : \nabla_\eta \Phi(\xi,\eta) = 0\big\}$ – stationary in η.
- Space-time resonant set $\mathcal{R} := \mathcal{T} \cap \mathcal{S}$ – stationary in both s and η.

Therefore the main contribution of (3.2.32) comes from those points where both Φ and $\nabla_\eta \Phi$ vanish.

By (3.2.31), (3.2.17) and (3.2.18), we can verify that for $\Phi_{wa}^{\iota_1\iota_2}$:

$$\begin{cases} \mathcal{R} = \mathcal{T} = \varnothing \subset \mathcal{S} = \{\xi = 2\eta\} & \text{for } \ \iota_1\iota_2 = ++, --\,, \\ \mathcal{R} = \mathcal{T} = \mathcal{S} = \{\xi = 0\} & \text{for } \ \iota_1\iota_2 = +-, -+\,, \end{cases} \tag{3.2.33}$$

and for $\Phi_{kg}^{\iota_1\iota_2}$:

$$\begin{cases} \mathcal{R} = \mathcal{T} = \varnothing \subset \mathcal{S} = \{\eta = 0\} & \text{for } \iota_1\iota_2 = -+, --\,, \\ \mathcal{R} = \mathcal{T} = \mathcal{S} = \{\eta = 0\} & \text{for } \iota_1\iota_2 = ++, +-\ . \end{cases} \tag{3.2.34}$$

We call it the "non-resonant" case if $\mathcal{R} = \mathcal{T} = \varnothing$, for which we can handle using the normal form method (integration by parts in time). The rest ($\iota_1\iota_2 = +-, -+$ for Wave and $\iota_1\iota_2 = ++, +-$ for KG) are called the "resonant" case, which is the part that contributes to the "resonant system".

Extraction of the "Resonant System"

Our general approach of obtaining the principal part of $\widehat{V}(t,\xi)$ will be to first restrict the integral in the Duhamel's formula near the resonant set and then analyzing the resulting contribution: the "resonant system" emerges from the reduction of (3.2.15) and (3.2.16) to resonant interactions. Below we give a schematic description of our main steps.

Step 1: Low-Frequency Outputs for Wave Component. As discussed in the previous section, for the wave equation, we will focus on the resonant case $\iota_1\iota_2 = +-, -+$ (when the two inputs have the opposite signs) and look at the contribution of low-frequency outputs:

$$|\xi| \lesssim \langle t\rangle^{-1+}. \tag{3.2.35}$$

To be precise, we may take a cutoff function (see the rigorous definition of $\varphi_{\leq 0}$ in Section 3.2.5)

$$\varphi_{\leq 0}\big(\xi \langle t\rangle^{\frac{7}{8}}\big). \tag{3.2.36}$$

The cutoff is made time-dependent (shrinking as $t \to \infty$) in order to trade size of support for time decay. Here the exponent $\frac{7}{8}$ is a convenient number slightly smaller than 1 (which is chosen from later proofs).

We start from (3.2.15) with $(\iota_1, \iota_2) = (+,-)$ or $(-,+)$ when $|\xi|$ is small. By Taylor expansion with respect to ξ up to the first order (here $\iota_2 = -\iota_1$),

$$\begin{aligned} \Phi^{\iota_1\iota_2}_{wa}(\xi,\eta) &= |\xi| - \iota_1\langle \xi-\eta\rangle - \iota_2\langle\eta\rangle = |\xi| - \iota_1\big\{\langle\xi-\eta\rangle - \langle\eta\rangle\big\} \\ &\rightsquigarrow |\xi| + \iota_1 \frac{\xi\cdot\eta}{\langle\eta\rangle} =: \Phi^{\iota_1\iota_2,0}_{wa}(\xi,\eta). \end{aligned} \tag{3.2.37}$$

On the other hand, taking $\xi = 0$, we get

$$a_{\iota_1\iota_2}(\xi,\eta) \rightsquigarrow 2, \tag{3.2.38}$$

$$\widehat{F}(t,\xi-\eta) \rightsquigarrow \widehat{F}(t,-\eta). \tag{3.2.39}$$

Substitute all the elements above into (3.2.17) and let

$$\mathbf{I}^{+-,0}_{wa}(t,\xi) := \frac{1}{2}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{+-,0}_{wa}(\xi,\eta)} \widehat{V^{kg,+}}(t,-\eta)\widehat{V^{kg,-}}(t,\eta)\,\mathrm{d}\eta, \tag{3.2.40}$$

$$\mathbf{I}^{-+,0}_{wa}(t,\xi) := \frac{1}{2}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{-+,0}_{wa}(\xi,\eta)} \widehat{V^{kg,-}}(t,-\eta)\widehat{V^{kg,+}}(t,\eta)\,\mathrm{d}\eta. \tag{3.2.41}$$

By change of variable $\eta \mapsto -\eta$ in $\mathbf{I}^{+-,0}_{wa}$ and noticing that $\widehat{V^{kg,-}}(t,-\eta) = \overline{\widehat{V^{kg,+}}(t,\eta)}$, we see that

$$\mathbf{I}_{wa}^{+-,0}(t,\xi) = \frac{1}{2}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \widehat{V^{kg,+}}(t,\eta)\widehat{V^{kg,-}}(t,-\eta)\,\mathrm{d}\eta \tag{3.2.42}$$

$$= \frac{1}{2}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \left|\widehat{V^{kg}}(t,\eta)\right|^2 \mathrm{d}\eta,$$

$$\mathbf{I}_{wa}^{-+,0}(t,\xi) = \frac{1}{2}(2\pi)^{-\frac{3}{2}} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \left|\widehat{V^{kg}}(t,\eta)\right|^2 \mathrm{d}\eta. \tag{3.2.43}$$

Finally, tagging the low-frequency cutoff, we arrive at

$$h(t,\xi) := \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\left\{\mathbf{I}_{wa}^{+-,0}(t,\xi) + \mathbf{I}_{wa}^{-+,0}(t,\xi)\right\} \tag{3.2.44}$$

$$= (2\pi)^{-\frac{3}{2}}\varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \left|\widehat{V^{kg}}(t,\eta)\right|^2 \mathrm{d}\eta.$$

We further define

$$H(t,\xi) := \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\left\{\widehat{V^{wa}}(0,\xi) + \int_0^t h(s,\xi)\,\mathrm{d}s\right\} \tag{3.2.45}$$

$$= \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\left\{\widehat{V^{wa}}(0,\xi) + (2\pi)^{-\frac{3}{2}}\int_0^t \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}s\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \left|\widehat{V^{kg}}(s,\eta)\right|^2 \mathrm{d}\eta\mathrm{d}s\right\}.$$

Remark 3.2.2. We expect that, roughly speaking, $\widehat{V^{wa}}(t,\xi) \approx H(t,\xi)$ and

$$\partial_t\widehat{V^{wa}}(t,\xi) = h(t,\xi) + R^{wa}(t,\xi). \tag{3.2.46}$$

Here R^{wa} should be a remainder which has sufficiently fast time decay in some suitable norm (see (3.2.98) and (3.2.99)) to guarantee convergence to the resonant system. In other words, the low-frequency outputs of wave component contribute a bulk term h that decays at a critical rate $\langle t\rangle^{-1}$ (see Lemma 3.4.15).

Step 2: High-Low Interactions and "Phase Shift" for KG Component. The next step is to measure the feedback contribution of the low-frequency bulk $h(t,\xi)$ to the nonlinear interactions of the Klein-Gordon equation.

This time for the Klein-Gordon equation, we focus on the resonant case $\iota_1\iota_2 = ++, +-$ (when one of the inputs is $V^{kg,+}$) and look at the high-low interactions:

$$|\eta| \lesssim \langle t\rangle^{-1+} (\ll |\xi|). \tag{3.2.47}$$

We start from (3.2.16) with $(\iota_1,\iota_2) = (+,+)$ or $(+,-)$ when $|\eta|$ is small. By Taylor expansion with respect to η up to the first order (here $\iota_1 = +$),

$$\begin{aligned}\Phi^{\iota_1\iota_2}_{kg}(\xi,\eta) &= \langle\xi\rangle - \langle\xi-\eta\rangle - \iota_2\,|\eta| = \big\{\langle\xi\rangle - \langle\xi-\eta\rangle\big\} - \iota_2\,|\eta| \\ &\rightsquigarrow \frac{\xi\cdot\eta}{\langle\xi\rangle} - \iota_2\,|\eta| =: \Phi^{+\iota_2,0}_{kg}(\xi,\eta).\end{aligned} \tag{3.2.48}$$

On the other hand, taking $\eta = 0$, we get

$$b_{+\iota_2}(\xi,\eta) \rightsquigarrow \iota_2\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}, \tag{3.2.49}$$

$$\widehat{V^{kg,+}}(t,\xi-\eta) \rightsquigarrow \widehat{V^{kg}}(t,\xi). \tag{3.2.50}$$

Moreover, we formally replace the low-frequency part of $\widehat{V^{wa,\pm}}(t,\eta)$ by the contribution coming from $H^{\pm}(t,\eta)$, where $H^+(t,\eta) = H(t,\eta) = \overline{H^-(t,-\eta)}$ is given by (3.2.45). Note that $H(t,\eta)$ contains a cutoff $\varphi_{\le 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)$. Let

$$\begin{aligned}\mathbf{I}^{+\iota_2,1}_{kg,0}(t,\xi) &:= \frac{1}{4}(2\pi)^{-\frac{3}{2}}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{+\iota_2,0}_{kg}(\xi,\eta)}\iota_2\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\widehat{V^{kg}}(t,\xi)H^{\iota_2}(t,\eta)\,\mathrm{d}\eta, \\ &= \iota_2\frac{1}{4}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\widehat{V^{kg}}(t,\xi)\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\iota_2|\eta|\right)}\frac{1}{|\eta|}H^{\iota_2}(t,\eta)\,\mathrm{d}\eta.\end{aligned} \tag{3.2.51}$$

By change of variable $\eta \mapsto -\eta$ in $\mathbf{I}^{+-,1}_{kg,0}$, we have

$$\begin{aligned}\mathbf{I}^{+-,1}_{kg,0}(t,\xi) &= -\frac{1}{4}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\widehat{V^{kg}}(t,\xi)\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}+|\eta|\right)}\frac{1}{|\eta|}H^-(t,\eta)\,\mathrm{d}\eta \\ &= -\frac{1}{4}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\widehat{V^{kg}}(t,\xi)\int_{\mathbb{R}^3}\mathrm{e}^{-\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\overline{H(t,\eta)}\,\mathrm{d}\eta.\end{aligned} \tag{3.2.52}$$

Hence,

$$\mathbf{I}^{++,1}_{kg,0}(t,\xi)+\mathbf{I}^{+-,1}_{kg,0}(t,\xi) = \frac{\mathrm{i}}{2}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\widehat{V^{kg}}(t,\xi)\cdot\mathrm{Im}\left\{\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}H(t,\eta)\,\mathrm{d}\eta\right\} \tag{3.2.53}$$

$$=: \mathrm{i}\,\mathcal{C}(t,\xi)\cdot\widehat{V^{kg}}(t,\xi),$$

where we let

$$\mathcal{C}(t,\xi) := \frac{1}{2}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\cdot\mathrm{Im}\left\{\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}H(t,\eta)\,\mathrm{d}\eta\right\}. \tag{3.2.54}$$

Remark 3.2.3. We will see that (3.2.16) can be written as

$$\partial_t\widehat{V^{kg}}(t,\xi) = \mathrm{i}\,\mathcal{C}(t,\xi)\cdot\widehat{V^{kg}}(t,\xi)+R^{kg}(t,\xi), \tag{3.2.55}$$

where R^{kg} is a remainder which decays sufficiently fast in time in some suitable norm (see (3.2.100) and (3.2.101)). In particular, $\mathcal{C}(t,\xi)$ is not integrable in time (see Lemma 3.4.19). This shows that the KG component is essentially transported by the low frequencies of wave bulk. Eventually, we will renormalize V^{kg} to incorporate this modification via a phase correction/shift using the fact that $\mathcal{C}(t,\xi)\in\mathbb{R}$. This is a similar phenomenon to the one occurring in the scattering-critical equations, such as the critical nonlinear Schrödinger equations and 2D water waves (see [39, 51, 38, 48, 49]).

Step 3: The Resonant System and Long-Time Asymptotics. Collecting all the contributions from previous steps (see (3.2.46) and (3.2.55)), we extract the "resonant system" as

$$\partial_t \widehat{V^{wa}}(t,\xi) \approx h(t,\xi), \tag{3.2.56}$$

$$\partial_t \widehat{V^{kg}}(t,\xi) \approx \mathrm{i}\,\mathcal{C}(t,\xi)\cdot \widehat{V^{kg}}(t,\xi), \tag{3.2.57}$$

and further deduce that

$$\widehat{V^{wa}}(t,\xi) \approx \widehat{V^{wa}}(0,\xi) + \int_0^t h(s,\xi)\,\mathrm{d}s, \tag{3.2.58}$$

$$\widehat{V^{kg}}(t,\xi) \approx \mathrm{e}^{\mathrm{i}\mathcal{D}(t,\xi)}\widehat{V^{kg}}(0,\xi), \tag{3.2.59}$$

where the phase correction

$$\mathcal{D}(t,\xi) := \int_0^t \mathcal{C}(s,\xi)\,\mathrm{d}s \tag{3.2.60}$$

is a real-valued nonlinear function of $\widehat{V^{kg}}$ which takes into account the low-frequency part of wave component. In particular, it grows mildly to infinity in time (see Lemma 3.4.20), and therefore will lead to genuine nonlinear scattering for the KG component.

The asymptotic dynamics and modified scattering for the W-KG system can be explained heuristically as follows:

(1) The wave and Klein-Gordon components are generated linearly by initial data;

(2) The KG component interacts with itself in the wave equation to produce a large low-frequency wave component $|u(t,x)| \approx t^{-1}\varepsilon_0^2$ (if $t \gg 1$ and $|x| \leq ct$) supported at low frequency $|\xi| \lesssim 1/t$;

(3) In the Klein-Gordon equation, this large low-frequency wave component interacts again with the high frequencies of the Klein-Gordon component to produce a "phase shift".

The "Asymptotic Profile"

Now that we have identified the asymptotic system (the so-called "resonant system") that governs long-time behavior of the W-KG system, our next step is to construct from there an "asymptotic profile" (which still depends on t) as the limit of a nonlinear profile V as $t \to \infty$.

Let the $(V_\infty^{wa}, V_\infty^{kg})$ be the initial data, i.e.

$$\widehat{V^{wa}}(0,\xi) = \widehat{V_\infty^{wa}}(\xi), \quad \widehat{V^{kg}}(0,\xi) = \widehat{V_\infty^{kg}}(\xi). \tag{3.2.61}$$

Motivated by (3.2.58)(3.2.59), we define the "asymptotic profile" $(\mathscr{V}^{wa}, \mathscr{V}^{kg})$ (which can be understood as an asymptotic approximation of (V^{wa}, V^{kg})) as follows:

$$\widehat{\mathscr{V}^{wa}}(t,\xi) := \widehat{V_\infty^{wa}}(\xi) + \mathcal{H}_\infty(t,\xi), \tag{3.2.62}$$

$$\widehat{\mathscr{V}^{kg}}(t,\xi) := \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V_\infty^{kg}}(\xi), \tag{3.2.63}$$

where

$$h_\infty(t,\xi) := (2\pi)^{-\frac{3}{2}}\varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big) \cdot \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\left|\widehat{V_\infty^{kg}}(\eta)\right|^2 \mathrm{d}\eta\,, \tag{3.2.64}$$

$$\mathcal{H}_\infty(t,\xi) := \int_0^t h_\infty(s,\xi)\,\mathrm{d}s\,, \tag{3.2.65}$$

$$H_\infty(t,\xi) := \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\left\{\widehat{V_\infty^{wa}}(\xi) + \mathcal{H}_\infty(t,\xi)\right\}, \tag{3.2.66}$$

and

$$\mathcal{C}_\infty(t,\xi) := \frac{1}{2}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\cdot \operatorname{Im}\left\{\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}H_\infty(t,\eta)\,\mathrm{d}\eta\right\}, \tag{3.2.67}$$

$$\mathcal{D}_\infty(t,\xi) := \int_0^t \mathcal{C}_\infty(s,\xi)\,\mathrm{d}s\,. \tag{3.2.68}$$

Remark 3.2.4. For all functions above, we also define its positive and negative counterpart in the same way as $\widehat{V^\pm}$: for a complex-valued function $\widehat{S}(t,\xi)$, let $\widehat{S^+}(t,\xi) := \widehat{S}(t,\xi)$ and $\widehat{S^-}(t,\xi) := \overline{\widehat{S}(t,-\xi)}$.

3.2.4 Equations of Perturbation

In this subsection, we will take the difference between the profile (V^{wa}, V^{kg}) of solutions and the asymptotic profile $(\mathscr{V}^{wa}, \mathscr{V}^{kg})$ as our perturbation unknown, and derive the equations for this new unknown. We represent the perturbation $G^{wa}(t,x)$, $G^{kg}(t,x)$ by

$$\widehat{G^{wa}}(t,\xi) := \widehat{V^{wa}}(t,\xi) - \left(\widehat{V^{wa}_\infty}(\xi) + \mathcal{H}_\infty(t,\xi)\right), \tag{3.2.69}$$

$$\widehat{G^{kg}}(t,\xi) := \widehat{V^{kg}}(t,\xi) - \left(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi) + \mathfrak{B}_\infty(t,\xi)\right). \tag{3.2.70}$$

Here $\mathfrak{B}_\infty(t,\xi)$ comes from contributions of the non-resonant part and is given by

$$\mathfrak{B}_\infty(t,\xi) := -\int_t^\infty \mathrm{e}^{\mathrm{i}(\mathcal{D}_\infty(t,\xi)-\mathcal{D}_\infty(s,\xi))}\mathfrak{b}_\infty(s,\xi)\,\mathrm{d}s, \tag{3.2.71}$$

$$\mathfrak{b}_\infty(t,\xi) := \sum_{\iota_2\in\{+,-\}} \widehat{\mathbf{I}^{-\iota_2}_{kg}}\left[\mathrm{e}^{-\mathrm{i}\mathcal{D}^-_\infty(t,\xi)}\widehat{V^{kg,-}_\infty}, H^{\iota_2}_\infty\right]. \tag{3.2.72}$$

Remark 3.2.5. From (3.2.69) and (3.2.70) we see that $\widehat{G^{wa}} = \widehat{V^{wa}} - \widehat{\mathscr{V}^{wa}}$ and $\widehat{G^{kg}} \approx$

$\widehat{V^{kg}} - \widehat{\mathscr{V}^{kg}}$; the latter is slightly different compared with the expression (3.2.63) of $\mathscr{V}^{kg}$. In particular, it contains a non-resonant contribution $\mathfrak{B}_\infty$. This is due to technical difficulties in nonlinear estimates (see Section 3.6). Roughly speaking, though being non-resonant, $\mathfrak{b}_\infty$ decays like $\langle t\rangle^{-1}$ and will be captured by the asymptotic analysis. However, it will not affect the final result since the time integral $\mathfrak{B}_\infty$ decays like $\langle t\rangle^{-\frac{7}{8}}$ (see Lemma 3.4.25). In other words, the the estimate of non-resonant part will improve upon taking time integration.

Our goal is to show that given any resonant system starting from $(V_\infty^{wa}, V_\infty^{kg})$, there exists a unique solution (V^{wa}, V^{kg}) to the W-KG system such that their difference goes to zero under some proper topology (which we will specify in Section 3.2.5) as $t \to \infty$, i.e.

$$\lim_{t\to\infty} \widehat{G^{wa}}(t,\xi) = 0, \quad \lim_{t\to\infty} \widehat{G^{kg}}(t,\xi) = 0. \tag{3.2.73}$$

Inserting (3.2.69)(3.2.70) into (3.2.15)(3.2.16), we obtain the system for (G^{wa}, G^{kg})

$$\begin{aligned}\partial_t \widehat{G^{wa}} = & -h_\infty \\ & + \sum_{\iota_1,\iota_2\in\{+,-\}} \mathbf{I}_{wa}^{\iota_1\iota_2}\Big[\widehat{G^{kg,\iota_1}} + e^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}} + \mathfrak{B}_\infty^{\iota_1},\ \widehat{G^{kg,\iota_2}} + e^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}}\widehat{V_\infty^{kg,\iota_2}} + \mathfrak{B}_\infty^{\iota_2}\Big],\end{aligned} \tag{3.2.74}$$

and

$$\begin{aligned}\partial_t \widehat{G^{kg}} = & -\mathrm{i}\,\mathcal{C}_\infty\Big(e^{\mathrm{i}\mathcal{D}_\infty}\widehat{V_\infty^{kg}}\Big) - \mathrm{i}\,\mathcal{C}_\infty\mathfrak{B}_\infty - \mathfrak{b}_\infty \\ & + \sum_{\iota_1,\iota_2\in\{+,-\}} \mathbf{I}_{kg}^{\iota_1\iota_2}\Big[\widehat{G^{kg,\iota_1}} + e^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}} + \mathfrak{B}_\infty^{\iota_1},\ \widehat{G^{wa,\iota_2}} + \widehat{V_\infty^{wa,\iota_2}} + \mathcal{H}_\infty^{\iota_2}\Big],\end{aligned} \tag{3.2.75}$$

with final data

$$\widehat{G^{wa}}(\infty,\xi) = 0\,, \quad \widehat{G^{kg}}(\infty,\xi) = 0\,. \tag{3.2.76}$$

It suffices to consider the well-posedness of (G^{wa}, G^{kg}) as solutions to (3.2.74)(3.2.75) (3.2.76).

3.2.5 Definition and Notation

Throughout the chapter, C will generally denote a universal constant that may vary from line to line. The notation $A \lesssim B$ means that $A \leq CB$ for some universal constant $C > 0$; we will use $\gtrsim$ and $\simeq$ in a similar standard way.

Littlewood-Paley Projections

In order to make the following analysis more precise, we need a good way of localization in Fourier space and physical space, which then allows us to quantify the variables in terms of time.

We first use a standard (inhomogeneous) dyadic decomposition of the indicator function $\mathbf{1}^t_{[0,+\infty)}$ to localize in t. Fix a smooth cutoff function $\tau : \mathbb{R}_+ \to [0,1]$ supported in $[0,2]$ and equal to 1 in $[0,1]$, and define for any $m \in \mathbb{N}$,

$$\begin{aligned} \tau_m(t) &:= \tau(t/2^m) - \tau(t/2^{m-1})\,, \quad m \geq 1\,; \\ \tau_0(t) &:= 1 - \sum_{m=1}^{+\infty} \tau_m(t) = \tau(t)\,, \end{aligned} \tag{3.2.77}$$

so that the sequence of functions $\{\tau_m(t)\}_{m\in\mathbb{N}}$ has the properties

$$\operatorname{supp}\tau_0 \subseteq [0,2]\,, \quad \operatorname{supp}\tau_m \subseteq [2^{m-1}, 2^{m+1}] \ \ (m \geq 1)\,, \quad \sum_{m=0}^{+\infty} \tau_m(t) \equiv 1\,. \tag{3.2.78}$$

In other words, in the support of τ_m we have $t \approx 2^m$. Also, we choose the function τ in such a way that $|\tau_m'(t)| \lesssim 2^{-m}$.

Also, we introduce the (homogeneous) dyadic decomposition[2] in three dimensions. Fix a smooth radial cutoff function $\varphi : \mathbb{R}^3 \to [0,1]$ that equals 1 for $|z| \leq 1$ and vanishes for $|z| \geq 2$. For any $k \in \mathbb{Z}$, denote by $k^+ := \max(k,0)$ and $k^- := \min(k,0)$. Let

$$\varphi_k(z) := \varphi(z/2^k) - \varphi(z/2^{k-1})\,, \quad k \in \mathbb{Z} \tag{3.2.79}$$

be a sequence of functions with the properties

$$\operatorname{supp}\varphi_k \subseteq \left\{z \in \mathbb{R}^3 : |z| \in \left[2^{k-1}, 2^{k+1}\right]\right\}\,, \quad \sum_{k\in\mathbb{Z}} \varphi_k(z) \equiv 1 \quad (z \in \mathbb{R}^3\backslash\{0\})\,, \tag{3.2.80}$$

so that in the support of φ_k we have $|z| \approx 2^k$. We define also

$$\varphi_I(z) := \sum_{k\in I\cap\mathbb{Z}} \varphi_k(z) \quad \text{for any } I \subset \mathbb{R}\,,$$

$$\varphi_{\leq A} := \varphi_{(-\infty,A]}\,, \quad \varphi_{\geq A} := \varphi_{[A,+\infty)}\,, \quad \varphi_{<A} := \varphi_{(-\infty,A)}\,, \quad \varphi_{>A} := \varphi_{(A,+\infty)} \quad \text{for any } A \in \mathbb{R}\,.$$

Let

$$\mathcal{J} := \left\{(k,j) \in \mathbb{Z}\times\mathbb{Z}_+ :\ k+j \geq 0 \ \text{ i.e. } \ j \geq -k^-\right\}.$$

[2] The homogeneous dyadic decomposition differs from the inhomogeneous one in that it decomposes also near the origin.

Given any $k \in \mathbb{Z}$, for any $(k, j) \in \mathcal{J}$, we further define

$$\varphi_j^{(k)} := \begin{cases} \varphi_j\,, & j > -k^-\,; \\ \varphi_{\le j}\,, & j = -k^-\,, \end{cases} \tag{3.2.81}$$

and notice that, for any fixed $k \in \mathbb{Z}$,

$$\sum_{j \ge -k^-} \varphi_j^{(k)}(z) \equiv 1 \quad (z \in \mathbb{R}^3 \backslash \{0\})\,.$$

Definition 3.2.1 (Littlewood-Paley Projections)**.** (1) Let P_k $(k \in \mathbb{Z})$ denote the standard Littlewood-Paley projection on $\mathbb{R}^3$ defined by the Fourier multiplier $\xi \mapsto \varphi_k(\xi)$:

$$P_k := \mathscr{F}^{-1} \varphi_k(\xi)\, \mathscr{F}\,, \qquad \widehat{P_k f}(\xi) := \varphi_k(\xi) \widehat{f}(\xi)\,. \tag{3.2.82}$$

Similarly, for any $A \in \mathbb{R}$, define the operator $P_{\le A}$ (resp. $P_{>A}$) on $\mathbb{R}^3$ by the Fourier multiplier $\xi \mapsto \varphi_{\le A}(\xi)$ (resp. $\xi \mapsto \varphi_{>A}(\xi)$).

(2) For $(k, j) \in \mathcal{J}$, let the operator Q_{jk} be defined as

$$Q_{jk} f(x) := \varphi_j^{(k)} \cdot P_k f(x)\,. \tag{3.2.83}$$

(3) For $(k, j) \in \mathcal{J}$, let the operator $\mathscr{Q}_{jk}$ be defined as

$$\mathscr{Q}_{jk} f(x) := P_{[k-2,\, k+2]} \left\{ \varphi_j^{(k)} \cdot P_k f \right\}(x)\,. \tag{3.2.84}$$

(4) Furthermore, let

$$Q_{\leq jk} f = \sum_{-k^- \leq j' \leq j} Q_{j'k} f, \tag{3.2.85}$$

$$\mathscr{Q}_{\leq jk} f = \sum_{-k^- \leq j' \leq j} \mathscr{Q}_{j'k} f. \tag{3.2.86}$$

Remark 3.2.6. (i) In our case we have $\Lambda(\xi) = \omega(|\xi|)$, so we will focus on quantifying the "energy" $|\xi|$. The direction is also important, but can usually be recovered by commuting with the rotation vector fields.

(ii) In view of the Heisenberg's uncertainty principle[3], the operators Q_{jk} are relevant only when $2^j 2^k \gtrsim 1$ i.e. $j + k \gtrsim 0$, which explains the definitions above.

(iii) The Littlewood–Paley operators give a decomposition of the identity (at least formally):

$$\begin{aligned} \sum_{k\in\mathbb{Z}} P_k &= \text{id}\,, & \sum_{k\in\mathbb{Z}} P_k f &= f\,; \\ \sum_{j\geq -k^-} Q_{jk} &= P_k\,, & \sum_{(k,j)\in\mathcal{J}} Q_{jk} f &= \sum_{k\in\mathbb{Z}} P_k f = f\,; \\ \sum_{j\geq -k^-} \mathscr{Q}_{jk} &= P_k\,, & \sum_{(k,j)\in\mathcal{J}} \mathscr{Q}_{jk} f &= \sum_{k\in\mathbb{Z}} P_k f = f\,. \end{aligned} \tag{3.2.87}$$

[3] It states that the more precisely the position (in physical space) of some particle is determined, the less precisely its momentum (frequency) can be known, and vice versa.

Norms and Function Spaces

Set the parameters

$$N_0 := 40, \quad N_1 := 3, \quad d := 10, \quad \delta = 10^{-10}$$
$$N(0) := N_0, \quad N(n) := N_0 - dn,$$
$$H(0) := 800, \quad H(n) := 800 - 200n,$$

and

$$H''_{wa}(n) = H''_{kg}(n) = H(n+1),$$
$$N''_{wa}(n) = N''_{kg}(n) = N(n) - 5.$$

Time-Independent Norms. We define the norms characterizing time-independent data (designed for $(V_\infty^{wa}, V_\infty^{kg})$). Assume $f(x)$ is a function of certain regularity defined for $x \in \mathbb{R}^3$. Define $f_{\mathcal{L}}$ as in (3.2.25) and (3.2.26).

Define the "data spaces"

$$Y_i := \{ f : \|f\|_{Y_i} < \infty \} \tag{3.2.88}$$

for $i = 1, 2$, with the norms

$$\|f\|_{Y_1} := \sup_{n \le N_1} \sup_{\mathcal{L} \in \mathcal{V}_n} \left\| |\nabla|^{-\frac{1}{2}} f_{\mathcal{L}} \right\|_{H^{N(n-3)}} + \sup_{n \le N_1 - 1} \sup_{\mathcal{L} \in \mathcal{V}_n, \ell \in \{1,2,3\}} \sup_{k \in \mathbb{Z}} \left(2^{N(n-2)k^+} 2^{\frac{k}{2}} \left\| \varphi_k \partial_{\xi_\ell} \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right), \tag{3.2.89}$$

$$\|f\|_{Y_2} := \sup_{n \le N_1} \sup_{\mathcal{L} \in \mathcal{V}_n} \|f_{\mathcal{L}}\|_{H^{N(n-3)}} + \sup_{n \le N_1 - 1} \sup_{\mathcal{L} \in \mathcal{V}_n, \ell \in \{1,2,3\}} \sup_{k \in \mathbb{Z}} \left(2^{N(n-2)k^+} 2^{k^+} \left\| \varphi_k \partial_{\xi_\ell} \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right). \tag{3.2.90}$$

Time-Dependent Norms. Next we define the time-dependent norms. Assume $f(t,x)$ is a function of certain regularity defined for $t \in [1, \infty)$ and $x \in \mathbb{R}^3$.

