Regularity for p-Laplace Type Equations with Applications in Composite Materials

by

Hanye Zhu

B.Sc., Nanjing University, Nanjing, Jiangsu, P. R. China, 2018

M.Sc., Brown University, Providence, RI, USA, 2019

A dissertation submitted in partial fulfillment of the
requirements for the degree of Doctor of Philosophy
in the Division of Applied Mathematics at Brown University

PROVIDENCE, RHODE ISLAND

May 2024

This dissertation by Hanye Zhu is accepted in its present form by the Division of Applied Mathematics as satisfying the dissertation requirement for the degree of Doctor of Philosophy.

Date ____________ ______________________________

Hongjie Dong, Ph.D., Advisor

Recommended to the Graduate Council

Date ____________ ______________________________

Yan Guo, Ph.D., Reader

Date ____________ ______________________________

Benoît Pausader, Ph.D., Reader

Approved by the Graduate Council

Date ____________ ______________________________

Thomas A. Lewis, Dean of the Graduate School

Curriculum Vitae

Hanye Zhu went to Nanjing University in September 2014 for his undergraduate study and obtained his B.Sc. degree in Mathematics and Applied Mathematics in June 2018. In September 2018, he started the Ph.D. program in the Division of Applied Math at Brown University, and obtained his M.Sc. degree in Applied Math in May 2019. While at Brown, he has worked under the supervision of Prof. Hongjie Dong. His publications and preprints include [33, 35, 36], and [32].

Acknowledgments

First and foremost, I would like to express my deepest gratitude to my advisor, Professor Hongjie Dong, for his consistent guidance and support throughout my doctoral journey. His insightful advice, endless encouragement, and rigorous attitude towards research have been essential to my growth as a scholar. I feel so fortunate to have him as my advisor.

It is my privilege to have Professor Yan Guo and Professor Benoit Pausader in my thesis committee. Both Yan and Benoit taught me more than one PDE courses at Brown, and has been consistently helping me.

Special thanks to Professor YanYan Li for helpful discussions and suggestions on my research.

I would like to thank my teachers, fellow students, and friends, including Walter Strauss, Constantine Dafermos, Chi-Wang Shu, Yanze Liu, Bekarys Bekmaganbetov, Hyunwoo Kwon, Zhen Zhang, Ziyao Xu, Sining Gong, Changjie Chen, and many others. In particular, I want to thank Zhuolun, Zongyuan, and Timur for generously sharing their knowledge, ideas, and experiences in academia.

Last, but not the least, I would like to express my love and gratitude to my parents, Ping Zhu and Xia Tan, and my girlfriend Shiyu Zhou. Their long-time love and support always pull me through the dark moments.

Contents

Acknowledgments **v**

1 Introduction **1**

1.1 Nonlinear potential theory 2

1.1.1 p-Laplace type equations with measure data 3

1.1.2 Parabolic p-Laplace type equations with measure data 5

1.2 Nonlinear high contrast composite materials 6

1.2.1 Nonlinear insulated conductivity problem 7

1.2.2 Nonlinear perfect conductivity problem 10

2 Gradient estimates for singular p-Laplace type equations with measure data **13**

2.1 Introduction 13

2.2 An oscillation estimate 23

2.3 Interior pointwise gradient estimates 28

2.4 Interior Lipschitz estimate and modulus of continuity estimate of the gradient 39

2.5 Global gradient estimates for the p-Laplacian equations 56

2.5.1 Global pointwise gradient estimates 64

2.5.2 Global Lipschitz estimates and modulus of continuity estimates of the gradient 67

3 Gradient estimates for singular parabolic p-Laplace type equations with measure data 80

3.1 Introduction 81

3.1.1 Pointwise gradient estimates 82

3.1.2 Gradient continuity results 85

3.2 Notation and basic inequalities 89

3.2.1 Notation 89

3.2.2 Definition of the L^q-mean oscillation and some basic inequalities 91

3.3 Gradient estimates for homogeneous equations 93

3.4 Pointwise gradient estimates 107

3.4.1 Comparison results 107

3.4.2 Proof of pointwise gradient estimates. 120

3.5 Gradient continuity results 129

3.5.1 Preliminary choices of constants and the geometry 129

3.5.2 Proof of gradient continuity results 130

4 The insulated conductivity problem with p-Laplacian 143

4.1 Introduction and Main results 143

4.2 Mean oscillation estimates 149

4.2.1 Mean oscillation estimates for small r 150

4.2.2 Mean oscillation estimates for intermediate r 151

4.2.3 Mean oscillation estimates for large r 156

4.2.4 Proof of Theorem 4.1.1 161

4.3 Improved gradient estimates 169

4.4 The $p > n + 1$ case 181

4.5 A two dimensional example 186
4.6 Bernstein type argument 190

5 Asymptotics of the solution to the perfect conductivity problem with p-Laplacian 200

5.1 Introduction and Main results 200
5.2 Preliminaries 211
5.3 Mean oscillation estimates 222
5.3.1 Mean oscillation estimates for small r 222
5.3.2 Mean oscillation estimates for intermediate r 223
5.3.3 Mean oscillation estimates for large r 229
5.3.4 Mean oscillation decay estimates 234
5.3.5 Proof of Proposition 5.1.5 239
5.4 Estimates of $U_1^\varepsilon - U_2^\varepsilon$ 247
5.5 Proofs of Theorem 5.1.1 and Proposition 5.1.4 252

6 Appendix 254

6.1 Proof of Proposition 5.4.2 254
6.2 Proof of Eq. (5.4.12) 255

List of Figures

1.1 Domain with closely spaced inclusions 7

5.1 An illustration of $\Gamma^0_{-,s} \cup \eta$ 221

CHAPTER 1

Introduction

In this thesis, we study the regularity theory for p-Laplace type equations. Physically, such equations arise from various materials, including dielectrics, plastic moulding, plasticity phenomena, viscous flows in glaciology, electro-rheological and thermo-rheological fluids. See e.g. [5, 14, 40, 46, 61, 81, 82]. Due to their degenerate and nonlinear nature, regularity theory for these equations is still not complete.

In the remaining part of this chapter, we introduce our main results on the nonlinear potential theory for p-Laplace type equations with measure data and stress concentration for nonlinear high contrast composite materials involving the p-Laplacian. We also present some important ideas and methods used to obtain those results. Details of the nonlinear potential theory can be found in Chapter 2 and 3, which are based on [36] and [35]. Detailed analysis on the nonlinear composite materials will be given in Chapter 4 and 5, which are based on [33] and [32].

1.1 Nonlinear potential theory

In this section, we introduce the potential theory for the gradient of solutions to elliptic and parabolic p-Laplace type equations with measure data. We consider

$$-\operatorname{div}(|Du|^{p-2}Du) = \mu \quad \text{in} \quad \Omega \subset \mathbb{R}^n, \quad \text{(elliptic case)}$$

and

$$u_t - \operatorname{div}(|Du|^{p-2}Du) = \mu \quad \text{in} \quad \Omega_T := \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}, \quad \text{(parabolic case)}$$

in the singular range $p \in (1,2)$, where μ is a locally finite signed Radon measure. A major difficulty for these equations in the range $p \in (1, 2 - \frac{1}{n}]$ (for the elliptic case) and $p \in (1, 2 - \frac{1}{n+1}]$ (for the parabolic case), is that the gradient of solutions (even when μ is the Dirac measure) fail to belong to L^1_{loc}. To overcome this difficulty, we develop iteration schemes to estimate the so-called L^q-mean oscillation of the gradient of solutions with $q \in (0,1)$, namely,

$$\inf_{\Theta \in \mathbb{R}^n} \left(\fint_{\mathcal{B}} |Du - \Theta|^q \right)^{1/q}$$

for domains $\mathcal{B} \subset \Omega$ (or Ω_T). With the help of comparison estimates with homogeneous equations, reverse Hölder inequalities, and stopping time arguments, we are able to establish pointwise gradient estimates and optimal conditions ensuring boundedness or continuity of the gradient in a singular range of p.

1.1.1 p-Laplace type equations with measure data

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) and μ be a locally finite signed Radon measure supported in Ω. Consider the following quasi-linear elliptic equation with measure data:

$$-\operatorname{div}(a(x, Du)) = \mu \quad \text{in } \Omega, \tag{1.1.1}$$

where the nonlinear tensor $a(\cdot)$ satisfies Ladyzhenskaya & Ural'tseva's conditions in the gradient variable, i.e.:

$$\begin{cases} |a(x,z)| + |z||\partial_z a(x,z)| \lesssim |z|^{p-1} \\ \langle \partial_z a(x,z)\xi, \xi \rangle \gtrsim |z|^{p-2}|\xi|^2 \end{cases}$$

and the oscillation with respect to the space variable can be quantified via a modulus of continuity of Dini type:

$$|a(x_1, z) - a(x_2, z)| \lesssim \omega(|x_1 - x_2|)|z|^{p-1},$$

with $\omega(\cdot)$ concave, non-decreasing, $\lim_{\rho \to 0_+} \omega(\rho) = \omega(0) = 0$ and $\int_0^1 \omega(t)\frac{dt}{t} < \infty$. In recent decades, people are interested in establishing a potential theory for the gradient of solutions, namely estimating Du by some potentials of μ.

(a) For the linear growth case $p = 2$, in the pioneering work [74], Mingione proved a pointwise upper bound of the gradient of solutions in terms of the linear Riesz potential $\mathbf{I}_1^{|\mu|}$.

(b) Duzzar-Mingione [38] and Kuusi-Mingione [56] showed that the pointwise upper bound of the gradient in terms of the linear Riesz potential $\mathbf{I}_1^{|\mu|}$ also holds when $p \in (2 - \frac{1}{n}, 2)$ and $p \in (2, \infty)$ (where solutions are $W^{1,1}_{\text{loc}}$).

(c) For the singular case when $p \in (\frac{3n-2}{2n-1}, 2 - \frac{1}{n}]$, Nguyen-Phuc [77] secured a pointwise upper bound of the gradient in terms of nonlinear Wolff potentials $\mathbf{P}_\gamma^{|\mu|}$ for $\gamma \in (\frac{n}{2n-1}, \frac{n(p-1)}{n-1}) \subset (0,1)$, under some extra assumptions on ω and the continuity of $\partial_z a$.

Here

$$\mathbf{I}_1^{|\mu|}(x,R) := \int_0^R \frac{|\mu|(B_t(x))}{t^{n-1}} \frac{dt}{t}$$

is the linear (truncated) Riesz potential, and

$$\mathbf{P}_\gamma^{|\mu|}(x,R) := \int_0^R \Big(\frac{|\mu|(B_t(x))}{t^{n-1}}\Big)^\gamma \frac{dt}{t}$$

is a nonlinear (truncated) Wolff potential.

In a joint work with H. Dong [36], we extended the pointwise gradient estimates via linear potential in (a) and (b) to the singular range $p \in (\frac{3n-2}{2n-1}, 2-\frac{1}{n})$ and obtained Lipschitz estimates for the more singular case when $p \in \big(1, \frac{3n-2}{2n-1}\big]$.

Theorem 1.1.1 (Chapter 2, Theorems 2.1.1, 2.1.3)**.** *Let $p \in (1,2)$, u be a solution to (1.1.1), and $B_R(x) \subset \Omega$. Then we have the following Lipschitz estimate*

$$\|Du\|_{L^\infty(B_{R/2}(x))} \lesssim \|\mathbf{I}_1^{|\mu|}(\cdot\,,R)\|_{L^\infty(B_R(x))}^{\frac{1}{p-1}} + R^{-\frac{n}{2-p}}\|Du\|_{L^{2-p}(B_R(x))}.$$

Moreover, if $p \in (\frac{3n-2}{2n-1}, 2)$, we also have pointwise gradient estimate

$$|Du(x)| \lesssim [\mathbf{I}_1^{|\mu|}(x,R)]^{1/(p-1)} + \Big(\fint_{B_R(x)} |Du|^{2-p} dy\Big)^{\frac{1}{2-p}}. \tag{1.1.2}$$

We also extended the optimal gradient continuity criterion for $p \geq 2$ in [56] to the singular range $p \in (1,2)$.

Theorem 1.1.2 (Chapter 2, Theorem 2.1.4)**.** *Let $p \in (1,2)$ and u be a solution to*

(1.1.1). *If*

$$\lim_{R\to 0} \mathbf{I}_1^{|\mu|}(x,R) = 0 \quad \textit{locally uniformly in } x \in \Omega, \tag{1.1.3}$$

then Du is continuous in Ω. In particular, assumption (1.1.3) holds when μ belongs to the Lorentz space $L(n,1)$ locally in Ω.

We should mention that the pointwise gradient estimates (1.1.2) has been recently proved by Nguyen-Phuc [79] for the range $p \in (1, \frac{3n-2}{2n-1}]$ and their proof merges improved comparison estimates with the techniques developed in the our work [36]. To conclude, a complete (local) potential theory for the gradient of solutions to p-Laplace type equations is now available for all exponents $p > 1$.

1.1.2 Parabolic p-Laplace type equations with measure data

Let $\Omega_T := \Omega \times (-T, 0) \subset \mathbb{R}^{n+1}$ $(n \geq 2)$ and and μ be a locally finite signed Radon measure supported in Ω_T. Consider the following parabolic p-Laplace type equation with measure data

$$u_t - \operatorname{div}(a(x,t,Du)) = \mu \quad \text{in } \Omega_T, \tag{1.1.4}$$

where the nonlinear tensor $a(\cdot)$ satisfies the same Ladyzhenskaya & Ural'tseva's conditions in the gradient variable and Dini condition in the space variable as in Section 1.1.1, but can be merely measurable in the time variable. The parabolic setting is more involved due to the lack of certain scaling properties. We denote the standard parabolic cylinder by $Q_r(x,t) := B_r(x) \times (t - r^2, t)$, and the (truncated) parabolic Riesz potential by

$$\mathbf{I}_1^{|\mu|}(x,t;r) := \int_0^r \frac{|\mu|(Q_\rho(x,t))}{\rho^{n+1}} \frac{d\rho}{\rho}.$$

For the case when $p > 2 - \frac{1}{n+1}$ (where the gradient of solutions belong to L^1_{loc}), Kuusi-Mingione [55, 59] proved pointwise upper bounds of the gradient of solutions in terms of the parabolic Riesz potential $\mathbf{I}_1^{|\mu|}$, and fine criteria for continuity of the gradient. In a joint work with H. Dong [35], we extended their results to the sub-critical range $p \in (\frac{2n}{n+1}, 2 - \frac{1}{n+1}]$. We overcame the difficulties by using intrinsic geometries and estimating the L^q-mean oscillation of the gradient of solutions with $q \in (0,1)$ as mentioned above.

Theorem 1.1.3 (Chapter 3, Theorems 3.1.2, 3.1.3). *Let $p \in (\frac{2n}{n+1}, 2 - \frac{1}{n+1}]$, u be a solution to (1.1.4), and $Q_{2r}(x,t) \subset \Omega_T$. Then we have*

$$|Du(x,t)| \lesssim [\mathbf{I}_1^{|\mu|}(x,t;2r)]^{\frac{2}{(n+1)p-2n}} + \Big(\fint_{Q_r(x,t)} (|Du|+1)^q \, dxdt \Big)^{\frac{2q}{2q-n(2-p)}}.$$

Moreover, if

$$\lim_{r\to 0} \mathbf{I}_1^{|\mu|}(x,t;r) = 0 \quad \textit{locally uniformly in } (x,t) \in \Omega_T, \tag{1.1.5}$$

then Du is continuous in Ω. In particular, assumption (1.1.5) holds when μ belongs to the Lorentz space $L(n+2,1)$ locally in Ω_T.

We remark that $p > 2n/(n+1)$ is a natural assumption that guarantees the existence of fundamental solution to the parabolic p-Laplace equation (the Barenblatt solution; see [10], [87, Chapter 11], [55, Section 1.3]).

1.2 Nonlinear high contrast composite materials

In this section, we introduce the nonlinear conductivity problems arising from high contrast composite materials. We study the insulated (or perfect) conductivity problem with closely spaced insulators (or perfect conductors) embedded in a homogeneous matrix where the current-electric field relation is the power law $J = \sigma|E|^{p-2}E$. The electric field may blow up as ε, the distance between the inclusions approaches 0.

We quantitatively analyze the concentration of the electric field $E = -Du$ between the inclusions, where u is the voltage potential. The linear case ($p = 2$) is already interesting and has attracted a lot of attention in recent decades. See e.g. [3, 4, 8, 9, 28, 29, 62, 63, 88–91].

1.2.1 Nonlinear insulated conductivity problem

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 2$) be a bounded domain with C^2 boundary, and let $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ be two C^2 open sets inside Ω with $\operatorname{diam}(\mathcal{D}_i^0) > c > 0$ and $\operatorname{dist}(\mathcal{D}_1^0 \cup \mathcal{D}_2^0, \partial\Omega) > c > 0$, touching at the origin with the outer normal of $\partial\mathcal{D}_2^0$ being the positive x_n-axis. We write variable x as (x', x_n), where $x' \in \mathbb{R}^{n-1}$. For $\varepsilon > 0$, translating $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ by $\frac{\varepsilon}{2}$ along the x_n-axis, we denote

$$\mathcal{D}_1^\varepsilon := \mathcal{D}_1^0 + (0', \frac{\varepsilon}{2}), \quad \mathcal{D}_2^\varepsilon := \mathcal{D}_2^0 - (0', \frac{\varepsilon}{2}),$$

and $\widetilde{\Omega}^\varepsilon := \Omega \setminus \overline{(\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon)}$. See Figure 1.1.

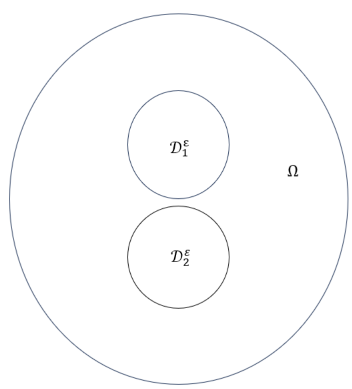

Figure 1.1: Domain with closely spaced inclusions

When the two inclusions $\mathcal{D}_1^\varepsilon, \mathcal{D}_2^\varepsilon$ are insulators, we set $\sigma = \mathbb{1}_{\widetilde{\Omega}^\varepsilon}$ and then the voltage potential u_ε is the unique solution to the following equation

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \partial\mathcal{D}_i^\varepsilon,\ i = 1, 2, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2.1}$$

where $\varphi \in C^2(\partial\Omega)$ is given, and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outer normal vector on $\partial\mathcal{D}_1^\varepsilon \cup \partial\mathcal{D}_2^\varepsilon$ (pointing towards $\widetilde{\Omega}^\varepsilon$). By the regularity of $\mathcal{D}_1^\varepsilon$ and $\mathcal{D}_2^\varepsilon$, we can assume that near the origin, the part of $\partial\mathcal{D}_1^\varepsilon$ and $\partial\mathcal{D}_2^\varepsilon$, denoted by Γ_+^ε and Γ_-^ε, are the graphs of two functions of x':

$$\Gamma_+^\varepsilon = \left\{x_n = \frac{\varepsilon}{2} + h_1(x'),\ |x'| < \frac{1}{2}\right\}, \quad \Gamma_-^\varepsilon = \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'),\ |x'| < \frac{1}{2}\right\}, \tag{1.2.2}$$

where h_1 and h_2 are strictly convex and strictly concave C^2 functions, respectively, satisfying

$$h_1(0') = h_2(0') = 0, \quad D_{x'}h_1(0') = D_{x'}h_2(0') = 0.$$

The goal is to estimate the electric field $-Du_\varepsilon$ in $\widetilde{\Omega}^\varepsilon$. By classical $C^{1,\alpha}$ estimates for the p-Laplace equation and the maximum principle, $Du_\varepsilon(x)$ should be uniformly bounded when $x \in \widetilde{\Omega}^\varepsilon \cap \{x \in \mathbb{R}^n : |x'| \geq 1/2\}$.