Define the "working spaces" (to run the fixed-point argument on G^{wa} and G^{kg})

$$X_i := \left\{ f : \|f\|_{X_i} < \infty \right\} \tag{3.2.91}$$

for $i = 1, 2$, with the norms

$$\|f\|_{X_1} := \|f\|_{S_1} + \|f\|_{T_1} \quad (\text{for } G^{wa}), \tag{3.2.92}$$

$$\|f\|_{X_2} := \|f\|_{S_2} + \|f\|_{T_2} \quad (\text{for } G^{kg}), \tag{3.2.93}$$

where

$$\|f\|_{S_1} := \sup_{t \in [1,\infty)} \sup_{n \le N_1} \sup_{\mathcal{L} \in \mathcal{V}_n} \left(\langle t \rangle^{H(n)\delta} \left\| |\nabla|^{-\frac{1}{2}} f_{\mathcal{L}} \right\|_{H^{N(n)}} \right), \tag{3.2.94}$$

$$\|f\|_{T_1} := \sup_{t \in [1,\infty)} \sup_{n \le N_1 - 1} \sup_{\mathcal{L} \in \mathcal{V}_n, \ell \in \{1,2,3\}} \sup_{k \in \mathbb{Z}} \left(\langle t \rangle^{H(n+1)\delta} 2^{N(n+1)k^+} 2^{\frac{k}{2}} \left\| \varphi_k \partial_{\xi_\ell} \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right), \tag{3.2.95}$$

and

$$\|f\|_{S_2} := \sup_{t\in[1,\infty)} \sup_{n\le N_1} \sup_{\mathcal{L}\in\mathcal{V}_n} \left(\langle t\rangle^{H(n)\delta} \|f_{\mathcal{L}}\|_{H^{N(n)}} \right), \tag{3.2.96}$$

$$\|f\|_{T_2} := \sup_{t\in[1,\infty)} \sup_{n\le N_1-1} \sup_{\mathcal{L}\in\mathcal{V}_n,\ell\in\{1,2,3\}} \sup_{k\in\mathbb{Z}} \left(\langle t\rangle^{H(n+1)\delta} 2^{N(n+1)k^+} 2^{k^+} \left\| \varphi_k \partial_{\xi_\ell} \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right). \tag{3.2.97}$$

We also need to introduce the auxiliary norms measuring corresponding variables with time derivatives (for $\partial_t G^{wa}$ and $\partial_t G^{kg}$):

$$\|f\|_{S_1'} := \sup_{t\in[1,\infty)} \sup_{n\le N_1} \sup_{\mathcal{L}\in\mathcal{V}_n} \left(\langle t\rangle^{1+H''_{wa}(n)\delta} 2^{N''_{wa}(n)k^+} 2^{-\frac{1}{2}k^-} \left\| \varphi_k \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right), \tag{3.2.98}$$

$$\|f\|_{T_1'} := \sup_{t\in[1,\infty)} \sup_{n\le N_1-1} \sup_{\mathcal{L}\in\mathcal{V}_n,\ell\in\{1,2,3\}} \sup_{k\in\mathbb{Z}} \left(\langle t\rangle^{H''_{wa}(n)\delta} 2^{N''_{wa}(n)k^+} 2^{-\frac{1}{2}k^-} \left\| \varphi_k \partial_{\xi_\ell} \left(\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} \widehat{f_{\mathcal{L}}} \right) \right\|_{L^2_\xi} \right), \tag{3.2.99}$$

and

$$\|f\|_{S_2'} := \sup_{t\in[1,\infty)} \sup_{n\le N_1} \sup_{\mathcal{L}\in\mathcal{V}_n} \left(\langle t\rangle^{1+H''_{kg}(n)\delta} 2^{N''_{kg}(n)k^+} \left\| \varphi_k \widehat{f_{\mathcal{L}}} \right\|_{L^2_\xi} \right), \tag{3.2.100}$$

$$\|f\|_{T_2'} := \sup_{t\in[1,\infty)} \sup_{n\le N_1-1} \sup_{\mathcal{L}\in\mathcal{V}_n,\ell\in\{1,2,3\}} \sup_{k\in\mathbb{Z}} \left(\langle t\rangle^{H''_{kg}(n)\delta} 2^{N''_{kg}(n)k^+} \left\| \varphi_k \partial_{\xi_\ell} \left(\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{f_{\mathcal{L}}} \right) \right\|_{L^2_\xi} \right). \tag{3.2.101}$$

Remark 3.2.7. Roughly speaking, the designed X_i norms indicate that the perturbation (G^{wa}, G^{kg}) decays mildly in time in certain Sobolev-type and weighted L^2 norms. Notice also that there is an energy hierarchy expressed in terms of the parameters $H(n)$ (for decay rate) and $N(n)$ (for regularity/weight), in the sense that the variables with more vector fields are allowed to decay slightly slower than those with fewer vector fields, in weaker Sobolev spaces.

3.2.6 Main Theorem and Proposition

As a counterpart of the forward problem, in this work we establish the existence of wave operators, namely that any (modified) asymptotic profile leads to a unique global nonlinear solution that converges to it.

Provided that $(V_\infty^{wa}, V_\infty^{kg}) \in Y_1 \times Y_2$ is given. Our goal is to justify that (3.2.74)(3.2.75) (3.2.76) can uniquely determine a solution (G^{wa}, G^{kg}). Below is our main theorem.

Theorem 3.2.1 (Main Theorem – Modified Wave Operator, precise version)**.** *There exists $\varepsilon_0 > 0$ such that if the scattering data $(V_\infty^{wa}, V_\infty^{kg}) \in Y_1 \times Y_2$ satisfies*

$$\left\|V_\infty^{wa}\right\|_{Y_1} + \left\|V_\infty^{kg}\right\|_{Y_2} \lesssim \varepsilon_0, \tag{3.2.102}$$

then there exists a unique solution (u, v) of the system (3.1.2) on $t \in [1, \infty)$ with the associated profile $\left(V^{wa}, V^{kg}\right) \in X_1 \times X_2$ satisfying

$$\left\|V^{wa} - \mathscr{V}^{wa}\right\|_{X_1} + \left\|V^{kg} - \mathscr{V}^{kg}\right\|_{X_2} \lesssim \varepsilon_0, \tag{3.2.103}$$

where the asymptotic profile $(\mathscr{V}^{wa}, \mathscr{V}^{kg})$ is explicitly given by $(V_\infty^{wa}, V_\infty^{kg})$ through (3.2.62) –(3.2.68).

Remark 3.2.8. Since X_i norms contain time growth, the above theorem actually implies that V and $\mathscr{V}$ demonstrate the same asymptotic behavior in certain norm as $t \to \infty$.

Theorem 3.2.1 follows from a fixed-point argument. Our main task is to prove the following bootstrap proposition.

Proposition 3.2.2. *There exists $\varepsilon_0 > 0$ (sufficiently small) such that for any $0 \leq$*

$\varepsilon \leq \varepsilon_0$, *if*

$$\left\|V_\infty^{wa}\right\|_{Y_1} \lesssim \varepsilon, \quad \left\|V_\infty^{kg}\right\|_{Y_2} \lesssim \varepsilon, \qquad (3.2.104)$$
$$\left\|G^{wa}\right\|_{X_1} \lesssim \varepsilon, \quad \left\|G^{kg}\right\|_{X_2} \lesssim \varepsilon,$$

then we have the improved bound

$$\left\|G^{wa}\right\|_{X_1} + \left\|G^{kg}\right\|_{X_2} \lesssim \varepsilon^{\frac{3}{2}}. \qquad (3.2.105)$$

The rest of this paper is concerned with the proof of Proposition 3.2.2, which consists of Proposition 3.7.4, Proposition 3.7.5, Proposition 3.8.4 and Proposition 3.8.5.

Remark 3.2.9. Based on Lemma 3.4.25, we know $\|\mathfrak{B}_\infty\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-\frac{7}{8}+}$. Hence, the presence of $\mathfrak{B}_\infty$ in (3.2.70) will not affect the main result.

3.2.7 Strategy of the Proof

In order to close the fixed-point argument, we plan to attack it in two steps:

- Step 1: Nonlinearity Estimates.

  We first estimate the right-hand side nonlinear terms in (3.2.74) and (3.2.75). The quadratic terms $\mathbf{I}[G, G]$ and linear terms $\mathbf{I}[G, \mathscr{V}]$ may be handled with standard dispersive estimates. However, the forcing terms $\mathbf{I}[\mathscr{V}, \mathscr{V}]$ is much more delicate. This relies on a frequency decomposition, where we have to utilize the resonant terms h_∞ and $\mathcal{C}_\infty$ to cancel out the slow-decaying low-frequency contributions, integration by parts in η for the high-frequency terms, and stationary-phase argument to handle the non-resonant terms. A direct consequence of this step is the estimate of time derivatives of the unknowns, i.e. S_i' and T_i' norms bound.

- Step 2: Solution Estimates.

  Then we turn back to the bound of the original solution itself in S_i and T_i norm. Here actually we treat them in completely different fashion.

  - Energy estimates: S_i norm is the classical energy norm, so we resort to the energy structure of the perturbation equation, which heavily relies on the trilinear estimates and S_i' norm bounds.
  - Vector fields: T_i norm is much trickier. We utilize the Lorentz vector fields to generate a relation in the mild formulation (note that we cannot directly take ∂_{ξ_ℓ} derivative in the mild formulation since it will hit the phase function and generate $\langle t\rangle$ growth), which is closely related to T_i' bounds.

In this procedure, we need to utilize both the normal forms and vector fields. Normal form method helps to improve estimates of time integrals; vector fields help improve dispersive estimates. We will delicately analyze all kinds of terms showing up in the resonant system and the perturbation equations, and these estimates actually constitute most part of this chapter.

Remark 3.2.10. Essentially, our proof mainly relies on the energy estimates in Sobolev space. Since we use the normal form transformation (integration by parts in time), in turn we must estimate the time derivatives. Since we need $\mathcal{Q}_{jk}$ estimate and integration by parts in η, we also require ξ_ℓ derivative estimates (Lemma 3.3.16) and vector fields (Lemma 3.3.11).

3.2.8 Organization

This chapter is organized as follows: in Section 3 and Section 4, we present some preliminary lemmas and a priori estimates on the solution and data; in Section 5 and Section 6, we discuss the nonlinear estimate for S_i' and T_i' norms using the Duhamel's formula without time integration; finally, in Section 7 and Section 8, we study the energy estimates for S_i norms and the regularity estimates for T_i norms.

3.3 Preliminaries: Some Lemmas

In this section, we record several preliminary lemmas and analytical tools. Most of them can be found in Ionescu-Pausader [45], so we will omit the proofs.

3.3.1 Basic Analytic Tools

Lemma 3.3.1 (Stationary Phase)**.** *Let $\phi(x) \in C^2(\mathbb{R}^n)$ and $a(x) \in C(\mathbb{R}^n)$. x_0 is the only non-degenerate critical point of ϕ, that is $\nabla_x \phi(x_0) = 0$ and $\det\left(\nabla_x^2 \phi(x_0)\right) \neq 0$. Then for $t \in \mathbb{R}^+$, we have*

$$\int_{\mathbb{R}^n} \mathrm{e}^{\mathrm{i}t\phi(x)} a(x) \mathrm{d}x = \left(\frac{2\pi}{t}\right)^{\frac{n}{2}} \frac{\mathrm{e}^{\mathrm{i}t\phi(x_0)}}{\left|\det\left(\nabla_x^2 \phi(x_0)\right)\right|^{\frac{1}{2}}} \mathrm{e}^{\frac{\mathrm{i}\pi}{4} sgn\left(\nabla_x^2 \phi(x_0)\right)} \Big(a(x_0) + O(t^{-1})\Big) \quad as \quad t \to \infty. \tag{3.3.1}$$

Here $\mathrm{sgn(A)}$ *represents the number of positive eigenvalues minus the number of negative eigenvalues.*

Proof. This is basically an integration by parts argument. See Evans [19, Section 4.5.3] and Stein-Shakarchi [65, Section 8.2]. □

Lemma 3.3.2 (Non-Stationary Phase). *(i) For $\psi(\eta) \in L^2(\mathbb{R}^n)$ with $\psi \neq 0$, we have*

$$\mathrm{e}^{\mathrm{i}s\psi} = \frac{1}{\mathrm{i}\psi}\partial_s(\mathrm{e}^{\mathrm{i}s\psi}). \tag{3.3.2}$$

(ii) For $\psi(\eta) \in H^1(\mathbb{R}^n)$ with $\nabla_\eta \psi \neq 0$, we have

$$\mathrm{e}^{\mathrm{i}s\psi} = \frac{1}{\mathrm{i}s\left|\nabla_\eta \psi\right|^2}\nabla_\eta \psi \cdot \nabla_\eta(\mathrm{e}^{\mathrm{i}s\psi}), \tag{3.3.3}$$

Proof. These rely on direct computation, so we omit the proof. □

Remark 3.3.1. The identity (3.3.2) is mainly used to perform integration by parts in the temporal variable, and (3.3.3) is mainly used to perform integration by parts in the frequency variables.

Lemma 3.3.3. *For $f, g \in L^2$ satisfying*

$$f(x) = \sum_{k_1=-\infty}^{\infty} P_{k_1} f(x), \quad g(x) = \sum_{k_2=-\infty}^{\infty} P_{k_2} g(x), \tag{3.3.4}$$

we have that for $k \in \mathbb{Z}$,

$$\begin{aligned} P_k(fg) \simeq & \sum_{k_1 \simeq k_2 \geq k} P_k\Big((P_{k_1} f)(P_{k_2} g)\Big) \\ & + \sum_{k_1 \simeq k \geq k_2+2} P_k\Big((P_{k_1} f)(P_{k_2} g)\Big) + \sum_{k_2 \simeq k \geq k_1+2} P_k\Big((P_{k_1} f)(P_{k_2} g)\Big). \end{aligned} \tag{3.3.5}$$

The summation is over all possible combination of $k_1, k_2 \in \mathbb{Z}$.

Proof. Let $h(x) = f(x)g(x)$. Hence,

$$\widehat{h}(\xi) = \big(\widehat{f} * \widehat{g}\big)(\xi) = \int_{\mathbb{R}^3} \widehat{f}(\xi - \eta)\widehat{g}(\eta)\mathrm{d}\eta. \tag{3.3.6}$$

Hence, we actually want to find the region that

$$|\xi| \in [2^{k-1}, 2^{k+1}], \quad |\xi - \eta| \in [2^{k_1-1}, 2^{k_1+1}], \quad |\eta| \in [2^{k_2-1}, 2^{k_2+1}]. \tag{3.3.7}$$

Then if $k_1 \simeq k_2$, we know

$$|\xi - \eta| + |\eta| \sim \big[2^{\max(k_1,k_2)+1}, 0\big], \tag{3.3.8}$$

which means $k \simeq k_1 \simeq k_2$. Otherwise, if $|k_1 - k_2| \geq 2$, we have

$$|\xi - \eta| + |\eta| \sim [2^{\max(k_1,k_2)+1}, 2^{\max(k_1,k_2)-1}], \tag{3.3.9}$$

which means $k \simeq \max(k_1, k_2)$. □

Remark 3.3.2. This lemma provides the effective region of convolution in frequency space. For convenience, we denote this set for fixed $k \in \mathbb{Z}$ by

$$\chi_k := \Big\{(k_1, k_2) \in \mathbb{Z}^2 : k_1 \simeq k_2 \geq k \;\text{ or }\; k_1 \simeq k \geq k_2 + 2 \;\text{ or }\; k_2 \simeq k \geq k_1 + 2\Big\},$$

and we will frequently take the summation over $(k_1, k_2) \in \chi_k$ (sometimes we simply write for short as $\sum_{k_1,k_2}$).

We record the following basic inequalities used repeatedly in the nonlinear estimates.

Lemma 3.3.4 (Young's Inequality)**.** *Suppose that* $f \in L^p(\mathbb{R}^3)$ *and* $g \in L^q(\mathbb{R}^3)$ *and*

$$\frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r}, \tag{3.3.10}$$

with $1 \le p, q, r \le \infty$*. Then we have*

$$\|f * g\|_{L^r} \lesssim \|f\|_{L^p} \|q\|_{L^q} . \tag{3.3.11}$$

Proof. See Duoandikoetxea [15, Corollary 1.21]. □

Lemma 3.3.5 (Minkowski's Integral Inequality)**.** *Given measure spaces* (X, μ) *and* (Y, ν) *with* σ*-finite measures. Then for* $1 \le p \le \infty$*,*

$$\left(\int_X \left| \int_Y f(x,y) \mathrm{d}\nu(y) \right|^p \mathrm{d}\mu(x) \right)^{\frac{1}{p}} \le \int_Y \left(\int_X |f(x,y)|^p \, \mathrm{d}\mu(x) \right)^{\frac{1}{p}} \mathrm{d}\nu(y). \tag{3.3.12}$$

Proof. See Duoandikoetxea [15, Page xviii]. □

3.3.2 Preliminary Estimates

Lemma 3.3.6 (Bilinear Estimates)**.** *(i) Assume that* $\ell \ge 2$*,* $f_1, \cdots, f_\ell, f_{\ell+1} \in L^2(\mathbb{R}^3)$*, and* $M : (\mathbb{R}^3)^\ell \to \mathbb{C}$ *is a continuously compactly supported function. Then*

$$\begin{aligned} &\left| \int_{(\mathbb{R}^3)^\ell} M(\xi_1, \cdots, \xi_\ell) \widehat{f}_1(\xi_1) \cdots \widehat{f}_\ell(\xi_\ell) \widehat{f}_{\ell+1}(-\xi_1 \cdots - \xi_\ell) \mathrm{d}\xi_1 \cdots \mathrm{d}\xi_\ell \right| \\ &\lesssim \left\| \mathscr{F}^{-1} M \right\|_{L^1((\mathbb{R}^3)^\ell)} \|f_1\|_{L^{p_1}} \cdots \|f_\ell\|_{L^{p_\ell}} \|f_{\ell+1}\|_{L^{p_{\ell+1}}} , \end{aligned} \tag{3.3.13}$$

for any exponent $p_1, \cdots, p_\ell, p_{\ell+1} \in [1, \infty]$ *satisfying* $\dfrac{1}{p_1} + \cdots \dfrac{1}{p_\ell} + \dfrac{1}{p_{\ell+1}} = 1$.

(ii) If $q, p_1, p_2 \in [1,\infty]$ *satisfying* $\dfrac{1}{q} = \dfrac{1}{p_1} + \dfrac{1}{p_2}$*, then*

$$\left\| \mathscr{F}^{-1}\Big(\int_{\mathbb{R}^3} M(\xi,\eta)\widehat{f}(\eta)\widehat{g}(-\xi-\eta)\mathrm{d}\eta \Big) \right\|_{L^q} \lesssim \left\| \mathscr{F}^{-1} M \right\|_{L^1(\mathbb{R}^3\times\mathbb{R}^3)} \|f\|_{L^{p_1}} \|g\|_{L^{p_2}} . \tag{3.3.14}$$

Proof. This is Ionescu-Pausader [45, Lemma 3.2]. □

Lemma 3.3.7 (Lower Bound for Quadratic Phases)**.** *Let* $\Phi_{wa}^{\iota_1\iota_2}$ *and* $\Phi_{kg}^{\iota_1\iota_2}$ *be the quadratic phases given in* (3.2.17) *and* (3.2.18)*. Assume* $|\xi|, |\eta|, |\xi-\eta| \in [0,b]$ *for* $b \geq 1$*. Then we have*

$$\left|\Phi_{wa}^{\iota_1\iota_2}(\xi,\eta)\right| \geq \frac{|\xi|}{4b^2}, \quad \left|\Phi_{kg}^{\iota_1\iota_2}(\xi,\eta)\right| \geq \frac{|\eta|}{4b^2}. \tag{3.3.15}$$

Proof. This is Ionescu-Pausader [45, Lemma 3.3]. □

Lemma 3.3.8 (Lorentz Vector Fields)**.** *Assume* $\mu \in \{wa, kg\}$ *and*

$$(\partial_t + \mathrm{i}\Lambda_\mu)U = \mathcal{N}, \tag{3.3.16}$$

on $[1,\infty)\times\mathbb{R}^3$*. If* $V(t) = \mathrm{e}^{\mathrm{i}t\Lambda_\mu}U$ *and* $\ell \in \{1,2,3\}$*, then for any* $t \in [1,\infty)$*, we have*

$$\widehat{\Gamma_\ell U}(t,\xi) = \mathrm{i}\big(\partial_{\xi_\ell}\widehat{\mathcal{N}}\big)(t,\xi) + \mathrm{e}^{-\mathrm{i}t\Lambda_\mu(\xi)}\partial_{\xi_\ell}\Big[\Lambda_\mu(\xi)\widehat{V}(t,\xi)\Big]. \tag{3.3.17}$$

Proof. This is Ionescu-Pausader [45, Lemma 6.1]. □

3.3.3 Linear Estimates

This subsection is about the linear estimates either in the physical space or in the frequency space. In the following lemmas, $\sigma > 0$ denotes a sufficiently small constant. When it is present, the inequality constant might depend on σ. Also, for all rotation vector fields Ω,

$$\|f\|_{H^{0,1}_{\Omega}} := \sum_{|\alpha|\leq 1} \|\Omega^{\alpha} f\|_{L^2} . \tag{3.3.18}$$

Lemma 3.3.9. *For any $f \in L^2(\mathbb{R}^3)$, $(k,j) \in \mathcal{J}$ and $\alpha \in (\mathbb{Z}_+)^3$, we have*

$$\left\| D^{\alpha}_{\xi} \widehat{\mathscr{Q}_{jk} f} \right\|_{L^2_{\xi}} \lesssim 2^{|\alpha| j} \left\| \widehat{\mathscr{Q}_{jk} f} \right\|_{L^2_{\xi}}, \quad \left\| D^{\alpha}_{\xi} \widehat{\mathscr{Q}_{jk} f} \right\|_{L^{\infty}_{\xi}} \lesssim 2^{|\alpha| j} \left\| \widehat{\mathscr{Q}_{jk} f} \right\|_{L^{\infty}_{\xi}} . \tag{3.3.19}$$

Proof. This is Ionescu-Pausader [45, (3.17)]. □

Lemma 3.3.10. *For any $f \in L^2(\mathbb{R}^3)$ and $(k,j) \in \mathcal{J}$, we have*

$$\left\| \widehat{Q_{jk} f} - \widehat{\mathscr{Q}_{jk} f} \right\|_{L^{\infty}_{\xi}} \lesssim 2^{\frac{3}{2} j} 2^{-4(j+k)} \|P_k f\|_{L^2} . \tag{3.3.20}$$

Proof. This is Ionescu-Pausader [45, (3.21)]. □

Lemma 3.3.11. *For any $f \in L^2(\mathbb{R}^3)$ and $(k,j) \in \mathcal{J}$, we have*

$$\left\| \widehat{\mathscr{Q}_{jk} f} \right\|_{L^{\infty}_{\xi}} \lesssim \min\left\{ 2^{\frac{3j}{2}} \|Q_{jk} f\|_{L^2} , \, 2^{\frac{j}{2}} 2^{-k} 2^{\frac{\sigma(j+k)}{8}} \|Q_{jk} f\|_{H^{0,1}_{\Omega}} \right\} . \tag{3.3.21}$$

Proof. This is Ionescu-Pausader [45, (3.18)]. □

Lemma 3.3.12 (Dispersive Estimates)**.** *For any $f \in L^2(\mathbb{R}^3)$ and $(k,j) \in \mathcal{J}$, we have*

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{jk} f \right\|_{L^\infty} \lesssim \min\left\{ 2^{\frac{3k}{2}},\, 2^{\frac{3k}{2}} 2^j \langle t \rangle^{-1} \right\} \left\| Q_{jk} f \right\|_{L^2}, \tag{3.3.22}$$

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} f \right\|_{L^\infty} \lesssim \min\left\{ 2^{\frac{3k}{2}},\, 2^{3k^+} 2^{\frac{3j}{2}} \langle t \rangle^{-\frac{3}{2}} \right\} \left\| Q_{jk} f \right\|_{L^2}. \tag{3.3.23}$$

Proof. This is Ionescu-Pausader [45, (3.24),(3.28)]. □

Lemma 3.3.13. *For any $f \in L^2(\mathbb{R}^3)$ and $(k,j) \in \mathcal{J}$, if $|t| \geq 1$, we have*

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^{\frac{k}{2}} \left(1 + 2^k \langle t \rangle \right)^{\frac{\sigma}{8}} \left\| Q_{jk} f \right\|_{H^{0,1}_\Omega}, \quad \text{if } \ 2^j \leq \langle t \rangle, \tag{3.3.24}$$

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{\leq jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^{2k} \left\| \widehat{Q_{\leq jk} f} \right\|_{L^\infty_\xi}, \quad \text{if } \ 2^j \lesssim \langle t \rangle^{\frac{1}{2}} 2^{-\frac{k}{2}}. \tag{3.3.25}$$

Proof. This is Ionescu-Pausader [45, (3.26),(3.27)]. □

Lemma 3.3.14. *For any $f \in L^2(\mathbb{R}^3)$ and $(k,j) \in \mathcal{J}$, if $1 \leq 2^{2k^- - 20} \langle t \rangle$, we have*

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} 2^{5k^+} 2^{-k^-} 2^{\frac{j}{2}} \left(1 + 2^{2k^-} \langle t \rangle \right)^{\frac{\sigma}{8}} \left\| Q_{jk} f \right\|_{H^{0,1}_\Omega}, \quad \text{if } \ 2^j \leq 2^{k^-} \langle t \rangle, \tag{3.3.26}$$

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{\leq jk} f \right\|_{L^\infty} \lesssim \langle t \rangle^{-\frac{3}{2}} 2^{5k^+} \left\| \widehat{Q_{\leq jk} f} \right\|_{L^\infty_\xi}, \quad \text{if } \ 2^j \lesssim \langle t \rangle^{\frac{1}{2}}. \tag{3.3.27}$$

Proof. This is Ionescu-Pausader [45, (3.29),(3.30)]. □

Lemma 3.3.15. *For any $1 \leq p \leq \infty$, we have*

$$\left\| \widehat{Q_{jk} f} \right\|_{L^p} \lesssim \left\| \widehat{P_k f} \right\|_{L^p}, \tag{3.3.28}$$

$$\left\| \widehat{Q_{\leq jk} f} \right\|_{L^p} \lesssim \left\| \widehat{P_k f} \right\|_{L^p}. \tag{3.3.29}$$

Proof. Since $Q_{jk}f = \varphi_j^{(k)} \cdot P_k f$, we know

$$\widehat{Q_{jk}f}(\xi) = \int_{\mathbb{R}^3} \widehat{\varphi_j^{(k)}}(\xi - \eta)\widehat{P_k f}(\eta)\,\mathrm{d}\eta. \tag{3.3.30}$$

Hence,

$$\left\|\widehat{Q_{jk}f}\right\|_{L^p} \lesssim \left\|\widehat{P_k f}\right\|_{L^p} \left\|\widehat{\varphi_j^{(k)}}\right\|_{L^1}. \tag{3.3.31}$$

In particular, $\|\widehat{\varphi_j^{(k)}}\|_{L^1} \lesssim 1$ uniformly in j by scaling invariance. □

Remark 3.3.3. This lemma indicates that we can use P_k to bound Q_{jk}. A natural corollary is that we can use Q_{jk} to bound $\mathscr{Q}_{jk}$.

Lemma 3.3.16. *For $f \in L^2(\mathbb{R}^3)$ and $k \in \mathbb{Z}$, let*

$$A_k := \|P_k f\|_{L^2} + \sum_{\ell=1}^{3} \left\|\varphi_k\big(\partial_{\xi_\ell}\widehat{f}\big)\right\|_{L^2_\xi}, \tag{3.3.32}$$

$$B_k := \left(\sum_{j \geq -k^-} 2^{2j} \|Q_{jk}f\|_{L^2}^2\right)^{\frac{1}{2}}. \tag{3.3.33}$$

Then for any $k \in \mathbb{Z}$,

$$A_k \lesssim \sum_{|k'-k| \leq 4} B_{k'}, \tag{3.3.34}$$

and

$$B_k \lesssim \begin{cases} \sum_{|k'-k|\leq 4} A_{k'} & \textit{if} \quad k \geq 0, \\ \sum_{k' \in \mathbb{Z}} A_{k'} 2^{-\frac{|k-k'|}{2}} & \textit{if} \quad k \leq 0. \end{cases} \tag{3.3.35}$$

Proof. This is Ionescu-Pausader [45, Lemma 3.5]. □

Remark 3.3.4. Roughly speaking, this lemma means that $A_k \sim B_k$, i.e. they are comparable.

3.3.4 Nonlinear Estimates

Lemma 3.3.17. *For $f, g \in L^2(\mathbb{R}^3)$, we have*

$$\left\| \varphi_k \mathbf{I}_{wa}^{\iota_1\iota_2} \left[P_{k_1} f, P_{k_2} g \right] \right\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}\min\{k_1,k_2,k\}} \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} g \right\|_{L^2}, \tag{3.3.36}$$

and

$$\left\| \varphi_k \mathbf{I}_{wa}^{\iota_1\iota_2} \left[P_{k_1} f, P_{k_2} g \right] \right\|_{L^2_\xi} \lesssim \min\left\{ \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_1} f \right\|_{L^\infty} \left\| P_{k_2} g \right\|_{L^2}, \left\| P_{k_1} f \right\|_{L^2} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_2} g \right\|_{L^\infty} \right\}. \tag{3.3.37}$$

Proof. Note that

$$\mathbf{I}_{wa}^{\iota_1\iota_2} \left[P_{k_1} f, P_{k_2} g \right] \simeq \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{wa}^{\iota_1\iota_2}(\xi,\eta)} a_{\iota_1\iota_2}(\xi,\eta) \Big(\varphi_{k_1}(\xi-\eta) \widehat{f}(t,\xi-\eta) \Big) \Big(\varphi_{k_2}(\eta) \widehat{g}(t,\eta) \Big) \, \mathrm{d}\eta. \tag{3.3.38}$$

Since $|a_{\iota_1\iota_2}(\xi,\eta)| \lesssim 1$, $a_{\iota_1\iota_2}$ will not play a role in the estimate.

Using Young's inequality with $(p,q,r) = (2,2,\infty)$, we have

$$\begin{aligned} \left\| \varphi_k \mathbf{I}_{wa}^{\iota_1\iota_2} \left[P_{k_1} f, P_{k_2} g \right] \right\|_{L^2_\xi} &\lesssim \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} g \right\|_{L^2} \left(\int_{\mathbb{R}^3} \varphi_k(\xi) \mathrm{d}\xi \right)^{\frac{1}{2}} \\ &\lesssim 2^{\frac{3}{2}k} \left\| P_{k_1} f \right\|_{L^2} \left\| P_{k_2} g \right\|_{L^2}, \end{aligned} \tag{3.3.39}$$

and with $(p,q,r)=(2,1,2)$, we have

$$\left\|\varphi_k \mathbf{I}_{wa}^{\iota_1\iota_2}\left[P_{k_1}f, P_{k_2}g\right]\right\|_{L^2_\xi} \lesssim \|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^1} \lesssim 2^{\frac{3}{2}k_2}\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^2}. \tag{3.3.40}$$

Similarly, we may show that

$$\left\|\varphi_k \mathbf{I}_{wa}^{\iota_1\iota_2}\left[P_{k_1}f, P_{k_2}g\right]\right\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}k_1}\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^2}. \tag{3.3.41}$$

Hence, (3.3.36) is justified. On the other hand, (3.3.37) follows from Lemma 3.3.6 with $(p_1,p_2,q)=(\infty,2,2)$ or $(p_1,p_2,q)=(2,\infty,2)$. ☐

Lemma 3.3.18. *For $f,g\in L^2(\mathbb{R}^3)$, we have*

$$\left\|\varphi_k \mathbf{I}_{kg}^{\iota_1\iota_2}\left[P_{k_1}f, P_{k_2}g\right]\right\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}\min\{k_1,k_2,k\}}2^{k_1}2^{-k_2}\|P_{k_1}f\|_{L^2}\|P_{k_2}g\|_{L^2}, \tag{3.3.42}$$

and

$$\begin{aligned}&\left\|\varphi_k \mathbf{I}_{kg}^{\iota_1\iota_2}\left[P_{k_1}f, P_{k_2}g\right]\right\|_{L^2_\xi} \qquad (3.3.43)\\ &\lesssim 2^{k_1}2^{-k_2}\min\Big\{\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_1}f\right\|_{L^\infty}\|P_{k_2}g\|_{L^2}, \|P_{k_1}f\|_{L^2}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}P_{k_2}g\right\|_{L^\infty}\Big\}.\end{aligned}$$

Proof. Note that for $|\xi-\eta|\approx 2^{k_1}$ and $|\eta|\approx 2^{k_2}$, $|b_{\iota_1\iota_2}(\xi,\eta)|\lesssim 2^{k_1}2^{-k_2}$. Then the rest is similar to the proofs of (3.3.36) and (3.3.37). ☐

Lemma 3.3.19. *For $\mathfrak{f},\mathfrak{g},\mathfrak{h}\in L^2(\mathbb{R}^3)$, we have*

$$\begin{aligned}&\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\Phi_{wa}}a(\xi,\eta)\widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\widehat{P_k\mathfrak{h}}(\xi)\,\mathrm{d}\eta\mathrm{d}\xi\right| \qquad (3.3.44)\\ &\lesssim 2^{\frac{3}{2}\min(k,k_1,k_2)}\|P_{k_1}\mathfrak{f}\|_{L^2}\|P_{k_2}\mathfrak{g}\|_{L^2}\|P_k\mathfrak{h}\|_{L^2},\end{aligned}$$

and

$$\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{wa}} a(\xi,\eta)\widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\widehat{P_k\mathfrak{h}}(\xi)\,\mathrm{d}\eta\mathrm{d}\xi\right| \tag{3.3.45}$$
$$\lesssim \min\Big\{ \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_1}\mathfrak{f}\right\|_{L^\infty} \|P_{k_2}\mathfrak{g}\|_{L^2} \|P_k\mathfrak{h}\|_{L^2}\,,\ \|P_{k_1}\mathfrak{f}\|_{L^2} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_2}\mathfrak{g}\right\|_{L^\infty} \|P_k\mathfrak{h}\|_{L^2}\,,$$
$$\|P_{k_1}\mathfrak{f}\|_{L^2} \|P_{k_2}\mathfrak{g}\|_{L^2} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} P_k\mathfrak{h}\right\|_{L^\infty} \Big\}.$$

Proof. This is a direct consequence of Lemma 3.3.17. □

Lemma 3.3.20. *For* $\mathfrak{f},\mathfrak{g},\mathfrak{h}\in L^2(\mathbb{R}^3)$, *we have*

$$\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{kg}} b(\xi,\eta)\widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\widehat{P_k\mathfrak{h}}(\xi)\,\mathrm{d}\eta\mathrm{d}\xi\right| \tag{3.3.46}$$
$$\lesssim 2^{\frac{3}{2}\min(k,k_1,k_2)} 2^{k_1} 2^{-k_2} \|P_{k_1}\mathfrak{f}\|_{L^2} \|P_{k_2}\mathfrak{g}\|_{L^2} \|P_k\mathfrak{h}\|_{L^2}\,,$$

and

$$\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{kg}} b(\xi,\eta)\widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\widehat{P_k\mathfrak{h}}(\xi)\,\mathrm{d}\eta\mathrm{d}\xi\right| \tag{3.3.47}$$
$$\lesssim 2^{k_1}2^{-k_2}\min\Big\{ \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_1}\mathfrak{f}\right\|_{L^\infty} \|P_{k_2}\mathfrak{g}\|_{L^2} \|P_k\mathfrak{h}\|_{L^2}\,,\ \|P_{k_1}\mathfrak{f}\|_{L^2} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_2}\mathfrak{g}\right\|_{L^\infty} \|P_k\mathfrak{h}\|_{L^2}\,,$$
$$\|P_{k_1}\mathfrak{f}\|_{L^2} \|P_{k_2}\mathfrak{g}\|_{L^2} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_k\mathfrak{h}\right\|_{L^\infty} \Big\}.$$

Proof. This is a direct consequence of Lemma 3.3.18. □

3.4 A Priori Estimates

Let

$$X(t;n) := \langle t\rangle^{-H(n)\delta}, \tag{3.4.1}$$

$$Y(k,t;n) := \langle t\rangle^{-H(n+1)\delta} 2^{-N(n+1)k^+}, \tag{3.4.2}$$

$$Z(k;n) := 2^{-N(n-2)k^+}. \tag{3.4.3}$$

3.4.1 L^2 Estimates of G^{wa} and G^{kg}

Lemma 3.4.1. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1$,

$$\sup_{\mathcal{L}\in\mathcal{V}_n} \left(\left\| |\nabla|^{-\frac{1}{2}} G^{wa}_{\mathcal{L}}(t) \right\|_{H^{N(n)}} + \left\| G^{kg}_{\mathcal{L}}(t) \right\|_{H^{N(n)}} \right) \lesssim \varepsilon X(t;n), \tag{3.4.4}$$

and for any $0 \le n \le N_1 - 1$,

$$\sup_{\mathcal{L}\in\mathcal{V}_n, \ell\in\{1,2,3\}} \sup_{k\in\mathbb{Z}} \left(2^{\frac{1}{2}k} \left\| \varphi_k \partial_{\xi_\ell} \widehat{G^{wa}_{\mathcal{L}}}(t) \right\|_{L^2_\xi} + 2^{k^+} \left\| \varphi_k \partial_{\xi_\ell} \widehat{G^{kg}_{\mathcal{L}}}(t) \right\|_{L^2_\xi} \right) \lesssim \varepsilon Y(k,t;n). \tag{3.4.5}$$

Proof. This can be directly obtained from (3.2.104) based on the definition of X_1 and X_2 norms. □

Remark 3.4.1. Using the definition of Sobolev spaces with Fourier transform, this lemma actually yields the L^2 bound for G^{wa} and G^{kg}. For any $k \in \mathbb{Z}$ and for any

$0 \leq n \leq N_1$,

$$\left\|P_k G_{\mathcal{L}}^{wa}(t)\right\|_{L^2} \lesssim \varepsilon X(t;n) 2^{\frac{1}{2}k} 2^{-N(n)k^+}, \tag{3.4.6}$$

$$\left\|P_k G_{\mathcal{L}}^{kg}(t)\right\|_{L^2} \lesssim \varepsilon X(t;n) 2^{-N(n)k^+}, \tag{3.4.7}$$

and for any $0 \leq n \leq N_1 - 1$,

$$\left\|\varphi_k \partial_{\xi_\ell} \widehat{G_{\mathcal{L}}^{wa}}(t)\right\|_{L^2_\xi} \lesssim \varepsilon Y(k,t;n) 2^{-\frac{1}{2}k}, \tag{3.4.8}$$

$$\left\|\varphi_k \partial_{\xi_\ell} \widehat{G_{\mathcal{L}}^{kg}}(t)\right\|_{L^2_\xi} \lesssim \varepsilon Y(k,t;n) 2^{-k^+}. \tag{3.4.9}$$

Lemma 3.4.2. *Assume* (3.2.104) *holds. Then for any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1 - 1$ and $(k,j) \in \mathcal{J}$, we have*

$$2^j \left\|Q_{jk} G_{\mathcal{L}}^{wa}(t)\right\|_{L^2} \lesssim \varepsilon Y(k,t;n) 2^{-\frac{1}{2}k}, \tag{3.4.10}$$

$$2^j \left\|Q_{jk} G_{\mathcal{L}}^{kg}(t)\right\|_{L^2} \lesssim \varepsilon Y(k,t;n) 2^{-k^+}, \tag{3.4.11}$$

and

$$\left\|P_k G_{\mathcal{L}}^{wa}(t)\right\|_{L^2} \lesssim \varepsilon Y(k,t;n) 2^{-\frac{1}{2}k} 2^{k^-}, \tag{3.4.12}$$

$$\left\|P_k G_{\mathcal{L}}^{kg}(t)\right\|_{L^2} \lesssim \varepsilon Y(k,t;n) 2^{-k^+} 2^{k^-}. \tag{3.4.13}$$

Proof. (3.4.10)(3.4.11) are direct consequences of Remark 3.4.1 and (3.3.35). Summing over $j \geq -k^-$, we get (3.4.12)(3.4.13). $\square$

Lemma 3.4.3. *Assume* (3.2.104) *holds. Then for any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1 - 1$*

and $(k,j)\in\mathcal{J}$*, we have*

$$\left\|P_k G_{\mathcal{L}}^{wa}(t)\right\|_{L^2}\lesssim\varepsilon\langle t\rangle^{-H(n+1)\delta}2^{-\left(N(n)-\frac{1}{2}\right)k^+}2^{\frac{1}{2}k^-},\tag{3.4.14}$$

$$\left\|P_k G_{\mathcal{L}}^{kg}(t)\right\|_{L^2}\lesssim\varepsilon\langle t\rangle^{-H(n+1)\delta}2^{-N(n)k^+}2^{k^-}.\tag{3.4.15}$$

Proof. These follow from a combination of Remark 3.4.1 (if $k\geq 0$) and Lemma 3.4.2 (if $k\leq 0$). □

Remark 3.4.2. Lemma 3.4.3 is an improved version for Remark 3.4.1. In particular, "2^{k^-}" in (3.4.15) is favorable when $k\leq 0$. We will use these bounds in Lemma 3.4.3 repeatedly in the following energy estimates.

3.4.2 L^∞ Estimates of G^{wa} and G^{kg}

Lemma 3.4.4. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\leq n\leq N_1-1$ *and* $(k,j)\in\mathcal{J}$*, we have*

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G_{\mathcal{L}}^{wa}(t)\right\|_{L^\infty}\lesssim\varepsilon Y(k,t;n)\min\left\{2^{-j},\ \langle t\rangle^{-1}\right\}2^k,\tag{3.4.16}$$

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G_{\mathcal{L}}^{kg}(t)\right\|_{L^\infty}\lesssim\varepsilon Y(k,t;n)\min\left\{2^{\frac{3}{2}k},\ 2^{3k^+}2^{\frac{3}{2}j}\langle t\rangle^{-\frac{3}{2}}\right\}2^{-j}2^{-k^+}.\tag{3.4.17}$$

Proof. Combining Lemma 3.3.12 and Lemma 3.4.2 yields these bounds. □

Lemma 3.4.5. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L}\in\mathcal{V}_n$ *with* $n\leq N_1-1$ *and* $(k,j)\in\mathcal{J}$*, we have*

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G_{\mathcal{L}}^{wa}(t)\right\|_{L^\infty}\lesssim\varepsilon Y(k,t;n)\min\left\{2^{k^-},\ \ln\langle t\rangle\langle t\rangle^{-1}\right\}2^k,\tag{3.4.18}$$

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G_{\mathcal{L}}^{kg}(t)\right\|_{L^\infty}\lesssim\varepsilon Y(k,t;n)\min\left\{2^{\frac{1}{2}k^+}2^{\frac{5}{2}k^-},\ 2^{\frac{3}{2}k^+}2^{\frac{1}{2}k^-}\langle t\rangle^{-1}\right\}.\tag{3.4.19}$$

Proof. Summing up (3.4.16) in Lemma 3.4.4, we have

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G^{wa}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n)\sum_{j\geq -k^-}\min\left\{2^{-j},\langle t\rangle^{-1}\right\}2^k. \tag{3.4.20}$$

For the first term in the minimum of (3.4.18), we naturally have

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G^{wa}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n)\sum_{j\geq -k^-}2^{-j}2^k\lesssim \varepsilon Y(k,t;n)2^{k^-}2^k. \tag{3.4.21}$$

For the second term in the minimum, we divide it into two cases:

- If $2^{-j}\geq \langle t\rangle^{-1}$, then since $-k^-\geq 0$, there are at most $\ln\langle t\rangle$ such j. Hence, by (3.4.20) we have the summation

$$\sum_{2^{-j}\geq \langle t\rangle^{-1}}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G^{wa}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n)\ln\langle t\rangle\langle t\rangle^{-1}2^k. \tag{3.4.22}$$

- If $2^{-j}\leq \langle t\rangle^{-1}$, then we naturally have

$$\sum_{2^{-j}\leq \langle t\rangle^{-1}}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}G^{wa}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n)\sum_{2^{-j}\leq \langle t\rangle^{-1}}2^{-j}2^k\lesssim \varepsilon Y(k,t;n)\langle t\rangle^{-1}2^k. \tag{3.4.23}$$

Summarizing the above two cases, we arrive at the desired result (3.4.18).

On the other hand, for (3.4.19), we have from (3.4.17) that

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n)\sum_{j\geq -k^-}\min\left\{2^{\frac{3k}{2}}2^{-j}2^{-k^+},\,2^{2k^+}2^{\frac{j}{2}}\langle t\rangle^{-\frac{3}{2}}\right\}. \tag{3.4.24}$$

For the first term in the minimum of (3.4.24), we have

$$\begin{aligned}\sum_{j\geq -k^-}\left\|e^{-it\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty} &\lesssim \varepsilon Y(k,t;n)\sum_{j\geq -k^-}2^{\frac{3k}{2}}2^{-j}2^{-k^+} \\ &\lesssim \varepsilon Y(k,t;n)2^{\frac{3k}{2}}2^{k^-}2^{-k^+}\lesssim \varepsilon Y(k,t;n)2^{\frac{k^+}{2}}2^{\frac{5k^-}{2}}.\end{aligned} \tag{3.4.25}$$

For the second term in the minimum of (3.4.24), j power is positive, so we cannot directly sum up over j. The key is to combine both bounds in (3.4.24). We further divide it into two cases:

- If $2^j \leq \langle t\rangle 2^k 2^{-2k^+}$, then $2^{\frac{3k}{2}}2^{-j}2^{-k^+} \geq 2^{2k^+}2^{\frac{j}{2}}\langle t\rangle^{-\frac{3}{2}}$. Hence, we choose the second term in the minimum of (3.4.24) to sum up over j and get

$$\sum_{j\geq -k^-}2^{2k^+}2^{\frac{j}{2}}\langle t\rangle^{-\frac{3}{2}} \leq 2^{2k^+}\langle t\rangle^{-\frac{3}{2}}\left(\langle t\rangle^{\frac{1}{2}}2^{\frac{1}{2}k}2^{-k^+}\right)\leq \langle t\rangle^{-1}2^{\frac{3}{2}k^+}2^{\frac{1}{2}k^-}. \tag{3.4.26}$$

- If $2^j \geq \langle t\rangle 2^k 2^{-2k^+}$, then $2^{\frac{3k}{2}}2^{-j}2^{-k^+} \leq 2^{2k^+}2^{\frac{j}{2}}\langle t\rangle^{-\frac{3}{2}}$. Hence, we choose the first term in the minimum of (3.4.24) to sum up over j and get

$$\sum_{j\geq -k^-}2^{\frac{3k}{2}}2^{-j}2^{-k^+} \leq 2^{\frac{3k}{2}}2^{-k^+}\left(\langle t\rangle^{-1}2^{-k}2^{2k^+}\right)\leq \langle t\rangle^{-1}2^{\frac{3}{2}k^+}2^{\frac{1}{2}k^-}. \tag{3.4.27}$$

Summarizing the above two cases, we have that

$$\sum_{j\geq -k^-}\left\|e^{-it\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \langle t\rangle^{-1}2^{\frac{3}{2}k^+}2^{\frac{1}{2}k^-}. \tag{3.4.28}$$

$\square$

Lemma 3.4.6. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\leq n\leq N_1-2$

and $(k,j)\in\mathcal{J}$ *with* $1\le 2^{2k^-}\langle t\rangle$*, we have for some sufficiently small* $\sigma>0$*,*

$$\sum_{j\ge -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}(t)\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n+1)\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{4k^+}2^{-k^-(\frac{1}{2}-\frac{\sigma}{4})}. \tag{3.4.29}$$

Proof. Based on Lemma 3.3.14, we know

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \langle t\rangle^{-\frac{3}{2}}2^{5k^+}2^{-k^-}2^{\frac{j}{2}}\left(2^{2k^-}\langle t\rangle\right)^{\frac{\sigma}{8}}\left\|Q_{jk}G^{kg}_{\mathcal{L}}\right\|_{H^{0,1}_{\Omega}}. \tag{3.4.30}$$

From Lemma 3.4.2, we have

$$\left\|Q_{jk}G^{kg}_{\mathcal{L}}\right\|_{H^{0,1}_{\Omega}}\lesssim \varepsilon Y(k,t;n+1)2^{-j}2^{-k^+}. \tag{3.4.31}$$

Combining the above two, we have

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n+1)\langle t\rangle^{-\frac{3}{2}}2^{4k^+}2^{-k^-}2^{-\frac{j}{2}}\left(2^{2k^-}\langle t\rangle\right)^{\frac{\sigma}{8}}. \tag{3.4.32}$$

Summing up over $j\ge -k^-$, we have

$$\sum_{j\ge -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathscr{Q}_{jk}G^{kg}_{\mathcal{L}}\right\|_{L^\infty}\lesssim \varepsilon Y(k,t;n+1)\langle t\rangle^{-\frac{3}{2}}2^{4k^+}2^{-\frac{k^-}{2}}\left(2^{2k^-}\langle t\rangle\right)^{\frac{\sigma}{8}}. \tag{3.4.33}$$

Hence, our result naturally follows. □

Remark 3.4.3. Roughly speaking, Lemma 3.4.5 and Lemma 3.4.6 provide the dispersive estimates

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}G^{wa}_{\mathcal{L}}(t)\right\|_{L^\infty}\sim\langle t\rangle^{-1}, \tag{3.4.34}$$
$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}}(t)\right\|_{L^\infty}\sim\langle t\rangle^{-\frac{3}{2}}. \tag{3.4.35}$$

This is consisent with the standard dispersive estimates for the wave equation and

Klein-Gordon equations. However, there is a delicate trade-off between time decay and the weights of k, k^+, k^-.

3.4.3 Estimates of V_∞^{wa} and V_∞^{kg}

Lemma 3.4.7. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1$,

$$\left\| P_k V_{\mathcal{L},\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k; n-1) 2^{\frac{1}{2}k}, \tag{3.4.36}$$

$$\left\| P_k V_{\mathcal{L},\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k; n-1), \tag{3.4.37}$$

and for any $0 \le n \le N_1 - 1$,

$$\left\| \varphi_k \partial_{\xi_\ell} \widehat{V_{\mathcal{L},\infty}^{wa}} \right\|_{L_\xi^2} \lesssim \varepsilon Z(k; n) 2^{-\frac{1}{2}k}, \tag{3.4.38}$$

$$\left\| \varphi_k \partial_{\xi_\ell} \widehat{V_{\mathcal{L},\infty}^{kg}} \right\|_{L_\xi^2} \lesssim \varepsilon Z(k; n) 2^{-k^+}. \tag{3.4.39}$$

Proof. Based on the definition of Y_1 and Y_2 norms, this is similar to the proof of Lemma 3.4.1 and Remark 3.4.1. □

Lemma 3.4.8. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 1$ *and* $(k, j) \in \mathcal{J}$, *we have*

$$2^j \left\| Q_{jk} V_{\mathcal{L},\infty}^{wa} \right\|_{L^2} \lesssim \varepsilon Z(k; n) 2^{-\frac{1}{2}k}, \tag{3.4.40}$$

$$2^j \left\| Q_{jk} V_{\mathcal{L},\infty}^{kg} \right\|_{L^2} \lesssim \varepsilon Z(k; n) 2^{-k^+}, \tag{3.4.41}$$

and

$$\left\|P_k V^{wa}_{\mathcal{L},\infty}\right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-\frac{1}{2}k} 2^{k^-}, \tag{3.4.42}$$

$$\left\|P_k V^{kg}_{\mathcal{L},\infty}\right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{-k^+} 2^{k^-}. \tag{3.4.43}$$

Proof. See the proof of Lemma 3.4.2. □

Lemma 3.4.9. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1 - 1$ *and* $(k,j) \in \mathcal{J}$*, we have*

$$\left\|P_k V^{wa}_{\mathcal{L},\infty}\right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{\frac{1}{2}k^-}, \tag{3.4.44}$$

$$\left\|P_k V^{kg}_{\mathcal{L},\infty}\right\|_{L^2} \lesssim \varepsilon Z(k;n) 2^{k^-}. \tag{3.4.45}$$

Proof. See the proof of Lemma 3.4.3. □

Lemma 3.4.10. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1 - 1$ *and* $(k,j) \in \mathcal{J}$*, we have*

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathcal{Q}_{jk} V^{wa}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \varepsilon Z(k;n) \min\left\{2^{-j},\, \langle t\rangle^{-1}\right\} 2^k, \tag{3.4.46}$$

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathcal{Q}_{jk} V^{kg}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \varepsilon Z(k;n) \min\left\{2^{\frac{3}{2}k},\, 2^{3k^+} 2^{\frac{3}{2}j} \langle t\rangle^{-\frac{3}{2}}\right\} 2^{-j} 2^{-k^+}. \tag{3.4.47}$$

Proof. See the proof of Lemma 3.4.4. □

Lemma 3.4.11. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1 - 1$ *and* $(k,j) \in \mathcal{J}$*, we have*

$$\sum_{j \geq -k^-} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathcal{Q}_{jk} V^{wa}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \varepsilon Z(k;n) \min\left\{2^{k^-}, \ln\langle t\rangle \langle t\rangle^{-1}\right\} 2^k, \tag{3.4.48}$$

$$\sum_{j \geq -k^-} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathcal{Q}_{jk} V^{kg}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \varepsilon Z(k;n) \min\left\{2^{\frac{1}{2}k^+} 2^{\frac{5}{2}k^-}, 2^{\frac{3}{2}k^+} 2^{\frac{1}{2}k^-} \langle t\rangle^{-1}\right\}. \tag{3.4.49}$$

Proof. See the proof of Lemma 3.4.5. □

Remark 3.4.4. Note that we have $\ln\langle t\rangle$ term in the estimate. Unlike the G^{wa} estimate in Lemma 3.4.5 which has $\langle t\rangle^{-H(n+1)}$ decay to kill the logarithmic term, this is a major troublemaker for later proofs since it does not give exact one order decay. We need an improved version (see Lemma 3.4.14).

Lemma 3.4.12. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 2$ *and* $(k,j) \in \mathcal{J}$ *with* $1 \le 2^{2k^-}\langle t\rangle$*, we have for some sufficiently small* $\sigma > 0$*,*

$$\sum_{j \ge -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} V^{kg}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \varepsilon Z(k; n+1) \langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}} 2^{4k^+} 2^{-k^-(\frac{1}{2}-\frac{\sigma}{4})}. \tag{3.4.50}$$

Proof. See the proof of Lemma 3.4.6. □

Lemma 3.4.13. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1$*, we have that for some sufficiently small* $\sigma > 0$*,*

$$\left\| \widehat{\mathscr{Q}_{jk} V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k; n) 2^{\frac{\sigma(j+k)}{8}} 2^{-\frac{j}{2}} 2^{-\frac{3}{2}k}, \tag{3.4.51}$$

$$\left\| \widehat{P_k V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k; n) 2^{-k^-} 2^{-k^+}, \tag{3.4.52}$$

and

$$\left\| \widehat{\mathscr{Q}_{jk} V^{kg}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k; n) 2^{\frac{\sigma(j+k)}{8}} 2^{-\frac{j}{2}} 2^{-k} 2^{-k^+}, \tag{3.4.53}$$

$$\left\| \widehat{P_k V^{kg}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k; n) 2^{-\frac{1}{2}k^-} 2^{-k^+}. \tag{3.4.54}$$

Proof. Based on (3.3.21) in Lemma 3.3.11, we know

$$\left\| \widehat{\mathscr{Q}_{jk} V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim 2^{\frac{j}{2}} 2^{-k} 2^{\frac{\sigma(j+k)}{8}} \left\| Q_{jk} V^{wa}_{\mathcal{L},\infty} \right\|_{H^{0,1}_\Omega}. \tag{3.4.55}$$

From Lemma 3.4.8, we know

$$\left\|Q_{jk}V^{wa}_{\mathcal{L},\infty}\right\|_{H^{0,1}_{\Omega}} \lesssim \varepsilon Z(k;n)2^{-j}2^{-\frac{k}{2}}. \tag{3.4.56}$$

Combining the above two bounds, we have

$$\left\|\widehat{\mathscr{Q}_{jk}V^{wa}_{\mathcal{L},\infty}}\right\|_{L^{\infty}_{\xi}} \lesssim \varepsilon Z(k;n)2^{\frac{\sigma(j+k)}{8}}2^{-\frac{j}{2}}2^{-\frac{3}{2}k}. \tag{3.4.57}$$

Summing up over $j \geq -k^-$, we have

$$\left\|\widehat{P_kV^{wa}_{\mathcal{L},\infty}}\right\|_{L^{\infty}_{\xi}} \lesssim \varepsilon Z(k;n)2^{(\frac{1}{2}-\frac{\sigma}{8})k^-}2^{-(\frac{3}{2}-\frac{\sigma}{8})k}. \tag{3.4.58}$$

For $k \geq 0$, $k = k^+$ and for $k \leq 0$, $k = k^-$. Hence, we arrive at (3.4.52).

Similarly, based on (3.3.21) in Lemma 3.3.11, we know

$$\left\|\widehat{\mathscr{Q}_{jk}V^{kg}_{\mathcal{L},\infty}}\right\|_{L^{\infty}_{\xi}} \lesssim 2^{\frac{j}{2}}2^{-k}2^{\frac{\sigma(j+k)}{8}}\left\|Q_{jk}V^{kg}_{\mathcal{L},\infty}\right\|_{H^{0,1}_{\Omega}}. \tag{3.4.59}$$

From Lemma 3.4.8, we know

$$\left\|Q_{jk}V^{kg}_{\mathcal{L},\infty}\right\|_{H^{0,1}_{\Omega}} \lesssim \varepsilon Z(k;n)2^{-j}2^{-k^+}. \tag{3.4.60}$$

Combining the above two bounds, we have

$$\left\|\widehat{\mathscr{Q}_{jk}V^{kg}_{\mathcal{L},\infty}}\right\|_{L^{\infty}_{\xi}} \lesssim \varepsilon Z(k;n)2^{\frac{\sigma(j+k)}{8}}2^{-\frac{j}{2}}2^{-k}2^{-k^+}. \tag{3.4.61}$$

Summing up over $j \geq -k^-$, we have

$$\left\|\widehat{P_kV^{kg}_{\mathcal{L},\infty}}\right\|_{L^{\infty}_{\xi}} \lesssim \varepsilon Z(k;n)2^{(\frac{1}{2}-\frac{\sigma}{8})k^-}2^{-(1-\frac{\sigma}{8})k}2^{-k^+}. \tag{3.4.62}$$

For $k \geq 0$, $k = k^+$ and for $k \leq 0$, $k = k^-$. Hence, we arrive at (3.4.54). $\square$

Lemma 3.4.14. *Assume* (3.2.104) *holds. Then for any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1 - 2$ *and* $(k, j) \in \mathcal{J}$*, we have*

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{jk} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \varepsilon Z(k; n+1) \langle t \rangle^{-1} 2^k. \tag{3.4.63}$$

Proof. Let $2^{J_0} = \langle t \rangle^{\frac{1}{2}} 2^{-\frac{k}{2}}$. We divide it into the following cases:

- If $2^{k^-} \leq \langle t \rangle^{-1}$, then by Lemma 3.4.11, we have

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{jk} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \varepsilon Z(k; n) 2^{k^-} 2^k \lesssim \varepsilon Z(k; n) \langle t \rangle^{-1} 2^k. \tag{3.4.64}$$

- If $2^{k^-} \geq \langle t \rangle^{-1}$ and $j \leq J_0$, by (3.3.25) in Lemma 3.3.13, we have

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{\leq J_0 k} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \langle t \rangle^{-1} 2^{2k} \left\| Q_{\leq J_0 k} \widehat{V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi}. \tag{3.4.65}$$

Based on Lemma 3.3.10 and Lemma 3.4.9, we have

$$\begin{aligned} \left\| Q_{\leq J_0 k} \widehat{V^{wa}_{\mathcal{L},\infty}} - \mathscr{Q}_{\leq J_0 k} \widehat{V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} &\lesssim \sum_{j \geq -k^-} 2^{-\frac{5}{2}j} 2^{-4k} \left\| P_k V^{wa}_{\mathcal{L},\infty} \right\|_{L^2} \\ &\lesssim 2^{-\frac{3}{2}k^-} \left\| P_k V^{wa}_{\mathcal{L},\infty} \right\|_{L^2} \lesssim \varepsilon Z(k; n) 2^{-k^-}. \end{aligned} \tag{3.4.66}$$

Also, based on the proof of Lemma 3.4.13, we know

$$\left\| \mathscr{Q}_{\leq J_0 k} \widehat{V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \left\| \widehat{P_k V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k; n) 2^{-k^-}. \tag{3.4.67}$$

In total, we have

$$\left\| \widehat{Q_{\leq J_0 k} V^{wa}_{\mathcal{L},\infty}} \right\|_{L^\infty_\xi} \lesssim \varepsilon Z(k;n) 2^{-k^-}. \tag{3.4.68}$$

Hence, we have

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{\leq J_0 k} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \varepsilon Z(k;n) \langle t \rangle^{-1} 2^{2k} 2^{-k^-}. \tag{3.4.69}$$

For $k \geq 0$, $k^- = 0$, so the desired result naturally holds. For $k \leq 0$, $k = k^-$, so we know $2^{2k} 2^{-k^-} = 2^{k^-}$. Hence, our desired result is true.

- If $2^{k^-} \geq \langle t \rangle^{-1}$ and $j \geq J_0$ and $2^k \lesssim \langle t \rangle$ and $2^j \leq \langle t \rangle$, by (3.3.24) in Lemma 3.3.13, we have

$$\left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{\geq J_0 k} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} \lesssim \sum_{j \geq J_0} \langle t \rangle^{-1} 2^{\frac{k}{2}} \left(1 + 2^k \langle t \rangle \right)^{\frac{\sigma}{8}} \left\| Q_{jk} V^{wa}_{\mathcal{L},\infty} \right\|_{H^{0,1}_\Omega}. \tag{3.4.70}$$

Lemma 3.4.8 implies

$$\left\| Q_{jk} V^{wa}_{\mathcal{L},\infty} \right\|_{H^{0,1}_\Omega} \lesssim \varepsilon Z(k;n+1) 2^{-j} 2^{-\frac{k}{2}}. \tag{3.4.71}$$

Combining them together, we have

$$\begin{aligned} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} \mathscr{Q}_{\geq J_0 k} V^{wa}_{\mathcal{L},\infty} \right\|_{L^\infty} &\lesssim \sum_{j \geq J_0} \varepsilon Z(k;n+1) \langle t \rangle^{-1} 2^{-j} \left(1 + 2^k \langle t \rangle \right)^{\frac{\sigma}{8}} \\ &\lesssim \sum_{j \geq J_0} \varepsilon Z(k;n+1) \langle t \rangle^{-1-\frac{\sigma}{8}} 2^{\frac{\sigma}{8}k} 2^{-j} \\ &\lesssim \varepsilon Z(k;n+1) \langle t \rangle^{-\frac{3}{2}-\frac{\sigma}{8}} 2^{(\frac{1}{2}+\frac{\sigma}{8})k}. \end{aligned} \tag{3.4.72}$$

Then for $\sigma > 0$ small, our result naturally holds.