For the linear case ($p = 2$), Ammari-Kang-Lim [4] and Bao-Li-Yin [9] proved that the optimal blow-up rate of the gradient is $\varepsilon^{-1/2}$ when $n = 2$ and recently Dong-Li-Yang [28, 29] obtained the optimal blow-up rate $\varepsilon^{-1/2+\gamma}$ when $n \geq 3$, where $\gamma > 0$ is a explicit constant depending on n and the principal curvatures of $\partial\mathcal{D}_1^0$ and $\partial\mathcal{D}_2^0$.

However, their arguments heavily rely on the linear structure and do not work in the nonlinear case ($p \neq 2$).

In a joint work with H. Dong and Z. Yang [33], in dimension $n = 2$ we obtained almost optimal blow-up rate of the gradient, and for the case when $n \geq 3$, we proved an upper bound of the gradient to be of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$, and showed that the singularity of the gradient diminishes as $p \to \infty$ or $n \to \infty$. We overcame the difficulties due to nonlinearity by different flattenings and extensions at different scales, delicate iteration arguments using mean oscillation estimates, and constructing suitable super-solutions and sub-solutions.

Theorem 1.2.1 (Chapter 4, Theorems 4.1.1, 4.1.3)**.** *Let $n = 2$, $\varepsilon \in (0,1)$, u_ε be the solution to (1.2.1), and $x \in \widetilde{\Omega}$. When $p \in (1,3]$, we have*

$$|Du_\varepsilon(x)| \lesssim (\varepsilon + |x'|^2)^{-1/2} \|\varphi\|_{L^\infty(\partial\Omega)}.$$

When $p > 3$, then for any $\delta > 0$, we have

$$|Du_\varepsilon(x)| \lesssim_\delta (\varepsilon + |x'|^2)^{-1/(p-1)-\delta} \|\varphi\|_{L^\infty(\partial\Omega)}.$$

Theorem 1.2.2 (Chapter 4, Theorems 4.1.2, 4.1.3, 4.1.5)**.** *Let $p > 1$, $n \geq 3$, $\varepsilon \in (0,1)$, u_ε be the solution to (1.2.1), and $x \in \widetilde{\Omega}$. Then there exists $\beta = \beta(n, p, D^2 h_1(\cdot), D^2 h_2(\cdot)) > 0$ such that*

$$|Du_\varepsilon(x)| \lesssim (\varepsilon + |x'|^2)^{-1/2+\beta} \|\varphi\|_{L^\infty(\partial\Omega)}.$$

Moreover, $\beta \to 1/2$ as $p \to \infty$ and $\beta \to 1/2$ as $n \to \infty$.

When $n = 2$, the blow-up rates $\varepsilon^{-1/2}$ for $p \in (1,3]$ and $\varepsilon^{-1/(p-1)-\delta}$ for $p > 3$ in Theorem 1.2.1 are almost optimal in the sense that we constructed the following example:

Theorem 1.2.3 (Chapter 4, Theorem 4.1.4)**.** *Let $n = 2$, $\varepsilon \in (0,1)$, $\Omega = B_5$, $\mathcal{D}_1, \mathcal{D}_2$ be the unit balls centered at $(0, 1+\varepsilon/2)$ and $(0, -1-\varepsilon/2)$, respectively. Let $\varphi = x_1$*

and u_ε be the solution to (1.2.1)*. Then for any $\delta > 0$, when $p \in (1,3]$,*

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}\cap B_{8\sqrt{\varepsilon/\delta}})} \gtrsim_\delta \varepsilon^{\frac{-1+\delta}{2}},$$

and when $p > 3$,

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}\cap B_{8\sqrt{\varepsilon/\delta}})} \gtrsim_\delta \varepsilon^{\frac{-1+\delta}{p-1}}.$$

1.2.2 Nonlinear perfect conductivity problem

We adopt the same notation for the domains Ω, $\mathcal{D}_1^0$, $\mathcal{D}_2^0$, $\mathcal{D}_1^\varepsilon$, $\mathcal{D}_2^\varepsilon$, $\widetilde{\Omega}^\varepsilon$ as in Section 1.2.1. When the two inclusions $\mathcal{D}_1^\varepsilon, \mathcal{D}_2^\varepsilon$ are perfect conductors, the voltage potential u_ε is the unique solution to the following equation

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ u_\varepsilon = U_i^\varepsilon & \text{on } \partial\mathcal{D}_i^\varepsilon,\ i = 1,2, \\ \displaystyle\int_{\partial\mathcal{D}_i^\varepsilon} |Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu = 0 & i = 1,2, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2.3}$$

where $\varphi \in C^2(\partial\Omega)$ is given, $\nu = (\nu_1, \ldots, \nu_n)$ is the outer normal vector, and U_1^ε, U_2^ε are two constants determined by $(1.2.3)_3$.

When $p \geq (n+1)/2$, Gorb-Novikov [43] showed that as $\varepsilon \to 0_+$, u_ε converges (both weakly in $W^{1,p}(\Omega)$ and strongly in $C^1(K)$ for any $K \subset\subset \widetilde{\Omega}^0$) to the unique solution of

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2}Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_0 & \text{on } \partial\mathcal{D}_1^0 \cup \partial\mathcal{D}_2^0, \\ \displaystyle\sum_{i=1}^2 \int_{\partial\mathcal{D}_i^0} |Du_0|^{p-2}Du_0\cdot\nu = 0, & \\ u_0 = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2.4}$$

where $\widetilde{\Omega}^0 := \Omega \setminus \overline{(\mathcal{D}_1^0 \cup \mathcal{D}_2^0)}$ and U_0 is a constant determined by (1.2.4)$_3$. In this case, the flux along $\partial\mathcal{D}_1^0$, denoted by

$$\mathcal{F} := \int_{\partial\mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu,$$

can be nonzero.

When $p \in (1, (n+1)/2)$, in a joint work with H. Dong and Z. Yang [32], we proved that as $\varepsilon \to 0_+$, u_ε converges (in the same sense) to the unique solution of

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2} Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_i & \text{on } \partial\mathcal{D}_i^0 \setminus \{0\},\ i = 1, 2, \\ \displaystyle\int_{\partial\mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu = 0, & i = 1, 2, \\ u_0 = \varphi & \text{on } \partial\Omega, \end{cases} \tag{1.2.5}$$

where U_1 and U_2 are two constants determined by (1.2.5)$_3$.

For the linear case $p = 2$ and $n \geq 2$, the optimal blow-up rate was obtained in [3, 4, 8]. Li-Li-Yang [63] and Li [62] established asymptotic expansions of $Du_\varepsilon(x)$ for sufficiently small ε and x', when $n = 2, 3$.

For the nonlinear case when $p \geq (n+1)/2$ and $\mathcal{F} \neq 0$, Gorb-Novikov [43] proved that the optimal blow up rate of Du_ε is of order $\Theta(\varepsilon)/\varepsilon$, where

$$\Theta(\varepsilon) := \begin{cases} \varepsilon^{\frac{2p-n-1}{2(p-1)}}, & p > \dfrac{n+1}{2}, \\ |\ln \varepsilon|^{-\frac{1}{p-1}}, & p = \dfrac{n+1}{2}. \end{cases}$$

In the joint work with H. Dong and Z. Yang [32], for any $p > 1$ and $n \geq 2$, we obtained an asymptotic expansion of $Du_\varepsilon(x)$ for sufficiently small ε and x'.

Theorem 1.2.4 (Chapter 5, Theorem 5.1.1)**.** *Let $n \geq 2$, u_ε, $\mathcal{F}$, U_1, U_2, $\Theta(\varepsilon)$ be as* *above,* $x \in \widetilde{\Omega}^{\varepsilon}$*, and* $\eta(x) := h_1(x') - h_2(x') + \varepsilon$*, where* h_1*,* h_2 *are defined in* (1.2.2)*. Then the following holds:*

(i) If $p \geq (n+1)/2$*, when* ε *and* $|x'|$ *are sufficiently small, we have*

$$Du_{\varepsilon}(x) = \big(0', \eta(x)^{-1}\Theta(\varepsilon)\,\mathrm{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)}\big) + R_1(x,\varepsilon),$$

where $K > 0$ *is an explicitly computable constant in terms of* n*,* p*, and* $D^2_{x'}(h_1 - h_2)(0')$*.*

(ii) If $p \in (1, (n+1)/2)$*, when* ε *and* $|x'|$ *are sufficiently small, we have*

$$Du_{\varepsilon}(x) = \big(0', \eta(x)^{-1}(U_1 - U_2)\big) + R_2(x,\varepsilon).$$

The remainders R_1 *and* R_2 *are of higher order with respect to* $\eta(x)^{-1}\Theta(\varepsilon)$ *and* $\eta(x)^{-1}$*, respectively.*

Our proof of Theorem 1.2.4 involves a delicate $C^{1,\alpha}$ estimate, which implies a better estimate for the oscillation of Du_{ε} in the x_n direction.

CHAPTER 2

Gradient estimates for singular p-Laplace type equations with measure data

We are concerned with interior and global gradient estimates for solutions to a class of singular quasilinear elliptic equations with measure data, whose prototype is given by the p-Laplace equation $-\Delta_p u = \mu$ with $p \in (1,2)$. The cases when $p \in \left(2-\frac{1}{n}, 2\right)$ and $p \in \left(\frac{3n-2}{2n-1}, 2-\frac{1}{n}\right]$ were studied in [38] and [77], respectively. In this chapter, we improve the results in [77] and address the open case when $p \in \left(1, \frac{3n-2}{2n-1}\right]$. Interior and global modulus of continuity estimates of the gradients of solutions are also established.

2.1 Introduction

We consider the quasilinear elliptic equation with measure data

$$-\mathrm{div}(A(x,Du)) = \mu \tag{2.1.1}$$

in a domain $\Omega \subset \mathbb{R}^n$, where $n \geq 2$. Here μ is a locally finite signed Radon measure in Ω, namely, $|\mu|(B_R(x) \cap \Omega) < \infty$ for any ball $B_R(x) \subset \mathbb{R}^n$. By setting $|\mu|(\mathbb{R}^n \backslash \Omega)$=0, we will always assume that μ is defined in the whole space $\mathbb{R}^n$. The vector field $A = (A_1, \ldots, A_n) : \Omega \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following growth, ellipticity, and continuity conditions: there exist constants $\lambda \geq 1$, $s \geq 0$, and $p > 1$ such that

$$|A(x,\xi)| \leq \lambda(s^2 + |\xi|^2)^{(p-1)/2}, \quad |D_\xi A(x,\xi)| \leq \lambda(s^2 + |\xi|^2)^{(p-2)/2}, \tag{2.1.2}$$

$$\langle D_\xi A(x,\xi)\eta, \eta\rangle \geq \lambda^{-1}(s^2 + |\xi|^2)^{(p-2)/2}|\eta|^2, \tag{2.1.3}$$

and

$$|A(x,\xi) - A(x_0,\xi)| \leq \lambda\omega(|x - x_0|)(s^2 + |\xi|^2)^{(p-1)/2} \tag{2.1.4}$$

hold for every $x, x_0 \in \Omega$ and every $(\xi, \eta) \in \mathbb{R}^n \times \mathbb{R}^n \backslash \{(0,0)\}$, and $\omega : [0,\infty) \to [0,1]$ is a concave non-decreasing function satisfying

$$\lim_{r \to 0^+} \omega(r) = \omega(0) = 0$$

and the Dini condition

$$\int_0^1 \omega(r)\,\frac{dr}{r} < +\infty. \tag{2.1.5}$$

A typical model equation is given by the (possibly nondegenerate) p-Laplace equation with measure data and $s \geq 0$:

$$-\mathrm{div}\left(a(x)(|Du|^2 + s^2)^{\frac{p-2}{2}} Du\right) = \mu \quad \text{in } \Omega, \tag{2.1.6}$$

where $a(\cdot)$ is a Dini continuous function in Ω, satisfying

$$0 < \lambda^{-1} \leq a(x) \leq \lambda \tag{2.1.7}$$

and

$$|a(x) - a(x_0)| \leq \lambda\omega(|x - x_0|) \tag{2.1.8}$$

for every $x, x_0 \in \Omega$.

By a (weak) solution to Eq. (2.1.1), we mean a function $u \in W^{1,p}_{\mathrm{loc}}(\Omega)$ such that the distributional relation

$$\int_\Omega \langle A(x, Du), D\varphi\rangle\, dx = \int_\Omega \varphi\, d\mu$$

holds whenever $\varphi \in C_0^\infty(\Omega)$ has compact support in Ω. We use $B_R(x)$ to denote the open ball of radius R centered at x and we set

$$B_R = B_R(0), \quad \Omega_R(x) = \Omega \cap B_R(x).$$

The gradient estimates for the super-quadratic case when $p \geq 2$ were well studied in the literature. See [37, 39, 56, 74, 75]. However, the corresponding results for the singular case when $p \in (1, 2)$ are far from complete.

In this chapter, we are concerned with only the singular case when $p \in (1, 2)$.

For singular quasi-linear equations, the case when $p \in \left(2 - \frac{1}{n}, 2\right)$ was considered in the pioneering work [38], in which the authors proved that under the conditions (2.1.2)-(2.1.5), if $u \in C^1(\Omega)$ solves (2.1.1), then it holds that

$$|Du(x)| \leq C\left[\mathbf{I}_1^R(|\mu|)(x)\right]^{\frac{1}{p-1}} + C \fint_{B_R(x)} (|Du(y)| + s)\, dy$$

for every ball $B_R(x) \subset \Omega$ with $R \in (0, 1]$, where $C = C(n, p, \lambda, \omega)$. Here $\fint_E$ stands for the integral average over a measurable set E and

$$\mathbf{I}_1^R(|\mu|)(x) = \int_0^R \frac{|\mu|(B_t(x))}{t^{n-1}}\,\frac{dt}{t} \tag{2.1.9}$$

is the truncated first-order Riesz potential. Later, the case when $p \in \left(\frac{3n-2}{2n-1}, 2-\frac{1}{n}\right]$ was treated in [77], in which the authors obtained a pointwise gradient bound involving the Wolff potential under stronger assumptions on A and ω. Namely, under the conditions (2.1.2)-(2.1.4) and further assuming that

$$\begin{aligned}
&|D_\xi A(x,\xi) - D_\xi A(x,\eta)| \\
&\le \lambda(s^2+|\xi|^2)^{(p-2)/2}(s^2+|\eta|^2)^{(p-2)/2}(s^2+|\xi|^2+|\eta|^2)^{(2-p-\alpha_0)/2}|\xi-\eta|^{\alpha_0}
\end{aligned} \tag{2.1.10}$$

and

$$\int_0^1 \omega^\gamma(r)\,\frac{dr}{r} < +\infty$$

for some $\alpha_0 \in (0, 2-p)$ and $\gamma \in \left(\frac{n}{2n-1}, \frac{n(p-1)}{n-1}\right) \subset (0,1)$, if $u \in C^1(\Omega)$ is a solution to (2.1.1), then it holds that

$$|Du(x)| \le C[\mathbf{P}_\gamma^R(|\mu|)(x)]^{\frac{1}{\gamma(p-1)}} + C\Big(\fint_{B_R(x)} (|Du(y)|+s)^\gamma\,dy\Big)^{\frac{1}{\gamma}},$$

where $C = C(n,p,\lambda,\alpha_0,\omega,\gamma)$ and

$$\mathbf{P}_\gamma^R(|\mu|)(x) = \int_0^R \Big(\frac{|\mu|(B_t(x))}{t^{n-1}}\Big)^\gamma\,\frac{dt}{t}$$

is a truncated nonlinear Wolff potential. We recall that in general, the truncated Wolff potential is defined as

$$\mathbf{W}_{\beta,p}^R(|\mu|)(x) = \int_0^R \left(\frac{|\mu|(B_t(x))}{t^{n-\beta p}}\right)^{\frac{1}{p-1}}\,\frac{dt}{t}, \quad \beta \in (0, n/p]. \tag{2.1.11}$$

Our first main result is stated as follows.

Theorem 2.1.1 (Interior pointwise gradient estimate)**.** *Let $p \in \left(\frac{3n-2}{2n-1}, 2\right)$ and suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution to (2.1.1). Then under the assumptions (2.1.2)-(2.1.5), there exists a constant $C = C(n, p, \lambda, \omega)$ such that the estimate*

$$|Du(x)| \leq C\big[\mathbf{I}_1^R(|\mu|)(x)\big]^{\frac{1}{p-1}} + C\left(\fint_{B_R(x)} (|Du(y)| + s)^{2-p}\, dy\right)^{\frac{1}{2-p}} \tag{2.1.12}$$

holds for any Lebesgue point x of the vector-valued function Du, with $B_R(x) \subset \Omega$ and $R \in (0, 1]$.

Remark 2.1.2. Our pointwise bound in Theorem 2.1.1 using Riesz potential $\mathbf{I}_1^R(|\mu|)$ is an improvement of the bound in [77, Theorem 1.1] which contains the Wolff potential $\mathbf{P}_\gamma^R(|\mu|)$, since

$$\mathbf{I}_1^R(|\mu|) \leq C\, \mathbf{P}_\gamma^{2R}(|\mu|)^{\frac{1}{\gamma}} \quad \forall\, \gamma < 1.$$

The conditions on ω and A in Theorem 2.1.1 are also weaker. In particular, (2.1.10) is not assumed.

For the more singular case when $p \in \left(1, \frac{3n-2}{2n-1}\right]$, which was open, we obtain the following Lipschitz estimate.

Theorem 2.1.3 (Interior Lipschitz estimate)**.** *Let $p \in (1, 2)$ and suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution to (2.1.1). Then under the assumptions (2.1.2)-(2.1.5), there exists a constant $C = C(n, p, \lambda, \omega)$ such that the estimate*

$$\|Du\|_{L^\infty(B_{R/2}(x))} \leq C\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(B_R(x))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\||Du| + s\|_{L^{2-p}(B_R(x))} \tag{2.1.13}$$

holds for any $B_R(x) \subset \Omega$ and $R \in (0, 1]$.

We also obtain a modulus of continuity estimate of Du in Theorem 2.4.3, which directly implies the following sufficient condition for the continuity of Du.

Theorem 2.1.4 (Gradient continuity via Riesz potential)**.** *Let $p \in (1,2)$ and $u \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.1.1). Assume that (2.1.2)-(2.1.5) are satisfied and the functions*

$$x \to \mathbf{I}_1^R(|\mu|)(x) \text{ converge locally uniformly to zero in } \Omega \text{ as } R \to 0. \tag{2.1.14}$$

Then Du is continuous in Ω.

Recall the Lorentz space $L^{n,1}$ is the collection of measurable functions f such that

$$\int_0^\infty |\{x : |f(x)| \geq t\}|^{\frac{1}{n}} \, dt < \infty.$$

Theorem 2.1.4 has the following corollary.

Corollary 2.1.5 (Gradient continuity via Lorentz spaces)**.** *Let $p \in (1,2)$ and $u \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.1.1). Assume that (2.1.2)-(2.1.5) are satisfied and*

$$\mu \in L^{n,1} \text{ holds locally in } \Omega. \tag{2.1.15}$$

Then Du is continuous in Ω.

We remark that the Lorentz-space result above was proved in [58] for a p-Laplacian system similar to (2.1.6) when $p \in (1,\infty)$.

A further, actually immediate, corollary of Theorem 2.1.4 concerns measures with certain density properties.

Corollary 2.1.6 (Gradient continuity via density)**.** *Let $p \in (1,2)$ and $u \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.1.1). Assume that (2.1.2)-(2.1.5) are satisfied and μ satisfies*

$$|\mu|(B_\rho(x)) \leq C\rho^{n-1}h(\rho) \tag{2.1.16}$$

for every ball $B_\rho(x) \subset\subset \Omega$, where C is a positive constant and $h : [0, \infty) \to [0, \infty)$ is a function satisfying the Dini condition

$$\int_0^R h(r)\,\frac{dr}{r} < \infty \text{ for some } R > 0. \tag{2.1.17}$$

Then Du is continuous in Ω.