- If $2^{k^-} \geq \langle t \rangle^{-1}$ and $j \geq J_0$ and $2^k \lesssim \langle t \rangle$ and $2^j \geq \langle t \rangle$, then $2^{-j} \leq \langle t \rangle^{-1}$. Using

Lemma 3.4.10, we have

$$\left\|e^{-it\Lambda_{wa}}\mathscr{Q}_{\geq J_0 k}V^{wa}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \varepsilon Z(k;n)\sum_{2^{-j}\leq\langle t\rangle^{-1}} 2^{-j}2^k \lesssim \varepsilon Z(k;n)\langle t\rangle^{-1}2^k. \quad (3.4.73)$$

- If $2^{k^-}\geq\langle t\rangle^{-1}$ and $j\geq J_0$ and $2^k\gtrsim\langle t\rangle$, we know $k\geq 0$ and $k=k^+$. Hence, by Sobolev embedding theorem, we have

$$\begin{aligned}\left\|e^{-it\Lambda_{wa}}\mathscr{Q}_{\geq J_0 k}V^{wa}_{\mathcal{L},\infty}\right\|_{L^\infty} &\lesssim \left\|P_k V^{wa}_{\mathcal{L},\infty}\right\|_{L^\infty} \lesssim \left\|P_k V^{wa}_{\mathcal{L},\infty}\right\|_{H^2} \\ &\lesssim \varepsilon 2^{2k^+}\left\||\nabla|^{-\frac{1}{2}}V^{wa}_{\mathcal{L},\infty}\right\|_{H^{N(0)}} \lesssim \varepsilon Z(k;n)\langle t\rangle^{-N_0}.\end{aligned} \quad (3.4.74)$$

This suffices to close the proof.

□

Remark 3.4.5. Compared with Lemma 3.4.11, the key improvement here is that we get rid of $\ln\langle t\rangle$. This is crucial for later estimates.

3.4.4 Estimates of h_∞ and $\mathcal{H}_\infty$

The low-frequency part $h_\infty(t,\xi)$ and $\mathcal{H}_\infty(t,\xi)$ of the wave component are defined in (3.2.64) and (3.2.65). Note that the low-frequency cutoff function in h_∞ and $\mathcal{H}_\infty$ implies that $|\xi|\lesssim\langle t\rangle^{-\frac{7}{8}}\leq 1$. Hence, all estimates in this subsection do not have k^+ part.

Lemma 3.4.15. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k h_\infty\|_{L^2_\xi} \lesssim \varepsilon^2\langle t\rangle^{-1}2^{\frac{1}{2}k^-}, \quad (3.4.75)$$

and

$$\|\varphi_k h_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-1} 2^{-k}. \tag{3.4.76}$$

Also, we have

$$\|\varphi_k \partial_{\xi_\ell} h_\infty\|_{L^2_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k^-}, \tag{3.4.77}$$

and

$$\|\varphi_k \partial_{\xi_\ell} h_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-k}. \tag{3.4.78}$$

Proof. Based on (3.2.64), we know

$$h_\infty(t,\xi) \simeq \varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big) \cdot \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}\right)} \left|\widehat{V^{kg}_\infty}(\eta)\right|^2 \mathrm{d}\eta. \tag{3.4.79}$$

Let $\psi(\xi,\eta) := |\xi| - \frac{\xi\cdot\eta}{\langle\eta\rangle}$. Then $\nabla_\eta\psi = \frac{\eta(\xi\cdot\eta) - \xi\langle\eta\rangle^2}{\langle\eta\rangle^3}$ and $\nabla^2_\eta\psi = \frac{6(\xi\cdot\eta)\langle\eta\rangle^2 - 3(\xi\cdot\eta)|\eta|^2}{\langle\eta\rangle^5}$, which yields $|\nabla_\eta\psi| \simeq |\xi|\,\langle\eta\rangle^{-1}$ and $\left|\nabla^2_\eta\psi\right| \simeq |\xi|\,\langle\eta\rangle^{-2}$. Using integration by parts in η (see (3.3.3)) and estimates in Lemma 3.4.7, we obtain

$$\|\varphi_k h_\infty\|_{L^\infty_\xi} \lesssim \langle t\rangle^{-1} 2^{-k} \left\|\widehat{V^{kg}_\infty}\right\|_{L^2_\eta} \left\|\widehat{V^{kg}_\infty}\right\|_{H^1_\eta} \lesssim \varepsilon^2 2^{-k} \langle t\rangle^{-1}. \tag{3.4.80}$$

Also, by Hölder's inequality and taking into account the measure of the support of φ_k, we have

$$\|\varphi_k h_\infty\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}k} \|\varphi_k h_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k} \langle t\rangle^{-1}. \tag{3.4.81}$$

Noting that

$$\partial_{\xi_\ell} h_\infty(t,\xi) \simeq \langle t\rangle \varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\Big|\widehat{V_\infty^{kg}}(\eta)\Big|^2 \mathrm{d}\eta, \tag{3.4.82}$$

the $\partial_{\xi_\ell} h_\infty$ estimates follow. $\square$

Lemma 3.4.16. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k \mathcal{H}_\infty\|_{L^2_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k^-}, \tag{3.4.83}$$

and

$$\|\varphi_k \mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-k}. \tag{3.4.84}$$

Also, we have

$$\|\varphi_k \partial_{\xi_\ell}\mathcal{H}_\infty\|_{L^2_\xi} \lesssim \varepsilon^2 2^{-\frac{1}{2}k^-}, \tag{3.4.85}$$

and

$$\|\varphi_k \partial_{\xi_\ell}\mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-2k}. \tag{3.4.86}$$

Proof. We first use integration in time (see (3.3.2)) to obtain

$$\begin{aligned}\mathcal{H}_\infty(t,\xi) &\simeq \int_0^t \varphi_{\leq 0}\big(\xi\langle s\rangle^{\frac{7}{8}}\big)\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}s\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\Big|\widehat{V_\infty^{kg}}(\eta)\Big|^2 \mathrm{d}\eta\mathrm{d}s \\ &\simeq \int_{\mathbb{R}^3} \frac{1}{\mathrm{i}\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\left(\mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}-1\right)\Big|\widehat{V_\infty^{kg}}(\eta)\Big|^2\mathrm{d}\eta.\end{aligned} \tag{3.4.87}$$

Here the cutoff $\varphi_{\leq 0}\big(\xi\langle s\rangle^{\frac{7}{8}}\big)$ only sets an upper bound $|\xi|^{-\frac{8}{7}}$ for the time integration limit. It will not intervene the integral estimate. Notice that

$$\left|\frac{1}{\mathrm{i}\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\left(\mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}-1\right)\right| \lesssim |\xi|^{-1}\langle\eta\rangle. \tag{3.4.88}$$

Hence, using Lemma 3.4.7, we obtain

$$\|\varphi_k\mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim 2^{-k}\int_{\mathbb{R}^3}\langle\eta\rangle\left|\widehat{V^{kg}_\infty}(\eta)\right|^2\mathrm{d}\eta \lesssim \varepsilon^2 2^{-k}, \tag{3.4.89}$$

and

$$\|\varphi_k\mathcal{H}_\infty\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}k}\|\varphi_k\mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{\frac{1}{2}k^-}. \tag{3.4.90}$$

On the other hand, note that

$$\partial_{\xi_\ell}\mathcal{H}_\infty \simeq \int_0^t \varphi_{\le 0}\big(\xi\langle s\rangle^{\frac{7}{8}}\big)\int_{\mathbb{R}^3}\langle s\rangle\mathrm{e}^{\mathrm{i}s\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\left|\widehat{V^{kg}_\infty}(\eta)\right|^2\mathrm{d}\eta\mathrm{d}s. \tag{3.4.91}$$

The key is to handle the extra $\langle s\rangle$. Integration by parts in η as in the proof of Lemma 3.4.15 creates $\langle s\rangle^{-1}|\xi|^{-1}$ (thus the net effect is to lose $|\xi|^{-1}$). We then apply the above argument to obtain

$$\|\varphi_k\partial_{\xi_\ell}\mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim 2^{-2k}\left\|\widehat{V^{kg}_\infty}\right\|^2_{H^1_\eta} \lesssim \varepsilon^2 2^{-2k}, \tag{3.4.92}$$

and

$$\|\varphi_k\partial_{\xi_\ell}\mathcal{H}_\infty\|_{L^2_\xi} \lesssim 2^{\frac{3}{2}k}\|\varphi_k\partial_{\xi_\ell}\mathcal{H}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon^2 2^{-\frac{1}{2}k^-}. \tag{3.4.93}$$

□

Lemma 3.4.17. *Assume* (3.2.104) *holds. Then for* $(k,j)\in\mathcal{J}$, *we have*

$$\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}\big(\mathscr{F}^{-1}\mathcal{H}_\infty\big)\right\|_{L^\infty}\lesssim\varepsilon^2\min\Big(2^{-j},\langle t\rangle^{-1}\Big)2^k,\tag{3.4.94}$$

and

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}\big(\mathscr{F}^{-1}\mathcal{H}_\infty\big)\right\|_{L^\infty}\lesssim\varepsilon^2\min\Big(2^{k^-},\ \ln\langle t\rangle\langle t\rangle^{-1}\Big)2^k\tag{3.4.95}$$

Proof. It is similar to the proof of Lemma 3.4.10 and Lemma 3.4.11 using Lemma 3.4.16. □

Lemma 3.4.18. *Assume* (3.2.104) *holds. Then we have*

$$\sum_{j\geq -k^-}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathscr{Q}_{jk}\big(\mathscr{F}^{-1}\mathcal{H}_\infty\big)\right\|_{L^\infty}\lesssim\varepsilon^2\langle t\rangle^{-1}2^k.\tag{3.4.96}$$

Proof. It is similar to the proof of Lemma 3.4.14 using Lemma 3.4.16. □

3.4.5 Estimates of $\mathcal{C}_\infty$ and $\mathcal{D}_\infty$

The definitions of $\mathcal{C}_\infty(t,\xi)$ and $\mathcal{D}_\infty(t,\xi)$ are given in (3.2.67) and (3.2.68).

Lemma 3.4.19. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k\mathcal{C}_\infty\|_{L^\infty_\xi}\lesssim\varepsilon 2^{3k}\langle t\rangle^{-1}\ln\langle t\rangle,\tag{3.4.97}$$

and

$$\|\varphi_k\partial_{\xi_\ell}\mathcal{C}_\infty\|_{L^\infty_\xi}\lesssim\varepsilon 2^{5k}\langle t\rangle^{-1}\ln\langle t\rangle.\tag{3.4.98}$$

Proof. Recall that

$$\mathcal{C}_\infty(t,\xi) \simeq \frac{|\xi|^2}{\langle\xi\rangle}\mathrm{Im}\left\{\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}H_\infty(t,\eta)\,\mathrm{d}\eta\right\}, \tag{3.4.99}$$

where

$$H_\infty(t,\xi) = \varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\left\{\widehat{V_\infty^{wa}}(\xi) + \mathcal{H}_\infty(t,\xi)\right\}. \tag{3.4.100}$$

We will focus on $\mathcal{H}_\infty$ part, and the similar techniques can apply to $\widehat{V_\infty^{wa}}$ part. We look at

$$\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\varphi_{\leq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta \simeq \int_{|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta\,. \tag{3.4.101}$$

From Lemma 3.4.16, we know

$$|\mathcal{H}_\infty(t,\xi)| \lesssim \varepsilon^2\,|\xi|^{-1}\,, \tag{3.4.102}$$

$$|\partial_{\xi_\ell}\mathcal{H}_\infty(t,\xi)| \lesssim \varepsilon^2\,|\xi|^{-2}\,. \tag{3.4.103}$$

For $|\eta| \lesssim \langle t\rangle^{-1}$, we directly integrate to obtain

$$\left|\int_{|\eta|\lesssim\langle t\rangle^{-1}} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta\right| \lesssim \varepsilon^2\int_{|\eta|\lesssim\langle t\rangle^{-1}}|\eta|^{-2}\,\mathrm{d}\eta \lesssim \varepsilon^2\langle t\rangle^{-1}. \tag{3.4.104}$$

For $\langle t\rangle^{-1} \lesssim |\eta| \lesssim \langle t\rangle^{-\frac{7}{8}}$, we use integration by parts in η following (3.3.3) with

$\psi(\xi,\eta) := \frac{\xi\cdot\eta}{\langle\xi\rangle} - |\eta|$. Since $|\nabla_\eta\psi| \simeq |\xi|^{-1}$ and $\left|\nabla_\eta^2\psi\right| \simeq |\eta|^{-1}$, we get

$$\begin{aligned} &\left|\int_{\langle t\rangle^{-1}\lesssim|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta\right| \\ &\lesssim \varepsilon^2\,|\xi|^2\,\langle t\rangle^{-1}\int_{\langle t\rangle^{-1}\lesssim|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}} |\eta|^{-3}\,\mathrm{d}\eta \lesssim \varepsilon^2\,|\xi|^2\,\langle t\rangle^{-1}\ln\langle t\rangle. \end{aligned} \tag{3.4.105}$$

Therefore, from the definition of $\mathcal{C}_\infty$, we have

$$|\mathcal{C}_\infty(t,\xi)| \lesssim \varepsilon^2\,|\xi|^3\,\langle t\rangle^{-1}\ln\langle t\rangle. \tag{3.4.106}$$

Next we turn to the bound of $\nabla_\xi\mathcal{C}_\infty$. If ∇_ξ hits $\frac{\xi}{\langle\xi\rangle^2}$, then following the proof of Lemma 3.4.19, we get the desired result. Hence, we focus on the case where ∇_ξ hits the integral:

$$\begin{aligned} \nabla_\xi\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta &\simeq \int_{\mathbb{R}^3}|\langle t\rangle\eta|\,\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta \\ &\simeq \int_{\mathbb{R}^3}\langle t\rangle\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta. \end{aligned} \tag{3.4.107}$$

Note that now we do not have the singularity of $|\eta|^{-1}$ anymore. Using again integration by parts in η, we obtain

$$\begin{aligned} &\int_{\mathbb{R}^3}\langle t\rangle\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta \\ &\simeq |\xi|^2\int_{|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta + |\xi|^2\int_{|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\nabla_\eta\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta. \end{aligned} \tag{3.4.108}$$

Then for both terms, we may follow the above discussion for $\mathcal{C}_\infty$ estimate to get

$$\left|\int_{|\eta|\lesssim\langle t\rangle^{-\frac{7}{8}}}\langle t\rangle\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\mathcal{H}_\infty(t,\eta)\,\mathrm{d}\eta\right| \lesssim \varepsilon^2\,|\xi|^5\,\langle t\rangle^{-1}\ln\langle t\rangle, \tag{3.4.109}$$

which concludes the proof of (3.4.98). □

Lemma 3.4.20. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k \mathcal{D}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon 2^{3k}\big(\ln\langle t\rangle\big)^2, \tag{3.4.110}$$

and

$$\|\varphi_k \partial_{\xi_\ell} \mathcal{D}_\infty\|_{L^\infty_\xi} \lesssim \varepsilon 2^{5k}\big(\ln\langle t\rangle\big)^2. \tag{3.4.111}$$

Proof. It follows from taking time integral of Lemma 3.4.19. □

Lemma 3.4.21. *Assume* (3.2.104) *holds. Then for* $1 \le n \le N_1$*, we have*

$$\left\|\varphi_k \Big[\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)\Big]_{\mathcal{L}}\right\|_{L^2_\xi} \lesssim \varepsilon\big(\ln\langle t\rangle\big)^2 Z(k;n-1)2^{k^-}. \tag{3.4.112}$$

Also, for $1 \le n \le N_1 - 1$*, we have*

$$\left\|\varphi_k \Big[\partial_{\xi_\ell}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)\Big)\Big]_{\mathcal{L}}\right\|_{L^2_\xi} \lesssim \varepsilon\big(\ln\langle t\rangle\big)^2 Z(k;n-1)2^{k^-}. \tag{3.4.113}$$

Proof. Let $\widehat{W^{kg}}(t,\xi) := \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)$. Based on (3.2.26), we have

$$\widehat{W^{kg}_{\mathcal{L}}}(t,\xi) \sim \mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)}\widehat{\mathcal{L}}\Big[\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\widehat{W^{kg}}(t,\xi)\Big]. \tag{3.4.114}$$

(Here we use $\widehat{\mathcal{L}}$ to denote the corresponding vector field under Fourier transform.)

We first consider $n = 1$ case. For rotation vector fields, we know

$$\widehat{\Omega}_{jk}\Big(\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\widehat{W^{kg}}(t,\xi)\Big) = \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\bigg[\Big(\widehat{\Omega}_{jk}\,\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\Big)\widehat{V^{kg}_\infty}(\xi) + \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\Big(\widehat{\Omega}_{jk}\widehat{V^{kg}_\infty}(\xi)\Big)\bigg]. \tag{3.4.115}$$

Hence,

$$\Big\|\varphi_k\widehat{W^{kg}_{\mathcal{L}}}(t,\xi)\Big\|_{L^2_\xi}\lesssim\big\|\widehat{\Omega}_{jk}\mathcal{C}_\infty(t,\xi)\big\|_{L^\infty_\xi}\Big\|\widehat{V^{kg}_\infty}(\xi)\Big\|_{L^2_\xi}+\Big\|\widehat{\Omega}_{jk}\widehat{V^{kg}_\infty}(\xi)\Big\|_{L^2_\xi}\lesssim\varepsilon\ln\langle t\rangle Z(k;n-1)2^{k^-}. \tag{3.4.116}$$

For Lorentz vector fields, we know

$$\begin{aligned}
&\widehat{\Gamma}_j\left(\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\widehat{W^{kg}}(t,\xi)\right) \qquad (3.4.117)\\
&=\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\bigg\{\mathrm{i}\langle\xi\rangle\Big[\partial_{\xi_j}\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)+\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\partial_{\xi_j}\widehat{V^{kg}_\infty}(\xi)\Big]\\
&\quad+\mathrm{i}\frac{\xi_j}{\langle\xi\rangle}\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)+\mathrm{i}t\frac{\xi_j}{\langle\xi\rangle}\mathcal{C}_\infty(t,\xi)\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)\\
&\quad+\Big[\partial_{\xi_j}\mathcal{C}_\infty(t,\xi)\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)+\mathcal{C}_\infty(t,\xi)\partial_{\xi_j}\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\widehat{V^{kg}_\infty}(\xi)+\mathcal{C}_\infty(t,\xi)\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)}\partial_{\xi_j}\widehat{V^{kg}_\infty}(\xi)\Big]\bigg\}.
\end{aligned}$$

Hence,

$$\begin{aligned}
&\Big\|\varphi_k\widehat{W^{kg}_{\mathcal{L}}}(t,\xi)\Big\|_{L^2_\xi} \qquad (3.4.118)\\
&\lesssim\Big(1+\big\|\nabla_\xi\mathcal{D}_\infty(t,\xi)\big\|_{L^\infty_\xi}+\big\|\nabla_\xi\mathcal{C}_\infty(t,\xi)\big\|_{L^\infty_\xi}+\big\|\langle t\rangle\mathcal{C}_\infty(t,\xi)\big\|_{L^\infty_\xi}\Big)\\
&\quad\times\left(\Big\|\widehat{V^{kg}_\infty}(\xi)\Big\|_{L^2_\xi}+\Big\|\nabla_\xi\widehat{V^{kg}_\infty}(\xi)\Big\|_{L^2_\xi}\right)\\
&\lesssim\varepsilon\big(\ln\langle t\rangle\big)^2Z(k;n-1)2^{k^-}.
\end{aligned}$$

In summary, we have verified that for $n=1$,

$$\Big\|\varphi_k\widehat{W^{kg}_{\mathcal{L}}}(t,\xi)\Big\|_{L^2_\xi}\lesssim\varepsilon\big(\ln\langle t\rangle\big)^2Z(k;n-1)2^{k^-}. \tag{3.4.119}$$

For $n>1$, we may continue the above process. However, the newly created terms are always combination of the above. With the techniques in Lemma 3.4.19 and Lemma 3.4.20 in hand, the result will follow. The similar argument also applies to the ∂_{ξ_ℓ}

estimate. □

Lemma 3.4.22. *Assume* (3.2.104) *holds. Then for* $1 \leq n \leq N_1$ *and* $(k,j) \in \mathcal{J}$, *we have*

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} \left\{ \mathscr{F}^{-1} \left[\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)} \widehat{V_\infty^{kg}}(\xi) \right] \right\}_{\mathcal{L}} \right\|_{L^\infty} \tag{3.4.120}$$
$$\lesssim \varepsilon \big(\ln\langle t \rangle \big)^2 Z(k; n-1) \min \Big(2^{\frac{1}{2}k^+} 2^{\frac{5}{2}k^-}, 2^{\frac{3}{2}k^+} 2^{\frac{1}{2}k^-} \langle t \rangle^{-1} \Big).$$

Proof. It is similar to Lemma 3.4.11 using Lemma 3.4.21. □

Lemma 3.4.23. *Assume* (3.2.104) *holds. Then for* $0 < n \leq N_1$ *and* $(k,j) \in \mathcal{J}$ *with* $1 \leq 2^{2k^-}\langle t \rangle$, *we have for some* σ *sufficiently small,*

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} \left\{ \mathscr{F}^{-1} \left[\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)} \widehat{V_\infty^{kg}}(\xi) \right] \right\}_{\mathcal{L}} \right\|_{L^\infty} \tag{3.4.121}$$
$$\lesssim \varepsilon \langle t \rangle^{-\frac{3}{2}+\frac{\sigma}{8}} \big(\ln\langle t \rangle \big)^2 Z(k; n) 2^{4k^+} 2^{-k^-(\frac{1}{2}-\frac{\sigma}{4})}.$$

Proof. It is similar to Lemma 3.4.12 using Lemma 3.4.21. □

Remark 3.4.6. For $n = 0$ case, the result also holds, but it does not have $\ln\langle t \rangle$ since there is no vector fields applied.

3.4.6 Estimates of $\mathfrak{b}_\infty$ and $\mathfrak{B}_\infty$

Recall that $\mathfrak{b}_\infty$ and $\mathfrak{B}_\infty$ are non-resonant contributions defined in (3.2.72) and (3.2.71).

Lemma 3.4.24. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k \mathfrak{b}_\infty\|_{L^2} \lesssim \varepsilon^2 \langle t \rangle^{-1} 2^{-N(n)k}, \tag{3.4.122}$$

and

$$\|\varphi_k \partial_{\xi_\ell} \mathfrak{b}_\infty\|_{L^2} \lesssim \varepsilon^2 2^{-N(n)k}. \tag{3.4.123}$$

Proof. Note that

$$\begin{aligned}\mathfrak{b}_\infty \simeq & \sum_{\iota_2 \in \{+,-\}} \sum_{k_1,k_2} \int_{\mathbb{R}^3} \varphi_{\leq 0}\big(\eta \langle t\rangle^{\frac{7}{8}}\big) \mathrm{e}^{\mathrm{i}t(\langle\xi\rangle + \langle\xi-\eta\rangle - \iota_2|\eta|)} \frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|} \\ & \times \left(\varphi_{k_1} \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V_\infty^{kg,-}}\right)(t,\xi-\eta) \varphi_{k_2} H_\infty^{t,\iota_2}(\eta)\,\mathrm{d}\eta.\end{aligned} \tag{3.4.124}$$

Since H_∞ has a cutoff function $\varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)$, using Lemma 3.4.7 and Lemma 3.4.18, we have

$$\begin{aligned}\|\varphi_k \mathfrak{b}_\infty\|_{L^2} & \lesssim \sum_{k_1,k_2} 2^{k_1} 2^{-k_2} \left\|\varphi_{k_1} \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V_\infty^{kg,-}}\right\|_{L^2} \left\|\mathrm{e}^{\mathrm{i}t\Lambda_{wa}} P_{k_2}\big(\mathscr{F}^{-1} H_\infty^{\iota_2}\big)\right\|_{L^\infty} \\ & \lesssim \sum_{k_1,k_2} 2^{k_1} 2^{-k_2} \left(\varepsilon 2^{-N(N_0)k_1^+}\right)\left(\varepsilon \langle t\rangle^{-1} 2^{-N(N_0)k_2^+} 2^{k_2}\right) \\ & \lesssim \sum_{k_1,k_2} \varepsilon^2 \langle t\rangle^{-1} 2^{k_1} 2^{-N_0 k_1^+ - N_0 k_2^+}.\end{aligned}$$

Then the result naturally follows from Lemma 3.3.3. The ξ derivative estimates follow with an extra $\langle t\rangle$ created. $\square$

Lemma 3.4.25. *Assume* (3.2.104) *holds. Then we have*

$$\|\varphi_k \mathfrak{B}_\infty\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-\frac{7}{8}} \ln\langle t\rangle 2^{-N(n)k}, \tag{3.4.125}$$

and

$$\|\varphi_k \partial_{\xi_\ell} \mathfrak{B}_\infty\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{\frac{1}{8}} \ln\langle t\rangle 2^{-N(n)k}. \tag{3.4.126}$$

Proof. Note that

$$\mathfrak{B}_\infty \simeq \mathrm{e}^{\mathrm{i}\mathcal{D}_\infty(t,\xi)} \int_t^\infty \mathrm{e}^{-\mathrm{i}\mathcal{D}_\infty(s,\xi)} \mathfrak{b}_\infty(s,\xi)\,\mathrm{d}s. \tag{3.4.127}$$

For the non-resonant case, $\psi(\xi,\eta) := \langle\xi\rangle + \langle\xi-\eta\rangle - \iota_2\,|\eta|$ satisfies $|\psi| \gtrsim 1$. Noticing that $\mathfrak{b}_\infty(\infty,\xi) = 0$ due to the cutoff function, we then integrate by parts in time using (3.3.2) to obtain

$$\begin{aligned}
&\int_t^\infty \mathrm{e}^{-\mathrm{i}\mathcal{D}_\infty(s,\xi)} \mathfrak{b}_\infty(s,\xi)\,\mathrm{d}s \qquad (3.4.128)\\
&= \int_t^\infty \int_{\mathbb{R}^3} \varphi_{\le 0}\big(\eta\langle s\rangle^{\frac{7}{8}}\big) \mathrm{e}^{-\mathrm{i}\mathcal{D}_\infty(s,\xi)} \mathrm{e}^{\mathrm{i}s\psi} \frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|} \Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V_\infty^{kg,-}}\Big)(s,\xi-\eta)\big(V_\infty^{wa,\iota_2} + \mathcal{H}_\infty^{\iota_2}\big)(s,\eta)\,\mathrm{d}\eta\mathrm{d}s\\
&= \int_{\mathbb{R}^3} \varphi_{\le 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big) \frac{1}{\mathrm{i}\psi} \mathrm{e}^{-\mathrm{i}\mathcal{D}_\infty(t,\xi)} \mathrm{e}^{\mathrm{i}t\psi} \frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|} \Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V_\infty^{kg,-}}\Big)(t,\xi-\eta)\big(V_\infty^{wa,\iota_2} + \mathcal{H}_\infty^{\iota_2}\big)(t,\eta)\,\mathrm{d}\eta\\
&\quad - \int_t^\infty \int_{\mathbb{R}^3} \frac{1}{\mathrm{i}\psi} \frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|} \mathrm{e}^{\mathrm{i}s\psi}\\
&\qquad\quad \times \partial_s\Big(\varphi_{\le 0}\big(\eta\langle s\rangle^{\frac{7}{8}}\big) \mathrm{e}^{-\mathrm{i}\mathcal{D}_\infty(s,\xi)} \Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V_\infty^{kg,-}}\Big)(s,\xi-\eta)\big(V_\infty^{wa,\iota_2} + \mathcal{H}_\infty^{\iota_2}\big)(s,\eta)\Big)\,\mathrm{d}\eta\mathrm{d}s.
\end{aligned}$$

In the following estimates, we will focus on the $\mathcal{H}_\infty^{\iota_2}$ part and the bounds for V_∞^{wa,ι_2} is similar and even simpler. on the right-hand side of (3.4.128), we call the first integral M_1, and split the second integral into M_2, M_3, M_4 when ∂_s hits different parts. The first term M_1 in (3.4.128) does not have time integral, so we may directly bound its L^2 norm by Lemma 3.4.7 and Lemma 3.4.16 as

$$\begin{aligned}
\|\varphi_k M_1\|_{L^2} &\lesssim \sum_{k_1,k_2} 2^{\frac{3}{2}\min(k,k_1,k_2)} 2^{k_1} 2^{-k_2} \Big\|\varphi_{k_1} \widehat{V_\infty^{kg,-}}\Big\|_{L^2} \|\varphi_{k_2}\mathcal{H}_\infty^{\iota_2}\|_{L^2} \qquad (3.4.129)\\
&\lesssim \sum_{k_1,k_2} 2^{\frac{1}{2}k_2} 2^{k_1} \Big(\varepsilon Z(k_1;n) 2^{-k_1^+}\Big)\Big(\varepsilon^2 2^{\frac{1}{2}k_2^-}\Big) \lesssim \varepsilon^3 \langle t\rangle^{-\frac{7}{8}} Z(k;n).
\end{aligned}$$

For the second term in (3.4.128), we further discuss the effects of ∂_s. Note that $\partial_s\varphi_{\le 0}\big(\eta\langle s\rangle^{\frac{7}{8}}\big) \simeq |\eta|\langle s\rangle^{-\frac{1}{8}}\varphi_{\le 0}\big(\eta\langle s\rangle^{\frac{7}{8}}\big)$. Then the resulting quantity M_2 is bounded by

Lemma 3.4.7 and Lemma 3.4.16 as

$$\begin{aligned}\|\varphi_k M_2\|_{L^2} &\lesssim \sum_{k_1,k_2}\int_t^\infty \langle s\rangle^{-\frac{1}{8}} 2^{\frac{3}{2}\min(k,k_1,k_2)} 2^{k_1}\left\|\varphi_{k_1}\widehat{V_\infty^{kg,-}}\right\|_{L^2}\|\varphi_{k_2}\mathcal{H}_\infty^{\iota_2}\|_{L^2} \\ &\lesssim \sum_{k_1,k_2}\int_t^\infty \langle s\rangle^{-\frac{1}{8}} 2^{\frac{3}{2}k_2}2^{k_1}\left(\varepsilon Z(k_1;n)2^{-k_1^+}\right)\left(\varepsilon^2 2^{\frac{1}{2}k_2^-}\right) \\ &\lesssim \int_t^\infty \varepsilon^3\langle s\rangle^{-\frac{15}{8}} Z(k;n) \lesssim \varepsilon^3\langle t\rangle^{-\frac{7}{8}}Z(k;n).\end{aligned} \tag{3.4.130}$$

Next, since $\partial_s e^{i\mathcal{D}_\infty(s,\xi)} = i\,\mathcal{C}_\infty(s,\xi)e^{i\mathcal{D}_\infty(s,\xi)}$, the resulting quantity M_3 is bounded using Lemma 3.4.7, Lemma 3.4.19 and Lemma 3.4.16 as

$$\begin{aligned}\|\varphi_k M_3\|_{L^2} &\lesssim \sum_{k_1,k_2}\int_t^\infty 2^{\frac{3}{2}\min(k,k_1,k_2)}2^{k_1}2^{-k_2}\left(\varepsilon 2^{3k_1}\langle s\rangle^{-1}\ln\langle s\rangle\right)\left\|\varphi_{k_1}\widehat{V_\infty^{kg,-}}\right\|_{L^2}\|\varphi_{k_2}\mathcal{H}_\infty^{\iota_2}\|_{L^2} \\ &\lesssim \sum_{k_1,k_2}\int_t^\infty \varepsilon\langle s\rangle^{-1}\ln\langle s\rangle 2^{\frac{1}{2}k_2}2^{4k_1}\left(\varepsilon Z(k_1;n)2^{-k_1^+}\right)\left(\varepsilon^2 2^{\frac{1}{2}k_2^-}\right) \\ &\lesssim \int_t^\infty \varepsilon^4\langle s\rangle^{-\frac{15}{8}}\ln\langle s\rangle Z(k;n) \lesssim \varepsilon^4\langle t\rangle^{-\frac{7}{8}}\ln\langle t\rangle Z(k;n).\end{aligned} \tag{3.4.131}$$

Finally, since $\partial_s\mathcal{H}_\infty^{\iota_2} = h_\infty^{\iota_2}$, the resulting quantity M_4 is bounded using Lemma 3.4.7 and Lemma 3.4.15 as

$$\begin{aligned}\|\varphi_k M_4\|_{L^2} &\lesssim \sum_{k_1,k_2}\int_t^\infty 2^{\frac{3}{2}\min(k,k_1,k_2)}2^{k_1}2^{-k_2}\left\|\varphi_{k_1}\widehat{V_\infty^{kg,-}}\right\|_{L^2}\|\varphi_{k_2}h_\infty^{\iota_2}\|_{L^2} \\ &\lesssim \sum_{k_1,k_2}\int_t^\infty 2^{\frac{1}{2}k_2}2^{k_1}\left(\varepsilon Z(k_1;n)2^{-k_1^+}\right)\left(\varepsilon^2\langle s\rangle^{-1}2^{\frac{1}{2}k_2^-}\right) \\ &\lesssim \int_t^\infty \varepsilon^3\langle s\rangle^{-\frac{15}{8}}Z(k;n) \lesssim \varepsilon^3\langle t\rangle^{-\frac{7}{8}}Z(k;n).\end{aligned} \tag{3.4.132}$$

Summarizing all above, we have that

$$\|\varphi_k\mathfrak{B}_\infty\|_{L^2} \lesssim \varepsilon^3\langle t\rangle^{-\frac{7}{8}}\ln\langle t\rangle Z(k;n). \tag{3.4.133}$$

The ξ derivative estimates follow with an extra $\langle t\rangle$ created. $\square$

Remark 3.4.7. Comparing Lemma 3.4.24 and Lemma 3.4.25, we can clearly see that the estimate of $\mathfrak{B}_\infty$ is much better than direct integrating over time in the estimate of $\mathfrak{b}_\infty$. The main reason is that $\mathfrak{B}_\infty$ has time integration, which allows us to make use of integration by parts in time (normal forms) for the non-resonant case. However, without time integration, $\mathfrak{b}_\infty$ estimate is similar to that of the resonant terms, which decays rather slowly.