We remark that Theorem 2.1.4 and Corollaries 2.1.5 and 2.1.6 above are indeed the sub-quadratic $-p \in (1, 2)-$ counterparts of [56, Theorem 1.5 and Corollaries 1.6, 1.7]. See also [37, Theorems 1, 3, and 4] and [75, Theorems 5.5 and 5.6]. We refer the reader to [37, 56, 75] for a discussion of the borderline nature of the assumptions in these results.

Another interesting consequence of Theorem 2.4.3 is the following gradient Hölder continuity result.

Corollary 2.1.7 (Gradient Hölder continuity via density)**.** *Let $p \in (1, 2)$ and $u \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.1.1). Then under the assumptions (2.1.2)-(2.1.5), there exists a constant $\alpha \in (0, 1)$ depending only on n, p, and λ, such that if $\omega(r) \leq Cr^\beta$ whenever $r > 0$ and $|\mu|(B_\rho(x)) \leq C\rho^{n-1+\beta}$ whenever $B_\rho(x) \subset\subset \Omega$, for some constants $C > 0$ and $\beta \in (0, \alpha)$, then $u \in C^{1,\beta}_{loc}(\Omega)$.*

Remark 2.1.8. We should stress that the constant α in Corollary 2.1.7 is the natural Hölder exponent of the gradients of solutions to corresponding homogeneous equations with x-independent nonlinearities (cf. Lemma 2.2.2). Therefore, our result in Corollary 2.1.7 provides the best possible Hölder exponent for the gradient of the solution. The previous Corollary is an improvement of the gradient Hölder regularity result by Liebermann [70, Theorem 5.3], who proved $u \in C^{1,\beta_1}_{\mathrm{loc}}$ for some $\beta_1 = \beta_1(n, p, \lambda, \beta) \in (0, 1)$ under the same assumptions.

We also obtain up-to-boundary gradient estimates for the p-Laplace equations with measure data in domains with $C^{1,\mathrm{Dini}}$ boundaries.

Definition 2.1.9. Let Ω be a domain in $\mathbb{R}^n$. We say that Ω has $C^{1,\mathrm{Dini}}$ boundary if there exists a constant $R_0 \in (0,1]$ and a non-decreasing function $\omega_0 : [0,1] \to [0,1]$ satisfying the Dini condition

$$\int_0^1 \omega_0(r)\,\frac{dr}{r} < +\infty,$$

such that the following holds: for any $x_0 = (x_{01}, x_0') \in \partial\Omega$, there exists a $C^{1,\mathrm{Dini}}$ function (i.e., C^1 function whose first derivatives are uniformly Dini continuous) $\chi : \mathbb{R}^{n-1} \to \mathbb{R}$ and a coordinate system depending on x_0 such that

$$\sup_{|x_1'-x_2'|\le r} |D_{x'}\chi(x_1') - D_{x'}\chi(x_2')| \le \omega_0(r), \quad \forall r \in (0, R_0),$$

and that in the new coordinate system, we have

$$|D_{x'}\chi(x_0')| = 0, \quad \Omega_{R_0}(x_0) = \{x \in B_{R_0}(x_0) : x_1 > \chi(x')\}.$$

Our global pointwise gradient estimate and Lipschitz estimate are stated as follows.

Theorem 2.1.10 (Boundary pointwise gradient estimate)**.** *Let $p \in \left(\frac{3n-2}{2n-1}, 2\right)$ and suppose that $u \in W_0^{1,p}(\Omega)$ is a solution to (2.1.6) with Dirichlet boundary data $u = 0$ on $\partial\Omega$. Assuming that (2.1.5), (2.1.7), and (2.1.8) are satisfied and Ω has a $C^{1,Dini}$ boundary characterized by R_0 and ω_0 as in Definition 2.1.9, then there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0)$ such that estimate*

$$|Du(x)| \le C[\mathbf{I}_1^R(|\mu|)(x)]^{\frac{1}{p-1}} + C\left(\fint_{\Omega_R(x)} (|Du(y)| + s)^{2-p}\,dy\right)^{\frac{1}{2-p}} \tag{2.1.18}$$

holds for any Lebesgue point $x \in \Omega$ of the vector-valued function Du and $R \in (0,1]$. Moreover, if $u \in C^1(\bar{\Omega})$, the estimate (2.1.18) holds for any $x \in \bar{\Omega}$.

Theorem 2.1.11 (Boundary Lipschitz estimate)**.** *Let $p \in (1,2)$ and suppose that $u \in W_0^{1,p}(\Omega)$ is a solution to (2.1.6) with Dirichlet boundary data $u = 0$ on $\partial\Omega$. Assuming that (2.1.5), (2.1.7), and (2.1.8) are satisfied and Ω has a $C^{1,Dini}$ boundary characterized by R_0 and ω_0 as in Definition 2.1.9, then there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0)$ such that the estimate*

$$\|Du\|_{L^\infty(\Omega_{R/2}(x))} \leq C\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(\Omega_R(x))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\||Du| + s\|_{L^{2-p}(\Omega_R(x))} \tag{2.1.19}$$

holds for any $x \in \bar{\Omega}$ and $R \in (0,1]$.

As a corollary, we also obtain the global Lipschitz estimate when Ω is bounded.

Corollary 2.1.12. *Let $\Omega \subset \mathbb{R}^n$ be a bounded domain. Under the assumptions of Theorem 2.1.11, there exists a constant $C = C(n, p, \lambda, \omega, R_0, \omega_0, diam(\Omega))$ such that*

$$\|Du\|_{L^\infty(\Omega)} \leq C\big\|\mathbf{I}_1^1(|\mu|)\big\|_{L^\infty(\Omega)}^{\frac{1}{p-1}} + Cs.$$

A global modulus of continuity estimate is established in Theorem 2.5.10 under the same assumptions of Theorem 2.1.11. One may also deduce corresponding up-to-boundary gradient continuity results from Theorem 2.5.10 similar to Theorem 2.1.4 and Corollaries 2.1.5-2.1.7 from Theorem 2.4.3.

Let us give a brief description of the proofs. We first apply an iteration argument to get an L^{γ_0}-mean oscillation estimate of the gradients of solutions to the homogeneous equation with x-independent nonlinearities

$$-\text{div}(A_0(Dv)) = 0$$

in Section 2.2, where $\gamma_0 \in (0,1)$. Our proofs of the interior gradient estimates are then based on a comparison estimate between the original solution u of (2.1.1) and the solution to the homogeneous equation $-\mathrm{div}(A(x, Dw)) = 0$ in a ball B_R with the boundary condition $u = w$ on ∂B_R. The outcome is the inequality

$$\begin{aligned}
&\left(\fint_{B_R} |Du - Dw|^{\gamma_0}\, dx\right)^{1/\gamma_0} \\
&\le C\left[\frac{|\mu|(B_R)}{R^{n-1}}\right]^{\frac{1}{p-1}} + C\frac{|\mu|(B_R)}{R^{n-1}} \fint_{B_R} (|Du| + s)^{2-p}\, dx,
\end{aligned} \tag{2.1.20}$$

which holds for some constant $\gamma_0 \in (0, 1)$. The details can be found in Lemma 2.3.2. For the case when $p \in \left(\frac{3n-2}{2n-1}, 2\right)$, we can choose $\gamma_0 = 2 - p$, which is the same integral exponent on the right hand side. We then borrow an idea in [25] by estimating the L^{γ_0}-mean oscillation to adapt the iteration scheme used, for instance, in [38]. However, for the case when $p \in \left(1, \frac{3n-2}{2n-1}\right]$, we are only able to prove the comparison estimate (2.1.20) for some $\gamma_0 < 2 - p$ and that is the reason why we only obtain Lipschitz estimates instead of pointwise gradient estimates in this case.

For the gradient estimates up to the boundary, we use the technique of flattening the boundary and generalize the interior oscillation estimates to half balls. We adapt an idea in [19] to establish the global L^{γ_0}-mean oscillation estimates by a delicate combination of the interior estimates and the estimates near a flat boundary. To this end, we also apply an odd extension argument to derive an L^{γ_0}-mean oscillation estimate on half balls for homogeneous equations with x-independent nonlinearities. This argument only works for equations in diagonalized form, such as the p-Laplace equation, so the global estimates for general equations remain open. As a partial result in this direction, we refer the reader to [77] for a weighted pointwise boundary estimate under the condition that $\partial\Omega$ is sufficiently flat in the sense of Reifenberg. We also refer the reader to [15, 68] for boundary regularity results for quasi-linear equations with sufficiently regular right-hand side.

The rest of the chapter is organized as follows. In the next section, we derive an L^{γ_0}-mean oscillation estimate of solutions to the homogeneous equation with x-independent nonlinearities. In Section 2.3, we give the proof of Theorem 2.1.1. Section 2.4 is devoted to the Lipschitz estimate and the interior modulus of continuity estimate of the gradient of solutions as well as its corollaries. Finally, in Section 2.5 we consider the corresponding boundary estimates.

2.2 An oscillation estimate

This section is devoted to the proof of the following interior oscillation estimate for solutions to the homogeneous equation

$$-\operatorname{div}(A_0(Dv)) = 0 \quad \text{in } \Omega, \tag{2.2.1}$$

where $A_0 = A_0(\xi)$ is a vector field independent of x satisfying conditions (2.1.2) and (2.1.3) for some $s \geq 0$, $\lambda \geq 1$, and $p > 1$. In this section, we denote the integral average over $B_R(x)$ by $(\cdot)_{B_R(x)}$.

Theorem 2.2.1. *Let $v \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.2.1) and $\gamma_0 \in (0,1)$. Then there exist constants $\alpha \in (0,1)$ depending on n, p, and λ, and $C > 1$ depending on n, p, λ, and γ_0, such that for every $B_R(x_0) \subset \Omega$ and $\rho \in (0,R)$, we have*

$$\inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho(x_0)} |Dv - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0} \leq C \left(\frac{\rho}{R}\right)^{\alpha} \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_R(x_0)} |Dv - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}. \tag{2.2.2}$$

To prove the above theorem, we first recall a classical oscillation estimate. Estimates of this type, with different exponents involved, were developed in [24, 38, 69].

Lemma 2.2.2. *Let $v \in W^{1,p}_{loc}(\Omega)$ be a solution to (2.2.1). There exist constants $C > 1$ and $\alpha \in (0,1)$ depending only on n, p, and λ, such that $v \in C^{1,\alpha}_{loc}(\Omega)$ and for every*

$B_R(x_0) \subset \Omega$ *and* $r \in (0, R)$*, we have*

$$\fint_{B_r(x_0)} |Dv - (Dv)_{B_r(x_0)}|^p \, dx \le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_R(x_0)} |Dv - (Dv)_{B_R(x_0)}|^p \, dx.$$

The lemma above directly implies the following corollary.

Corollary 2.2.3. *Under the conditions of Lemma 2.2.2, there exist constants* $C > 1$ *and* $\alpha \in (0, 1)$ *depending only on* n*,* p*, and* λ*, such that for every* $B_R(x_0) \subset \Omega$*, we have*

$$R^\alpha [Dv]_{C^\alpha(B_{R/2}(x_0))} \le C \left(\fint_{B_R(x_0)} |Dv - (Dv)_{B_R(x_0)}|^p \right)^{1/p}, \tag{2.2.3}$$

and for any $r \in [R/2, R)$*,*

$$[Dv]_{C^\alpha(B_r(x_0))} \le C \frac{R^{n/p+1-\alpha}}{(R-r)^{n/p+1}} \left(\fint_{B_R(x_0)} |Dv - (Dv)_{B_R(x_0)}|^p \right)^{1/p}. \tag{2.2.4}$$

Proof. Without loss of generality, we assume $x_0 = 0$. For any $x \in B_{R/2}$ and $r \le R/2$, by Lemma 2.2.2 we have

$$\begin{aligned}
\fint_{B_r(x)} |Dv - (Dv)_{B_r(x)}|^p &\le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_{R/2}(x)} |Dv - (Dv)_{B_{R/2}(x)}|^p \\
&\le C \left(\frac{r}{R}\right)^{\alpha p} \fint_{B_R} |Dv - (Dv)_{B_R}|^p.
\end{aligned}$$

By Campanato's characterization of Hölder continuous functions, we obtain (2.2.3).

Now for any $r > R/2$ and $z \in B_r$, using (2.2.3) and the triangle inequality, we have

$$\begin{aligned}
[Dv]_{C^\alpha(B_{(R-r)/2}(z))} &\le C(R-r)^{-\alpha} \left(\fint_{B_{R-r}(z)} |Dv - (Dv)_{B_{R-r}(z)}|^p \right)^{1/p} \\
&\le C \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\fint_{B_R} |Dv - (Dv)_{B_R}|^p \right)^{1/p}.
\end{aligned} \tag{2.2.5}$$

Thus for any $x, y \in B_r$, let $N = \min\left\{ m \in \mathbb{Z} : m > \frac{2|x-y|}{R-r} \right\}$. We can divide the line segment connecting x and y into N equal segments using $x_1, ..., x_{N-1}$, $x_0 = x$, and $x_N = y$, such that

$$|x_k - x_{k+1}| = \frac{|x-y|}{N} < \frac{R-r}{2}.$$

Then by the triangle inequality and (2.2.5), we have

$$\begin{aligned}
&|Dv(x) - Dv(y)| \leq \sum_{k=0}^{N-1} |Dv(x_k) - Dv(x_{k+1})| \\
&\leq C \sum_{k=0}^{N-1} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |Dv - (Dv)_{B_R}|^p \right)^{1/p} \left| \frac{x-y}{N} \right|^\alpha \\
&\leq C N^{1-\alpha} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |Dv - (Dv)_{B_R}|^p \right)^{1/p} |x-y|^\alpha \\
&\leq C \left(\frac{R}{R-r} \right)^{1-\alpha} \frac{R^{n/p}}{(R-r)^{n/p+\alpha}} \left(\int_{B_R} |Dv - (Dv)_{B_R}|^p \right)^{1/p} |x-y|^\alpha,
\end{aligned}$$

which directly implies (2.2.4). The corollary is proved. $\square$

Now we are ready to give the proof of Theorem 2.2.1.

Proof of Theorem 2.2.1. As before, without loss of generality, we assume $x_0 = 0$. Clearly for any $B_\rho = B_\rho(0) \subset \Omega$, there exists $\mathbf{q}_\rho = (q_\rho^{(1)}, ..., q_\rho^{(n)}) \in \mathbb{R}^n$ such that

$$\left(\fint_{B_\rho} |Dv - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0} = \inf_{\mathbf{q} \in \mathbb{R}^n} \left(\fint_{B_\rho} |Dv - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}.$$

Also, it is easily seen that

$$q_\rho^{(i)} \in \mathrm{Range}(D_i v \,|_{B_\rho}). \tag{2.2.6}$$

We claim that there exists a constant C depending only on n, p, λ, and γ_0, such that

$$\|Dv - \mathbf{q}_{\rho/2}\|_{L^\infty(B_{\rho/2})} \leq C \left(\fint_{B_\rho} |Dv - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0} \tag{2.2.7}$$

holds for any $B_\rho(x_0) \subset \Omega$.

We prove the claim by using Corollary 2.2.3 and iteration.

For any $R/2 < r < R \le \operatorname{dist}(x_0, \partial\Omega)$, using (2.2.4) and the triangle inequality, we get

$$
\begin{aligned}
&\|Dv - \mathbf{q}_r\|_{L^\infty(B_r)} \le r^\alpha [Dv]_{C^\alpha(B_r)} \\
&\le C \frac{R^{n/p+1}}{(R-r)^{n/p+1}} \left(\fint_{B_R} |Dv - (Dv)_{B_R}|^p \right)^{1/p} \\
&\le C \frac{R^{n/p+1}}{(R-r)^{n/p+1}} \left(\fint_{B_R} |Dv - \mathbf{q}_R|^p \right)^{1/p} \\
&\le C \frac{R^{n/p+1}}{(R-r)^{n/p+1}} \|Dv - \mathbf{q}_R\|_{L^\infty(B_R)}^{\frac{p-\gamma_0}{p}} \left(\fint_{B_R} |Dv - \mathbf{q}_R|^{\gamma_0} \right)^{1/p} \\
&\le \varepsilon \|Dv - \mathbf{q}_R\|_{L^\infty(B_R)} + C_\varepsilon \left(\frac{R^{n/p+1}}{(R-r)^{n/p+1}} \right)^{\frac{p}{\gamma_0}} \left(\fint_{B_R} |Dv - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0},
\end{aligned}
$$

where we used Young's inequality with exponents p/γ_0 and $p/(p-\gamma_0)$ in the last line.

Now taking $r_k = (1-2^{-k})\rho$, $r = r_k$, and $R = r_{k+1}$, we have

$$
\begin{aligned}
&\|Dv - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})} \\
&\le \varepsilon \|Dv - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} + C_\varepsilon 2^{k\beta} \left(\fint_{B_{r_{k+1}}} |Dv - \mathbf{q}_{r_{k+1}}|^{\gamma_0} \right)^{1/\gamma_0} \\
&\le \varepsilon \|Dv - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} + C_\varepsilon 2^{k\beta + n/\gamma_0} \left(\fint_{B_\rho} |Dv - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0},
\end{aligned}
$$

where $\beta = (n+p)/\gamma_0$. Taking $\varepsilon = 3^{-\beta}$, multiplying both sides by ε^k and summing in

k, we get

$$\sum_{k=1}^{\infty} \varepsilon^k \|Dv - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})} \leq \sum_{k=1}^{\infty} \varepsilon^{k+1} \|Dv - \mathbf{q}_{r_{k+1}}\|_{L^\infty(B_{r_{k+1}})} + C \left(\fint_{B_\rho} |Dv - \mathbf{q}_\rho|^{\gamma_0} \right)^{1/\gamma_0},$$

where the summations are finite and $C = C(n, p, \lambda, \gamma_0)$. By subtracting

$$\sum_{k=2}^{\infty} \varepsilon^k \|Dv - \mathbf{q}_{r_k}\|_{L^\infty(B_{r_k})}$$

from both sides of the above inequality, we obtain (2.2.7). The claim is proved.

Now we are ready to prove (2.2.2). If $r \leq R/4$, by (2.2.3), (2.2.6), and (2.2.7) we get

$$\begin{aligned}
&\left(\fint_{B_r} |Dv - \mathbf{q}_r|^{\gamma_0} \right)^{1/\gamma_0} \leq C r^\alpha [Dv]_{C^\alpha(B_{\frac{R}{4}})} \\
&\leq C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_{\frac{R}{2}}} |Dv - (Dv)_{B_{\frac{R}{2}}}|^p \right)^{1/p} \leq C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_{\frac{R}{2}}} |Dv - \mathbf{q}_{\frac{R}{2}}|^p \right)^{1/p} \\
&\leq C \left(\frac{r}{R} \right)^\alpha \|Dv - \mathbf{q}_{\frac{R}{2}}\|_{L^\infty(B_{\frac{R}{2}})} \leq C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_R} |Dv - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0}.
\end{aligned}$$

If $r > R/4$, we have

$$\begin{aligned}
\left(\fint_{B_r} |Dv - \mathbf{q}_r|^{\gamma_0} \right)^{1/\gamma_0} &\leq \left(\fint_{B_r} |Dv - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0} \\
&\leq C \left(\frac{r}{R} \right)^\alpha \left(\fint_{B_R} |Dv - \mathbf{q}_R|^{\gamma_0} \right)^{1/\gamma_0}.
\end{aligned}$$

The theorem is proved. $\square$

2.3 Interior pointwise gradient estimates

In order to prove the interior pointwise gradient estimates, we follow the outline of arguments given in [77] while replacing their oscillation estimates with our new oscillation estimate in Section 2.2. We also borrow an idea in [25] by estimating the L^{γ_0}-mean oscillations of solutions, where $\gamma_0 \in (0,1)$.

Let $u \in W^{1,p}_{\rm loc}(\Omega)$ be a solution to (2.1.1) and $B_{2r}(x_0) \subset\subset \Omega$. We consider the unique solution $w \in u + W^{1,p}_0(B_{2r}(x_0))$ to the equation

$$\begin{cases} -\mathrm{div}(A(x,Dw)) = 0 & \text{in} \quad B_{2r}(x_0), \\ w = u & \text{on} \quad \partial B_{2r}(x_0). \end{cases} \tag{2.3.1}$$

We first recall an interior reverse Hölder inequality that can be found in [41, Theorem 6.7].

Lemma 2.3.1 ([77], Lemma 3.1)**.** *Let w be a solution to (2.3.1). There exists a constant $\theta_1 > p$ depending only on n, p, and λ such that for any $t > 0$, the estimate*

$$\left(\fint_{B_{\rho/2}(y)} (|Dw| + s)^{\theta_1} \, dx \right)^{\frac{1}{\theta_1}} \le C \left(\fint_{B_\rho(y)} (|Dw| + s)^t \, dx \right)^{\frac{1}{t}} \tag{2.3.2}$$

holds for all $B_\rho(y) \subset B_{2r}(x_0)$, where $C = C(n,p,\lambda,t) > 0$.

We also have the following comparison result, which generalizes and refines similar results in [38, 76, 78].

Lemma 2.3.2. *Let w be a solution to (2.3.1) and assume that $p \in (1,2)$. Then for any $\gamma_0 \in (0, 2-p]$ when $p \in \left(\frac{3n-2}{2n-1}, 2\right)$ or $\gamma_0 \in \left(0, \frac{(p-1)n}{n-1}\right)$ when $p \in \left(1, \frac{3n-2}{2n-1}\right]$, it holds* *that*

$$\left(\fint_{B_{2r}(x_0)} |Du - Dw|^{\gamma_0}\, dx\right)^{1/\gamma_0}$$
$$\leq C\left[\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}}\fint_{B_{2r}(x_0)} (|Du| + s)^{2-p}\, dx,$$

where C is a constant depending only on n, p, λ, and γ_0.

Proof. The case when $p \in \left(1, \frac{3n-2}{2n-1}\right]$ and $s = 0$ was proved in [78, Lemma 2.1] and their proof also works for the case when $p \in \left(1, \frac{3n-2}{2n-1}\right]$ and $s > 0$. Therefore, we only focus on the case when $p \in \left(\frac{3n-2}{2n-1}, 2\right)$ and $s \geq 0$. By scaling invariance (see [38, Remark 4.1] for example), we may assume that $B_{2r}(x_0) = B_2$ and $|\mu|(B_2) = 1$. For $k > 0$, using

$$\varphi_1 = T_{2k}(u - w) := \max\big\{\min\{u - w, 2k\}, -2k\big\}$$

as a test function in (2.1.1) and (2.3.1) and recalling (2.1.3), we have

$$\int_{B_2 \cap \{|u-w|<2k\}} g^s(u, w) \leq Ck \quad \text{with} \quad g^s(u, w) = \frac{|D(u - w)|^2}{(|Dw| + |Du| + s)^{2-p}}. \tag{2.3.3}$$

By the triangle inequality, we have

$$\begin{aligned}|D(u - w)| &= g^s(u, w)^{\frac{1}{2}}(|Dw| + |Du| + s)^{\frac{2-p}{2}}\\ &\leq g^s(u, w)^{\frac{1}{2}}(|D(u - w)| + 2|Du| + s)^{\frac{2-p}{2}}\\ &\leq Cg^s(u, w)^{\frac{1}{2}}|D(u - w)|^{\frac{2-p}{2}} + Cg^s(u, w)^{\frac{1}{2}}(|Du| + s)^{\frac{2-p}{2}}.\end{aligned}$$

Using Young's inequality with exponents $\frac{2}{p}$ and $\frac{2}{2-p}$, we obtain

$$|D(u - w)| \leq Cg^s(u, w)^{\frac{1}{p}} + Cg^s(u, w)^{\frac{1}{2}}(|Du| + s)^{\frac{2-p}{2}}. \tag{2.3.4}$$

Now we set

$$E_k = B_2 \cap \{k < |u-w| < 2k\} \quad \text{and} \quad F_k = B_2 \cap \{|u-w| > k\}.$$

Using the Sobolev inequality, Hölder's inequality, and (2.3.4), we obtain

$$\begin{aligned}
&k|\{x : |u-w| > 2k\} \cap B_2|^{\frac{n-1}{n}} \le C\Big(\int_{B_2} |T_{2k}(u-w) - T_k(u-w)|^{\frac{n}{n-1}}\Big)^{\frac{n-1}{n}} \\
&\le C\int_{E_k} |D(u-w)| \\
&\le C\int_{E_k} \Big(g^s(u,w)^{\frac{1}{p}} + g^s(u,w)^{\frac{1}{2}}(|Du|+s)^{\frac{2-p}{2}}\Big) \\
&\le C|E_k|^{\frac{p-1}{p}}\Big(\int_{E_k} g^s(u,w)\Big)^{\frac{1}{p}} + C\Big(\int_{E_k} g^s(u,w)\Big)^{\frac{1}{2}}\Big(\int_{E_k}(|Du|+s)^{2-p}\Big)^{\frac{1}{2}}.
\end{aligned} \tag{2.3.5}$$

From (2.3.3) and (2.3.5), we get

$$k^{\frac{1}{2}}|F_{2k}|^{\frac{n-1}{n}} \le Ck^{-\frac{1}{2}+\frac{1}{p}}|F_k|^{\frac{p-1}{p}} + CQ_1^{\frac{2-p}{2}},$$

where $Q_1 := \||Du|+s\|_{L^{2-p}(B_2)}$. Therefore, by taking the sup in $k \in (0,\infty)$, we obtain

$$\|u-w\|^{1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C\|u-w\|^{1/p-1/2}_{L^{\frac{2-p}{2(p-1)},\infty}(B_2)} + CQ_1^{\frac{2-p}{2}}.$$

Since $\frac{3n-2}{2n-1} < p < 2$, we have

$$\frac{n}{2(n-1)} > \frac{2-p}{2(p-1)},$$

which implies

$$\|u-w\|^{1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C\|u-w\|^{1/p-1/2}_{L^{\frac{n}{2(n-1)},\infty}(B_2)} + CQ_1^{\frac{2-p}{2}}.$$

Thus, by Young's inequality, we obtain

$$\|u-w\|_{L^{\frac{n}{2(n-1)},\infty}(B_2)} \le C + CQ_1^{\frac{2-p}{2}}. \tag{2.3.6}$$

Let $k,\ l>0$ and $q=\frac{n}{2(n-1)}$. By the Chebyshev inequality and (2.3.3), we have

$$\begin{aligned}
&|\{x : g^s(u,w)>l\}\cap B_2| \\
&\le |\{x : |u-w|>k\}\cap B_2| + |\{x : |u-w|\le k,\ g^s(u,w)>l\}\cap B_2| \\
&\le Ck^{-q}\|u-w\|^q_{L^{q,\infty}(B_2)} + \frac{1}{l}\int_{B_2\cap\{x:|u-w|\le k\}} g^s(u,w)\,dx \\
&\le Ck^{-q}\|u-w\|^q_{L^{q,\infty}(B_2)} + \frac{Ck}{l}.
\end{aligned}$$

By choosing

$$k=\Big[l\|u-w\|^q_{L^{q,\infty}(B_2)}\Big]^{\frac{1}{1+q}},$$

we get

$$l^{\frac{q}{1+q}}|\{x : g^s(u,w)>l\}\cap B_2| \le C\|u-w\|^{\frac{q}{1+q}}_{L^{q,\infty}(B_2)},$$

Therefore, by taking the sup in $l\in(0,\infty)$, we obtain

$$\|g^s(u,w)\|_{L^{\frac{q}{1+q},\infty}(B_2)} \le C\|u-w\|_{L^{q,\infty}(B_2)}. \tag{2.3.7}$$

Let $\gamma_0\in(0,2-p]$. By (2.3.4) and Hölder's inequality with exponents $2/p$ and $2/(2-p)$, we get

$$\begin{aligned}
&\int_{B_2}|D(u-w)|^{\gamma_0} \le C\int_{B_2}\Big(g^s(u,w)^{\frac{\gamma_0}{p}} + g^s(u,w)^{\frac{\gamma_0}{2}}(|Du|+s)^{\frac{\gamma_0(2-p)}{2}}\Big) \\
&\le C\Big(\int_{B_2} g^s(u,w)^{\frac{\gamma_0}{p}}\Big) + C\Big(\int_{B_2} g^s(u,w)^{\frac{\gamma_0}{p}}\Big)^{\frac{p}{2}}\Big(\int_{B_2}(|Du|+s)^{\gamma_0}\Big)^{\frac{2-p}{2}} \\
&\le C\|g^s(u,w)\|^{\gamma_0/p}_{L^{\frac{q}{1+q},\infty}(B_2)} + C\|g^s(u,w)\|^{\gamma_0/2}_{L^{\frac{q}{1+q},\infty}(B_2)}Q_1^{\frac{\gamma_0(2-p)}{2}}.
\end{aligned} \tag{2.3.8}$$

In the last inequality, we used the fact that

$$\frac{\gamma_0}{p} < \frac{q}{1+q}, \quad \gamma_0 \le 2 - p.$$

Combining (2.3.6), (2.3.7), and (2.3.8), we have

$$\int_{B_2} |D(u-w)|^{\gamma_0} \le C + CQ_1^{\gamma_0(2-p)},$$

which implies the desired result. $\square$

We now let $v \in w + W_0^{1,p}(B_r(x_0))$ be the unique solution to

$$\begin{cases} -\mathrm{div}(A(x_0, Dv)) = 0 & \text{in} \quad B_r(x_0), \\ v = w & \text{on} \quad \partial B_r(x_0). \end{cases} \tag{2.3.9}$$

By testing (2.3.1) and (2.3.9) with $v - w$, we obtain an estimate for the difference $Dv - Dw$:

$$\fint_{B_r(x_0)} |Dv - Dw|^p \, dx \le C\omega(r)^p \fint_{B_r(x_0)} (|Dw| + s)^p \, dx. \tag{2.3.10}$$

Detailed proof of this result can be found in [38, Eq. (4.35)]. Thus by (2.3.2) and Hölder's inequality, we get

$$\fint_{B_r(x_0)} |Dv - Dw|^{\gamma_0} \, dx \le C\omega(r)^{\gamma_0} \fint_{B_{2r}(x_0)} (|Dw| + s)^{\gamma_0} \, dx. \tag{2.3.11}$$

For a ball $B_\rho(x) \subset\subset \Omega$ and a function $f \in W^{1,p}_{\mathrm{loc}}(\Omega)$, there exists $\mathbf{q}_{x,\rho}(f) \in \mathbb{R}^n$ such that

$$\left(\fint_{B_\rho(x)} |Df - \mathbf{q}_{x,\rho}(f)|^{\gamma_0} \right)^{1/\gamma_0} = \inf_{\mathbf{q} \in \mathbb{R}^n} \left(\fint_{B_\rho(x)} |Df - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}.$$

We denote $\mathbf{q}_{x,\rho} = \mathbf{q}_{x,\rho}(u)$ and

$$\phi(x,\rho) = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho(x)} |Du - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}.$$

Since

$$|\mathbf{q}_{x,\rho} - Du(x)|^{\gamma_0} \le |\mathbf{q}_{x,\rho} - Du(z)|^{\gamma_0} + |Du(z) - Du(x)|^{\gamma_0},$$

by taking the average over $z \in B_\rho(x)$ and then taking the γ_0-th root, we obtain

$$|\mathbf{q}_{x,\rho} - Du(x)| \le C\phi(x,\rho) + C\Big(\fint_{B_\rho(x)} |Du(z) - Du(x)|^{\gamma_0}\, dz \Big)^{1/\gamma_0}.$$

Therefore, from the definition of ϕ and the fact that $0 < \gamma_0 < 1$, we obtain that

$$\lim_{\rho\to 0} \mathbf{q}_{x,\rho} = Du(x) \tag{2.3.12}$$

holds for any Lebesgue point $x \in \Omega$ of the vector-valued function Du.

Proposition 2.3.3. *Suppose that $u \in W^{1,p}_{loc}(\Omega)$ is a solution to (2.1.1). Then for any $\varepsilon \in (0,1)$ and $B_{2r}(x_0) \subset\subset \Omega$, we have*

$$\begin{aligned}\phi(x_0, \varepsilon r) \le & C\varepsilon^\alpha \phi(x_0, r) + C_\varepsilon \left(\frac{|\mu|(B_{2r}(x_0))}{r^{n-1}} \right)^{\frac{1}{p-1}} \\ & + C_\varepsilon \frac{|\mu|(B_{2r}(x_0))}{r^{n-1}} \fint_{B_{2r}(x_0)} (|Du| + s)^{2-p} \\ & + C_\varepsilon \omega(r) \left(\fint_{B_{2r}(x_0)} (|Du| + s)^{2-p} \right)^{\frac{1}{2-p}},\end{aligned} \tag{2.3.13}$$

where $\alpha \in (0,1)$ is the constant in Theorem 2.2.1, γ_0 is the same constant as in Lemma 2.3.2, C_ε is a constant depending on ε, n, p, λ, and γ_0, and C is a constant depending on n, p, λ, and γ_0.

Proof. By Theorem 2.2.1 and the definition of $\mathbf{q}_{x,\rho}(\cdot)$, we have

$$
\begin{aligned}
&\left(\fint_{B_{\varepsilon r}(x_0)} |Du - \mathbf{q}_{x_0,\varepsilon r}(u)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} \le \left(\fint_{B_{\varepsilon r}(x_0)} |Du - \mathbf{q}_{x_0,\varepsilon r}(v)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} \\
&\le C\left(\fint_{B_{\varepsilon r}(x_0)} |Dv - \mathbf{q}_{x_0,\varepsilon r}(v)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} + C\left(\fint_{B_{\varepsilon r}(x_0)} |Du - Dv|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |Dv - \mathbf{q}_{x_0,r}(v)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} + C\varepsilon^{-\frac{n}{\gamma_0}}\left(\fint_{B_r(x_0)} |Du - Dv|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} \qquad (2.3.14) \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |Dv - \mathbf{q}_{x_0,r}(u)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} + C\varepsilon^{-\frac{n}{\gamma_0}}\left(\fint_{B_r(x_0)} |Du - Dv|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} \\
&\le C\varepsilon^{\alpha}\left(\fint_{B_r(x_0)} |Du - \mathbf{q}_{x_0,r}(u)|^{\gamma_0}\right)^{\frac{1}{\gamma_0}} + C\varepsilon^{-\frac{n}{\gamma_0}}\left(\fint_{B_r(x_0)} |Du - Dv|^{\gamma_0}\right)^{\frac{1}{\gamma_0}}.
\end{aligned}
$$

Moreover, by (2.3.11) and the fact that $|\omega(r)| \le 1$, one has

$$
\begin{aligned}
&\fint_{B_r(x_0)} |Du - Dv|^{\gamma_0} \le \fint_{B_r(x_0)} |Du - Dw|^{\gamma_0} + \fint_{B_r(x_0)} |Dw - Dv|^{\gamma_0} \\
&\le C\fint_{B_{2r}(x_0)} |Du - Dw|^{\gamma_0} + C\omega(r)^{\gamma_0}\fint_{B_{2r}(x_0)} (|Dw| + s)^{\gamma_0} \qquad (2.3.15) \\
&\le C\fint_{B_{2r}(x_0)} |Du - Dw|^{\gamma_0} + C\omega(r)^{\gamma_0}\fint_{B_{2r}(x_0)} (|Du| + s)^{\gamma_0}.
\end{aligned}
$$

Thus from (2.3.14) and (2.3.15), we have

$$
\begin{aligned}
\phi(x_0, \varepsilon r) \le C\varepsilon^{\alpha}\phi(x_0, r) + C_{\varepsilon}\left(\fint_{B_{2r}(x_0)} |Du - Dw|^{\gamma_0}\right)^{1/\gamma_0} \\
+ C_{\varepsilon}\omega(r)\left(\fint_{B_{2r}(x_0)} (|Du| + s)^{\gamma_0}\right)^{1/\gamma_0}.
\end{aligned} \qquad (2.3.16)
$$

Now we can apply Lemma 2.3.2 to bound the second term on the right-hand side of (2.3.16) to conclude the proof. □

Now we are ready to prove Theorem 2.1.1.

Proof of Theorem 2.1.1. We prove the theorem at a Lebesgue point $x = x_0$ of the vector-valued function Du, assuming that $B_R(x_0) \subset \Omega$. Since $p \in \left(\frac{3n-2}{2n-1}, 2\right)$, we choose

$\gamma_0 = 2 - p$ in Lemma 2.3.2. Choose $\varepsilon = \varepsilon(n, p, \lambda, \alpha) \in (0, 1/4)$ sufficiently small so that $C\varepsilon^\alpha \le 1/4$, where C is the constant in (2.3.13).

For an integer $j \ge 0$, set $r_j = \varepsilon^j R$, $B^j = B_{2r_j}(x_0)$,

$$T_j = \left(\fint_{B^j} (|Du| + s)^{2-p} \, dx \right)^{\frac{1}{2-p}}, \quad \phi_j = \phi(x_0, r_j), \quad \mathbf{q}_j = \mathbf{q}_{x_0, r_j}.$$

Applying (2.3.13) yields

$$\phi_{j+1} \le \frac{1}{4}\phi_j + C\left(\frac{|\mu|(B^j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C\frac{|\mu|(B^j)}{r_j^{n-1}} T_j^{2-p} + C\omega(r_j)T_j.$$

Let j_0 and m be positive integers to be specified later such that $j_0 \le m$. Summing the above inequality over $j = j_0, j_0 + 1, \ldots, m$, we obtain

$$\begin{aligned} \sum_{j=j_0}^{m+1} \phi_j \le C\phi_{j_0} &+ C \sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\ &+ C \sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_0}^{m} \omega(r_j) T_j. \end{aligned} \tag{2.3.17}$$

Since

$$|\mathbf{q}_{j+1} - \mathbf{q}_j|^{\gamma_0} \le |\mathbf{q}_{j+1} - Du(x)|^{\gamma_0} + |Du(x) - \mathbf{q}_j|^{\gamma_0},$$

by taking the average over $x \in B_{r_{j+1}}(x_0)$ and then taking γ_0-th root, we obtain

$$|\mathbf{q}_{j+1} - \mathbf{q}_j| \le C\phi_j + C\phi_{j+1}.$$

Then, by iterating, we get

$$|\mathbf{q}_{m+1} - \mathbf{q}_{j_0}| \le C \sum_{j=j_0}^{m+1} \phi_j,$$

which together with (2.3.17) implies

$$
\begin{aligned}
|\mathbf{q}_{m+1}| + \sum_{j=j_0}^{m+1} \phi_j \leq & C\phi_{j_0} + |\mathbf{q}_{j_0}| + C\sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\
& + C\sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} T_j^{2-p} + C\sum_{j=j_0}^{m} \omega(r_j) T_j.
\end{aligned}
\tag{2.3.18}
$$

By the definition of ϕ_{j_0}, we have

$$
\phi_{j_0} \leq C \left(\fint_{B^{j_0}} |Du|^{\gamma_0} \, dx \right)^{\frac{1}{\gamma_0}} \leq CT_{j_0}.
$$

Since

$$
|\mathbf{q}_{j_0}|^{\gamma_0} \leq |Du(x) - \mathbf{q}_{j_0}|^{\gamma_0} + |Du(x)|^{\gamma_0},
$$

by taking the average over $x \in B_{r_{j_0}}(x_0)$ and taking the γ_0-th root, we obtain

$$
|\mathbf{q}_{j_0}| \leq C\phi_{j_0} + C \left(\fint_{B^{j_0}} |Du|^{\gamma_0} \, dx \right)^{\frac{1}{\gamma_0}} \leq CT_{j_0}.
$$

Therefore, (2.3.18) implies that

$$
\begin{aligned}
|\mathbf{q}_{m+1}| + \sum_{j=j_0}^{m+1} \phi_j \leq & CT_{j_0} + C\sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\
& + C\sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} T_j^{2-p} + C\sum_{j=j_0}^{m} \omega(r_j) T_j.
\end{aligned}
\tag{2.3.19}
$$

By (2.1.5) and the comparison principle for Riemann integrals, there exists $j_0 = j_0(n, p, \varepsilon, C, \omega) > 1$ sufficiently large such that

$$
4^{\frac{1}{\gamma_0}} (2\varepsilon)^{-\frac{n}{\gamma_0}} C \sum_{j=j_0}^{\infty} \omega(r_j) \leq \frac{1}{10}, \tag{2.3.20}
$$

where C is the constant in (2.3.19).