Lemma 3.4.26. *Assume* (3.2.104) *holds. Then for* $(k, j) \in \mathcal{J}$*, we have*

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} \mathfrak{B}_\infty \right\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-\frac{7}{8}} \ln\langle t \rangle Z(k; n-1) \min\left(2^{\frac{1}{2}k^+} 2^{\frac{5}{2}k^-},\ 2^{\frac{3}{2}k^+} 2^{\frac{1}{2}k^-} \langle t \rangle^{-1} \right). \tag{3.4.134}$$

Proof. It is similar to the proof of Lemma 3.4.11 using Lemma 3.4.25. □

Lemma 3.4.27. *Assume* (3.2.104) *holds. Then for* $(k, j) \in \mathcal{J}$ *with* $1 \leq 2^{2k^-}\langle t \rangle$*, we have for some* $\sigma > 0$*,*

$$\sum_{j \geq -k^-} \left\| \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} \mathscr{Q}_{jk} \mathfrak{B}_\infty \right\|_{L^\infty} \lesssim \varepsilon \langle t \rangle^{-\frac{7}{8} - \frac{3}{2} + \frac{\sigma}{8}} \ln\langle t \rangle Z(k; n) 2^{4k^+} 2^{-k^-(\frac{1}{2} - \frac{\sigma}{4})}. \tag{3.4.135}$$

Proof. It is similar to the proof of Lemma 3.4.12 using Lemma 3.4.25. □

3.5 Bounds on the Nonlinearities of Wave Equation

Based on (3.2.74), in order to control $\partial_t G^{wa}_{\mathcal{L}}$, it suffices to bound for any $\mathcal{L} \in \mathcal{V}_n$ with $0 \le n \le N_1$,

$$
\begin{aligned}
&\mathrm{e}^{\mathrm{i}t\Lambda_{wa}(\xi)}\widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - h_{\mathcal{L},\infty} \qquad (3.5.1)\\
=&\sum_{\mathcal{L}_1,\mathcal{L}_2}\Bigg\{\sum_{\iota_1,\iota_2\in\{+,-\}} \mathbf{I}^{\iota_1\iota_2}_{wa}\left[G^{kg,\iota_1}_{\mathcal{L}_1}, G^{kg,\iota_2}_{\mathcal{L}_2}\right]\\
&+\sum_{\iota_1,\iota_2\in\{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}, \left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2} + \mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right]\\
&+\sum_{\iota_1,\iota_2\in\{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \widehat{G^{kg,\iota_2}_{\mathcal{L}_2}}\right]\Bigg\}\\
&+\Bigg\{\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_1,\iota_2\in\{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2} + \mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right] - h_{\mathcal{L},\infty}\Bigg\}.
\end{aligned}
$$

$\sum_{\mathcal{L}_1,\mathcal{L}_2}$ is a summation over all combinations of vector fields $\mathcal{L}_1 \in \mathcal{V}_{n_1}$ and $\mathcal{L}_2 \in \mathcal{V}_{n_2}$ with $n_1 + n_2 = n$ and $\mathcal{L} = \mathcal{L}_1 \circ \mathcal{L}_2$ (due to Leibniz rule). Also, all variables here with $\mathcal{L}_1$ or $\mathcal{L}_2$ should be understood in the sense of (3.2.25) and (3.2.26).

3.5.1 S_1' Estimates

Lemma 3.5.1. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \le n \le N_1$, we have*

$$
\left\| \sum_{\iota_1,\iota_2\in\{+,-\}} \varphi_k \mathbf{I}^{\iota_1\iota_2}_{wa}\left[G^{kg,\iota_1}_{\mathcal{L}_1}, G^{kg,\iota_2}_{\mathcal{L}_2}\right]\right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H''_{wa}(n)\delta)} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \qquad (3.5.2)
$$

Proof. Without loss of generality, we assume $n_1 \leq n_2$. For simplicity, we temporarily ignore ι_1 and ι_2 superscripts. Our proof mainly relies on two types of bounds (3.3.36) and (3.3.37).

In principle, based on Remark 3.4.1 (the worst scenario is that all $\mathcal{L}$ hit the same G^{kg}, then we cannot use Lemma 3.4.3), we always have

$$\left\| P_{k_2} G^{kg}_{\mathcal{L}_2} \right\|_{L^2} \lesssim \varepsilon \langle t \rangle^{-H(n_2)\delta} 2^{-N(n_2)k_2^+}. \tag{3.5.3}$$

Then we focus on the bounds of $P_{k_1} G^{kg}_{\mathcal{L}_1}$. Note that we always have $n_1 \leq N_1 - 2$ since $n_1 \leq n_2$ and $n_1 + n_2 \leq n \leq N_1$, and that $k^+ \lesssim \max\{k_1^+, k_2^+\}$ due to Lemma 3.3.3.

We first discuss the case when $n = N_1 = 3$. We divide it into several cases:

- Case I: For $2^{k^-} \lesssim \langle t \rangle^{-1}$, (3.3.36) and Lemma 3.4.3 justify that

$$\begin{aligned}
&\sum_{k_1,k_2} \left\| \varphi_k \mathbf{I}_{wa}\left[P_{k_1} G^{kg}_{\mathcal{L}_1}, P_{k_2} G^{kg}_{\mathcal{L}_2} \right] \right\|_{L^2} \qquad (3.5.4)\\
\lesssim &\sum_{k_1,k_2} 2^{\frac{3}{2}\min\{k_1,k_2,k\}} \left\| P_{k_1} G^{kg}_{\mathcal{L}_1} \right\|_{L^2} \left\| P_{k_2} G^{kg}_{\mathcal{L}_2} \right\|_{L^2}\\
\lesssim &\sum_{k_1,k_2} 2^{\frac{1}{2}k^-} \langle t \rangle^{-1} \left(\varepsilon \langle t \rangle^{-H(n_1)\delta} 2^{-N(n_1)k_1^+} 2^{k_1^-} \right) \left(\varepsilon \langle t \rangle^{-H(n_2)\delta} 2^{-N(n_2)k_2^+} \right)\\
\lesssim &\sum_{k_1,k_2} \varepsilon^2 2^{\frac{1}{2}k^-} \langle t \rangle^{-1} \langle t \rangle^{-(H(n_1)+H(n_2))\delta} 2^{-N(n_1)k_1^+ - N(n_2)k_2^+} 2^{k_1^-}\\
\lesssim &\, \varepsilon^2 2^{\frac{1}{2}k^-} \langle t \rangle^{-(1+H(n_1)\delta + H(n_2)\delta)} 2^{-\min\{N(n_1),N(n_2)\}k^+}.
\end{aligned}$$

Note that

$$H(n_1) + H(n_2) \geq H''_{wa}(n), \quad \min\Big\{N(n_1), N(n_2)\Big\} \geq N''_{wa}(n). \tag{3.5.5}$$

- Case II: For $2^{k^-} \gtrsim \langle t\rangle^{-1}$ and $2^{k_1^-} \lesssim 2^{k^-}$, (3.3.37) and Lemma 3.4.5 imply that

$$\begin{aligned}
&\sum_{k_1,k_2} \Big\| \varphi_k \mathbf{I}_{wa}\Big[P_{k_1} G^{kg}_{\mathcal{L}_1}, P_{k_2} G^{kg}_{\mathcal{L}_2}\Big]\Big\|_{L^2} \qquad (3.5.6)\\
&\lesssim \sum_{k_1,k_2} \Big\| \mathrm{e}^{\mathrm{i}t\Lambda_{kg}} P_{k_1} G^{kg}_{\mathcal{L}_1}\Big\|_{L^\infty} \Big\| P_{k_2} G^{kg}_{\mathcal{L}_2}\Big\|_{L^2}\\
&\lesssim \sum_{k_1,k_2} \Big(\varepsilon\langle t\rangle^{-H(n_1+1)\delta} 2^{-(N(n_1+1)-\frac{5}{2})k_1^+} 2^{\frac{1}{2}k_1^-}\langle t\rangle^{-1}\Big)\Big(\varepsilon\langle t\rangle^{-H(n_2)\delta} 2^{-N(n_2)k_2^+}\Big)\\
&\lesssim \sum_{k_1,k_2} \varepsilon^2\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1+1)+H(n_2))\delta} 2^{-(N(n_1+1)-\frac{5}{2})k_1^+ - N(n_2)k_2^+} 2^{\frac{1}{2}k_1^-}\\
&\lesssim \varepsilon^2\langle t\rangle^{-(1+H(n_1)\delta+H(n_2)\delta)} 2^{-\min\{N(n_1)-\frac{5}{2}, N(n_2)\}k^+} 2^{\frac{1}{2}k^-}.
\end{aligned}$$

Note that summation over k_1 will result in $2^{\frac{1}{2}k^-}$. Also, we have

$$H(n_1+1) + H(n_2) \geq H''_{wa}(n), \quad \min\Big\{N(n_1+1) - \frac{5}{2}, N(n_2)\Big\} \geq N''_{wa}(n). \tag{3.5.7}$$

- Case III: For $2^{k^-} \gtrsim \langle t\rangle^{-1}$ and $2^{k_1^-} \lesssim 2^{-k^-}\langle t\rangle^{-1}$, (3.3.36) and Lemma 3.4.3 justify that

$$\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{wa}\Big[P_{k_1}G^{kg}_{\mathcal{L}_1},P_{k_2}G^{kg}_{\mathcal{L}_2}\Big]\right\|_{L^2} \tag{3.5.8}$$

$$\lesssim\sum_{k_1,k_2}2^{\frac{3}{2}\min\{k_1,k_2,k\}}\left\|P_{k_1}G^{kg}_{\mathcal{L}_1}\right\|_{L^2}\left\|P_{k_2}G^{kg}_{\mathcal{L}_2}\right\|_{L^2}$$

$$\lesssim\sum_{k_1,k_2}2^{\frac{3}{2}k^-}\Big(\varepsilon\langle t\rangle^{-H(n_1)\delta}2^{-N(n_1)k_1^+}2^{k_1^-}\Big)\Big(\varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}\Big)$$

$$\lesssim\sum_{k_1,k_2}\varepsilon^2 2^{\frac{3}{2}k^-}\langle t\rangle^{-(H(n_1)+H(n_2))\delta}2^{-N(n_1)k_1^+-N(n_2)k_2^+}2^{k_1^-}$$

$$\lesssim\varepsilon^2\langle t\rangle^{-(1+H(n_1)\delta+H(n_2)\delta)}2^{-\min\{N(n_1),N(n_2)\}k^+}2^{\frac{1}{2}k^-}.$$

Note that summation over k_1 will result in $2^{-k^-}\langle t\rangle^{-1}$, and that

$$H(n_1)+H(n_2)\geq H''_{wa}(n),\quad \min\Big\{N(n_1),N(n_2)\Big\}\geq N''_{wa}(n). \tag{3.5.9}$$

- Case IV: For $2^{k^-}\gtrsim\langle t\rangle^{-1}$ and $2^{k_1^-}\gtrsim 2^{k^-}$ and $2^{k_1^-}\gtrsim 2^{-k^-}\langle t\rangle^{-1}$, (3.3.37) and Lemma 3.4.6 implies (in this case, we naturally have $2^{2k_1^-}\langle t\rangle\gtrsim 2^{k_1^-}2^{k^-}\langle t\rangle\gtrsim 1$)

$$\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{wa}\Big[P_{k_1}G^{kg}_{\mathcal{L}_1},P_{k_2}G^{kg}_{\mathcal{L}_2}\Big]\right\|_{L^2} \tag{3.5.10}$$

$$\lesssim\sum_{k_1,k_2}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_1}G^{kg}_{\mathcal{L}_1}\right\|_{L^\infty}\left\|P_{k_2}G^{kg}_{\mathcal{L}_2}\right\|_{L^2}$$

$$\lesssim\sum_{k_1,k_2}\Big(\varepsilon\langle t\rangle^{-H(n_1+2)\delta}2^{-(N(n_1+2)-4)k_1^+}2^{-k_1^-(\frac{1}{2}-\frac{\sigma}{4})}\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}\Big)\Big(\varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}\Big)$$

$$\lesssim\sum_{k_1,k_2}\varepsilon^2 2^{\frac{1}{2}k^-}\langle t\rangle^{-\frac{3}{2}}\langle t\rangle^{-(H(n_1+2)+H(n_2))\delta+\frac{\sigma}{8}}2^{-(N(n_1+2)-4)k_1^+-N(n_2)k_2^+}2^{\frac{\sigma}{4}k_1^-}\Big(2^{-\frac{1}{2}k_1^-}2^{-\frac{1}{2}k^-}\Big).$$

Note that $2^{-\frac{1}{2}k_1^-}2^{-\frac{1}{2}k^-} \lesssim \langle t\rangle^{\frac{1}{2}}$. Hence,

$$\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k \mathbf{I}_{wa}\Big[P_{k_1}G^{kg}_{\mathcal{L}_1}, P_{k_2}G^{kg}_{\mathcal{L}_2}\Big]\right\|_{L^2} \qquad (3.5.11)\\
&\lesssim \sum_{k_1,k_2} \varepsilon^2 2^{\frac{1}{2}k^-}\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1+2)+H(n_2))\delta+\frac{\sigma}{8}}2^{-(N(n_1+2)-4)k_1^+-N(n_2)k_2^+}2^{\frac{\sigma}{4}k_1^-}\\
&\lesssim \varepsilon^2\langle t\rangle^{-(1+H(n_1)\delta+H(n_2)\delta)-\frac{\delta}{8}}2^{-\min\{N(n_1+2)-4,N(n_2)\}k^+}2^{\frac{1}{2}k^-}.
\end{aligned}$$

Here we have for $\sigma \simeq \delta$,

$$H(n_1+2)+H(n_2)-\frac{1}{8} \geq H''_{wa}(n), \quad \min\Big\{N(n_1+2)-4, N(n_2)\Big\} \geq N''_{wa}(n). \qquad (3.5.12)$$

The above discussion works perfectly well for $n = N_1 = 3$ (i.e. $(n_1, n_2) = (0, 3)$ or $(n_1, n_2) = (1, 2)$). For the other n, there are subtle issues (a naive application of the above recipe does not give sufficient control to N''_{wa}), so we need a detailed discussion:

- $n = 2$ (i.e. $(n_1, n_2) = (0, 2)$ or $(n_1, n_2) = (1, 1)$): If $(n_1, n_2) = (0, 2)$, then the above recipe works well. However, if $(n_1, n_2) = (1, 1)$, we need to discuss based on k, k_1, k_2 values. Based on Lemma 3.3.3, if $k \simeq k_2 \geq k_1$ or $k_1 \simeq k_2 \geq k$, we can still use the above recipe; if $k \simeq k_1 \geq k_2$, then in Case I through Case IV, we interchange the status of $G^{kg}_{\mathcal{L}_1}$ and $G^{kg}_{\mathcal{L}_2}$, which can improve N''_{wa}.
- $n = 1$ (i.e. $(n_1, n_2) = (0, 1)$) or $n = 0$ (i.e. $(n_1, n_2) = (0, 0)$): Based on Lemma 3.3.3, if $k \simeq k_2 \geq k_1$ or $k_1 \simeq k_2 \geq k$, we can still use the above recipe; if $k \simeq k_1 \geq k_2$, then in Case I through Case IV, we interchange the status of $G^{kg}_{\mathcal{L}_1}$ and $G^{kg}_{\mathcal{L}_2}$.

□

Lemma 3.5.2. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1$*, we have*

$$\left\| \sum_{\iota_1,\iota_2\in\{+,-\}} \varphi_k \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \left(\mathrm{e}^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}} \widehat{V_\infty^{kg,\iota_2}} \right)_{\mathcal{L}_2} + \mathfrak{B}_{\mathcal{L}_2,\infty}^{\iota_2} \right] \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+} 2^{\frac{1}{2}k^-}, \tag{3.5.13}$$

$$\left\| \sum_{\iota_1,\iota_2\in\{+,-\}} \varphi_k \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \widehat{G_{\mathcal{L}_2}^{kg,\iota_2}} \right] \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.5.14}$$

Proof. Basically, the proof is similar to that of Lemma 3.5.1. The only difference is that we do not have $H(n_2)$ time decay, so we should always assign L^2 to $G_{\mathcal{L}_1}^{kg,\iota_1}$ or $G_{\mathcal{L}_2}^{kg,\iota_2}$. Then $\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}$ or $\left(\mathrm{e}^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}} \widehat{V_\infty^{kg,\iota_2}} \right)_{\mathcal{L}_2} + \mathfrak{B}_{\mathcal{L}_2,\infty}$ may take L^2 or L^∞ with Lemma 3.4.7, Lemma 3.4.11, Lemma 3.4.25 and Lemma 3.4.27 following the argument in the proof of Lemma 3.5.1.

- If only partial $\mathcal{L}$ hits G^{kg}, then we have sufficient time decay and k^+ decay since less $\mathcal{L}$ yields faster decay rate.
- If the full $\mathcal{L}$ hits G^{kg}, then this term already achieve the critical time decay and k^+ decay in the desired result.

In both cases, we can get the desired result. $\square$

Lemma 3.5.3. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1$*, we have*

$$\left\| \varphi_k \left\{ \sum_{\mathcal{L}_1,\mathcal{L}_2} \sum_{\iota_1,\iota_2\in\{+,-\}} \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \left(\mathrm{e}^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}} \widehat{V_\infty^{kg,\iota_2}} \right)_{\mathcal{L}_2} + \mathfrak{B}_{\mathcal{L}_2,\infty}^{\iota_2} \right] - h_{\mathcal{L},\infty} \right\} \right\|_{L^2_\xi}$$

$$\lesssim \varepsilon^2 \langle t\rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.5.15}$$

Proof. Based on the proof of Deng-Pusateri [11, Lemma 4.2], we have (up to fast-decaying terms)

$$
\begin{aligned}
h_{\mathcal{L},\infty} &\sim \sum_{\mathcal{L}_1,\mathcal{L}_2} h_{\mathcal{L}_1,\mathcal{L}_2,\infty} \qquad (3.5.16)\\
&:= \sum_{\mathcal{L}_1,\mathcal{L}_2} (2\pi)^{-\frac{3}{2}}\varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}(-\eta)\Big(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}_\infty^{\iota_2}}\widehat{V_\infty^{kg,\iota_2}}\Big)_{\mathcal{L}_2}(\eta)\,\mathrm{d}\eta,
\end{aligned}
$$

In the following estimates, let $\widehat{g_1} = \Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}$ and $\widehat{g_2} = \Big(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}_\infty^{\iota_2}}\widehat{V_\infty^{kg,\iota_2}}\Big)_{\mathcal{L}_2}$.

Step 1. Resonant Case: $(\iota_1,\iota_2)=(+,-)$ *or* $(\iota_1,\iota_2)=(-,+)$

We further decompose $\widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}$ into low-frequency and high-frequency parts:

$$
\widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\big[\widehat{g_1},\widehat{g_2}\big] = \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\big[\widehat{g_1},\widehat{g_2}\big] + \varphi_{\ge 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\big[\widehat{g_1},\widehat{g_2}\big]. \qquad (3.5.17)
$$

Step 1-1. Resonant Case - Low Frequency

We may decompose

$$
\begin{aligned}
&\varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}_{wa}^{-+}}\big[\widehat{g_1},\widehat{g_2}\big] - \frac{1}{2}h_{\mathcal{L}_1,\mathcal{L}_2,\infty} \qquad (3.5.18)\\
\simeq{}& \varphi_{\le 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\Big\{\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\Phi_{wa}^{-+}(\xi,\eta)}a_{-+}(\xi,\eta)\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta\\
&- 2\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\widehat{g_1}(t,-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta\Big\}\\
={}& \mathscr{S}_1+\mathscr{S}_2+\mathscr{S}_3,
\end{aligned}
$$

where

$$\mathscr{S}_1 := \varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{-+}_{wa}(\xi,\eta)}\big[a_{-+}(\xi,\eta)-2\big]\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta, \tag{3.5.19}$$

$$\mathscr{S}_2 := 2\varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\int_{\mathbb{R}^3} \Big[\mathrm{e}^{\mathrm{i}t\Phi^{-+}_{wa}(\xi,\eta)} - \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\Big]\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta, \tag{3.5.20}$$

$$\mathscr{S}_3 := 2\varphi_{\leq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\Big[\widehat{g_1}(t,\xi-\eta)-\widehat{g_1}(t,-\eta)\Big]\widehat{g_2}(t,\eta)\,\mathrm{d}\eta. \tag{3.5.21}$$

For $\mathscr{S}_1$, noticing that

$$\big|a_{-+}(\xi,\eta)-2\big| = \left|1-\frac{(\xi-\eta)\cdot\eta-1}{\langle\xi-\eta\rangle\langle\eta\rangle}-2\right| \lesssim |\xi|\,, \tag{3.5.22}$$

we may use (3.3.36) and Lemma 3.4.21 to obtain

$$\|\varphi_k\mathscr{S}_1\|_{L^2} \lesssim 2^{\frac{3}{2}k}2^k\varphi_{\leq 0}\big(2^k\langle t\rangle^{\frac{7}{8}}\big)\,\|\widehat{g_1}\|_{L^2}\,\|\widehat{g_2}\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-\frac{7}{4}}\,2^{\frac{1}{2}k^-}. \tag{3.5.23}$$

For $\mathscr{S}_2$, noticing that

$$\left|\mathrm{e}^{\mathrm{i}t\Phi^{-+}_{wa}(\xi,\eta)} - \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\right| = \left|\mathrm{e}^{\mathrm{i}t\big(|\xi|+\langle\xi-\eta\rangle-\langle\eta\rangle\big)} - \mathrm{e}^{\mathrm{i}t\left(|\xi|-\frac{\xi\cdot\eta}{\langle\eta\rangle}\right)}\right| \lesssim |\xi|^2\,\langle t\rangle, \tag{3.5.24}$$

we may use (3.3.36) and Lemma 3.4.21 to obtain

$$\|\varphi_k\mathscr{S}_2\|_{L^2} \lesssim 2^{\frac{3}{2}k}2^{2k}\langle t\rangle\varphi_{\leq 0}\big(2^k\langle t\rangle^{\frac{7}{8}}\big)\,\|\widehat{g_1}\|_{L^2}\,\|\widehat{g_2}\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-\frac{13}{8}}\,2^{\frac{1}{2}k^-}. \tag{3.5.25}$$

$\mathscr{S}_3$ is a bit more delicate. Minkowski's integral inequality and Lemma 3.4.7 imply

$$\begin{aligned} &\big\|\widehat{g_1}(t,-\eta)-\widehat{g_1}(t,\xi-\eta)\big\|_{L^2} \\ \lesssim{}& \left\|\int_0^{|\xi|}\partial_{\widehat{\xi}}\widehat{g_1}(t,-\eta+c\,\widehat{\xi})\,\mathrm{d}c\right\|_{L^2_\eta} \lesssim \int_0^{|\xi|}\big\|\partial_{\widehat{\xi}}\widehat{g_1}(t,-\eta+c\,\widehat{\xi})\big\|_{L^2_\eta}\mathrm{d}c \\ \lesssim{}& \int_0^{|\xi|}\big\|\partial_\eta\widehat{g_1}(t,-\eta)\big\|_{L^2_\eta}\mathrm{d}c \lesssim |\xi|\,\big\|\partial_\eta\widehat{g_1}(t,\eta)\big\|_{L^2_\eta}. \end{aligned} \tag{3.5.26}$$

Then we may use (3.3.36) and Lemma 3.4.21 to obtain

$$\|\varphi_k \mathscr{S}_3\|_{L^2} \lesssim 2^{\frac{3}{2}k} \varphi_{\leq 0}\big(2^k \langle t\rangle^{\frac{7}{8}}\big)\, \|\widehat{g}_1\|_{H^1_\eta}\, \|\widehat{g}_2\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-\frac{13}{8}}\, 2^{\frac{1}{2}k^-}. \tag{3.5.27}$$

In total, we have for $(\iota_1, \iota_2) = (-, +)$ case

$$\left\| \varphi_{\leq 0}\big(\xi \langle t\rangle^{\frac{7}{8}}\big) \cdot \widehat{\mathbf{I}_{wa}^{-+}}\,[\widehat{g}_1, \widehat{g}_2] - \frac{1}{2} h_{\mathcal{L}_1, \mathcal{L}_2, \infty} \right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-\frac{7}{4}}\, 2^{\frac{1}{2}k^-}. \tag{3.5.28}$$

By a similar argument, we may justify $(\iota_1, \iota_2) = (+, -)$ case.