Note that by the comparison principle for Riemann integrals,

$$\sum_{j=j_0}^{m} \frac{|\mu|(B^j)}{r_j^{n-1}} \leq C \int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \tag{2.3.21}$$

and since $p < 2$ we also have

$$\sum_{j=j_0}^{m} \left(\frac{|\mu|(B^j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \leq C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}}. \tag{2.3.22}$$

To prove (2.1.12) at $x = x_0$, it is sufficient to show that

$$|Du(x_0)| \leq CT_{j_0} + C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}}. \tag{2.3.23}$$

To this end, we consider the following possibilities.

Case 1: If $|Du(x_0)| \leq T_{j_0}$, then (2.3.23) easily follows.

Case 2: If $T_j < |Du(x_0)|$, $\forall j_0 \leq j \leq j_1$, and $|Du(x_0)| \leq T_{j_1+1}$, then since $\gamma_0 = 2 - p < 1$, we have

$$\begin{aligned} |Du(x_0)| &\leq \left(\fint_{B^{j_1+1}} (|Du| + s)^{\gamma_0} \, dx \right)^{1/\gamma_0} \\ &\leq 2^{\frac{1}{\gamma_0}} \left(\fint_{B^{j_1+1}} |Du|^{\gamma_0} \, dx \right)^{1/\gamma_0} + 2^{\frac{1}{\gamma_0}} s \\ &\leq 2^{\frac{1}{\gamma_0}} (2\varepsilon)^{-\frac{n}{\gamma_0}} \left(\fint_{B_{r_{j_1}}(x_0)} |Du|^{\gamma_0} \, dx \right)^{1/\gamma_0} + 2^{\frac{1}{\gamma_0}} s \\ &\leq 4^{\frac{1}{\gamma_0}} (2\varepsilon)^{-\frac{n}{\gamma_0}} (\phi_{j_1} + |\mathbf{q}_{j_1}|) + 2^{\frac{1}{\gamma_0}} s, \end{aligned} \tag{2.3.24}$$

where the last inequality follows from the definitions of ϕ_{j_1} and $\mathbf{q}_{j_1}$. Now applying

(2.3.19) with $m = j_1 - 1$ and using (2.3.21) and (2.3.22), from (2.3.24) we get

$$\begin{aligned}
|Du(x_0)| \le & C' T_{j_0} + C'' \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}} \\
& + C'' \int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \cdot |Du(x_0)|^{2-p} \\
& + C' \sum_{j=j_0}^{m} \omega(r_j) |Du(x_0)| + 2^{\frac{1}{\gamma_0}} s,
\end{aligned}$$

where $C' = 4^{\frac{1}{\gamma_0}} (2\varepsilon)^{-\frac{n}{\gamma_0}} C$, C is the constant in (2.3.19), and C'' is a constant depending on n, p, and λ. Hence using (2.3.20) and Young's inequality, we find

$$|Du(x_0)| \le CT_{j_0} + C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}} + \frac{1}{5} |Du(x_0)| + Cs.$$

This implies (2.3.23) as desired.

Case 3: If $T_j < |Du(x_0)|$ for any $j \ge j_0$, then from (2.3.19), (2.3.21), and (2.3.22) we have for any $m > j_0$,

$$\begin{aligned}
|\mathbf{q}_{m+1}| \le & CT_{j_0} + C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}} \\
& + C \int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \cdot |Du(x_0)|^{2-p} + C \sum_{j=j_0}^{m} \omega(r_j) |Du(x_0)| \\
\le & CT_{j_0} + C \left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \right)^{\frac{1}{p-1}} \\
& + C \int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}} \frac{d\rho}{\rho} \cdot |Du(x_0)|^{2-p} + \frac{1}{10} |Du(x_0)|.
\end{aligned}$$

Here we used (2.3.20) in the last inequality. Letting $m \to \infty$ and using (2.3.12), we

get

$$
\begin{aligned}
|Du(x_0)| \leq & CT_{j_0} + C\left(\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\frac{d\rho}{\rho}\right)^{\frac{1}{p-1}} \\
& + C\int_0^{2r_{j_0-1}} \frac{|\mu|(B_\rho(x_0))}{\rho^{n-1}}\frac{d\rho}{\rho}\cdot |Du(x_0)|^{2-p} + \frac{1}{10}|Du(x_0)|.
\end{aligned}
$$

Then using Young's inequality, we deduce (2.3.23). The proof is completed. □

2.4 Interior Lipschitz estimate and modulus of continuity estimate of the gradient

In this section, we give the proof of the Lipschitz estimate in Theorem 2.1.3 and derive an interior modulus of continuity estimate of Du under the same conditions. We first adapt the argument in [25] to obtain some decay estimates from Proposition 2.3.3. Let $\alpha \in (0,1)$ be the same constant as in Theorem 2.2.1, $\alpha_1 \in (0,\alpha)$, $R \in (0,1]$ and $B_R(x_0) \subset \Omega$. Choose $\varepsilon = \varepsilon(n,p,\lambda,\gamma_0,\alpha,\alpha_1) > 0$ sufficiently small such that

$$
C\varepsilon^{\alpha-\alpha_1} < 1 \quad \text{and} \quad \varepsilon^{\alpha_1} < 1/4,
$$

where C is the constant in (2.3.13).

Proposition 2.3.3 implies that for any $B_{2r}(x) \subset\subset B_R(x_0)$,

$$
\begin{aligned}
\phi(x,\varepsilon r) \leq & \varepsilon^{\alpha_1}\phi(x,r) + C\left(\frac{|\mu|(B_{2r}(x))}{r^{n-1}}\right)^{\frac{1}{p-1}} \\
& + C\frac{|\mu|(B_{2r}(x))}{r^{n-1}}\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right)^{2-p} \\
& + C\omega(r)\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right).
\end{aligned} \tag{2.4.1}
$$

Denote

$$
g(x,r) = \frac{|\mu|(B_r(x))}{r^{n-1}}, \quad h(x,r) = g(x,r)^{\frac{1}{p-1}}. \tag{2.4.2}
$$

By iteration, from (2.4.1) we get

$$
\begin{aligned}
\phi(x,\varepsilon^j r) \leq & \varepsilon^{\alpha_1 j}\phi(x,r) + C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}h(x,2\varepsilon^{j-i}r) \\
& + C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}g(x,2\varepsilon^{j-i}r)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)^{2-p} \\
& + C\sum_{i=1}^{j}\varepsilon^{\alpha_1(i-1)}\omega(\varepsilon^{j-i}r)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)
\end{aligned}
$$

for any $B_{2r}(x) \subset\subset B_R(x_0)$ with $r \in (0, R/4)$. Thus,

$$
\begin{aligned}
\phi(x,\varepsilon^j r) \leq & \varepsilon^{\alpha_1 j}\phi(x,r) + C\tilde{h}(x,2\varepsilon^j r) + C\tilde{g}(x,2\varepsilon^j r)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)^{2-p} \\
& + C\tilde{\omega}(\varepsilon^j r)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right),
\end{aligned}
\tag{2.4.3}
$$

where we defined

$$
\begin{aligned}
\tilde{h}(x,t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\left(h(x,\varepsilon^{-i}t)[\varepsilon^{-i}t \leq R/2] + h(x,R/2)[\varepsilon^{-i}t > R/2]\right), \\
\tilde{g}(x,t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\left(g(x,\varepsilon^{-i}t)[\varepsilon^{-i}t \leq R/2] + g(x,R/2)[\varepsilon^{-i}t > R/2]\right), \\
\tilde{\omega}(t) &:= \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}\left(\omega(\varepsilon^{-i}t)[\varepsilon^{-i}t \leq R/2] + \omega(R/2)[\varepsilon^{-i}t > R/2]\right).
\end{aligned}
\tag{2.4.4}
$$

Here we used the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$ otherwise. We obtain the following lemma from (2.4.3).

Lemma 2.4.1. *Let $B_{2r}(x) \subset\subset B_R(x_0) \subset \Omega$ with $r \leq R/4$. There exists a constant C depending only on ε, n, p, λ, γ_0, and α_1, such that for any $\rho \in (0, r]$, we have*

(i)

$$\phi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\tilde{h}(x,2\rho)$$
$$+ C\tilde{g}(x,2\rho)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)^{2-p} + C\tilde{\omega}(\rho)\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right), \tag{2.4.5}$$

(ii)

$$\begin{aligned}\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \le& C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\ &+ C\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt \\ &+ C\left(\|Du\|_{L^\infty(B_{2r}(x))}+s\right)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt.\end{aligned} \tag{2.4.6}$$

To prove Lemma 2.4.1, we need the following technical lemma.

Lemma 2.4.2. *Let $B_R(x_0) \subset \Omega$. Then there exist constants c_1 and c_2 depending on ε, n, p, and α_1, such that for any fixed $x \in B_R(x_0)$ and any $f \in \{\tilde{\omega}, \tilde{g}(x,\cdot), \tilde{h}(x,\cdot)\}$, it holds that $c_1 f(t) \le f(s) \le c_2 f(t)$, whenever $0 < \varepsilon t \le s \le t$.*

Proof. We will only show the proof for g since the other cases are similar. For fixed $x \in B_{R/4}(x_0)$, we set

$$G(x,r) = \begin{cases} g(x,r) & \text{if} \quad 0 < r \le R/2, \\ g(x,R/2) & \text{if} \quad r > R/2, \end{cases}$$

and observe that by (2.4.4),

$$\tilde{g}(x,r) = \sum_{i=1}^{\infty}\varepsilon^{\alpha_1 i}G(x,\varepsilon^{-i}r).$$

Suppose that $0 < \varepsilon t \le s \le t$. It is easy to see from the definitions of g and G that $G(x,s) \le \varepsilon^{1-n}G(x,t)$ and therefore $\tilde{g}(x,s) \le \varepsilon^{1-n}\tilde{g}(x,t)$. Also the fact that

$0 < \varepsilon s \le \varepsilon t \le s$ implies that $\tilde{g}(x, \varepsilon t) \le \varepsilon^{1-n}\tilde{g}(x,s)$. On the other hand,

$$\tilde{g}(x,t) = \varepsilon^{-\alpha_1}\sum_{i=1}^{\infty}\varepsilon^{\alpha_1(i+1)}G(x,\varepsilon^{-(i+1)}\varepsilon t) \le \varepsilon^{-\alpha_1}\tilde{g}(x,\varepsilon t).$$

The lemma is proved. □

Now we are ready to prove Lemma 2.4.1.

Proof of Lemma 2.4.1. We first prove Assertion (i). For given $\rho \in (0, r]$, let j be an integer such that $\varepsilon^{j+1} < \rho/r \le \varepsilon^j$. Then by (2.4.3) with $\varepsilon^{-j}\rho$ in place of r, we get

$$\begin{aligned}
\phi(x,\rho) \le & \varepsilon^{\alpha_1 j}\phi(x,\varepsilon^{-j}\rho) + C\tilde{h}(x,2\rho) + C\tilde{g}(x,2\rho)\left(\|Du\|_{L^\infty(B_{2\varepsilon^{-j}\rho}(x))} + s\right)^{2-p} \\
& + C\tilde{\omega}(\rho)\left(\|Du\|_{L^\infty(B_{2\varepsilon^{-j}\rho}(x))} + s\right) \\
\le & C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\tilde{h}(x,2\rho) + C\tilde{g}(x,2\rho)\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right)^{2-p} \\
& + C\tilde{\omega}(\rho)\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right).
\end{aligned}$$

Therefore, Assertion (i) holds. Now applying (2.4.5) with $\varepsilon^j\rho$ in place of ρ and summing in j, we get

$$\begin{aligned}
\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \le & C\left(\frac{\rho}{r}\right)^{\alpha_1}\phi(x,r) + C\sum_{j=1}^{\infty}\tilde{h}(x,2\varepsilon^j\rho) \\
& + C\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right)^{2-p}\sum_{j=1}^{\infty}\tilde{g}(x,2\varepsilon^j\rho) \\
& + C\left(\|Du\|_{L^\infty(B_{2r}(x))} + s\right)\sum_{j=1}^{\infty}\tilde{\omega}(\varepsilon^j\rho).
\end{aligned}$$

Hence, by using Lemma 2.4.2 and the comparison principle for Riemann integrals, we can easily get (2.4.6). The lemma is proved. □

Recall the definition of $\mathbf{q}_{x,\rho}$ from Section 2.3. Since we have

$$|\mathbf{q}_{x,\varepsilon\rho} - \mathbf{q}_{x,\rho}|^{\gamma_0} \le |Du(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |Du(z) - \mathbf{q}_{x,\varepsilon\rho}|^{\gamma_0},$$

by taking the average over $z \in B_{\varepsilon\rho}(x)$ and then taking the γ_0-th root, we obtain

$$|\mathbf{q}_{x,\varepsilon\rho} - \mathbf{q}_{x,\rho}| \le C\phi(x, \varepsilon\rho) + C\phi(x, \rho).$$

Then, by iterating, we get

$$|\mathbf{q}_{x,\varepsilon^j\rho} - \mathbf{q}_{x,\rho}| \le C \sum_{i=0}^{j} \phi(x, \varepsilon^i \rho).$$

Therefore, by using (2.3.12), we obtain that

$$|Du(x) - \mathbf{q}_{x,\rho}| \le C \sum_{j=0}^{\infty} \phi(x, \varepsilon^j \rho) \tag{2.4.7}$$

holds for any Lebesgue point $x \in \Omega$ of the vector-valued function Du.

Now we are ready to prove the interior Lipschitz estimate.

Proof of Theorem 2.1.3. We prove the theorem around a given point $x = x_0$ assuming that $B_R(x_0) \subset \Omega$ with $R \in (0, 1]$ and

$$\left\| \mathbf{I}_1^R(|\mu|) \right\|_{L^\infty(B_R(x_0))} < \infty. \tag{2.4.8}$$

We first derive an a priori estimate for the case when $u \in C^1$ and then use approximation to prove the general case.

Step 1: The case when $u \in C^1\big(\overline{B_R(x_0)}\big)$.

Using (2.4.6) with $\rho = r$ and (2.4.7), we obtain that

$$\begin{aligned}|Du(x) - \mathbf{q}_{x,\rho}| \leq & C\phi(x,\rho) + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(B_{2\rho}(x))} + s\right)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(B_{2\rho}(x))} + s\right)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt\end{aligned}$$

holds for any $B_{2\rho}(x) \subset\subset B_R(x_0)$ with $\rho \in (0, R/4]$.

Note that

$$|\mathbf{q}_{x,\rho}| \leq C\phi(x,\rho) + C\rho^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(B_\rho(x))} \leq C\rho^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(B_\rho(x))}.$$

Combining the above two inequalities, we have

$$\begin{aligned}|Du(x)| \leq & C\rho^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(B_\rho(x))} + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(B_{2\rho}(x))} + s\right)^{2-p}\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(B_{2\rho}(x))} + s\right)\int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt.\end{aligned} \tag{2.4.9}$$

Note that $\tilde{\omega}$ also satisfies the Dini condition (2.1.5); see [31, Lemma 1]. Thus we can take $\rho_0 = \rho_0(n, p, \lambda, \omega, \alpha_1, \gamma_0, R) \in (0, R/4]$ sufficiently small such that

$$C\int_0^{\rho_0} \frac{\tilde{\omega}(t)}{t}\,dt \leq 3^{-1-n/\gamma_0},$$

where C is the constant in (2.4.9). Then for any $B_{2\rho}(x) \subset\subset B_R(x_0)$ with $0 < \rho \leq \rho_0$, by (2.4.9) and Young's inequality, we have

$$\begin{aligned}|Du(x)| \leq & C\rho^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(B_\rho(x))} + C\int_0^\rho \frac{\tilde{h}(x,t)}{t}\,dt \\ & + C\left(\int_0^\rho \frac{\tilde{g}(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + 3^{-n/\gamma_0}\left(\|Du\|_{L^\infty(B_{2\rho}(x))} + s\right).\end{aligned} \tag{2.4.10}$$

For $k \geq 1$, we denote $\rho_k = (1 - 2^{-k})R$. Since $\rho_{k+1} - \rho_k = 2^{-k-1}R$, we have $B_{2\rho}(x) \subset B_{\rho_{k+1}}(x_0)$ for any $x \in B_{\rho_k}(x_0)$ and $\rho = 2^{-k-2}R$. We take k_0 sufficiently large such that $2^{-k_0-2} \leq \rho_0$. Then by (2.4.10) with $\rho = 2^{-k-2}R$, we have for any $k \geq k_0$ that

$$\begin{aligned}
&\|Du\|_{L^\infty(B_{\rho_k}(x_0))} + s \\
&\leq 3^{-n/\gamma_0}(\|Du\|_{L^\infty(B_{\rho_{k+1}}(x_0))} + s) + C\left(\frac{2^{k+2}}{R}\right)^{n/\gamma_0} \|Du\|_{L^{\gamma_0}(B_{\rho_{k+1}}(x_0))} \\
&\quad + C \sup_{x \in B_{\rho_k}(x_0)} \int_0^{\frac{R}{2}} \frac{\tilde{h}(x,t)}{t}\, dt + C \sup_{x \in B_{\rho_k}(x_0)} \left(\int_0^{\frac{R}{2}} \frac{\tilde{g}(x,t)}{t}\, dt\right)^{\frac{1}{p-1}} + s
\end{aligned}$$

Multiplying the above inequality by $3^{-nk/\gamma_0}$, and summing the terms with respect to $k = k_0, k_0+1, \cdots$, we obtain that

$$\begin{aligned}
&\sum_{k=k_0}^{\infty} 3^{-nk/\gamma_0}(\|Du\|_{L^\infty(B_{\rho_k}(x_0))} + s) \\
&\leq \sum_{k=k_0+1}^{\infty} 3^{-nk/\gamma_0}(\|Du\|_{L^\infty(B_{\rho_k}(x_0))} + s) + CR^{-n/\gamma_0}(\|Du\|_{L^{\gamma_0}(B_R(x_0))} + s) \\
&\quad + C \sup_{x \in B_R(x_0)} \int_0^{\frac{R}{2}} \frac{\tilde{h}(x,t)}{t}\, dt + C \sup_{x \in B_R(x_0)} \left(\int_0^{\frac{R}{2}} \frac{\tilde{g}(x,t)}{t}\, dt\right)^{\frac{1}{p-1}} + Cs,
\end{aligned}$$

where each summation is finite. By subtracting

$$\sum_{k=k_0+1}^{\infty} 3^{-nk/\gamma_0}(\|Du\|_{L^\infty(B_{\rho_k}(x_0))} + s)$$

from both sides of the above inequality, we get the following L^∞-estimate for Du:

$$\begin{aligned}
\|Du\|_{L^\infty(B_{R/2}(x_0))} + s &\leq CR^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(B_R(x_0))} \\
&\quad + C \sup_{x \in B_R(x_0)} \int_0^{\frac{R}{2}} \frac{\tilde{h}(x,t)}{t}\, dt + C \sup_{x \in B_R(x_0)} \left(\int_0^{\frac{R}{2}} \frac{\tilde{g}(x,t)}{t}\, dt\right)^{\frac{1}{p-1}} + Cs,
\end{aligned} \tag{2.4.11}$$

We can simplify the terms in (2.4.11) to get

$$\|Du\|_{L^\infty(B_{R/2}(x_0))} \le C\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(B_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\||Du|+s\|_{L^{2-p}(B_R(x_0))}. \quad (2.4.12)$$

Indeed, by the definition of $\tilde{g}$ in (2.4.4), we have

$$\begin{aligned}\int_0^{R/2}\frac{\tilde{g}(x,t)}{t}\,dt = &\sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\int_0^{R/2}\frac{g(x,\varepsilon^{-i}t)}{t}[\varepsilon^{-i}t\le R/2]\,dt\\ &+\sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\int_0^{R/2}\frac{g(x,R/2)}{t}[\varepsilon^{-i}t> R/2]\,dt.\end{aligned}$$

The first term above is equal to

$$\sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\int_0^{\varepsilon^i R/2}\frac{g(x,\varepsilon^{-i}t)}{t}\,dt = \sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\int_0^{R/2}\frac{g(x,t)}{t}\,dt \le C\mathbf{I}_1^R(|\mu|)(x).$$

The second term is equal to

$$\sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\ln(\varepsilon^{-i})g(x,R/2)\le Cg(x,R/2)\le C\mathbf{I}_1^R(|\mu|)(x).$$

Therefore,

$$\int_0^{R/2}\frac{\tilde{g}(x,t)}{t}\,dt \le C\mathbf{I}_1^R(|\mu|)(x). \quad (2.4.13)$$

We can similarly get

$$\begin{aligned}&\int_0^{R/2}\frac{\tilde{h}(x,t)}{t}\,dt = \sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\int_0^{R/2}\frac{h(x,t)}{t}\,dt + \sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\ln(\varepsilon^{-i})h(x,R/2)\\ &\le C\Big(\int_0^R\frac{g(x,t)}{t}\,dt\Big)^{\frac{1}{p-1}} + C\left(\mathbf{I}_1^R(|\mu|)(x)\right)^{\frac{1}{p-1}} \le C\left(\mathbf{I}_1^R(|\mu|)(x)\right)^{\frac{1}{p-1}}.\end{aligned} \quad (2.4.14)$$

Recalling the fact that $\gamma_0\le 2-p$ (cf. Lemma 2.3.2), using (2.4.13), (2.4.14), and Hölder's inequality, from (2.4.11) we obtain (2.4.12).

Step 2: The general case.

We take $r_1 \in (0, R)$, $r_2 = (R + r_1)/2$, and a sequence of standard mollifiers $\{\varphi_k\}$ such that for any positive integer k,

$$\varphi_k \in C_0^\infty(B_{1/k}(0)), \quad \varphi_k \geq 0, \quad \text{and} \fint_{B_{1/k}(0)} \varphi_k = 1.$$

Then we mollify μ and A by setting

$$\mu_k(x) = (\mu * \varphi_k)(x), \quad A_k(x, \xi) = (A(\cdot, \xi) * \varphi_k)(x), \quad x \in B_{r_2}(x_0).$$

We note that A_k is well defined and satisfies the growth, ellipticity and continuity assumptions (2.1.2)-(2.1.4) in $B_{r_2}(x_0)$ for $k > 1/(R - r_2)$. By the corollary after [45, Theorem 1], (2.4.8) implies $\mu \in W^{-1,p'}(B_{r_2}(x_0))$, where $p' = p/(p-1)$, and therefore

$$\|\mu_k - \mu\|_{W^{-1,p'}(B_{r_2}(x_0))} \to 0. \tag{2.4.15}$$

Next we let $u_k \in u + W_0^{1,p}(B_{r_2}(x_0))$ be the unique solution to

$$\begin{cases} -\operatorname{div}(A_k(x, Du_k)) = \mu_k & \text{in} \quad B_{r_2}(x_0), \\ u_k = u & \text{on} \quad \partial B_{r_2}(x_0). \end{cases} \tag{2.4.16}$$

Choosing $u_k - u$ as a test function in (2.4.16), we obtain

$$\begin{aligned} &\int_{B_{r_2}(x_0)} \langle A(x, Du_k), Du_k \rangle dx \\ &= \int_{B_{r_2}(x_0)} \langle A(x, Du_k), Du \rangle \, dx + \int_{B_{r_2}(x_0)} (u_k - u) \, d\mu_k. \end{aligned} \tag{2.4.17}$$

Using the fundamental theorem of calculus, (2.1.2), (2.1.3), and Young's inequality with exponents p and $p/(p-1)$, we have

$$\begin{aligned}
&\int_{B_{r_2}(x_0)} \langle A(x, Du_k), Du_k\rangle \, dx \\
&= \int_{B_{r_2}(x_0)} \Big(\int_0^1 \langle D_\xi A(x, tDu_k) Du_k, Du_k\rangle \, dt + \langle A(x, 0), Du_k\rangle \Big) \, dx \\
&\geq \int_{B_{r_2}(x_0)} \Big(\int_0^1 \lambda^{-1} (s^2 + |Du_k|^2 t^2)^{\frac{p-2}{2}} |Du_k|^2 \, dt - \lambda s^{p-1} |Du_k| \Big) \, dx \\
&\geq \lambda^{-1} \int_{B_{r_2}(x_0)} (s^2 + |Du_k|^2)^{\frac{p-2}{2}} |Du_k|^2 \, dx - \int_{B_{r_2}(x_0)} \lambda s^{p-1} |Du_k| \, dx \\
&\geq c(\lambda, p) \int_{B_{r_2}(x_0)} |Du_k|^p \, dx - C'(\lambda, p) \int_{B_{r_2}(x_0)} s^p \, dx.
\end{aligned}$$

On the other hand, using (2.1.2) and Young's inequality, we obtain

$$\begin{aligned}
&\int_{B_{r_2}(x_0)} \langle A(x, Du_k), Du\rangle \, dx + \int_{B_{r_2}(x_0)} (u_k - u) \, d\mu_k \\
&\leq \lambda \int_{B_{r_2}(x_0)} (s^2 + |Du_k|^2)^{\frac{p-1}{2}} |Du| \, dx + \|u_k - u\|_{W_0^{1,p}(B_{r_2}(x_0))} \|\mu_k\|_{W^{-1,p'}(B_{r_2}(x_0))} \\
&\leq \frac{1}{4} c(\lambda, p) \int_{B_{r_2}(x_0)} (|Du_k| + s)^p \, dx + C''(\lambda, p) \int_{B_{r_2}(x_0)} |Du|^p \, dx \\
&\quad + \frac{1}{8} c(\lambda, p) \|Du_k - Du\|^p_{L^p(B_{r_2}(x_0))} + C''(\lambda, p) \, \|\mu_k\|^{p'}_{W^{-1,p'}(B_{r_2}(x_0))}.
\end{aligned}$$

Therefore, (2.4.17) implies that

$$\|Du_k\|^p_{L^p(B_{r_2}(x_0))} \leq C \| |Du| + s\|^p_{L^p(B_{r_2}(x_0))} + C \|\mu_k\|^{p'}_{W^{-1,p'}(B_{r_2}(x_0))}, \tag{2.4.18}$$

where C is a constant not depending on k.

Now we recall a well-known inequality

$$c^{-1}\big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} \leq \frac{|V(\xi_2) - V(\xi_1)|^2}{|\xi_2 - \xi_1|^2} \leq c\big(s^2 + |\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}}, \tag{2.4.19}$$

where $c = c(n,p) > 1$ is a positive constant and the mapping $V(\cdot)$ is defined as

$$V(\xi) = (|\xi|^2 + s^2)^{\frac{p-2}{4}}\xi, \quad \xi \in \mathbb{R}^n.$$

Combining (2.1.3) and (2.4.19) yields

$$c_0^{-1}|V(\xi_2) - V(\xi_1)|^2 \leq \langle A(x,\xi_2) - A(x,\xi_1),\, \xi_2 - \xi_1\rangle,$$

for some positive constant $c_0 = c_0(n,p,\lambda)$. We note that the above inequality also holds for A_k. Then choosing $(u_k - u)1_{B_{r_2}(x_0)}$ as a test function in (2.1.1) and (2.4.16), we have

$$\begin{aligned}
&c_0^{-1}\int_{B_{r_2}(x_0)} |V(Du_k) - V(Du)|^2\,dx \\
&\leq \int_{B_{r_2}(x_0)} \langle A_k(x,Du_k) - A_k(x,Du), Du_k - Du\rangle\,dx \\
&= \int_{B_{r_2}(x_0)} \langle A(x,Du) - A_k(x,Du), Du_k - Du\rangle\,dx + \int_{B_{r_2}(x_0)} (u_k - u)\,d(\mu_k - \mu) \\
&\leq \|A(\cdot,Du) - A_k(\cdot,Du)\|_{L^{p/(p-1)}(B_{r_2}(x_0))}\cdot\|u_k - u\|_{W_0^{1,p}(B_{r_2}(x_0))} \\
&\quad + \|u_k - u\|_{W_0^{1,p}(B_{r_2}(x_0))}\cdot\|\mu_k - \mu\|_{W^{-1,p'}(B_{r_2}(x_0))}.
\end{aligned}$$

From the definition of A_k, by using the Minkowski inequality and (2.1.4), we obtain

$$\|A(\cdot,Du) - A_k(\cdot,Du)\|_{L^{p/(p-1)}(B_{r_2}(x_0))} \leq \lambda\omega(1/k)\Big(\int_{B_{r_2}(x_0)} (s^2 + |Du|^2)^{\frac{p}{2}}\,dx\Big)^{\frac{p-1}{p}},$$

which together with (2.4.15) and (2.4.18), yields

$$\int_{B_{r_2}(x_0)} |V(Du_k) - V(Du)|^2\,dx \to 0.$$

By (2.4.19), we have

$$|Du_k - Du|^p \le c|V(Du_k) - V(Du)|^p \big(|Du_k|^2 + |Du|^2 + s^2\big)^{\frac{p(2-p)}{4}}$$

and therefore using Hölder's inequality with exponents $2/p$ and $2/(2-p)$, we obtain that

$$\begin{aligned}
&\int_{B_{r_2}(x_0)} |Du_k - Du|^p \, dx \\
&\le c\Big(\int_{B_{r_2}(x_0)} |V(Du_k) - V(Du)|^2 \, dx\Big)^{\frac{p}{2}} \Big(\int_{B_{r_2}(x_0)} \big(|Du_k|^2 + |Du|^2 + s^2\big)^{\frac{p}{2}} \, dx\Big)^{\frac{2-p}{2}},
\end{aligned}$$

which implies that

$$Du_k \to Du \quad \text{strongly in } L^p(B_{r_2}(x_0)).$$

Thus there exists a subsequence $\{k_j\}$ such that $Du_{k_j} \to Du$ almost everywhere in $B_{r_2}(x_0)$.

Since A_k and μ_k are smooth in x, by the classical regularity theory (see, for instance, [22],[83]), we know that $u_k \in C^{1,\alpha}_{\mathrm{loc}}(B_{r_2}(x_0))$. Therefore the Lipschitz estimate (2.4.12) from Step 1 holds for u_k in $B_{r_1/2}(x_0)$. Namely,

$$\|Du_k\|_{L^\infty(B_{r_1/2}(x_0))} \le C\big\|\mathbf{I}_1^{r_1}(|\mu_k|)\big\|_{L^\infty(B_{r_1}(x_0))}^{\frac{1}{p-1}} + Cr_1^{\frac{n}{2-p}}\||Du_k| + s\|_{L^{2-p}(B_{r_1}(x_0))}.$$

Note that by direct computation, for any $t > 0$, it follows that

$$\mu_k(B_t(x)) = \int_{\mathbb{R}^n} \mu(B_t(x-y))\varphi_k(y)dy.$$

Therefore, for sufficiently large k, by the Fubini–Tonelli theorem we have

$$\big\|\mathbf{I}_1^{r_1}(|\mu_k|)\big\|_{L^\infty(B_{r_1}(x_0))} \le \big\|\mathbf{I}_1^{r_1}(|\mu|)\big\|_{L^\infty(B_{r_2}(x_0))}.$$

Thus by taking $k = k_j \nearrow \infty$ and then $r_1 \nearrow R$, we obtain the Lipschitz estimate (2.1.13) around $x = x_0$. $\square$

In the rest of the section, we derive an interior modulus of continuity estimate of Du under the conditions of Theorem 2.1.3. Recall that we fixed an $\varepsilon \in (0, 1/4)$ sufficiently small such that

$$C\varepsilon^{\alpha-\alpha_1} < 1 \quad \text{and} \quad \varepsilon^{\alpha_1} < 1/4,$$

where C is the constant in (2.3.13), $\alpha \in (0,1)$ is the same constant as in Theorem 2.2.1 and $\alpha_1 \in (0, \alpha)$. We also took a ball $B_R(x_0) \subset \Omega$ with $R \in (0, 1]$. Similar to (2.4.4), we define

$$\begin{aligned}
&\tilde{\omega}(t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(\omega(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega(R/2)[\varepsilon^{-i}t > R/2] \right), \\
&\tilde{\mathbf{I}}_1^{\rho}(|\mu|)(x) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2] \right), \\
&\tilde{\mathbf{W}}_{1/p,p}^{\rho}(|\mu|)(x) \\
&\quad = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(\mathbf{W}_{1/p,p}^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{W}_{1/p,p}^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2] \right).
\end{aligned} \tag{2.4.20}$$

Here we also used the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$ otherwise, and $\mathbf{I}_1$ and $\mathbf{W}_{1/p,p}$ are the Riesz and Wolff potentials defined in (2.1.9) and (2.1.11), respectively. We note that since $1/(p-1) > 1$, we have

$$\mathbf{W}_{1/p,p}^{\rho}(|\mu|)(x) \le C\big(\mathbf{I}_1^{2\rho}(|\mu|)(x)\big)^{\frac{1}{p-1}}, \tag{2.4.21}$$

so that $\tilde{\mathbf{W}}_{1/p,p}^{\rho}(|\mu|)(x)$ and $\tilde{\mathbf{I}}_1^{\rho}(|\mu|)(x)$ are bounded and converge to zero as $\rho \to 0$ as long as $\mathbf{I}_1^{R}(|\mu|)(x)$ is finite.

Our interior modulus of continuity estimate is stated as follows.

Theorem 2.4.3. *Assume the conditions of Theorem 2.1.3 and $\alpha_1 \in (0,\alpha)$, where α is the constant in Theorem 2.2.1. Then there exist a constant $C = C(n,p,\lambda,\alpha_1,\omega)$, such that for any $R \in (0,1]$, $B_R(x_0) \subset \Omega$, $x, y \in B_{R/4}(x_0)$ being Lebesgue points of the vector-valued function Du, it holds that*

$$\begin{aligned} &|Du(x) - Du(y)| \\ &\le C\,\mathbf{M}\Big[\Big(\frac{\rho}{R}\Big)^{\alpha_1} + \int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt\Big] + C\,\big\|\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)\big\|_{L^\infty(B_{R/4}(x_0))} \\ &\quad + C\,\mathbf{M}^{2-p}\big\|\tilde{\mathbf{I}}^{\rho}_1(|\mu|)\big\|_{L^\infty(B_{R/4}(x_0))} \end{aligned} \tag{2.4.22}$$

where $\rho = |x-y|$, $\tilde{\omega}$, $\tilde{\mathbf{W}}_{1/p,p}$, and $\tilde{\mathbf{I}}_1$ are defined in (2.4.20), and

$$\mathbf{M} := R^{-\frac{n}{2-p}}\||Du| + s\|_{L^{2-p}(B_R(x_0))} + \big\|\mathbf{I}^R_1(|\mu|)\big\|^{\frac{1}{p-1}}_{L^\infty(B_R(x_0))}.$$

Note that due to (2.4.21), the term $\big\|\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)\big\|_{L^\infty(B_{R/4}(x_0))}$ in (2.4.22) can be replaced with the sup norm of a summation of the truncated Riesz potentials similar to (2.4.20).

Proof of Theorem 2.4.3. For any $x, y \in B_{R/4}(x_0)$ being Lebesgue points of Du, by the triangle inequality, we have

$$\begin{aligned} &|Du(x) - Du(y)|^{\gamma_0} \\ &\le |Du(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |\mathbf{q}_{x,\rho} - \mathbf{q}_{y,\rho}|^{\gamma_0} + |Du(y) - \mathbf{q}_{y,\rho}|^{\gamma_0} \\ &\le 2\sup_{y_0 \in B_{R/4}(x_0)} |Du(y_0) - \mathbf{q}_{y_0,\rho}|^{\gamma_0} + |Du(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |Du(z) - \mathbf{q}_{y,\rho}|^{\gamma_0}. \end{aligned}$$

We set $\rho = |x-y|$, take the average over $z \in B(x,\rho) \cap B(y,\rho)$, and then take the

γ_0-th root to get

$$
\begin{aligned}
|Du(x) - Du(y)| &\le C \sup_{y_0 \in B_{R/4}(x_0)} |Du(y_0) - \mathbf{q}_{y_0,\rho}| + C\phi(x,\rho) + C\phi(y,\rho) \\
&\le C \sup_{y_0 \in B_{R/4}(x_0)} \sum_{j=0}^{\infty} \phi(y_0, \varepsilon^j \rho) + C \sup_{y_0 \in B_{R/4}(x_0)} \phi(y_0, \rho) \\
&\le C \sup_{y_0 \in B_{R/4}(x_0)} \sum_{j=0}^{\infty} \phi(y_0, \varepsilon^j \rho). \qquad (2.4.23)
\end{aligned}
$$

Here we used (2.4.7) in the second inequality.

If $\rho < R/8$, by using (2.4.23), (2.4.6) with $R/8$ in place of r, and the fact that

$$
B_{R/4}(y_0) \subset B_{R/2}(x_0) \quad \forall y_0 \in B_{R/4}(x_0),
$$

we obtain

$$
\begin{aligned}
&|Du(x) - Du(y)| \\
&\le C \left(\frac{\rho}{R}\right)^{\alpha_1} \|Du\|_{L^\infty(B_{R/2}(x_0))} + C \sup_{y_0 \in B_{R/4}(x_0)} \int_0^\rho \frac{\tilde{h}(y_0,t)}{t}\,dt \\
&\quad + C \Big(\|Du\|_{L^\infty(B_{R/2}(x_0))} + s\Big)^{2-p} \sup_{y_0 \in B_{R/4}(x_0)} \int_0^\rho \frac{\tilde{g}(y_0,t)}{t}\,dt \\
&\quad + C \Big(\|Du\|_{L^\infty(B_{R/2}(x_0))} + s\Big) \int_0^\rho \frac{\tilde{\omega}(t)}{t}\,dt.
\end{aligned} \qquad (2.4.24)
$$

Clearly, (2.4.24) still holds when $\rho \ge R/8$.