Step 1-2. Resonant Case - High Frequency

For the high-frequency part, we bound

$$\left\| \varphi_k \left\{ \varphi_{\geq 0}\big(\xi \langle t\rangle^{\frac{7}{8}}\big) \cdot \widehat{\mathbf{I}_{wa}^{\iota_1 \iota_2}}\,[\widehat{g}_1, \widehat{g}_2] \right\} \right\|_{L^2} \lesssim \sum_{k_1, k_2} \left\| \varphi_k \left\{ \varphi_{\geq 0}\big(\xi \langle t\rangle^{\frac{7}{8}}\big) \cdot \widehat{\mathbf{I}_{wa}^{\iota_1 \iota_2}}\,[\varphi_{k_1} \widehat{g}_1, \varphi_{k_2} \widehat{g}_2] \right\} \right\|_{L^2}. \tag{3.5.29}$$

If $2^{\min(k_1, k_2)} \leq \langle t\rangle^{-\frac{1}{2}}$, we integrate by parts in η using (3.3.3), and apply Lemma 3.4.21

and (3.3.36) to obtain

$$
\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\left\{\varphi_{\geq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\varphi_{k_1}\widehat{g_1},\varphi_{k_2}\widehat{g_2}\right]\right\}\right\|_{L^2} \qquad (3.5.30)\\
\lesssim&\sum_{k_1,k_2}2^{\frac{3}{2}k}\left\|\varphi_k\varphi_{\geq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\varphi_{k_1}\widehat{g_1},\varphi_{k_2}\widehat{g_2}\right]\right\|_{L^\infty}\\
\lesssim&\sum_{k_1,k_2}2^{\frac{3}{2}k}\langle t\rangle^{-1}2^{-k}\left\|\varphi_{k_1}\widehat{g_1}\right\|_{H^1_\eta}\left\|\varphi_{k_2}\widehat{g_2}\right\|_{H^1_\eta}\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2 2^{\frac{1}{2}k}\langle t\rangle^{-1}2^{-N(n_1)k_1^+-N(n_2)k_2^+}2^{k_1^-+k_2^-}\\
\lesssim&\,\varepsilon^2 2^{\frac{1}{2}k^-}\langle t\rangle^{-\frac{3}{2}}2^{-\min\{N(n_1),N(n_2)\}k^+}.
\end{aligned}
$$

If $2^{\min(k_1,k_2)}\geq\langle t\rangle^{-\frac{1}{2}}$ and $\max(k_1,k_2)\geq 0$ (without loss of generality, we assume $k_1\gtrsim k_2$), we apply (3.3.37) and Lemma 3.4.23, Lemma 3.4.12 to obtain

$$
\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\left\{\varphi_{\geq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big)\cdot\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\varphi_{k_1}\widehat{g_1},\varphi_{k_2}\widehat{g_2}\right]\right\}\right\|_{L^2} \qquad (3.5.31)\\
\lesssim&\sum_{k_1,k_2}\left\|\varphi_{k_1}\widehat{g_1}\right\|_{L^2}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_2}g_2\right\|_{L^\infty}\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{-N(n_1)k_1^+-N(n_2)k_2^+}2^{-k_1^-(\frac{1}{2}-\frac{\sigma}{4})}2^{k_2^-}\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{-N(n_1)k_1^+-N(n_2)k_2^+}2^{k_2^-}\\
\lesssim&\,\varepsilon^2 2^{\frac{1}{2}k^-}\langle t\rangle^{-\frac{17}{16}+\frac{\sigma}{8}}2^{-\min\{N(n_1),N(n_2)\}k^+}.
\end{aligned}
$$

If $2^{\min(k_1,k_2)}\geq\langle t\rangle^{-\frac{1}{2}}$ and $\max(k_1,k_2)\leq 0$, we integrate by parts in η using (3.3.3), and apply (3.3.37) and Lemma 3.4.22, Lemma 3.4.11 to obtain

$$\begin{aligned}
&\sum_{k_1,k_2} \left\| \varphi_k \left\{ \varphi_{\geq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big) \cdot \mathbf{I}_{wa}^{\iota_1\iota_2}\left[\varphi_{k_1}\widehat{g_1}, \varphi_{k_2}\widehat{g_2}\right] \right\} \right\|_{L^2} \qquad (3.5.32)\\
\lesssim &\sum_{k_1,k_2} \langle t\rangle^{-1} 2^{-k} \left\|\varphi_{k_1}\partial_{\xi_\ell}\widehat{g_1}\right\|_{L^2} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_2} g_2\right\|_{L^\infty}\\
\lesssim &\sum_{k_1,k_2} \varepsilon^2 \langle t\rangle^{-2} 2^{-N(n_1)k_1^+ - N(n_2)k_2^+} 2^{\frac{1}{2}k_1^-} 2^{k_2^-}\\
\lesssim &\,\varepsilon^2 2^{\frac{1}{2}k^-} \langle t\rangle^{-\frac{25}{16}} 2^{-\min\{N(n_1),N(n_2)\}k^+}.
\end{aligned}$$

In total, we have

$$\left\| \varphi_k \left\{ \varphi_{\geq 0}\big(\xi\langle t\rangle^{\frac{7}{8}}\big) \cdot \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\left[\varphi_{k_1}\widehat{g_1}, \varphi_{k_2}\widehat{g_2}\right] \right\} \right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-\frac{17}{16}+\frac{\sigma}{8}} 2^{-N(n)k} 2^{\frac{1}{2}k^-}. \qquad (3.5.33)$$

<u>Step 2</u>. Non-Resonant Case: $(\iota_1,\iota_2) = (+,+)$ *or* $(\iota_1,\iota_2) = (-,-)$

Using stationary phase argument in Lemma 3.3.1 (with $\eta_c = \xi/2$), we have

$$
\begin{aligned}
&\left\|\varphi_k \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\left[\widehat{g_1}, \widehat{g_2}\right]\right\|_{L^2} \qquad (3.5.34)\\
\lesssim &\sum_{k_1,k_2} \left\|\varphi_k \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\left[\varphi_{k_1}\widehat{g_1}, \varphi_{k_2}\widehat{g_2}\right]\right\|_{L^2}\\
\lesssim &\sum_{k_1,k_2} 2^{\frac{3}{2}k} \left\|\varphi_k \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}}\left[\varphi_{k_1}\widehat{g_1}, \varphi_{k_2}\widehat{g_2}\right]\right\|_{L^\infty}\\
\lesssim &\sum_{k_1,k_2} 2^{\frac{3}{2}k}\langle t\rangle^{-\frac{3}{2}} 2^{\frac{5}{2}k^+} \left\|\varphi_{k_1}\widehat{g_1}\right\|_{L^\infty_\eta} \left\|\varphi_{k_2}\widehat{g_2}\right\|_{L^\infty_\eta}\\
\lesssim &\,\varepsilon^2 2^{\frac{3}{2}k}\langle t\rangle^{-\frac{3}{2}} \sum_{k_1,k_2} 2^{\frac{5}{2}k^+ - N(n_1)k_1^+ - N(n_2)k_2^+} 2^{k_1^- + k_2^-}\\
\lesssim &\,\varepsilon^2 2^{\frac{1}{2}k}\langle t\rangle^{-\frac{19}{8}} 2^{-(N(n)-5)k^+}.
\end{aligned}
$$

Remark 3.5.1. The application of stationary phase argument highly depends on the non-resonant phase, which is not applicable to the resonant phase.

Step 3. $\mathfrak{B}_{\mathcal{L}_j,\infty}$ *Terms*

Note that all terms related to $\mathfrak{B}_{\mathcal{L}_j,\infty}$ can be bounded in a similar manner as Lemma 3.5.1 since Lemma 3.4.25 and Lemma 3.4.26 provide sufficient time decay. □

Combining Lemma 3.5.1, Lemma 3.5.2 and Lemma 3.5.3, we get the following proposition:

Proposition 3.5.4. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1$, *we have*

$$\left\|\varphi_k \partial_t \widehat{G_{\mathcal{L}}^{wa}}\right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H''_{wa}(n)\delta)} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \qquad (3.5.35)$$

3.5.2 T_1' Estimates

Lemma 3.5.5. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \le n \le N_1 - 1$ and $\ell \in \{1, 2, 3\}$, we have*

$$\left\| \varphi_k \partial_{\xi_\ell} \left\{ \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} \mathbf{I}^{\iota_1\iota_2}_{wa} \left[G^{kg,\iota_1}_{\mathcal{L}_1}, G^{kg,\iota_2}_{\mathcal{L}_2} \right] \right\} \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-H''_{wa}(n)\delta} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \quad (3.5.36)$$

Proof. The proof is similar to that of Lemma 3.5.1 and of [45, Lemma 4.2].

$$\mathbf{I}^{\iota_1\iota_2}_{wa} \left[G^{kg,\iota_1}_{\mathcal{L}_1}, G^{kg,\iota_2}_{\mathcal{L}_2} \right](t,\xi) \sim \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi^{\iota_1\iota_2}_{wa}} a_{\iota_1\iota_2}(\xi,\eta) \widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}(t,\xi-\eta) \widehat{G^{kg,\iota_2}_{\mathcal{L}_2}}(t,\eta)\,\mathrm{d}\eta\,. \quad (3.5.37)$$

Note that ξ_ℓ derivative may hit the phase $\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathrm{e}^{\mathrm{i}t\Phi^{\iota_1\iota_2}_{wa}}$, the multiplier $a_{\iota_1\iota_2}$, or $G^{kg,\iota_1}_{\mathcal{L}_1}(\xi-\eta)$.

- If ξ_ℓ derivative hits the phase $\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}\mathrm{e}^{\mathrm{i}t\Phi^{\iota_1\iota_2}_{wa}}$, then we have an extra $\langle t \rangle$ term popping out. Following exactly the same argument as in the proof of Lemma 3.5.1, we get the desired result.
- If ξ_ℓ derivative hits the multiplier $a_{\iota_1\iota_2}$, then nothing happens and we follow the same proof of Lemma 3.5.1.
- If ξ_ℓ derivative hits $\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}(\xi-\eta)$, then we simply use (3.3.36) and Lemma 3.4.1 to obtain

$$\begin{aligned} &\left\| \varphi_k \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}} \left[\varphi_{k_1}\left(\partial_{\xi_\ell} \widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}\right), \varphi_{k_2} \widehat{G^{kg,\iota_2}_{\mathcal{L}_2}} \right] \right\|_{L^2} \quad (3.5.38) \\ &\lesssim 2^{\frac{3}{2}\min\{k_1,k_2,k\}} \left\| \varphi_{k_1}\left(\partial_{\xi_\ell} \widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}\right) \right\|_{L^2} \left\| \varphi_{k_2} \widehat{G^{kg,\iota_2}_{\mathcal{L}_2}} \right\|_{L^2} \\ &\lesssim \varepsilon^2 \langle t \rangle^{-(H(n_1+1)+H(n_2))\delta} 2^{\frac{3}{2}\min\{k_1,k_2,k\}} 2^{-(N(n_1+1)+1)k_1^+ - N(n_2)k_2^+}. \end{aligned}$$

Here, through change of variables in the convolution, we can always assume $k_1 \le k_2$, so we know both $H''_{wa}(n)$ and $N''_{wa}(n)$ requirements can be met.

$\square$

Lemma 3.5.6. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 1$ *and* $\ell \in \{1, 2, 3\}$, *we have*

$$\left\| \varphi_k \partial_{\xi_\ell} \left\{ \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} \, \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \left(\mathrm{e}^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}} \widehat{V_\infty^{kg,\iota_2}} \right)_{\mathcal{L}_2} + \mathfrak{B}_{\mathcal{L}_2,\infty}^{\iota_2} \right] \right\} \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-H''_{wa}(n)\delta} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}, \tag{3.5.39}$$

$$\left\| \varphi_k \partial_{\xi_\ell} \left\{ \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} \, \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \widehat{G_{\mathcal{L}_2}^{kg,\iota_2}} \right] \right\} \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-H''_{wa}(n)\delta} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.5.40}$$

Proof. It is similar to the proofs of Lemma 3.5.2 and Lemma 3.5.5. $\square$

Lemma 3.5.7. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 1$ *and* $\ell \in \{1, 2, 3\}$, *we have*

$$\Bigg\| \varphi_k \partial_{\xi_\ell} \Bigg\{ \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} \tag{3.5.41}$$

$$\times \left(\sum_{\mathcal{L}_1, \mathcal{L}_2} \sum_{\iota_1,\iota_2 \in \{+,-\}} \widehat{\mathbf{I}_{wa}^{\iota_1\iota_2}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \left(\mathrm{e}^{\iota_2 \mathrm{i}\mathcal{D}_\infty^{\iota_2}} \widehat{V_\infty^{kg,\iota_2}} \right)_{\mathcal{L}_2} + \mathfrak{B}_{\mathcal{L}_2,\infty}^{\iota_2} \right] - h_{\mathcal{L},\infty} \right) \Bigg\} \Bigg\|_{L^2_\xi}$$

$$\lesssim \varepsilon^2 \langle t \rangle^{-H''_{wa}(n)\delta} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.5.42}$$

Proof. It is similar to the proofs of Lemma 3.5.3 and Lemma 3.5.5. $\square$

Combining Lemma 3.5.5, Lemma 3.5.6 and Lemma 3.5.7, we get the following proposition:

Proposition 3.5.8. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 1$ *and* $\ell \in \{1, 2, 3\}$*, we have*

$$\left\| \varphi_k \partial_{\xi_\ell} \left\{ \widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} h_{\mathcal{L},\infty} \right\} \right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t \rangle^{-H''_{wa}(n)\delta} 2^{-N''_{wa}(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.5.43}$$

3.6 Bounds on the Nonlinearities of Klein-Gordon Equation

Based on (3.2.75), in order to control $\partial_t G^{kg}_{\mathcal{L}}$, it suffices to bound for any $\mathcal{L} \in \mathcal{V}_n$ with $0 \le n \le N_1$,

$$\begin{aligned}
&\mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \left\{ \mathrm{i}\, \mathcal{C}_\infty \left(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V^{kg}_\infty} + \mathfrak{B}_\infty \right) \right\}_{\mathcal{L}} - \mathfrak{b}_{\mathcal{L},\infty} \qquad (3.6.1)\\
&= \sum_{\mathcal{L}_1,\mathcal{L}_2} \Bigg\{ \sum_{\iota_1,\iota_2 \in \{+,-\}} \mathbf{I}^{\iota_1\iota_2}_{kg} \left[G^{kg,\iota_1}_{\mathcal{L}_1}, G^{wa,\iota_2}_{\mathcal{L}_2} \right] \\
&\quad + \sum_{\iota_1,\iota_2 \in \{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}} \left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}, \widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}} + \mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty} \right] \\
&\quad + \sum_{\iota_1,\iota_2 \in \{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}^{\iota_1}_\infty} \widehat{V^{kg,\iota_1}_\infty} \right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \widehat{G^{wa,\iota_2}_{\mathcal{L}_2}} \right] \Bigg\} \\
&\quad + \Bigg\{ \sum_{\mathcal{L}_1,\mathcal{L}_2} \sum_{\iota_1,\iota_2 \in \{+,-\}} \widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}^{\iota_1}_\infty} \widehat{V^{kg,\iota_1}_\infty} \right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}} + \mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty} \right] \\
&\qquad - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\, \mathcal{C}_{\mathcal{L}_2,\infty} \left(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty} \widehat{V^{kg}_\infty} \right)_{\mathcal{L}_1} - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\, \mathcal{C}_{\mathcal{L}_2,\infty} \mathfrak{B}_{\mathcal{L}_1,\infty} - \mathfrak{b}_{\mathcal{L},\infty} \Bigg\},
\end{aligned}$$

$\sum_{\mathcal{L}_1,\mathcal{L}_2}$ is a summation over all combinations of vector fields $\mathcal{L}_1 \in \mathcal{V}_{n_1}$ and $\mathcal{L}_2 \in \mathcal{V}_{n_2}$ with $n_1+n_2=n$ and $\mathcal{L}=\mathcal{L}_1\circ\mathcal{L}_2$. Again, all variables here with $\mathcal{L}_1$ or $\mathcal{L}_2$ should be understood in the sense of (3.2.25) and (3.2.26).

3.6.1 S_2' Estimates

Lemma 3.6.1. *Assume* (3.2.104) *holds. For any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\le n\le N_1$, *we have*

$$\left\|\sum_{\iota_1,\iota_2\in\{+,-\}}\varphi_k\,\mathbf{I}_{kg}^{\iota_1\iota_2}\left[G_{\mathcal{L}_1}^{kg,\iota_1},G_{\mathcal{L}_2}^{wa,\iota_2}\right]\right\|_{L^2_\xi}\lesssim\varepsilon^2\langle t\rangle^{-(1+H''_{kg}(n)\delta)}2^{-N''_{kg}(n)k^+}. \tag{3.6.2}$$

Proof. For simplicity, we temporarily ignore ι_1 and ι_2 superscripts. Considering Lemma 3.3.3, it suffices to discuss the cases $k_1\le k_2$ and $k_2\le k_1$.

- Case I: $k_1\le k_2$ and $n_1\le N_1-1$. We use (3.3.43) combined with Lemma 3.4.5 and Lemma 3.4.1 to get

$$\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{kg}\left[P_{k_1}G_{\mathcal{L}_1}^{kg},P_{k_2}G_{\mathcal{L}_2}^{wa}\right]\right\|_{L^2} \qquad (3.6.3)\\
\lesssim&\sum_{k_1,k_2}2^{k_1}2^{-k_2}\left\|\mathrm{e}^{\mathrm{i}t\Lambda_{kg}}P_{k_1}G_{\mathcal{L}_1}^{kg}\right\|_{L^\infty}\left\|P_{k_2}G_{\mathcal{L}_2}^{wa}\right\|_{L^2}\\
\lesssim&\sum_{k_1,k_2}\left(\varepsilon\langle t\rangle^{-1}\langle t\rangle^{-H(n_1+1)\delta}2^{-(N(n_1+1)-\frac{5}{2})k_1^+}2^{\frac12k_1^-}\right)\left(\varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}2^{\frac12k_2^-}\right)\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1+1)+H(n_2))\delta}2^{-(N(n_1+1)-\frac52)k_1^+-N(n_2+1)k_2^+}2^{\frac12k_1^-+\frac12k_2^-}\\
\lesssim&\,\varepsilon^2\langle t\rangle^{-(1+H(n_1+1)\delta+H(n_2)\delta)}2^{-\min\{N(n_1+1)-\frac52,N(n_2+1)\}k^+}.
\end{aligned}$$

Note that

$$H(n_1+1)+H(n_2)\geq H''_{kg}(n),\quad \min\left\{N(n_1+1)-\frac{5}{2},N(n_2+1)\right\}\geq N''_{kg}(n). \tag{3.6.4}$$

- Case II: $k_1\leq k_2$ and $n_1=N_1$. We use (3.3.43) combined with Lemma 3.4.1 and Lemma 3.4.5 to obtain

$$\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{kg}\left[P_{k_1}G^{kg}_{\mathcal{L}},P_{k_2}G^{wa}\right]\right\|_{L^2} \qquad (3.6.5)\\
\lesssim&\sum_{k_1,k_2}2^{k_1}2^{-k_2}\left\|P_{k_1}G^{kg}_{\mathcal{L}}\right\|_{L^2}\left\|\mathrm{e}^{\mathrm{i}t\Lambda_{wa}}P_{k_2}G^{wa}\right\|_{L^\infty}\\
\lesssim&\sum_{k_1,k_2}\left(\varepsilon\langle t\rangle^{-H(N_1)\delta}2^{-N(N_1)k_1^+}\right)\left(\varepsilon\langle t\rangle^{-1}\langle t\rangle^{-H(1)\delta}\ln\langle t\rangle2^{-N(1)k_2^+}2^{k_2^-}\right)\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-1}\langle t\rangle^{-(H(N_1)+H(1))\delta}\ln\langle t\rangle2^{-N(N_1)k_1^+-N(1)k_2^+}2^{k_2^-}\\
\lesssim&\,\varepsilon^2\langle t\rangle^{-(1+H(N_1)\delta+H(1)\delta)}\ln\langle t\rangle2^{-\min\{N(N_1),N(1)\}k^+}.
\end{aligned}$$

Note that

$$H(N_1)+H(1)\geq H''_{kg}(N_1),\quad \min\left\{N(N_1),N(1)\right\}\geq N''_{kg}(N_1). \tag{3.6.6}$$

- Case III: $k_2\leq k_1$ and $n_2\leq N_1-1$. We use (3.3.43) combined with Lemma 3.4.5

and Lemma 3.4.1 to get

$$\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{kg}\Big[P_{k_1}G^{kg}_{\mathcal{L}_1},P_{k_2}G^{wa}_{\mathcal{L}_2}\Big]\right\|_{L^2} && (3.6.7)\\
&\lesssim\sum_{k_1,k_2}2^{k_1}2^{-k_2}\left\|P_{k_1}G^{kg}_{\mathcal{L}_1}\right\|_{L^2}\left\|\mathrm{e}^{\mathrm{i}t\Lambda_{wa}}P_{k_2}G^{wa}_{\mathcal{L}_2}\right\|_{L^\infty}\\
&\lesssim\sum_{k_1,k_2}2^{k_1}2^{-k_2}\left(\varepsilon\langle t\rangle^{-H(n_1)\delta}2^{-N(n_1)k_1^+}\right)\left(\varepsilon\langle t\rangle^{-1}\langle t\rangle^{-H(n_2+1)\delta}\ln\langle t\rangle 2^{-N(n_2+1)k_2^+}2^{k_2^-}\right)\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1)+H(n_2+1))\delta}\ln\langle t\rangle 2^{-(N(n_1)-1)k_1^+-N(n_2+1)k_2^+}2^{k_1^-}\\
&\lesssim\varepsilon^2\langle t\rangle^{-(1+H(n_1)\delta+H(n_2+1)\delta)}\ln\langle t\rangle 2^{-\min\{N(n_1)-1,N(n_2+1)\}k^+}.
\end{aligned}$$

Note that

$$H(n_1)+H(n_2+1)\ge H''_{kg}(n),\quad \min\Big\{N(n_1)-1,N(n_2+1)\Big\}\ge N''_{kg}(n). \tag{3.6.8}$$

- Case IV: $k_2\le k_1$ and $n_2=N_1$. This is the most complicated case. The key is to control the singular term $2^{-k_2^-}$. We need to bound

$$\sum_{k_1,k_2}\left\|\varphi_k\mathbf{I}_{kg}\Big[P_{k_1}G^{kg},P_{k_2}G^{wa}_{\mathcal{L}}\Big]\right\|_{L^2}. \tag{3.6.9}$$

The only possible control of $P_{k_2}G^{wa}_{\mathcal{L}}$ is through Lemma 3.4.1 as

$$\|P_{k_2}G^{wa}_{\mathcal{L}}\|_{L^2}\lesssim\varepsilon\langle t\rangle^{-H(N_1)\delta}2^{-N(N_1)k_2^+}2^{\frac12 k_2^-}. \tag{3.6.10}$$

This is not enough and only provides $2^{\frac12 k_2^-}$. Then it remains to bound $P_{k_1}G^{kg}$ part and provide another $2^{\frac12 k_2^-}$. Notice that this has been achieved in the wave nonlinear estimate (Lemma 3.5.1). We just need to discuss the four cases (replacing k^- by k_2^- there) to get the desired estimate

$$\sum_{k_1,k_2} \left\|\varphi_k \mathbf{I}_{kg}\left[P_{k_1}G^{kg}, P_{k_2}G^{wa}_{\mathcal{L}}\right]\right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H''_{kg}(n)\delta)} 2^{-N''_{kg}(n)k^+}. \tag{3.6.11}$$

This suffices to close the proof.

□

Lemma 3.6.2. *Assume* (3.2.104) *holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1$, we have*

$$\left\|\sum_{\iota_1,\iota_2\in\{+,-\}} \varphi_k \widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}}\left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}, \widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}} + \mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\right]\right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+}, \tag{3.6.12}$$

$$\left\|\sum_{\iota_1,\iota_2\in\{+,-\}} \varphi_k \widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}}\left[\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \widehat{G^{wa,\iota_2}_{\mathcal{L}_2}}\right]\right\|_{L^2_\xi} \lesssim \varepsilon^2 \langle t\rangle^{-(1+H(n)\delta)} 2^{-N(n)k^+}. \tag{3.6.13}$$

Proof. The proof is similar to that of Lemma 3.6.1. The only difference is that we do not have $H(n_2)$ time decay, so we should always assign L^2 to $G^{kg,\iota_1}_{\mathcal{L}_1}$ or $G^{wa,\iota_2}_{\mathcal{L}_2}$. The other part $V^{wa,\iota_2}_{\mathcal{L}_2,\infty} + \mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}$ or $\left(\mathrm{e}^{\iota_1 \mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}$ may take L^2 or L^∞ following the argument in the proof of Lemma 3.6.1. Then based on the similar analysis as in Lemma 3.5.2 and using the results from Lemma 3.4.18, Lemma 3.4.14 and Lemma 3.4.26, we get the desired result. □

Lemma 3.6.3. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1$, *we have*

$$\begin{aligned}\Bigg\|\varphi_k\Big\{&\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_1,\iota_2\in\{+,-\}}\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}+\mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}\,,\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big] \\ &-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V_\infty^{kg}}\Big)_{\mathcal{L}_1}-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}-\mathfrak{b}_{\mathcal{L},\infty}\Big\}\Bigg\|_{L^2_\xi} \\ &\lesssim \varepsilon^2\langle t\rangle^{-(1+H(n)\delta)}2^{-N(n)k^+}.\end{aligned} \tag{3.6.14}$$

Proof. Based on the proof of Deng-Pusateri [11, Lemma 4.2], we have (up to fast-decaying terms)

$$\mathcal{C}_{\mathcal{L},\infty}(t,\xi)\sim\frac{1}{2}(2\pi)^{-\frac{3}{2}}\frac{|\xi|^2}{\langle\xi\rangle}\cdot\mathrm{Im}\left\{\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-|\eta|\right)}\frac{1}{|\eta|}H_{\mathcal{L},\infty}(t,\eta)\,\mathrm{d}\eta\right\}. \tag{3.6.15}$$

We may decompose

$$\begin{aligned}&\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1},\Big(\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big)\Big] \\ &=\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1},\varphi_{\le 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\Big(\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big)\Big] \\ &\quad+\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1},\varphi_{\ge 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\Big(\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big)\Big].\end{aligned} \tag{3.6.16}$$

<u>Step 1</u>. *Low Frequency - Resonant Case* $(\iota_1,\iota_2)=(+,+)$ *or* $(\iota_1,\iota_2)=(+,-)$

Let $\widehat{g_1}=\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}$ and $\widehat{g_2}=\varphi_{\le 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\Big(\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big)=H^{\iota_2}_{\mathcal{L}_2,\infty}$. Then we have

$$\sum_{\iota_2\in\{+,-\}} \widehat{\mathbf{I}_{kg}^{+\iota_2}}\big[\widehat{g_1},\widehat{g_2}\big] - \mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\widehat{g_1} \tag{3.6.17}$$

$$\simeq \sum_{\iota_2\in\{+,-\}} \widehat{\mathbf{I}_{kg}^{+\iota_2}}\big[\widehat{g_1},\widehat{g_2}\big] - \widehat{g_1}\sum_{\iota_2\in\{+,-\}} \iota_2\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\iota_2\eta\right)}\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\widehat{g_2}(t,\eta)\,\mathrm{d}\eta.$$

We only focus on the $(\iota_1,\iota_2)=(+,+)$ case. The $(\iota_1,\iota_2)=(+,-)$ case can be handled in a similar way. It suffices to consider

$$\widehat{\mathbf{I}_{kg}^{++}}\big[\widehat{g_1},\widehat{g_2}\big] - \widehat{g_1}\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)}\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\widehat{g_2}(t,\eta)\,\mathrm{d}\eta \tag{3.6.18}$$

$$\simeq \int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{++}(\xi,\eta)}b_{++}(\xi,\eta)\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta - \widehat{g_1}(t,\xi)\int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)}\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\widehat{g_2}(t,\eta)\,\mathrm{d}\eta$$

$$= \mathscr{R}_1+\mathscr{R}_2+\mathscr{R}_3,$$

where

$$\mathscr{R}_1 := \int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{++}(\xi,\eta)}\left[b_{++}(\xi,\eta)-\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\right]\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta, \tag{3.6.19}$$

$$\mathscr{R}_2 := \int_{\mathbb{R}^3}\left[\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{++}(\xi,\eta)}-\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)}\right]\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\widehat{g_1}(t,\xi-\eta)\widehat{g_2}(t,\eta)\,\mathrm{d}\eta, \tag{3.6.20}$$

$$\mathscr{R}_3 := \int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)}\frac{|\xi|^2}{\langle\xi\rangle}\frac{1}{|\eta|}\Big[\widehat{g_1}(t,\xi-\eta)-\widehat{g_1}(t,\xi)\Big]\widehat{g_2}(t,\eta)\,\mathrm{d}\eta. \tag{3.6.21}$$

Note that due to the cutoff in $\widehat{g_2}$, all integral over η is restricted to $\varphi_{\le 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)$. For $\mathscr{R}_1$, note that

$$\left|\frac{|\xi|^2}{\langle\xi\rangle}-\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle}\right| \lesssim |\eta|\,. \tag{3.6.22}$$

Using (3.3.42), Lemma 3.4.21 and Lemma 3.4.16, we obtain

$$\|\varphi_k \mathscr{R}_1\|_{L^2} \lesssim 2^{\frac{3}{2}k_2} \|\widehat{g_1}\|_{L^2} \|\widehat{g_2}\|_{L^2} \lesssim \varepsilon^3 \langle t\rangle^{-\frac{21}{16}}. \tag{3.6.23}$$

For $\mathscr{R}_2$, note that

$$\left| \mathrm{e}^{\mathrm{i}t\Phi^{++}_{kg}(\xi,\eta)} - \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)} \right| \lesssim |\eta|^2. \tag{3.6.24}$$

Using (3.3.42), Lemma 3.4.21 and Lemma 3.4.16, we obtain

$$\|\varphi_k \mathscr{R}_2\|_{L^2} \lesssim 2^{\frac{3}{2}k_2} 2^{k_2} \|\widehat{g_1}\|_{L^2} \|\widehat{g_2}\|_{L^2} \lesssim \varepsilon^3 \langle t\rangle^{-\frac{35}{16}}. \tag{3.6.25}$$

$\mathscr{R}_3$ is a bit more delicate. Minkowski's integral inequality implies

$$\begin{aligned} &\|\widehat{g_1}(\xi-\eta) - \widehat{g_1}(\xi)\|_{L^2} \\ \lesssim &\left\| \int_0^{|\eta|} \partial_{\widehat{\eta}} \widehat{g_1}(\xi - c\widehat{\eta}) \mathrm{d}c \right\|_{L^2_\xi} \lesssim \int_0^{|\eta|} \|\partial_{\widehat{\eta}} \widehat{g_1}(\xi - c\widehat{\eta})\|_{L^2_\xi} \mathrm{d}c \\ \lesssim &\int_0^{|\eta|} \|\partial_\xi \widehat{g_1}\|_{L^2_\xi} \mathrm{d}c \lesssim |\eta| \, \|\partial_\xi \widehat{g_1}\|_{L^2_\xi}. \end{aligned} \tag{3.6.26}$$

Then we may use (3.3.42), Lemma 3.4.20 and Lemma 3.4.21 to obtain

$$\|\varphi_k \mathscr{R}_3\|_{L^2} \lesssim 2^{\frac{3}{2}k} \varphi_{\leq 0}\big(2^k \langle t\rangle^{\frac{7}{8}}\big) \|\widehat{g_1}\|_{H^1_\eta} \|\widehat{g_2}\|_{L^2} \lesssim \varepsilon^3 \langle t\rangle^{-\frac{21}{16}} \big(\ln\langle t\rangle\big)^2. \tag{3.6.27}$$

In total, we have

$$\left\| \widehat{\mathbf{I}^{++}_{kg}}[\widehat{g_1}, \widehat{g_2}] - \widehat{g_1} \int_{\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\left(\frac{\xi\cdot\eta}{\langle\xi\rangle}-\eta\right)} \frac{|\xi|^2}{\langle\xi\rangle} \frac{1}{|\eta|} \widehat{g_2}(t,\eta) \, \mathrm{d}\eta \right\|_{L^2} \lesssim \varepsilon^3 \langle t\rangle^{-\frac{21}{16}} \big(\ln\langle t\rangle\big)^2. \tag{3.6.28}$$

<u>*Step 2.*</u> *Low Frequency - Non-Resonant Case* $(\iota_1,\iota_2) = (-,+)$ *or* $(\iota_1,\iota_2) = (-,-)$

By the definition of $\mathfrak{b}_\infty$, we directly have

$$\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_2\in\{+,-\}}\widehat{\mathbf{I}_{kg}^{-\iota_2}}[\widehat{g}_1,\widehat{g}_2]-\mathfrak{b}_{\mathcal{L},\infty}=0. \tag{3.6.29}$$

<u>*Step 3*</u>*. High Frequency*

Since the high-frequency cutoff will eliminate $\mathcal{H}_\infty$, we let $\widehat{g}_1=\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_{\mathcal{L}_1,\infty}^{kg}}$ and $\widehat{g}_2=\widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}}$. Then it suffices to bound

$$\begin{aligned}&\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\geq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{g}_1(t,\xi-\eta)\widehat{g}_2(\eta)\,\mathrm{d}\eta\right\|_{L^2} \qquad (3.6.30)\\ \lesssim&\sum_{k_1,k_2}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\geq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}.\end{aligned}$$

- Case I: If $2^{2k_1^-}\langle t\rangle^{\frac{1}{8}}\lesssim 1$, then using (3.3.43), we have

$$\begin{aligned}&\sum_{k_1,k_2}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\geq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}\\ \lesssim&\sum_{k_1,k_2}2^{k_1^++2k_1^-}2^{-k_2}\left\|\widehat{P_{k_1}g_1}\right\|_{L^2}\left\|\mathrm{e}^{-\iota_2\mathrm{i}t\Lambda_{wa}}P_{k_2}g_2\right\|_{L^\infty}\\ \lesssim&\sum_{k_1,k_2}2^{k_1^++2k_1^-}2^{-k_2}\Big(\varepsilon\big(\ln\langle t\rangle\big)^2 2^{-N(n_1-4)k_1^+}2^{k_1^-}\Big)\Big(\varepsilon\langle t\rangle^{-1}2^{-N(n_2-2)k_2^+}2^{k_2}\Big)\\ \lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-1}\big(\ln\langle t\rangle\big)^2 2^{-N(n_1-4)k_1^++k_1^++3k_1^--N(n_2-2)k_2^+}\\ \lesssim&\,\varepsilon^2\langle t\rangle^{-\frac{9}{8}}\big(\ln\langle t\rangle\big)^2\cdot\sum_{\max(k_1,k_2)\geq k}2^{-N(n_1-4)k_1^++k_1-N(n_2-2)k_2^+}\\ \lesssim&\,\varepsilon^2\langle t\rangle^{-(1+H(n)\delta)}2^{-N(n)k^+}. \qquad (3.6.31)\end{aligned}$$

Here in the second inequality we use Lemma 3.4.7 and Lemma 3.4.14. In the last inequality we may take the summation over k_1, k_2 for $\max(k_1,k_2)\geq k$ in view of Lemma 3.3.3.