We can simplify the terms in (2.4.24) as follows. For any $y_0 \in B_{R/4}(x_0)$ and $\rho \in (0, R/2)$, by the definition of $\tilde{g}$ in (2.4.4), we have

$$
\begin{aligned}
\int_0^\rho \frac{\tilde{g}(y_0,t)}{t}\,dt = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^\rho \frac{g(y_0, \varepsilon^{-i} t)}{t} [\varepsilon^{-i} t \le R/2]\,dt \\
+ \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \int_0^\rho \frac{g(y_0, R/2)}{t} [\varepsilon^{-i} t > R/2]\,dt.
\end{aligned}
$$

Recalling the definition of $\tilde{\mathbf{I}}_1$ from (2.4.20), the first term above is equal to

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\Big(\int_0^{\rho} \frac{g(y_0,\varepsilon^{-i}t)}{t}\,dt[\varepsilon^{-i}\rho \le R/2] + \int_0^{\varepsilon^i R/2} \frac{g(y_0,\varepsilon^{-i}t)}{t}\,dt[\varepsilon^{-i}\rho > R/2]\Big) \\
&= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\Big(\int_0^{\varepsilon^{-i}\rho} \frac{g(y_0,t)}{t}\,dt[\varepsilon^{-i}\rho \le R/2] + \int_0^{R/2} \frac{g(y_0,t)}{t}\,dt[\varepsilon^{-i}\rho > R/2]\Big) \\
&= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\Big(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(y_0)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(y_0)[\varepsilon^{-i}\rho > R/2]\Big) \\
&= \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0).
\end{aligned}$$

The second term is equal to

$$\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}[\varepsilon^{-i}\rho > R/2]\ln(2\varepsilon^{-i}\rho/R)g(y_0,R/2).$$

Let K be the positive integer such that $\varepsilon^{-K}\rho > R/2$ and $\varepsilon^{-(K-1)}\rho \le R/2$. Then we have

$$\begin{aligned}
&\sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}[\varepsilon^{-i}\rho > R/2]\ln(2\varepsilon^{-i}\rho/R) = \sum_{i=K}^{\infty} \varepsilon^{\alpha_1 i}\ln(2\varepsilon^{-i}\rho/R) \\
&= \varepsilon^{\alpha_1 K}\sum_{i=K}^{\infty} \varepsilon^{\alpha_1(i-K)}\Big(\ln(2\varepsilon^{-K}\rho/R) + (i-K)\ln(\varepsilon^{-1})\Big) \\
&\le \Big(\frac{2\rho}{R}\Big)^{\alpha_1}\sum_{i=K}^{\infty} \varepsilon^{\alpha_1(i-K)}(i-K+1)\ln(\varepsilon^{-1}) \le C\Big(\frac{\rho}{R}\Big)^{\alpha_1}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
\int_0^{\rho} \frac{\tilde{g}(y_0,t)}{t}\,dt \le& \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1}g(y_0,R/2) \\
\le& \tilde{\mathbf{I}}_1^{\rho}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1}\mathbf{I}_1^{R}(|\mu|)(y_0).
\end{aligned} \tag{2.4.25}$$

We can similarly get the following estimate

$$\begin{aligned}\int_0^\rho \frac{\tilde{h}(y_0,t)}{t}\,dt &\le \sum_{i=1}^\infty \left(\varepsilon^{\alpha_1 i}\int_0^{\varepsilon^{-i}\rho} h(y_0,t)\,\frac{dt}{t}[\varepsilon^{-i}\rho \le R/2]\right.\\ &\quad\left.+\int_0^{R/2} h(y_0,t)\,\frac{dt}{t}[\varepsilon^{-i}\rho > R/2]\right) + C\left(\frac{\rho}{R}\right)^{\alpha_1} h(y_0,R/2)\\ &\le \tilde{\mathbf{W}}^\rho_{1/p,p}(|\mu|)(y_0) + C(\rho/R)^{\alpha_1}\left(\mathbf{I}^R_1(|\mu|)(y_0)\right)^{1/(p-1)}.\end{aligned} \tag{2.4.26}$$

Using (2.1.13), (2.4.25), and (2.4.26), from (2.4.24) we obtain (2.4.22). The theorem is proved. □

Proof of Theorem 2.1.4. Since the set of Lebesgue points of Du is dense in Ω, it suffices to show the right-hand side of (2.4.22) converges to zero when $\rho \to 0$. In fact, we have

$$\begin{aligned}\|\tilde{\mathbf{I}}^\rho_1(|\mu|)\|_{L^\infty(B_{R/4}(x_0))} &\le \sum_{i=1}^\infty \varepsilon^{\alpha_1 i}\left(\|\mathbf{I}^{\varepsilon^{-i}\rho}_1(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}[\varepsilon^{-i}\rho \le R/2]\right.\\ &\quad\left.+\|\mathbf{I}^{R/2}_1(|\mu|)\|_{L^\infty(B_{R/4}(x_0))}[\varepsilon^{-i}\rho > R/2]\right),\end{aligned}$$

which must converge to 0 by using (2.1.14) and the dominated convergence theorem. We can similarly prove the convergence of other terms in (2.4.22). □

Proof of Corollary 2.1.5. By [37, Lemma 3] with $p = 2$ and $k = 1$, the assumption (2.1.15) implies (2.1.14). Therefore Corollary 2.1.5 follows from Theorem 2.1.4. □

Proof of Corollary 2.1.6. This is an immediate corollary of Theorem 2.1.4 since the assumption (2.1.14) is verified by (2.1.16) and (2.1.17). □

Proof of Corollary 2.1.7. We choose $\alpha_1 \in (\beta, \alpha)$ in Theorem 2.4.3. Then Corollary 2.1.7 follows by a direct computation using (2.4.22). □

2.5 Global gradient estimates for the p-Laplacian equations

This section is devoted to the proof of the global pointwise gradient estimate in Theorem 2.1.10, the Lipschitz estimate in Theorem 2.1.11, Corollary 2.1.12, as well as the derivation of a global modulus of continuity estimate of Du stated in Theorem 2.5.10 for the following (possibly nondegenerate) p-Laplace equation with Dirichlet boundary condition:

$$\begin{cases} -\operatorname{div}\Big(a(x)(|Du|^2+s^2)^{\frac{p-2}{2}}Du\Big)=\mu & \text{in } \Omega, \\ \qquad\qquad\qquad\qquad\qquad u=0 & \text{on } \partial\Omega, \end{cases} \tag{2.5.1}$$

where $a(\cdot)$ satisfies (2.1.5), (2.1.7), and (2.1.8), and Ω has a $C^{1,\mathrm{Dini}}$ boundary characterized by R_0 and ω_0 as in Definition 2.1.9.

First, we derive a gradient estimate around any point $x_0 \in \partial\Omega$. Without loss of generality, we assume that $x_0 = 0 \in \partial\Omega$. Then we can choose a local coordinate around $x_0 = 0$ and a function χ as in Definition 2.1.9 such that $\chi(0') = 0$. Let

$$\Gamma(y) = (y_1 + \chi(y'), y') \quad \text{and} \quad \Lambda(x) = \Gamma^{-1}(x) = (x_1 - \chi(x'), x').$$

Note that the determinants of the Jacobian of $\Gamma(\cdot)$ and $\Lambda(\cdot)$ are equal to 1. Since Ω has $C^{1,\mathrm{Dini}}$ boundary, from the proof of [19, Lemma 2.2], there exists $R_1 = R_1(\omega_0, R_0) \in (0, R_0)$ such that

$$|D_{x'}\chi(x')| \le 1/2 \quad \text{if} \quad |x'| \le R_1, \tag{2.5.2}$$

$$\Omega_{r/2} \subset \Gamma(B_r^+) \subset \Omega_{2r} \quad \forall\, r \in (0, R_1/2]. \tag{2.5.3}$$

Therefore, there exist constants $c_1(n)$ and $c_2(n)$ depending only on n, such that for any $x \in \bar{\Omega}$ and $0 < r \leq R_1$,

$$c_1(n)r^n \leq |\Omega_r(x)| \leq c_2(n)r^n. \tag{2.5.4}$$

Now we use the technique of flattening the boundary. We denote $u_1(y) = u(\Gamma(y))$, $a_1(y) = a(\Gamma(y))$, and $\mu_1(A) = \mu(\Gamma(A))$ for any Borel set $A \subset \mathbb{R}^n$. Then u_1 satisfies

$$\begin{cases} \operatorname{div}_y \left(a_1(y)(|D\Lambda\, D_y u_1|^2 + s^2)^{\frac{p-2}{2}} (D\Lambda)^T D\Lambda\, D_y u_1 \right) = \mu_1 & \text{in } B^+_{R_1/2}, \\ u_1 = 0 & \text{on } B_{R_1/2} \cap \partial\mathbb{R}^n_+. \end{cases} \tag{2.5.5}$$

We set

$$A_1(y,\xi) = a_1(y)(|D\Lambda\, \xi|^2 + s^2)^{\frac{p-2}{2}} (D\Lambda)^T D\Lambda\, \xi.$$

By direct computations with (2.5.2) in hand, A_1 satisfies the following conditions with $\omega_1 = \omega + \omega_0$ and some constant $\lambda_1 = \lambda_1(n,p,\lambda)$:

$$|A_1(y,\xi)| \leq \lambda_1(s^2 + |\xi|^2)^{(p-1)/2}, \quad |D_\xi A_1(y,\xi)| \leq \lambda_1(s^2 + |\xi|^2)^{(p-2)/2}, \tag{2.5.6}$$

$$\langle D_\xi A_1(y,\xi)\eta, \eta\rangle \geq \lambda_1^{-1}(s^2 + |\xi|^2)^{(p-2)/2}|\eta|^2, \tag{2.5.7}$$

$$|A_1(y,\xi) - A_1(y_0,\xi)| \leq \lambda_1\omega_1(|y - y_0|)(s^2 + |\xi|^2)^{(p-1)/2} \tag{2.5.8}$$

for every $y, y_0 \in B^+_{R_1/2}$ and every $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$.

Suppose that $4r \leq R_1$. We now consider the unique solution $w \in u_1 + W^{1,p}_0(B^+_{2r})$ to the equation

$$\begin{cases} -\operatorname{div}_y(A_1(y, D_y w)) = 0 & \text{in } B^+_{2r}, \\ w = u_1 & \text{on } \partial B^+_{2r}. \end{cases} \tag{2.5.9}$$

We first derive a boundary version of the reverse Hölder's inequality.

Lemma 2.5.1. *Let w be a solution to (2.5.9). There exists a constant $\theta_1 > p$ depending only on n, p, and λ, such that for any $t > 0$, the estimate*

$$\left(\fint_{B^+_{\rho/2}(y_0)} (|D_y w| + s)^{\theta_1}\, dx\right)^{1/\theta_1} \leq C \left(\fint_{B^+_{\rho}(y_0)} (|D_y w| + s)^{t}\, dx\right)^{1/t} \tag{2.5.10}$$

holds for all $B^+_\rho(y_0) \subset B^+_{2r}$, where $C = C(n, p, \lambda, t) > 0$.

Proof. For simplicity, we still denote $D = D_y$ through this proof. First we prove a Caccioppoli type inequality in half balls. Suppose that $y_0 \in B_{2r} \cap \partial\mathbb{R}^n_+$ and $B_{2\rho}(y_0) \subset\subset B_{2r}$. Let ζ be a nonnegative smooth function satisfying $\zeta = 1$ in $B_\rho(y_0)$, $|D\zeta| \leq 2\rho^{-1}$, and $\zeta = 0$ outside $B_{2\rho}(y_0)$. Using $\zeta^p w$ as a test function in (2.5.9), we get the following

$$0 = \int_{B^+_{2r}} \langle A_1(y, Dw), \zeta^p Dw\rangle\, dy + p\int_{B^+_{2r}} \langle A_1(y, Dw), \zeta^{p-1} w D\zeta\rangle\, dy := \mathrm{I} + \mathrm{II}. \tag{2.5.11}$$

Using the fundamental theorem of calculus, (2.5.6), (2.5.7), and Young's inequality with exponents p and $p/(p-1)$, we have

$$\begin{aligned}
\mathrm{I} &= \int_{B^+_{2r}} \zeta^p \Big(\int_0^1 \langle D_\xi A_1(y, tDw)Dw, Dw\rangle\; dt + A_1(y, 0)Dw\Big) dy \\
&\geq \int_{B^+_{2r}} \zeta^p \Big(\int_0^1 \lambda_1^{-1}(s^2 + |Dw|^2 t^2)^{\frac{p-2}{2}} |Dw|^2\, dt - \lambda_1 s^{p-1}|Dw|\Big) dy \\
&\geq c(\lambda_1, p)\int_{B^+_{2r}} \zeta^p |Dw|^p\, dy - C'(\lambda_1, p)\int_{B^+_{2r}} \zeta^p s^p\, dy.
\end{aligned}$$

On the other hand, using (2.5.6) and Young's inequality, we have

$$\begin{aligned}|\mathrm{II}| &\leq p\lambda_1 \int_{B^+_{2\rho}(y_0)} (s^2+|Dw|^2)^{\frac{p-1}{2}} \zeta^{p-1} |wD\zeta|\, dy \\ &\leq \frac{1}{2} c(\lambda_1, p) \int_{B^+_{2r}} \zeta^p (s^2+|Dw|^2)^{\frac{p}{2}}\, dy + C''(\lambda_1, p) \int_{B^+_{2\rho}(y_0)} |wD\zeta|^p\, dy \\ &\leq \frac{1}{2} c(\lambda_1, p) \int_{B^+_{2r}} \zeta^p (s^p+|Dw|^p)\, dy + C''(\lambda_1, p) \int_{B^+_{2\rho}(y_0)} |wD\zeta|^p\, dy.\end{aligned}$$

Therefore, (2.5.11) implies the following Caccioppoli type inequality

$$\int_{B^+_{\rho}(y_0)} |Dw|^p\, dx \leq C\rho^n s^p + C\rho^{-p} \int_{B^+_{2\rho}(y_0)} |w|^p\, dx. \tag{2.5.12}$$

Since $w = u = 0$ on $B_{2r} \cap \partial\mathbb{R}^n_+$, by the Sobolev-Poincaré inequality,

$$\left(\int_{B^+_{2\rho}(y_0)} |w|^p\, dx \right)^{1/p} \leq C\rho^{1+\frac{n}{p}-\frac{n}{q}} \left(\int_{B^+_{2\rho}(y_0)} |Dw|^q\, dx \right)^{1/q} \tag{2.5.13}$$

for any q such that $\max\left\{1, \frac{np}{n+p}\right\} \leq q < p$. Thus by combining (2.5.12) and (2.5.13), we have

$$\left(\fint_{B^+_{\rho}(y_0)} (|Dw|+s)^p\, dx \right)^{1/p} \leq C \left(\fint_{B^+_{2\rho}(y_0)} (|Dw|+s)^q\, dx \right)^{1/q}.$$

Similarly, the following interior version of estimate

$$\left(\fint_{B_{\rho}(y_0)} (|Dw|+s)^p\, dx \right)^{1/p} \leq C \left(\fint_{B_{2\rho}(y_0)} (|Dw|+s)^q\, dx \right)^{1/q}$$

holds for all $B_{2\rho}(y_0) \subset\subset B^+_{2r}$. Therefore, by a standard covering argument and Gehring's lemma, we get (2.5.10). □

We also have a boundary comparison result analogous to Lemma 2.3.2 by following almost the same proof.

Lemma 2.5.2. *Let w be a solution to (2.5.9) and assume that $p \in (1,2)$. Then for any $\gamma_0 \in (0, 2-p]$ when $p \in \left(\frac{3n-2}{2n-1}, 2\right)$ and $\gamma_0 \in \left(0, \frac{(p-1)n}{n-1}\right)$ when $p \in \left(1, \frac{3n-2}{2n-1}\right]$, it holds that*

$$\begin{aligned}&\left(\fint_{B_{2r}^+} |D_y u_1 - D_y w|^{\gamma_0}\, dx\right)^{1/\gamma_0}\\&\le C\left[\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\right]^{\frac{1}{p-1}} + C\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\fint_{B_{2r}^+}(|D_y u_1| + s)^{2-p}\, dx,\end{aligned}$$

where C is a constant depending only on n, p, λ, and γ_0.

We now let $v \in w + W_0^{1,p}(B_r^+)$ be the unique solution to

$$\begin{cases} -\operatorname{div}_y(A_1(0, D_y v)) = 0 & \text{in } B_r^+,\\ v = w & \text{on } \partial B_r^+. \end{cases} \tag{2.5.14}$$

We also have an estimate for the difference $Dv - Dw$ analogous to (2.3.10) by following almost the same proof as that of [38, Eq. (4.35)]:

$$\fint_{B_r^+} |D_y v - D_y w|^p\, dx \le C\omega_0(r)^p \fint_{B_r^+}(|D_y w| + s)^p\, dx.$$

Thus by (2.5.10) and Hölder's inequality, we get

$$\fint_{B_r^+} |D_y v - D_y w|^{\gamma_0}\, dx \le C\omega_0(r)^{\gamma_0} \fint_{B_{2r}^+}(|D_y w| + s)^{\gamma_0}\, dx.$$

Next we prove an oscillation estimate for v.

Lemma 2.5.3. *Let v be a solution to (2.5.14). There exists constant $C > 1$ depending* *only on n, p, λ, and γ_0, such that for any half ball $B_\rho^+ \subset B_R^+ \subset B_r^+$, we have*

$$
\begin{aligned}
&\inf_{\theta\in\mathbb{R}} \left(\fint_{B_\rho^+} |D_{y_1} v - \theta|^{\gamma_0} + |D_{y'} v|^{\gamma_0} \right)^{1/\gamma_0} \\
&\le C \left(\frac{\rho}{R}\right)^{\alpha} \inf_{\theta\in\mathbb{R}} \left(\fint_{B_R^+} |D_{y_1} v - \theta|^{\gamma_0} + |D_{y'} v|^{\gamma_0} \right)^{1/\gamma_0},
\end{aligned}
\tag{2.5.15}
$$

where $\alpha \in (0,1)$ is the same constant as in Theorem 2.2.1.

Proof. Let $\bar{v}$ be an odd extension of v in B_r, namely,

$$
\bar{v}(y) = \begin{cases} v(y) & \text{if} \quad y_1 \ge 0, \\ -\,v(-y_1, y') & \text{if} \quad y_1 < 0. \end{cases}
$$

Then since

$$
A_1(0,\xi) = a_1(0)(|\xi|^2 + s^2)^{\frac{p-2}{2}} \xi,
$$

$\bar{v} \in W^{1,p}(B_r)$ is a solution to the equation

$$
-\mathrm{div}_y \left(a_1(0)(|D_y \bar{v}|^2 + s^2)^{\frac{p-2}{2}} D_y \bar{v} \right) = 0 \quad \text{in } B_r.
$$

Thus we can apply Theorem 2.2.1 to $\bar{v}$ to get

$$
\inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho} |D_y \bar{v} - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0} \le C \left(\frac{\rho}{R}\right)^{\alpha} \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_R} |D_y \bar{v} - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}.
$$

Since $\bar{v}$ is an odd function in y_1, by the triangle inequality, there exists $\theta_\rho \in \mathbb{R}$ such that

$$
\left(\fint_{B_\rho^+} |D_{y_1} v - \theta_\rho|^{\gamma_0} + |D_{y'} v|^{\gamma_0} \right)^{1/\gamma_0} \le C \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{B_\rho} |D_y \bar{v} - \mathbf{q}|^{\gamma_0} \right)^{1/\gamma_0}.
$$

By the triangle inequality again, it is easily seen that

$$\inf_{\mathbf{q}\in\mathbb{R}^n}\left(\fint_{B_R}|D_y\bar{v}-\mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}\le C\inf_{\theta\in\mathbb{R}}\left(\fint_{B_R^+}|D_{y_1}v-\theta|^{\gamma_0}+|D_{y'}v|^{\gamma_0}\right)^{1/\gamma_0}.$$

Then (2.5.15) is a direct consequence of the three inequalities above. ☐

Lemma 2.5.4. *Suppose that $u_1\in W^{1,p}(B_{R_1}^+)$ is a solution to (2.5.5). Then for any $\varepsilon\in(0,1)$ and $r\in(0,R_1/4]$, we have*

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B_{\varepsilon r}^+}|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0}\right)^{1/\gamma_0}\\
&\le C\varepsilon^{\alpha}\inf_{\theta\in\mathbb{R}}\left(\fint_{B_r^+}|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0}\right)^{1/\gamma_0}\\
&\quad+C_\varepsilon\left(\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\right)^{\frac{1}{p-1}}+C_\varepsilon\omega_1(r)\left(\fint_{B_{2r}^+}(|D_yu_1|+s)^{2-p}\right)^{\frac{1}{2-p}}\\
&\quad+C_\varepsilon\frac{|\mu_1|(B_{2r}^+)}{r^{n-1}}\fint_{B_{2r}^+}(|D_yu_1|+s)^{2-p},
\end{aligned}\tag{2.5.16}$$

where α and γ_0 are the same constants as in Proposition 2.3.3, C_ε is a constant depending on ε, n, p, λ, and γ_0, and C is a constant depending on n, p, λ, and γ_0.