- Case II: If $2^{2k_1^-}\langle t\rangle^{\frac{1}{8}} \gtrsim 1$ and $k_1 \gtrsim k_2$ and $k_2 \le 0$, then we use Lemma 3.4.23 and (3.4.48) to obtain

$$
\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\ge 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}\\
&\lesssim\sum_{k_1,k_2}2^{k_1^+}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_1}g_1\right\|_{L^\infty}\left\|\frac{1}{|\eta|}\widehat{P_{k_2}g_2}\right\|_{L^2_\eta}\\
&\lesssim\sum_{k_1,k_2}2^{k_1^+}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_1}g_1\right\|_{L^\infty}\left\|P_{k_2}\frac{1}{|\eta|}\right\|_{L^2_\eta}\left\|\widehat{P_{k_2}g_2}\right\|_{L^\infty_\eta}\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\big(\ln\langle t\rangle\big)^2\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{-(N(n_1)-5)k_1^+-N(n_2)k_2^+}2^{-k_1^-(\frac{1}{2}-\frac{\sigma}{4})-\frac{1}{2}k_2^-}\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\big(\ln\langle t\rangle\big)^2\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{-(N(n_1)-5)k_1^+-N(n_2)k_2^+}\langle t\rangle^{\frac{1}{16}(\frac{1}{2}-\frac{\sigma}{4})}\langle t\rangle^{-\frac{7}{16}}\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-\frac{33}{32}}\big(\ln\langle t\rangle\big)^2 2^{-(N(n_1)-5)k_1^+-N(n_2)k_2^+}.
\end{aligned}\tag{3.6.32}
$$

- Case III: If $2^{2k_1^-}\langle t\rangle^{\frac{1}{8}} \gtrsim 1$ and $k_1 \gtrsim k_2$ and $k_2 \ge 0$, then

$$
\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\ge 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}\\
&\lesssim\sum_{k_1,k_2}2^{k_1^+}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_1}g_1\right\|_{L^\infty}\left\|\widehat{P_{k_2}g_2}\right\|_{L^2}\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\big(\ln\langle t\rangle\big)^2\langle t\rangle^{-\frac{3}{2}+\frac{\sigma}{8}}2^{-(N(n_1)-5)k_1^+-N(n_2)k_2^+}2^{-k_1^-(\frac{1}{2}-\frac{\sigma}{4})}\\
&\lesssim\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-\frac{47}{32}}\big(\ln\langle t\rangle\big)^2 2^{-(N(n_1)-5)k_1^+-N(n_2)k_2^+}.
\end{aligned}\tag{3.6.33}
$$

- Case IV: If $2^{2k_1^-}\langle t\rangle^{\frac{1}{8}} \gtrsim 1$ and $k_1 \lesssim k_2$, then we have

$$
\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\lesssim 1.\tag{3.6.34}
$$

We first integrate by parts in η using (3.3.3), and then use Lemma 3.4.7, Lemma 3.4.21 and (3.4.48) to obtain that

$$
\begin{aligned}
&\sum_{k_1,k_2}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\geq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}\\
\lesssim&\sum_{k_1,k_2}\langle t\rangle^{-1}\left\|\varphi_k\int_{\mathbb{R}^3}\varphi_{\geq 0}\big(\eta\langle t\rangle^{\frac{7}{8}}\big)\mathrm{e}^{\mathrm{i}t(\langle\xi\rangle-\iota_1\langle\xi-\eta\rangle-\iota_2|\eta|)}\frac{|\xi-\eta|^2}{\langle\xi-\eta\rangle\,|\eta|^3}\widehat{P_{k_1}g_1}(t,\xi-\eta)\widehat{P_{k_2}g_2}(\eta)\,\mathrm{d}\eta\right\|_{L^2}\\
\lesssim&\sum_{k_1,k_2}\langle t\rangle^{-1}\left\|\widehat{P_{k_1}g_1}\right\|_{L^2}2^{-k_2}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}P_{k_2}g_2\right\|_{L^\infty}\\
\lesssim&\sum_{k_1,k_2}\varepsilon^2\langle t\rangle^{-2}\big(\ln\langle t\rangle\big)^3\,2^{-N(n_1)k_1^+-N(n_2)k_2^+}.
\end{aligned}\tag{3.6.35}
$$

For the term with derivative after integration by parts, we always assign L^2, and the other term with L^∞. Note that now 2^{-k_2}, even present, is not a big deal since it grows at most $\langle t\rangle^{\frac{1}{8}}$.

Step 4. $\mathfrak{B}_{\mathcal{L},\infty}$ *Terms*

In bounding

$$
\sum_{\mathcal{L}_1,\mathcal{L}_2}\varphi_k\Big\{\sum_{\iota_1,\iota_2\in\{+,-\}}\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1},\widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}}+\mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2}\Big]-\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}\Big\},\tag{3.6.36}
$$

all terms related to $\mathfrak{B}_{\mathcal{L}_1,\infty}$ can be controlled as in Lemma 3.6.1 since $\mathfrak{B}_{\mathcal{L}_1,\infty}$ has sufficient time decay as in Lemma 3.4.25 and Lemma 3.4.26. □

Combining Lemma 3.6.1, Lemma 3.6.2 and Lemma 3.6.3, we get the following proposition:

Proposition 3.6.4. *Assume* (3.2.104) *holds. For any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\leq n\leq N_1$*, we* *have*

$$\left\|\varphi_k\partial_t\widehat{G_{\mathcal{L}}^{kg}}\right\|_{L^2_\xi} \lesssim \varepsilon^2\langle t\rangle^{-(1+H''_{kg}(n)\delta)}2^{-N''_{kg}(n)k^+}. \tag{3.6.37}$$

3.6.2 T_2' Estimates

Lemma 3.6.5. *Assume* (3.2.104) *holds. For any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\le n\le N_1-1$ *and* $\ell\in\{1,2,3\}$, *we have*

$$\left\|\varphi_k\partial_{\xi_\ell}\left\{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\,\mathbf{I}_{kg}^{\iota_1\iota_2}\left[G_{\mathcal{L}_1}^{kg,\iota_1},G_{\mathcal{L}_2}^{wa,\iota_2}\right]\right\}\right\|_{L^2_\xi} \lesssim \varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}. \tag{3.6.38}$$

Proof. The proof is similar to that of Lemma 3.6.1 and of [45, Lemma 4.3]. Since

$$\mathbf{I}_{kg}^{\iota_1\iota_2}\left[G_{\mathcal{L}_1}^{kg,\iota_1},G_{\mathcal{L}_2}^{wa,\iota_2}\right](t,\xi)\sim\int_{\mathbb{R}^3}\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{\iota_1\iota_2}}b_{\iota_1\iota_2}(\xi,\eta)\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}(t,\xi-\eta)\widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}(t,\eta)\,\mathrm{d}\eta, \tag{3.6.39}$$

ξ_ℓ derivative may hit the phase $\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{\iota_1\iota_2}}$, the multiplier $b_{\iota_1\iota_2}$, or $G_{\mathcal{L}_1}^{kg,\iota_1}(\xi-\eta)$.

- If ξ_ℓ derivative hits the phase $\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}\mathrm{e}^{\mathrm{i}t\Phi_{kg}^{\iota_1\iota_2}}$, then we have an extra $\langle t\rangle$ term popping out. Following exactly the same argument as in the proof of Lemma 3.6.1, we get the desired result.
- If ξ_ℓ derivative hits the multiplier $b_{\iota_1\iota_2}$, then nothing happens and we follow the same proof of Lemma 3.6.1.
- If ξ_ℓ derivative hits $\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}(\xi-\eta)$, then we simply use (3.3.42) and Lemma 3.4.1

to obtain

$$
\begin{aligned}
&\left\|\varphi_k \widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\left[\varphi_{k_1}\partial_{\xi_\ell}\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \varphi_{k_2}\widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}\right]\right\|_{L^2} \qquad (3.6.40)\\
&\lesssim 2^{\frac{3}{2}\min\{k_1,k_2,k\}}2^{-k_2}\left\|\varphi_{k_1}\partial_{\xi_\ell}\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}\right\|_{L^2}\left\|\varphi_{k_2}\widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}\right\|_{L^2}\\
&\lesssim \varepsilon^2\langle t\rangle^{-(H(n_1+1)+H(n_2))\delta}2^{\frac{3}{2}\min\{k_1,k_2,k\}}2^{-\frac{1}{2}k_2}2^{-(N(n_1+1)+1)k_1^+-N(n_2)k_2^+}.
\end{aligned}
$$

This works for $k_1 \leq k_2$. If $k_1 \geq k_2$, then through change of variables in the convolution, we can always transfer the derivative to $G_{\mathcal{L}_2}^{wa,\iota_2}$, so we know both $H''_{kg}(n)$ and $N''_{kg}(n)$ requirements can be met.

□

Lemma 3.6.6. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1 - 1$ and $\ell \in \{1,2,3\}$, we have*

$$\left\|\varphi_k\partial_{\xi_\ell}\left\{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\,\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}} + \mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2}\right]\right\}\right\|_{L^2_\xi} \lesssim \varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}, \tag{3.6.41}$$

$$\left\|\varphi_k\partial_{\xi_\ell}\left\{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\,\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}\right]\right\}\right\|_{L^2_\xi} \lesssim \varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}. \tag{3.6.42}$$

Proof. It is similar to that of Lemma 3.6.2 and Lemma 3.6.5. □

Lemma 3.6.7. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1 - 1$ and*

$\ell \in \{1,2,3\}$, *we have*

$$\Bigg\|\varphi_k\partial_{\xi_\ell}\Bigg\{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Bigg(\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_1,\iota_2=\{+,-\}}\widehat{\mathbf{I}^{\iota_1\iota_2}_{kg}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\Big)_{\mathcal{L}_1}+\mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty},\widehat{V^{wa,\iota_2}_{\mathcal{L}_2,\infty}}+\mathcal{H}^{\iota_2}_{\mathcal{L}_2,\infty}\Big]$$
$$-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1}-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}-\mathfrak{b}_{\mathcal{L},\infty}\Bigg)\Bigg\}\Bigg\|_{L^2_\xi}$$
$$\lesssim \varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}. \tag{3.6.43}$$

Proof. It is similar to that of Lemma 3.6.3 and Lemma 3.6.5. □

Combining Lemma 3.6.5, Lemma 3.6.6 and Lemma 3.6.7, we get the following proposition:

Proposition 3.6.8. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \le n \le N_1 - 1$ *and* $\ell \in \{1,2,3\}$, *we have*

$$\Bigg\|\varphi_k\partial_{\xi_\ell}\Bigg\{\widehat{\mathcal{N}^{kg}_{\mathcal{L}}}-\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Bigg(\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1}+\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}+\mathfrak{b}_{\mathcal{L},\infty}\Bigg)\Bigg\}\Bigg\|_{L^2_\xi}$$
$$\lesssim \varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}. \tag{3.6.44}$$

3.7 Energy Estimates for Wave Equation

3.7.1 S_1 Estimates

We start from the equation (3.2.74) with vector fields:

$$\partial_t \widehat{G^{wa}_{\mathcal{L}}} = \mathrm{e}^{\mathrm{i}t\Lambda_{wa}(\xi)} \widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - h_{\mathcal{L},\infty}. \tag{3.7.1}$$

Multiplying $|\xi|^{-1}\langle\xi\rangle^{2N(n)}\overline{\widehat{G^{wa}_{\mathcal{L}}}}$ on both sides and integrating over $\xi \in \mathbb{R}^3$, we have

$$\partial_t \mathcal{E}^{\mathcal{L}}_{wa} = \mathcal{R}^{\mathcal{L}}_{wa}, \tag{3.7.2}$$

where

$$\mathcal{E}^{\mathcal{L}}_{wa}(t) := \frac{1}{2} \left\| |\nabla|^{-\frac{1}{2}} G^{wa}_{\mathcal{L}}(t) \right\|^2_{H^{N(n)}}, \tag{3.7.3}$$

and

$$\mathcal{R}^{\mathcal{L}}_{wa} := \int_{\mathbb{R}^3} |\xi|^{-1}\langle\xi\rangle^{2H(n)} \left(\mathrm{e}^{\mathrm{i}t\Lambda_{wa}(\xi)} \widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - h_{\mathcal{L},\infty} \right) \overline{\widehat{G^{wa}_{\mathcal{L}}}} \,\mathrm{d}\xi. \tag{3.7.4}$$

Hence, we have

$$\mathcal{E}^{\mathcal{L}}_{wa}(t) = -\int_t^\infty \mathcal{R}^{\mathcal{L}}_{wa}(s)\,\mathrm{d}s. \tag{3.7.5}$$

Similar to the nonlinear analysis, in order to estimate $\mathcal{E}^{\mathcal{L}}_{wa}(t)$, it suffices to bound the time integral of every term in (3.5.1) for $\mathrm{e}^{\mathrm{i}t\Lambda_{wa}(\xi)} \widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - h_{\mathcal{L},\infty}$.

Remark 3.7.1. It seems that we can directly apply nonlinear estimates in Lemma 3.5.1, Lemma 3.5.2 and Lemma 3.5.3 to obtain the desired result. However, this is not always workable since nonlinear estimates do not have sufficient time decay and k^+ weight, so we have to redo the estimates term by term. In particular, the time integral will play a key role.

Lemma 3.7.1. *Assume* (3.2.104) *holds. For any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\le n\le N_1$*, we have*

$$\left|\int_t^\infty\int_{\mathbb{R}^3}|\xi|^{-1}\langle\xi\rangle^{2H(n)}\mathbf{I}_{wa}^{\iota_1\iota_2}\left[G_{\mathcal{L}_1}^{kg,\iota_1},G_{\mathcal{L}_2}^{kg,\iota_2}\right]\cdot\overline{\widehat{G_{\mathcal{L}}^{wa}}}\,\mathrm{d}\xi\mathrm{d}s\right|\lesssim\varepsilon^3\langle t\rangle^{-2H(n)\delta}. \tag{3.7.6}$$

Proof. Without loss of generality, we assume $n_1\le n_2$ and $n_1+n_2=n$. Also, we ignore ι_1 and ι_2 when there is no confusion. We intend to bound

$$\begin{aligned}\mathcal{I}_{wa}\left[G_{\mathcal{L}_1}^{kg},G_{\mathcal{L}_2}^{kg},G_{\mathcal{L}}^{wa}\right]:=&\int_t^\infty\iint_{\mathbb{R}^3\times\mathbb{R}^3}\mathrm{e}^{\mathrm{i}s\Phi_{wa}}\,|\xi|^{-1}\langle\xi\rangle^{2N(n)}a(\xi,\eta)\\&\widehat{G_{\mathcal{L}_1}^{kg}}(s,\xi-\eta)\widehat{G_{\mathcal{L}_2}^{kg}}(s,\eta)\overline{\widehat{G_{\mathcal{L}}^{wa}}(s,\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}s\,.\end{aligned} \tag{3.7.7}$$

Here $a(\xi,\eta)$ will not play a role in the estimate since $|a(\xi,\eta)|\lesssim 1$. It is convenient to write

$$\begin{aligned}\mathcal{I}_{wa}^{m,k,k_1,k_2}\left[G_{\mathcal{L}_1}^{kg},G_{\mathcal{L}_2}^{kg},G_{\mathcal{L}}^{wa}\right]:=&\int_t^\infty\iint_{\mathbb{R}^3\times\mathbb{R}^3}\tau_m(s)\,\mathrm{e}^{\mathrm{i}s\Phi_{wa}}\,|\xi|^{-1}\langle\xi\rangle^{2N(n)}\\&\times\widehat{P_{k_1}G_{\mathcal{L}_1}^{kg}}(s,\xi-\eta)\widehat{P_{k_2}G_{\mathcal{L}_2}^{kg}}(s,\eta)\overline{\widehat{P_kG_{\mathcal{L}}^{wa}}(s,\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\mathrm{d}s\,.\end{aligned} \tag{3.7.8}$$

Here $\tau_m(s)$ is a time cutoff function defined in (3.2.77).

If $1 \le n_1 \le n_2 \le n-1$, then we integrate by parts in time using (3.3.2) to get

$$\begin{aligned}\mathcal{I}_{wa}^{m,k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{kg}, G_{\mathcal{L}}^{wa}\big] \simeq \int_t^\infty \tau_m' \mathcal{K}_{wa}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{kg}, G_{\mathcal{L}}^{wa}\big](s)\,\mathrm{d}s \\ + \int_t^\infty \tau_m \mathcal{K}_{wa}^{k,k_1,k_2}\big[\partial_s G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{kg}, G_{\mathcal{L}}^{wa}\big](s)\,\mathrm{d}s \\ + \int_t^\infty \tau_m \mathcal{K}_{wa}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, \partial_s G_{\mathcal{L}_2}^{kg}, G_{\mathcal{L}}^{wa}\big](s)\,\mathrm{d}s \\ + \int_t^\infty \tau_m \mathcal{K}_{wa}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{kg}, \partial_s G_{\mathcal{L}}^{wa}\big](s)\,\mathrm{d}s,\end{aligned} \tag{3.7.9}$$

where

$$\mathcal{K}_{wa}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{kg}, G_{\mathcal{L}}^{wa}\big] := \iint_{\mathbb{R}^3\times\mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}s\Phi_{wa}}}{\Phi_{wa}} |\xi|^{-1}\langle\xi\rangle^{2N(n)} \widehat{P_{k_1} G_{\mathcal{L}_1}^{kg}}(\xi-\eta)\widehat{P_{k_2} G_{\mathcal{L}_2}^{kg}}(\eta)\overline{\widehat{P_k G_{\mathcal{L}}^{wa}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\,. \tag{3.7.10}$$

Note that for $|\xi| \approx 2^k$ and $s \approx 2^m$,

$$\left|\frac{\mathrm{e}^{\mathrm{i}s\Phi_{wa}}}{\Phi_{wa}}\right| \lesssim |\xi|^{-1} \approx 2^{-k}, \tag{3.7.11}$$

and that

$$|\tau_m| \lesssim 1, \quad |\tau_m'| \lesssim 2^{-m} \approx s^{-1}. \tag{3.7.12}$$

From (3.3.44) in Lemma 3.3.19, we know

$$\begin{aligned}\big\|\mathcal{K}_{wa}^{k,k_1,k_2}[\mathfrak{f},\mathfrak{g},\mathfrak{h}]\big\|_{L^\infty} &\lesssim 2^{-k} 2^{2N(n)k^+} \left|\frac{\mathrm{e}^{\mathrm{i}s\Phi_{wa}}}{\Phi_{wa}}\right| \left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\overline{\widehat{P_k\mathfrak{h}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \\ &\lesssim 2^{-2k}2^{2N(n)k^+}2^{\frac{3}{2}\min(k,k_1,k_2)} \|P_{k_1}\mathfrak{f}\|_{L^2} \|P_{k_2}\mathfrak{g}\|_{L^2} \|P_k\mathfrak{h}\|_{L^2} \\ &\lesssim 2^{-\frac{1}{2}k}2^{2N(n)k^+} \|P_{k_1}\mathfrak{f}\|_{L^2} \|P_{k_2}\mathfrak{g}\|_{L^2} \|P_k\mathfrak{h}\|_{L^2}\,.\end{aligned} \tag{3.7.13}$$

Also, based on Lemma 3.4.1, we have

$$\left\|P_{k_1} G_{\mathcal{L}_1}^{kg}\right\|_{L^2} \lesssim \varepsilon \langle t\rangle^{-H(n_1)\delta} 2^{-N(n_1)k_1^+}, \tag{3.7.14}$$

$$\left\|P_{k_2} G_{\mathcal{L}_2}^{kg}\right\|_{L^2} \lesssim \varepsilon \langle t\rangle^{-H(n_2)\delta} 2^{-N(n_2)k_2^+}, \tag{3.7.15}$$

$$\left\|P_{k} G_{\mathcal{L}}^{wa}\right\|_{L^2} \lesssim \varepsilon \langle t\rangle^{-H(n)\delta} 2^{-N(n)k^+} 2^{\frac{1}{2}k}, \tag{3.7.16}$$

and based on Proposition 3.5.4 and Proposition 3.6.4, we have

$$\left\|P_{k_1} \partial_t G_{\mathcal{L}_1}^{kg}\right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-H''_{kg}(n_1)\delta} \langle t\rangle^{-1} 2^{-N''_{kg}(n_1)k_1^+}, \tag{3.7.17}$$

$$\left\|P_{k_2} \partial_t G_{\mathcal{L}_2}^{kg}\right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-H''_{kg}(n_2)\delta} \langle t\rangle^{-1} 2^{-N''_{kg}(n_2)k_2^+}, \tag{3.7.18}$$

$$\left\|P_{k} \partial_t G_{\mathcal{L}}^{wa}\right\|_{L^2} \lesssim \varepsilon^2 \langle t\rangle^{-H''_{wa}(n)\delta} \langle t\rangle^{-1} 2^{-N''_{kg}(n)k^+} 2^{\frac{1}{2}k^-}. \tag{3.7.19}$$

Note the following facts, which suffice to close the estimate:

- We have enough time decay

$$H(n_1) + H(n_2) + H(n) \geq 2H(n), \tag{3.7.20}$$

$$H''_{kg}(n_1) + H(n_2) + H(n) \geq 2H(n), \tag{3.7.21}$$

$$H(n_1) + H''_{kg}(n_2) + H(n) \geq 2H(n), \tag{3.7.22}$$

$$H(n_1) + H(n_2) + H''_{wa}(n) \geq 2H(n). \tag{3.7.23}$$

  This relies on the fact that $0 < n_1, n_2 < n$, and that terms with less derivatives have faster time decay.

- We have enough k^+ weight

$$\min\Big(N(n_1), N(n_2)\Big) + N(n) - 5 \geq 2N(n), \tag{3.7.24}$$

$$\min\Big(N''_{kg}(n_1), N(n_2)\Big) + N(n) - 5 \geq 2N(n), \tag{3.7.25}$$

$$\min\Big(N(n_1), N''_{kg}(n_2)\Big) + N(n) - 5 \geq 2N(n), \tag{3.7.26}$$

$$\min\Big(N(n_1), N(n_2)\Big) + N''_{wa}(n) - 5 \geq 2N(n). \tag{3.7.27}$$

  This also relies on the fact that $0 < n_1, n_2 < n$, and that terms with less derivatives have better k^+ weight.

- We have enough k^- weight. $\|P_k G^{wa}_{\mathcal{L}}\|_{L^2}$ and $\|P_k \partial_t G^{wa}_{\mathcal{L}}\|_{L^2}$ always provide an extra $2^{\frac{1}{2}k^-}$.

Now the only remaining case is when $n_1 = 0$ and $n_2 = n$. (If we continue using integration by parts in time, time decay is enough, but k^+ weight is not.) Instead, we use (3.3.45).

- Case $k_1 \simeq k \geq k_2$: For $n \leq 2$ (time decay is delicate), we use Lemma 3.4.5 to get

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{wa}}\, |\xi|^{-1}\langle\xi\rangle^{2N(n)}\, \widehat{P_{k_1}G^{kg}_{\mathcal{L}_1}}(\xi-\eta)\widehat{P_{k_2}G^{kg}_{\mathcal{L}_2}}(\eta)\overline{\widehat{P_kG^{wa}_{\mathcal{L}}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \qquad (3.7.28)\\
&\lesssim 2^{-k}2^{2N(n)k^+}\left\|P_{k_1}G^{kg}_{\mathcal{L}_1}\right\|_{L^2}\left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}P_{k_2}G^{kg}_{\mathcal{L}_2}\right\|_{L^\infty}\|P_kG^{wa}_{\mathcal{L}}\|_{L^2}\\
&\lesssim 2^{-k}2^{2N(n)k^+}\left(\varepsilon\langle t\rangle^{-H(n_1)\delta}2^{-N(n_1)k_1^+}\right)\left(\varepsilon\langle t\rangle^{-1}\langle t\rangle^{-H(n_2+1)\delta}2^{-(N(n_2+1)-\frac{5}{2})k_2^+}2^{\frac{1}{2}k_2^-}\right)\\
&\quad\times\left(\varepsilon\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}2^{\frac{1}{2}k}\right)\\
&\lesssim \varepsilon^3\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1)+H(n_2+1)+H(n))\delta}2^{-N(n_1)k_1^+-(N(n_2+1)-\frac{5}{2})k_2^++\frac{1}{2}k_2^-+N(n)k^++\frac{1}{2}k^--k}.
\end{aligned}$$

For $n \geq 2$ (k^+ weight is delicate), we use Lemma 3.4.5 to get

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{wa}} \, |\xi|^{-1}\langle\xi\rangle^{2N(n)} \widehat{P_{k_1} G^{kg}_{\mathcal{L}_1}}(\xi-\eta)\widehat{P_{k_2} G^{kg}_{\mathcal{L}_2}}(\eta)\overline{\widehat{P_k G^{wa}_{\mathcal{L}}}(\xi)} \,\mathrm{d}\eta\mathrm{d}\xi\right| \qquad (3.7.29)\\
&\lesssim 2^{-k}2^{2N(n)k^+} \left\|\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}} P_{k_1} G^{kg}_{\mathcal{L}_1}\right\|_{L^\infty} \left\|P_{k_2} G^{kg}_{\mathcal{L}_2}\right\|_{L^2} \|P_k G^{wa}_{\mathcal{L}}\|_{L^2}\\
&\lesssim 2^{-k}2^{2N(n)k^+}\left(\varepsilon\langle t\rangle^{-1}\langle t\rangle^{-H(n_1+1)\delta}2^{-(N(n_1+1)-\frac{5}{2})k_1^+}2^{\frac{1}{2}k_1^-}\right)\left(\varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}\right)\\
&\quad\times\left(\varepsilon\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}2^{\frac{1}{2}k}\right)\\
&\lesssim \varepsilon^3\langle t\rangle^{-1}\langle t\rangle^{-(H(n_1+1)+H(n_2)+H(n))\delta}2^{-(N(n_1+1)-\frac{5}{2})k_1^+ + \frac{1}{2}k_1^- - N(n_2)k_2^+ + N(n)k^+ + \frac{1}{2}k^-}.
\end{aligned}$$

- Case $k_2 \simeq k \geq k_1$: We choose $(\infty, 2, 2)$ in (3.3.45) and use Lemma 3.4.5 as above to close the estimate.
- Case $k_1 \simeq k_2 \geq k$: We may use integration by parts in time as in the case $0 < n_1, n_2 < n$ (note that now k^+ weight is sufficient).

□

Lemma 3.7.2. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \leq n \leq N_1$, we have*

$$\left|\int_t^\infty\int_{\mathbb{R}^3} |\xi|^{-1}\langle\xi\rangle^{2H(n)} \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}}, \left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2} + \mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right]\cdot\overline{\widehat{G^{wa}_{\mathcal{L}}}}\,\mathrm{d}\xi\mathrm{d}s\right| \lesssim \varepsilon^3\langle t\rangle^{-2H(n)\delta}, \tag{3.7.30}$$

$$\left|\int_t^\infty\int_{\mathbb{R}^3} |\xi|^{-1}\langle\xi\rangle^{2H(n)} \widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1} + \mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}, \widehat{G^{kg,\iota_2}_{\mathcal{L}_2}}\right]\cdot\overline{\widehat{G^{wa}_{\mathcal{L}}}}\,\mathrm{d}\xi\mathrm{d}s\right| \lesssim \varepsilon^3\langle t\rangle^{-2H(n)\delta}. \tag{3.7.31}$$

Proof. By Cauchy-Schwartz inequality, we have

$$\left|\int_t^\infty\int_{\mathbb{R}^3}|\xi|^{-1}\langle\xi\rangle^{2H(n)}\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}},\left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2}+\mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right]\cdot\overline{\widehat{G^{wa}_{\mathcal{L}}}}\,\mathrm{d}\xi\mathrm{d}s\right| \tag{3.7.32}$$
$$\lesssim\int_t^\infty\left\||\xi|^{-\frac12}\langle\xi\rangle^{H(n)}\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\widehat{G^{kg,\iota_1}_{\mathcal{L}_1}},\left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2}+\mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right]\right\|_{L^2_\xi}\left\||\xi|^{-\frac12}\langle\xi\rangle^{H(n)}\widehat{G^{wa}_{\mathcal{L}}}\right\|_{L^2_\xi}\mathrm{d}s\,,$$

and

$$\left|\int_t^\infty\int_{\mathbb{R}^3}|\xi|^{-1}\langle\xi\rangle^{2H(n)}\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1}+\mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}\,,\widehat{G^{kg,\iota_2}_{\mathcal{L}_2}}\right]\cdot\overline{\widehat{G^{wa}_{\mathcal{L}}}}\,\mathrm{d}\xi\mathrm{d}s\right| \tag{3.7.33}$$
$$\lesssim\int_t^\infty\left\||\xi|^{-\frac12}\langle\xi\rangle^{H(n)}\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1}+\mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}\,,\widehat{G^{kg,\iota_2}_{\mathcal{L}_2}}\right]\right\|_{L^2_\xi}\left\||\xi|^{-\frac12}\langle\xi\rangle^{H(n)}\widehat{G^{wa}_{\mathcal{L}}}\right\|_{L^2_\xi}\mathrm{d}s\,.$$

Then Lemma 3.5.2 and Lemma 3.4.1 will close the proof. □

Lemma 3.7.3. *Assume* (3.2.104) *holds. For any $\mathcal{L}\in\mathcal{V}_n$ with $0\le n\le N_1$, we have*

$$\Bigg|\int_t^\infty\int_{\mathbb{R}^3}|\xi|^{-1}\langle\xi\rangle^{2H(n)}\overline{\widehat{G^{wa}_{\mathcal{L}}}} \tag{3.7.34}$$
$$\times\left\{\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_1,\iota_2\in\{+,-\}}\widehat{\mathbf{I}^{\iota_1\iota_2}_{wa}}\left[\left(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}^{\iota_1}_\infty}\widehat{V^{kg,\iota_1}_\infty}\right)_{\mathcal{L}_1}+\mathfrak{B}^{\iota_1}_{\mathcal{L}_1,\infty}\,,\left(\mathrm{e}^{\iota_2\mathrm{i}\mathcal{D}^{\iota_2}_\infty}\widehat{V^{kg,\iota_2}_\infty}\right)_{\mathcal{L}_2}+\mathfrak{B}^{\iota_2}_{\mathcal{L}_2,\infty}\right]-h_{\mathcal{L},\infty}\right\}\mathrm{d}\xi\mathrm{d}s\Bigg|$$
$$\lesssim\varepsilon^3\langle t\rangle^{-2H(n)\delta}.$$

Proof. Similar to the proof of Lemma 3.7.2, we may use Lemma 3.5.3 and Lemma 3.4.1 to get the bound. □

Remark 3.7.2. Note that the nonlinear estimates in Lemma 3.5.2 and Lemma 3.5.3 are better than that of Lemma 3.5.1. This makes the proofs of Lemma 3.7.2 and Lemma 3.7.3 much shorter than that of Lemma 3.7.1.

Combining Lemma 3.7.1, Lemma 3.7.2 and Lemma 3.7.3, we get the following proposition:

Proposition 3.7.4. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1$, *we have*

$$\left|\mathcal{E}_{wa}^{\mathcal{L}}[G_{\mathcal{L}}^{wa}, G_{\mathcal{L}}^{wa}]\right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)\delta}. \tag{3.7.35}$$

This concludes the estimate of $\left\|G^{wa}\right\|_{S_1}$ in Proposition 3.2.2.

3.7.2 T_1 Estimates

We now estimate ξ derivatives of $\widehat{G_{\mathcal{L}}^{wa}}$. Deriving from (3.7.1), we get the equation

$$\left[\partial_t + \mathrm{i}\Lambda_{wa}(\xi)\right]\left(\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}\widehat{G_{\mathcal{L}}^{wa}}\right) = \widehat{\mathcal{N}_{\mathcal{L}}^{wa}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} h_{\mathcal{L},\infty}. \tag{3.7.36}$$

Taking $\mu = wa$ in Lemma 3.3.8, we know

$$\Gamma_\ell\Big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} G_{\mathcal{L}}^{wa}}\Big)(t,\xi) = \mathrm{i}\partial_{\xi_\ell}\Big[\widehat{\mathcal{N}_{\mathcal{L}}^{wa}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} h_{\mathcal{L},\infty}\Big](t,\xi) + \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}\partial_{\xi_\ell}\Big[\Lambda_{wa}(\xi)\widehat{G_{\mathcal{L}}^{wa}}(t,\xi)\Big]. \tag{3.7.37}$$

Note that $\Lambda_{wa}(\xi) = |\xi|$. Inserting

$$\partial_{\xi_\ell}\Big[\Lambda_{wa}(\xi)\widehat{G_{\mathcal{L}}^{wa}}(t,\xi)\Big] = \Lambda_{wa}(\xi)\partial_{\xi_\ell}\widehat{G_{\mathcal{L}}^{wa}}(t,\xi) + \frac{\xi_\ell}{|\xi|}\widehat{G_{\mathcal{L}}^{wa}}(t,\xi) \tag{3.7.38}$$

into (3.7.37) yields

$$\begin{aligned}\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}\Lambda_{wa}(\xi)\Big(\partial_{\xi_\ell}\widehat{G_{\mathcal{L}}^{wa}}(t,\xi)\Big) &= \Gamma_\ell\Big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}} G_{\mathcal{L}}^{wa}}\Big)(t,\xi) \\ &\quad - \mathrm{i}\partial_{\xi_\ell}\Big[\widehat{\mathcal{N}_{\mathcal{L}}^{wa}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)} h_{\mathcal{L},\infty}\Big](t,\xi) - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}\frac{\xi_\ell}{|\xi|}\widehat{G_{\mathcal{L}}^{wa}}(t,\xi).\end{aligned} \tag{3.7.39}$$

Multiplying the above equality by $2^{-\frac{1}{2}k}\varphi_k$ and estimating in L^2_ξ, we obtain

$$2^{\frac{1}{2}k}\left\|\varphi_k\partial_{\xi_\ell}\widehat{G^{wa}_{\mathcal{L}}}\right\|_{L^2} \lesssim 2^{-\frac{1}{2}k}\left\|\varphi_k\Gamma_\ell\big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}G^{wa}_{\mathcal{L}}}\big)\right\|_{L^2} + 2^{-\frac{1}{2}k}\left\|\varphi_k\partial_{\xi_\ell}\Big[\widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}h_{\mathcal{L},\infty}\Big]\right\|_{L^2} + 2^{-\frac{1}{2}k}\left\|\widehat{P_kG^{wa}_{\mathcal{L}}}\right\|_{L^2}. \tag{3.7.40}$$

Based on the energy estimate in Proposition 3.7.4, we have

$$2^{-\frac{1}{2}k}\left\|\varphi_k\Gamma_\ell\big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}G^{wa}_{\mathcal{L}}}\big)\right\|_{L^2} \lesssim \varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n+1)\delta}2^{-N(n+1)k^+}, \tag{3.7.41}$$

$$2^{-\frac{1}{2}k}\left\|\widehat{P_kG^{wa}_{\mathcal{L}}}\right\|_{L^2} \lesssim \varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}. \tag{3.7.42}$$

Based on the nonlinear estimate in Proposition 3.5.8, we have

$$2^{-\frac{1}{2}k}\left\|\varphi_k\partial_{\xi_\ell}\Big[\widehat{\mathcal{N}^{wa}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{wa}(\xi)}h_{\mathcal{L},\infty}\Big]\right\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-H''_{wa}(n)\delta}2^{-N''_{wa}(n)k^+}. \tag{3.7.43}$$

Combining (3.7.40)–(3.7.43), we get the following proposition:

Proposition 3.7.5. *Assume* (3.2.104) *holds. For any* $\mathcal{L}\in\mathcal{V}_n$ *with* $0\le n\le N_1-1$, *we have*

$$2^{\frac{1}{2}k}\left\|\varphi_k\partial_{\xi_\ell}\widehat{G^{wa}_{\mathcal{L}}}\right\|_{L^2} \lesssim \varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n+1)\delta}2^{-N(n+1)k^+}. \tag{3.7.44}$$

This concludes the estimate of $\left\|G^{wa}\right\|_{T_1}$ in Proposition 3.2.2.

3.8 Energy Estimates for Klein-Gordon Equation

3.8.1 S_2 Estimates

We start from the equation (3.2.75) with vector fields:

$$\partial_t \widehat{G^{kg}_{\mathcal{L}}} = \mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1} - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty} - \mathfrak{b}_{\mathcal{L},\infty}. \quad (3.8.1)$$

Multiplying $\langle\xi\rangle^{2N(n)}\overline{\widehat{G^{kg}_{\mathcal{L}}}}$ on both sides and integrating over $\xi \in \mathbb{R}^3$, we have

$$\partial_t \mathcal{E}^{\mathcal{L}}_{kg} = \mathcal{R}^{\mathcal{L}}_{kg}, \quad (3.8.2)$$

where

$$\mathcal{E}^{\mathcal{L}}_{kg}(t) := \frac{1}{2}\big\|G^{kg}_{\mathcal{L}}(t)\big\|^2_{H^{N(n)}}, \quad (3.8.3)$$

and

$$\begin{aligned}\mathcal{R}^{\mathcal{L}}_{kg} := \int_{\mathbb{R}^3} &\langle\xi\rangle^{2H(n)}\,\overline{\widehat{G^{kg}_{\mathcal{L}}}}(t,\xi) \quad (3.8.4)\\ &\times\Big\{\mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1} - \sum_{\mathcal{L}_1,\mathcal{L}_2} \mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty} - \mathfrak{b}_{\mathcal{L},\infty}\Big\}\,\mathrm{d}\xi\,.\end{aligned}$$

Hence, we have

$$\mathcal{E}^{\mathcal{L}}_{kg}(t) = -\int_t^\infty \mathcal{R}^{\mathcal{L}}_{kg}(s)\,\mathrm{d}s. \quad (3.8.5)$$

Similar to the nonlinear analysis, in order to estimate $\mathcal{E}^{\mathcal{L}}_{kg}(t)$, it suffices to bound the time integral of every term in (3.6.1) for $\mathrm{e}^{\mathrm{i}t\Lambda_{kg}(\xi)} \widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \Big\{\mathrm{i}\,\mathcal{C}_\infty\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty} + \mathfrak{B}_\infty\Big)\Big\}_{\mathcal{L}} - \mathfrak{b}_{\mathcal{L},\infty}$.

Lemma 3.8.1. *Assume (3.2.104) holds. For any $\mathcal{L} \in \mathcal{V}_n$ with $0 \le n \le N_1$, we have*

$$\left| \int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \mathbf{I}_{kg}^{\iota_1 \iota_2} \left[G_{\mathcal{L}_1}^{kg,\iota_1}, G_{\mathcal{L}_2}^{wa,\iota_2} \right] \cdot \overline{\widehat{G_{\mathcal{L}}^{kg}}}(s,\xi) \,\mathrm{d}\xi \mathrm{d}s \right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)\delta}. \tag{3.8.6}$$

Proof. We ignore ι_1 and ι_2 when there is no confusion. We focus on the bound of

$$\mathcal{I}_{kg}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big] := \int_t^\infty \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \mathrm{e}^{\mathrm{i}s\Phi_{kg}} \langle \xi \rangle^{2N(n)} b(\xi,\eta) \widehat{G_{\mathcal{L}_1}^{kg}}(s,\xi-\eta) \widehat{G_{\mathcal{L}_2}^{wa}}(s,\eta) \overline{\widehat{G_{\mathcal{L}}^{kg}}(s,\xi)} \,\mathrm{d}\eta \mathrm{d}\xi \mathrm{d}s\,. \tag{3.8.7}$$

Denote by

$$\begin{aligned} \mathcal{I}_{kg}^{m,k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big] := \int_t^\infty \iint_{\mathbb{R}^3 \times \mathbb{R}^3} & \tau_m(s)\, \mathrm{e}^{\mathrm{i}s\Phi_{kg}} \langle \xi \rangle^{2N(n)} b(\xi,\eta) \\ & \times \widehat{P_{k_1} G_{\mathcal{L}_1}^{kg}}(s,\xi-\eta) \widehat{P_{k_2} G_{\mathcal{L}_2}^{wa}}(s,\eta) \overline{\widehat{P_k G_{\mathcal{L}}^{kg}}(s,\xi)} \,\mathrm{d}\eta \mathrm{d}\xi \mathrm{d}s\,. \end{aligned} \tag{3.8.8}$$

For the case $1 \le n_1 \le n_2 \le n-1$, we integrate by parts in time using (3.3.2) to get

$$\begin{aligned} \mathcal{I}_{kg}^{m,k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big] \simeq & \int_t^\infty \tau_m' \mathcal{K}_{kg}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big](s) \,\mathrm{d}s \\ & + \int_t^\infty \tau_m \mathcal{K}_{kg}^{k,k_1,k_2}\big[\partial_s G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big](s) \,\mathrm{d}s \\ & + \int_t^\infty \tau_m \mathcal{K}_{kg}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, \partial_s G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big](s) \,\mathrm{d}s \\ & + \int_t^\infty \tau_m \mathcal{K}_{kg}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, \partial_s G_{\mathcal{L}}^{kg}\big](s) \,\mathrm{d}s, \end{aligned} \tag{3.8.9}$$

where

$$\mathcal{K}_{kg}^{k,k_1,k_2}\big[G_{\mathcal{L}_1}^{kg}, G_{\mathcal{L}_2}^{wa}, G_{\mathcal{L}}^{kg}\big] := \iint_{\mathbb{R}^3 \times \mathbb{R}^3} \frac{\mathrm{e}^{\mathrm{i}s\Phi_{kg}}}{\Phi_{kg}} \langle \xi \rangle^{2N(n)} b(\xi,\eta) \widehat{P_{k_1} G_{\mathcal{L}_1}^{kg}}(\xi-\eta) \widehat{P_{k_2} G_{\mathcal{L}_2}^{wa}}(\eta) \overline{\widehat{P_k G_{\mathcal{L}}^{kg}}(\xi)} \,\mathrm{d}\eta \mathrm{d}\xi\,. \tag{3.8.10}$$

Note that for $|\xi| \approx 2^k$ and $s \approx 2^m$,

$$\left|\frac{\mathrm{e}^{\mathrm{i}s\Phi_{kg}}}{\Phi_{kg}}\right| \lesssim |\eta|^{-1} \approx 2^{-k_2}, \tag{3.8.11}$$

$$|b(\xi,\eta)| \lesssim \frac{|\xi-\eta|}{|\eta|} \approx 2^{k_1}2^{-k_2}, \tag{3.8.12}$$

and that

$$|\tau_m| \lesssim 1, \quad |\tau_m'| \lesssim 2^{-m} \approx s^{-1}. \tag{3.8.13}$$

From (3.3.44) in Lemma 3.3.19, we know

$$\begin{aligned}
\left\|\mathcal{K}_{kg}^{k,k_1,k_2}[\mathfrak{f},\mathfrak{g},\mathfrak{h}]\right\|_{L^\infty} &\lesssim 2^{2N(n)k^+}2^{k_1}2^{-k_2}\left|\frac{\mathrm{e}^{\mathrm{i}s\Phi_{kg}}}{\Phi_{kg}}\right|\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3}\widehat{P_{k_1}\mathfrak{f}}(\xi-\eta)\widehat{P_{k_2}\mathfrak{g}}(\eta)\overline{\widehat{P_k\mathfrak{h}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \\
&\lesssim 2^{k_1}2^{-2k_2}2^{2N(n)k^+}2^{\frac{3}{2}\min(k,k_1,k_2)}\left\|P_{k_1}\mathfrak{f}\right\|_{L^2}\left\|P_{k_2}\mathfrak{g}\right\|_{L^2}\left\|P_k\mathfrak{h}\right\|_{L^2} \\
&\lesssim 2^{k_1}2^{-\frac{1}{2}k_2}2^{2N(n)k^+}\left\|P_{k_1}\mathfrak{f}\right\|_{L^2}\left\|P_{k_2}\mathfrak{g}\right\|_{L^2}\left\|P_k\mathfrak{h}\right\|_{L^2}.
\end{aligned} \tag{3.8.14}$$

Also, based on Lemma 3.4.1, we have

$$\left\|P_{k_1}G_{\mathcal{L}_1}^{kg}\right\|_{L^2} \lesssim \varepsilon\langle t\rangle^{-H(n_1)\delta}2^{-N(n_1)k_1^+}, \tag{3.8.15}$$

$$\left\|P_{k_2}G_{\mathcal{L}_2}^{wa}\right\|_{L^2} \lesssim \varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}2^{\frac{1}{2}k_2}, \tag{3.8.16}$$

$$\left\|P_kG_{\mathcal{L}}^{kg}\right\|_{L^2} \lesssim \varepsilon\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}, \tag{3.8.17}$$

and based on Proposition 3.5.4 and Proposition 3.6.4, we have

$$\left\|P_{k_1}\partial_tG_{\mathcal{L}_1}^{kg}\right\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-1-H''_{kg}(n_1)\delta}2^{-N''_{kg}(n_1)k_1^+}, \tag{3.8.18}$$

$$\left\|P_{k_2}\partial_tG_{\mathcal{L}_2}^{wa}\right\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-1-H''_{wa}(n_2)\delta}2^{-N''_{kg}(n_1)k^+}2^{\frac{1}{2}k_2^-}, \tag{3.8.19}$$

$$\left\|P_k\partial_tG_{\mathcal{L}}^{kg}\right\|_{L^2} \lesssim \varepsilon^2\langle t\rangle^{-1-H''_{kg}(n)\delta}2^{-N''_{kg}(n_1)k^+}. \tag{3.8.20}$$

Then following a similar argument as in the proof of Lemma 3.7.1, we may close the estimate.

When $n_1 = 0$ and $n_2 = n$, we use (3.3.47) and discuss the following cases:

- Case $k_1 \simeq k_2 \geq k$ or $k_2 \simeq k \geq k_1$: We use Lemma 3.4.5 to get

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{kg}} \langle\xi\rangle^{2N(n)} b(\xi,\eta)\widehat{P_{k_1}G^{kg}_{\mathcal{L}_1}}(\xi-\eta)\widehat{P_{k_2}G^{wa}_{\mathcal{L}_2}}(\eta)\overline{\widehat{P_k G^{kg}_{\mathcal{L}}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \qquad (3.8.21)\\
&\lesssim 2^{k_1}2^{-k_2}2^{2N(n)k^+}\left\|P_{k_1}\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}_1}\right\|_{L^\infty}\left\|P_{k_2}G^{wa}_{\mathcal{L}_2}\right\|_{L^2}\left\|P_kG^{kg}_{\mathcal{L}}\right\|_{L^2}\\
&\lesssim 2^{k_1}2^{-k_2}2^{2N(n)k^+}\left(\varepsilon\langle t\rangle^{-1-H(n_1+1)\delta}2^{-(N(n_1+1)-\frac{5}{2})k_1^+}2^{\frac{1}{2}k_1^-}\right)\left(\varepsilon\langle t\rangle^{-H(n_2)\delta}2^{-N(n_2)k_2^+}2^{\frac{1}{2}k_2^-}\right)\\
&\quad\times\left(\varepsilon\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}\right)\\
&\lesssim \varepsilon^3\langle t\rangle^{-1-(H(n_1+1)+H(n_2)+H(n))\delta}2^{-(N(n_1+1)-\frac{5}{2})k_1^+-N(n_2)k_2^++N(n)k^+}2^{\frac{1}{2}k_1^-+\frac{1}{2}k_2^-}2^{k_1}2^{-k_2}.
\end{aligned}$$

- Case $k_1 \simeq k \geq k_2$: If $n \leq 2$ (time decay is delicate), we use Lemma 3.4.5 to get

$$\begin{aligned}
&\left|\iint_{\mathbb{R}^3\times\mathbb{R}^3} \mathrm{e}^{\mathrm{i}t\Phi_{kg}} \langle\xi\rangle^{2N(n)} b(\xi,\eta)\widehat{P_{k_1}G^{kg}_{\mathcal{L}_1}}(\xi-\eta)\widehat{P_{k_2}G^{wa}_{\mathcal{L}_2}}(\eta)\overline{\widehat{P_k G^{kg}_{\mathcal{L}}}(\xi)}\,\mathrm{d}\eta\mathrm{d}\xi\right| \qquad (3.8.22)\\
&\lesssim 2^{k_1}2^{-k_2}2^{2N(n)k^+}\left\|P_{k_1}G^{kg}_{\mathcal{L}_1}\right\|_{L^2}\left\|P_{k_2}\mathrm{e}^{-\mathrm{i}t\Lambda_{wa}}G^{wa}_{\mathcal{L}_2}\right\|_{L^\infty}\left\|P_kG^{kg}_{\mathcal{L}}\right\|_{L^2}\\
&\lesssim 2^{k_1}2^{-k_2}2^{2N(n)k^+}\left(\varepsilon\langle t\rangle^{-H(n_1)\delta}2^{-N(n_1)k_1^+}\right)\left(\varepsilon\langle t\rangle^{-1}\ln\langle t\rangle\langle t\rangle^{-H(n_2+1)\delta}2^{-(N(n_2+1)-1)k_2^+}2^{k_2^-}\right)\\
&\quad\times\left(\varepsilon\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}\right)\\
&\lesssim \varepsilon^3\ln\langle t\rangle\langle t\rangle^{-1-(H(n_1)+H(n_2+1)+H(n))\delta}2^{-N(n_1)k_1^+-(N(n_2+1)-1)k_2^++N(n)k^+}2^{k_2^-}2^{k_1}2^{-k_2}.
\end{aligned}$$

  If $n \geq 2$ ($N(n)$ is relatively small, but we must find a term providing 2^{k_2}; however, G^{wa} in L^2 only provides $2^{\frac{1}{2}k_2}$), then we use integration by parts in time to close the estimate.

When $n_1 = n$ and $n_2 = 0$, we use (3.3.47) and discuss the following cases:

- Case $k_1 \simeq k \geq k_2$ or $k_1 \simeq k_2 \geq k$: We choose $(2, \infty, 2)$ in (3.3.47) and use Lemma 3.4.5 as above to close the estimate.
- Case $k_2 \simeq k \geq k_1$: If $n \leq 2$, we choose $(\infty, 2, 2)$ in (3.3.47) and use Lemma 3.4.5. If $n \geq 2$, we may use integration by parts in time to close the proof.

$\square$

Lemma 3.8.2. *Assume* (3.2.104) *holds. For any* $\mathcal{L} \in \mathcal{V}_n$ *with* $0 \leq n \leq N_1$, *we have*

$$\left| \int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \widehat{\mathbf{I}_{kg}^{\iota_1 \iota_2}} \left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}} + \mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2} \right] \cdot \overline{\widehat{G_{\mathcal{L}}^{kg}}} \,\mathrm{d}\xi \mathrm{d}s \right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)\delta}, \tag{3.8.23}$$

$$\left| \int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \widehat{\mathbf{I}_{kg}^{\iota_1 \iota_2}} \left[\left(\mathrm{e}^{\iota_1 \mathrm{i} \mathcal{D}_\infty^{\iota_1}} \widehat{V_\infty^{kg,\iota_1}} \right)_{\mathcal{L}_1} + \mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1}, \widehat{G_{\mathcal{L}_2}^{wa,\iota_2}} \right] \cdot \overline{\widehat{G_{\mathcal{L}}^{kg}}} \,\mathrm{d}\xi \mathrm{d}s \right| \lesssim \varepsilon^3 \langle t \rangle^{-2H(n)\delta}. \tag{3.8.24}$$

Proof. By Cauchy-Schwartz inequality, we have

$$\begin{aligned} &\left| \int_t^\infty \int_{\mathbb{R}^3} \langle \xi \rangle^{2H(n)} \widehat{\mathbf{I}_{kg}^{\iota_1 \iota_2}} \left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}} + \mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2} \right] \cdot \overline{\widehat{G_{\mathcal{L}}^{kg}}} \,\mathrm{d}\xi \mathrm{d}s \right| \qquad (3.8.25) \\ &\lesssim \int_t^\infty \left\| \langle \xi \rangle^{H(n)} \widehat{\mathbf{I}_{kg}^{\iota_1 \iota_2}} \left[\widehat{G_{\mathcal{L}_1}^{kg,\iota_1}}, \widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}} + \mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2} \right] \right\|_{L_\xi^2} \left\| \langle \xi \rangle^{H(n)} \widehat{G_{\mathcal{L}}^{kg}} \right\|_{L_\xi^2} \mathrm{d}s\,, \end{aligned}$$

and

$$\left|\int_t^\infty\int_{\mathbb{R}^3}\langle\xi\rangle^{2H(n)}\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}+\mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1},\widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}\Big]\cdot\overline{\widehat{G_{\mathcal{L}}^{kg}}}\,\mathrm{d}\xi\mathrm{d}s\right| \tag{3.8.26}$$
$$\lesssim\int_t^\infty\left\|\langle\xi\rangle^{H(n)}\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}+\mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1},\widehat{G_{\mathcal{L}_2}^{wa,\iota_2}}\Big]\right\|_{L_\xi^2}\left\|\langle\xi\rangle^{H(n)}\widehat{G_{\mathcal{L}}^{kg}}\right\|_{L_\xi^2}\mathrm{d}s\,.$$

Then Lemma 3.6.2 and Lemma 3.4.1 will close the proof. □

Lemma 3.8.3. *Assume* (3.2.104) *holds. For any $\mathcal{L}\in\mathcal{V}_n$ with $0\le n\le N_1$, we have*

$$\Bigg|\int_t^\infty\int_{\mathbb{R}^3}\langle\xi\rangle^{2H(n)}\overline{\widehat{G_{\mathcal{L}}^{kg}}}\Bigg\{\sum_{\mathcal{L}_1,\mathcal{L}_2}\sum_{\iota_1,\iota_2=\{+,-\}}\widehat{\mathbf{I}_{kg}^{\iota_1\iota_2}}\Big[\Big(\mathrm{e}^{\iota_1\mathrm{i}\mathcal{D}_\infty^{\iota_1}}\widehat{V_\infty^{kg,\iota_1}}\Big)_{\mathcal{L}_1}+\mathfrak{B}_{\mathcal{L}_1,\infty}^{\iota_1},\widehat{V_{\mathcal{L}_2,\infty}^{wa,\iota_2}}+\mathcal{H}_{\mathcal{L}_2,\infty}^{\iota_2}\Big]$$
$$-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V_\infty^{kg}}\Big)_{\mathcal{L}_1}-\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}-\mathfrak{b}_{\mathcal{L},\infty}\Bigg\}\mathrm{d}\xi\mathrm{d}s\Bigg|$$
$$\lesssim\varepsilon^3\langle t\rangle^{-2H(n)\delta}, \tag{3.8.27}$$

Proof. Similar to the proof of Lemma 3.8.2, we may use Lemma 3.6.3 and Lemma 3.4.1 to get the bound. □

Combining Lemma 3.8.1, Lemma 3.8.2 and Lemma 3.8.3, we get the following proposition:

Proposition 3.8.4. *Assume* (3.2.104) *holds. For any $\mathcal{L}\in\mathcal{V}_n$ with $0\le n\le N_1$, we have*

$$\left|\mathcal{E}_{kg}^{\mathcal{L}}[G_{\mathcal{L}}^{kg},G_{\mathcal{L}}^{kg}]\right|\lesssim\varepsilon^3\langle t\rangle^{-2H(n)\delta}. \tag{3.8.28}$$

This concludes the estimate of $\left\|G^{kg}\right\|_{S_2}$ in Proposition 3.2.2.

3.8.2 T_2 Estimates

We now estimate ξ derivatives of $\widehat{G^{kg}_{\mathcal{L}}}$. Deriving from (3.8.1), we get the equation

$$\begin{aligned}&\Big[\partial_t + \mathrm{i}\Lambda_{kg}(\xi)\Big]\Big(\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\widehat{G^{kg}_{\mathcal{L}}}\Big) \\ =\ &\widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Big\{\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1} + \sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty} + \mathfrak{b}_{\mathcal{L},\infty}\Big\}.\end{aligned} \tag{3.8.29}$$

Taking $\mu = kg$ in Lemma 3.3.8, we know

$$\begin{aligned}&\Gamma_\ell\Big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}}}\Big)(t,\xi) \\ =\ &\mathrm{i}\partial_{\xi_\ell}\Big\{\widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Big[\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1} + \sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty} + \mathfrak{b}_{\mathcal{L},\infty}\Big]\Big\}(t,\xi) \\ &+ \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\partial_{\xi_\ell}\Big[\Lambda_{kg}(\xi)\widehat{G^{kg}_{\mathcal{L}}}(t,\xi)\Big].\end{aligned} \tag{3.8.30}$$

Note that $\Lambda_{kg}(\xi) = \langle\xi\rangle$. Inserting

$$\partial_{\xi_\ell}\Big[\Lambda_{kg}(\xi)\widehat{G^{kg}_{\mathcal{L}}}(t,\xi)\Big] = \Lambda_{kg}(\xi)\partial_{\xi_\ell}\widehat{G^{kg}_{\mathcal{L}}}(t,\xi) + \frac{\xi_\ell}{\langle\xi\rangle}\widehat{G^{kg}_{\mathcal{L}}}(t,\xi) \tag{3.8.31}$$

into (3.8.30) yields

$$\begin{aligned}&\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Lambda_{kg}(\xi)\Big(\partial_{\xi_\ell}\widehat{G^{kg}_{\mathcal{L}}}(t,\xi)\Big) \\ =\ &\Gamma_\ell\Big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}}}\Big)(t,\xi) - \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\frac{\xi_\ell}{\langle\xi\rangle}\widehat{G^{kg}_{\mathcal{L}}}(t,\xi) \\ &- \mathrm{i}\partial_{\xi_\ell}\Big\{\widehat{\mathcal{N}^{kg}_{\mathcal{L}}} - \mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Big[\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1} + \sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty} + \mathfrak{b}_{\mathcal{L},\infty}\Big]\Big\}(t,\xi)\,.\end{aligned} \tag{3.8.32}$$

Multiplying the above equality by φ_k and estimating in L^2, we obtain

$$\begin{aligned}
&2^{k^+}\Big\|\varphi_k\partial_{\xi_\ell}\widehat{G^{kg}_{\mathcal{L}}}\Big\|_{L^2} \qquad (3.8.33)\\
&\lesssim \Big\|\varphi_k\Gamma_\ell\big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}}}\big)\Big\|_{L^2}+\Big\|\widehat{P_kG^{kg}_{\mathcal{L}}}\Big\|_{L^2}\\
&\quad+\Bigg\|\varphi_k\partial_{\xi_\ell}\Bigg\{\widehat{\mathcal{N}^{kg}_{\mathcal{L}}}-\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Bigg[\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1}+\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}+\mathfrak{b}_{\mathcal{L},\infty}\Bigg]\Bigg\}\Bigg\|_{L^2}.
\end{aligned}$$

Based on the energy estimate in Proposition 3.8.4, we have

$$\Big\|\varphi_k\Gamma_\ell\big(\widehat{\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}}G^{kg}_{\mathcal{L}}}\big)\Big\|_{L^2}\lesssim\varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n+1)\delta}2^{-N(n+1)k^+}, \qquad (3.8.34)$$

$$\Big\|\widehat{P_kG^{kg}_{\mathcal{L}}}\Big\|_{L^2}\lesssim\varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n)\delta}2^{-N(n)k^+}. \qquad (3.8.35)$$

Based on the nonlinear estimate in Proposition 3.6.8, we have

$$\begin{aligned}
&\Bigg\|\varphi_k\partial_{\xi_\ell}\Bigg\{\widehat{\mathcal{N}^{kg}_{\mathcal{L}}}-\mathrm{e}^{-\mathrm{i}t\Lambda_{kg}(\xi)}\Bigg[\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\Big(\mathrm{e}^{\mathrm{i}\mathcal{D}_\infty}\widehat{V^{kg}_\infty}\Big)_{\mathcal{L}_1}+\sum_{\mathcal{L}_1,\mathcal{L}_2}\mathrm{i}\,\mathcal{C}_{\mathcal{L}_2,\infty}\mathfrak{B}_{\mathcal{L}_1,\infty}+\mathfrak{b}_{\mathcal{L},\infty}\Bigg]\Bigg\}\Bigg\|_{L^2}\\
&\lesssim\varepsilon^2\langle t\rangle^{-H''_{kg}(n)\delta}2^{-N''_{kg}(n)k^+}. \qquad (3.8.36)
\end{aligned}$$

Combining (3.8.33)–(3.8.36), we get the following proposition:

Proposition 3.8.5. *Assume (3.2.104) holds. For any $\mathcal{L}\in\mathcal{V}_n$ with $0\le n\le N_1-1$, we have*

$$2^{k^+}\Big\|\varphi_k\partial_{\xi_\ell}\widehat{G^{kg}_{\mathcal{L}}}\Big\|_{L^2}\lesssim\varepsilon^{\frac{3}{2}}\langle t\rangle^{-H(n+1)\delta}2^{-N(n+1)k^+}. \qquad (3.8.37)$$

This concludes the estimate of $\|G^{kg}\|_{T_2}$ in Proposition 3.2.2.

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