Proof. By using Lemmas 2.5.1, 2.5.2, and 2.5.3, the proof is almost identical to that of Proposition 2.3.3, so we omit it. ☐

We now define

$$\psi(x_0,r)=\inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_r(x_0)}|D_1u-\theta|^{\gamma_0}+|D_{x'}u|^{\gamma_0}\right)^{1/\gamma_0}.$$

Let $\varepsilon\in(0,1)$ and $r\in(0,R_1/4]$. By using change of variables, (2.5.3), (2.5.4), and the triangle inequality, we have

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B^+_{\varepsilon r}}|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0}\right)^{1/\gamma_0}\\
&=\inf_{\theta\in\mathbb{R}}\left(\fint_{\Gamma(B^+_{\varepsilon r})}|D_1u-\theta|^{\gamma_0}+|D_1u\;D_{x'}\chi+D_{x'}u|^{\gamma_0}\right)^{1/\gamma_0}\\
&\geq C\inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_{\varepsilon r/2}}|D_1u-\theta|^{\gamma_0}+|D_{x'}u|^{\gamma_0}\right)^{1/\gamma_0}-C'\left(\fint_{\Omega_{\varepsilon r/2}}|D_1u\;D_{x'}\chi|^{\gamma_0}\right)^{1/\gamma_0}\\
&\geq C\psi(0,\varepsilon r/2)-C'\omega_1(\varepsilon r/2)\left(\fint_{\Omega_{\varepsilon r/2}}|Du|^{\gamma_0}\right)^{1/\gamma_0},
\end{aligned}\tag{2.5.17}$$

where C and C' are positive constants depending only on n and γ_0. Similarly,

$$\begin{aligned}
&\inf_{\theta\in\mathbb{R}}\left(\fint_{B^+_{r}}|D_{y_1}u_1-\theta|^{\gamma_0}+|D_{y'}u_1|^{\gamma_0}\right)^{1/\gamma_0}\\
&=\inf_{\theta\in\mathbb{R}}\left(\fint_{\Gamma(B^+_{r})}|D_1u-\theta|^{\gamma_0}+|D_1u\;D_{x'}\chi+D_{x'}u|^{\gamma_0}\right)^{1/\gamma_0}\\
&\leq C''\inf_{\theta\in\mathbb{R}}\left(\fint_{\Omega_{2r}}|D_1u-\theta|^{\gamma_0}+|D_{x'}u|^{\gamma_0}\right)^{1/\gamma_0}+C''\left(\fint_{\Omega_{2r}}|D_1u\;D_{x'}\chi|^{\gamma_0}\right)^{1/\gamma_0}\\
&\leq C''\psi(0,2r)+C''\omega_1(2r)\left(\fint_{\Omega_{2r}}|Du|^{\gamma_0}\right)^{1/\gamma_0},
\end{aligned}\tag{2.5.18}$$

where C'' is a positive constant depending only on n and γ_0. Therefore, by using (2.5.17), (2.5.18), (2.5.2), and (2.5.4), (2.5.16) implies that

$$\begin{aligned}
\psi(0,\varepsilon r/2)&\leq C\varepsilon^\alpha\psi(0,2r)+C_\varepsilon\left(\frac{|\mu|(\Omega_{4r})}{r^{n-1}}\right)^{\frac{1}{p-1}}\\
&\quad+C_\varepsilon\omega_1(2r)\left(\fint_{\Omega_{4r}}(|Du|+s)^{2-p}\right)^{\frac{1}{2-p}}+C_\varepsilon\frac{|\mu|(\Omega_{4r})}{r^{n-1}}\fint_{\Omega_{4r}}(|Du|+s)^{2-p}.
\end{aligned}$$

By replacing $\varepsilon/4$ and $2r$ with ε and r respectively, we obtain

Corollary 2.5.5. *Suppose that $u\in W^{1,p}_0(\Omega)$ is a solution to (2.5.1) and $x_0\in\partial\Omega$.*

Then for $\varepsilon \in (0,1/4)$, $r \le R_1/2$, *and* α, C, C_ε *as above, we have*

$$\psi(x_0,\varepsilon r) \le C\varepsilon^\alpha \psi(x_0,r) + C_\varepsilon \left(\frac{|\mu|(\Omega_{2r}(x_0))}{r^{n-1}}\right)^{\frac{1}{p-1}} + C_\varepsilon \omega_1(r) \left(\fint_{\Omega_{2r}(x_0)} (|Du|+s)^{2-p}\right)^{\frac{1}{2-p}} + C_\varepsilon \frac{|\mu|(\Omega_{2r}(x_0))}{r^{n-1}} \fint_{\Omega_{2r}(x_0)} (|Du|+s)^{2-p}. \tag{2.5.19}$$

As in Section 2.3, for any $x \in \bar{\Omega}$, we define

$$\phi(x,\rho) = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{\Omega_\rho(x)} |Du-\mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}$$

and choose $\mathbf{q}_{x,r} \in \mathbb{R}^n$ such that

$$\left(\fint_{\Omega_r(x)} |Du-\mathbf{q}_{x,r}|^{\gamma_0}\right)^{1/\gamma_0} = \inf_{\mathbf{q}\in\mathbb{R}^n} \left(\fint_{\Omega_r(x)} |Du-\mathbf{q}|^{\gamma_0}\right)^{1/\gamma_0}. \tag{2.5.20}$$

We remark that our definition of ϕ is invariant under any orthogonal change of coordinates as in Definition 2.1.9 and that (2.3.12) still holds for any Lebesgue point $x \in \Omega$ of the vector-valued function Du from the same argument as in Section 2.3. Moreover, if we assume $u \in C^1(\bar{\Omega})$, then (2.3.12) actually holds for any $x \in \bar{\Omega}$.

2.5.1 Global pointwise gradient estimates

To prove the pointwise gradient estimate for $p \in \left(\frac{3n-2}{2n-1}, 2\right)$, we choose $\gamma_0 = 2-p$ and $\varepsilon = \varepsilon(n,p,\lambda,\alpha) \in (0,1/4)$ sufficiently small such that $C\varepsilon^\alpha \le 1/4$ for both constants C in (2.3.13) and (2.5.19). Fix $x_0 \in \partial\Omega$ and $R \le R_1/2$. For $j \ge 0$, set $r_j = \varepsilon^j R$, $\Omega_j = \Omega_{2r_j}(x_0)$,

$$T_j = \left(\fint_{\Omega_j} (|Du|+s)^{2-p}\,dx\right)^{\frac{1}{2-p}}, \quad \phi_j = \phi(x_0,r_j), \quad \text{and } \psi_j = \psi(x_0,r_j).$$

Applying (2.5.19) yields

$$\psi_{j+1} \le \frac{1}{4}\psi_j + C\left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} + C\frac{|\mu|(\Omega_j)}{r_j^{n-1}}T_j^{2-p} + C\omega_1(r_j)T_j.$$

Let j_0 and m be positive integers such that $j_0 \le m$. Summing the above inequality over $j \in \{j_0, j_0+1, \dots, m\}$ and noting that $\phi_j \le \psi_j \le CT_j$, we obtain that

$$\begin{aligned}\sum_{j=j_0}^{m+1}\phi_j \le \sum_{j=j_0}^{m+1}\psi_j \le CT_{j_0} + C\sum_{j=j_0}^{m}\left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\ + C\sum_{j=j_0}^{m}\frac{|\mu|(\Omega_j)}{r_j^{n-1}}T_j^{2-p} + C\sum_{j=j_0}^{m}\omega_1(r_j)T_j\end{aligned} \tag{2.5.21}$$

for any $x_0 \in \partial\Omega$ and $R \le R_1/2$.

On the other hand, according to (2.3.17),

$$\begin{aligned}\sum_{j=j_0}^{m+1}\phi_j \le C\phi_{j_0} + C\sum_{j=j_0}^{m}\left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}}\right)^{\frac{1}{p-1}} \\ + C\sum_{j=j_0}^{m}\frac{|\mu|(\Omega_j)}{r_j^{n-1}}T_j^{2-p} + C\sum_{j=j_0}^{m}\omega(r_j)T_j\end{aligned} \tag{2.5.22}$$

holds for any $x_0 \in \Omega$ and $R > 0$ such that $r_{j_0} = \varepsilon^{j_0}R < \operatorname{dist}(x_0, \partial\Omega)/2$.

We now define

$$\Omega_j' = \Omega_{8r_j}(x_0), \quad Z_j = \left(\fint_{\Omega_j'}(|Du|+s)^{2-p}\,dx\right)^{\frac{1}{2-p}}.$$

Then we can obtain the following lemma.

Lemma 2.5.6. *Suppose that $u \in W_0^{1,p}(\Omega)$ is a solution to (2.5.1), $x_0 \in \bar{\Omega}$, and*

$R \le R_1/6$. Then we have

$$\begin{aligned}\sum_{j=j_0}^{m+1} \phi_j \le C Z_{j_0} + C \sum_{j=j_0}^{m} \left(\frac{|\mu|(\Omega'_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\ + C \sum_{j=j_0}^{m} \frac{|\mu|(\Omega'_j)}{r_j^{n-1}} Z_j^{2-p} + C \sum_{j=j_0}^{m+1} \omega_1(r_j) Z_j,\end{aligned} \tag{2.5.23}$$

where C is a constant depending only on n, p, λ, and γ_0.

Proof. First when $x_0 \in \partial\Omega$, since $\Omega_j \subset \Omega'_j$, we have $T_j \le C Z_j$. Thus (2.5.21) directly implies (2.5.23). It remains to prove the lemma for $x_0 \in \Omega$. Since (2.5.22) holds when $r_{j_0} < \mathrm{dist}(x_0, \partial\Omega)/2$, we only need to show that (2.5.23) holds when $r_{j_0} \ge \mathrm{dist}(x_0, \partial\Omega)/2$. Now assume $r_{j_1} \ge \mathrm{dist}(x_0, \partial\Omega)/2$ and $r_{j_1+1} < \mathrm{dist}(x_0, \partial\Omega)/2$. By (2.5.22), we have

$$\begin{aligned}\sum_{j=j_1+1}^{m+1} \phi_j \le C \phi_{j_1+1} + C \sum_{j=j_1+1}^{m} \left(\frac{|\mu|(\Omega_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\ + C \sum_{j=j_1+1}^{m} \frac{|\mu|(\Omega_j)}{r_j^{n-1}} T_j^{2-p} + C \sum_{j=j_1+1}^{m} \omega(r_j) T_j.\end{aligned} \tag{2.5.24}$$

By (2.5.4), we also have

$$\phi_{j_1+1} \le \left(\fint_{\Omega_{r_{j_1+1}}(x_0)} |Du - \mathbf{q}_{x_0, r_{j_1}}|^{\gamma_0} \right)^{1/\gamma_0} \le C \phi_{j_1}. \tag{2.5.25}$$

Now for any $j \in \{j_0, j_0+1, \dots, j_1\}$, $r_j \ge \mathrm{dist}(x_0, \partial\Omega)/2$. Choose $y_0 \in \partial\Omega$ such that $d := \mathrm{dist}(x_0, \partial\Omega) = |y_0 - x_0|$, so that $\Omega_{r_j}(x_0) \subset \Omega_{3r_j}(y_0)$ and $\Omega_{6r_j}(y_0) \subset \Omega_{8r_j}(x_0)$.

Thus by using (2.5.21) at $y_0 \in \partial\Omega$, we have

$$
\begin{aligned}
&\sum_{j=j_0}^{j_1} \phi_j \le C \sum_{j=j_0}^{j_1} \phi(y_0, 3r_j) \\
&\le CY_{j_0} + C \sum_{j=j_0}^{j_1} \left(\frac{|\mu|(\Omega_{6r_j}(y_0))}{r_j^{n-1}} \right)^{\frac{1}{p-1}} \\
&\quad + C \sum_{j=j_0}^{j_1} \frac{|\mu|(\Omega_{6r_j}(y_0))}{r_j^{n-1}} Y_j^{2-p} + C \sum_{j=j_0}^{j_1+1} \omega_1(3r_j) Y_j \\
&\le CZ_{j_0} + C \sum_{j=j_0}^{j_1} \left(\frac{|\mu|(\Omega'_j)}{r_j^{n-1}} \right)^{\frac{1}{p-1}} + C \sum_{j=j_0}^{j_1} \frac{|\mu|(\Omega'_j)}{r_j^{n-1}} Z_j^{2-p} + C \sum_{j=j_0}^{j_1+1} \omega_1(r_j) Z_j,
\end{aligned}
\tag{2.5.26}
$$

where

$$
Y_j := \left(\fint_{\Omega_{6r_j}(y_0)} (|Du| + s)^{2-p} \, dx \right)^{\frac{1}{2-p}}.
$$

Recall that $\omega_1 = \omega + \omega_0$. Combining (2.5.24), (2.5.25), and (2.5.26), we obtain (2.5.23). The lemma is proved. □

Proof of Theorem 2.1.10. With (2.5.23) in place of (2.3.17), we can easily get the global pointwise gradient estimate (2.1.18) using the same ideas as in the proof of Theorem 2.1.1. □

2.5.2 Global Lipschitz estimates and modulus of continuity estimates of the gradient

Let $x_0 \in \bar{\Omega}$ and $0 < R \le R_1$. For any fixed $\alpha_1 \in (0, \alpha)$, let $\alpha_2 = (\alpha_1 + \alpha)/2$, and choose $\varepsilon = \varepsilon(n, p, \lambda, \gamma_0, \alpha, \alpha_1) \in (0, 1/4)$ sufficiently small such that $\varepsilon^{\alpha_2} < 1/4$ and $C\varepsilon^{\alpha - \alpha_2} < 1$ for both constants C in (2.3.13) and (2.5.19). Next we define

$$
g_1(x, r) = \frac{|\mu|(B_r(x) \cap B_{R/2}(x_0))}{r^{n-1}}, \quad h_1(x, r) = g_1(x, r)^{\frac{1}{p-1}},
$$

$$
\omega_1^*(r) := \omega_1(r)[r \le R/2] + \omega_1(R/2)[r > R/2],
$$

and

$$
\begin{aligned}
\hat{g}_1(x,t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} g_1(x, \varepsilon^{-i}t), \quad \breve{g}_1(x,t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} g_1(x, \varepsilon^{-i}t), \\
\hat{h}_1(x,t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} h_1(x, \varepsilon^{-i}t), \quad \breve{h}_1(x,t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} h_1(x, \varepsilon^{-i}t), \\
\hat{\omega}_1(t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_2 i} \omega_1^*(\varepsilon^{-i}t), \quad \breve{\omega}_1(t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \omega_1^*(\varepsilon^{-i}t),
\end{aligned}
$$

Indeed, we have

$$
\begin{aligned}
\breve{\omega}_1(t) &= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(\omega_1(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega_1(R/2)[\varepsilon^{-i}t > R/2]\right) := \tilde{\omega}_1(t), \\
\breve{g}_1(x,t) &\le \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(g(x, \varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + g(x_0, R/2)[\varepsilon^{-i}t > R/2]\right), \\
\breve{h}_1(x,t) &\le \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i} \left(h(x, \varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + h(x_0, R/2)[\varepsilon^{-i}t > R/2]\right),
\end{aligned} \tag{2.5.27}
$$

where the functions g and h are defined in (2.4.2).

Using the same iteration technique as in Lemma 2.4.1, we can obtain from (2.5.19) that

$$
\begin{aligned}
\psi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_2} \psi(x,r) &+ C\hat{h}_1(x,2\rho) \\
&+ C\hat{g}_1(x,2\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)^{2-p} \\
&+ C\hat{\omega}_1(\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)
\end{aligned} \tag{2.5.28}
$$

holds for any $x \in \partial\Omega$, $B_{2r}(x) \subset B_{R/2}(x_0)$, and $0 < \rho \le r$.

Similarly, from (2.4.1) and the fact that $\omega \le \omega_1$, we have

$$
\begin{aligned}
\phi(x,\rho) \le C\left(\frac{\rho}{r}\right)^{\alpha_2} \phi(x,r) &+ C\hat{h}_1(x,2\rho) \\
&+ C\hat{g}_1(x,2\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)^{2-p} \\
&+ C\hat{\omega}_1(\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)
\end{aligned} \tag{2.5.29}
$$

for any $B_{2r}(x) \subset\subset \Omega$, $B_{2r}(x) \subset B_{R/2}(x_0)$, and $0 < \rho \leq r$.

By combining (2.5.28) and (2.5.29), we will show the following estimates.

Lemma 2.5.7. *Let $x \in \bar{\Omega}$ and $B_{2r}(x) \subset B_{R/2}(x_0)$. There exists a constant C depending only on ε, n, p, λ, γ_0, and α_1 such that, for any $0 < \rho \leq r \leq R_1$, we have*

(i)

$$\begin{aligned}\phi(x,\rho) \leq & C\left(\frac{\rho}{r}\right)^{\alpha_2} r^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(\Omega_r(x))} + C\breve{h}_1(x,\rho) \\ & + C\breve{g}_1(x,\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)^{2-p} \\ & + C\breve{\omega}_1(\rho)\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right),\end{aligned} \tag{2.5.30}$$

(ii)

$$\begin{aligned}\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho) \leq & C\left(\frac{\rho}{r}\right)^{\alpha_2} r^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(\Omega_r(x))} + C\int_0^\rho \frac{\breve{h}_1(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)^{2-p}\int_0^\rho \frac{\breve{g}_1(x,t)}{t}\,dt \\ & + C\left(\|Du\|_{L^\infty(\Omega_{2r}(x))} + s\right)\int_0^\rho \frac{\breve{\omega}_1(t)}{t}\,dt.\end{aligned} \tag{2.5.31}$$

To prove Lemma 2.5.7, we also need the following technical lemma.

Lemma 2.5.8. *Let $x, y \in \bar{\Omega}$ and $p \in (1,2)$. Then for $g_1, h_1, \hat{g}_1, \hat{h}_1, \breve{g}_1, \breve{h}_1, \hat{\omega}_1, \breve{\omega}_1$ defined as above, we have the following:*

(i) There exist constants $C_1, C_2 > 0$ depending on ε, n, p, α and α_1 such that for any fixed $x \in \bar{\Omega}$, and any $f \in \{\hat{g}_1(x,\cdot), \hat{h}_1(x,\cdot), \hat{\omega}_1, \breve{g}_1(x,\cdot), \breve{h}_1(x,\cdot), \breve{\omega}_1\}$, we have

$$C_1 f(t) \leq f(s) \leq C_2 f(t), \quad \textit{whenever} \quad 0 < \varepsilon t \leq s \leq t.$$

(ii) There exists a constant $C > 0$ depending on ε, n, p, α and α_1 such that for any

$0 < \varepsilon r \le \rho \le r$ with $\Omega_\rho(x) \subset \Omega_r(y)$, and any $F \in \{\hat{g}_1, \hat{h}_1, \breve{g}_1, \breve{h}_1\}$, we have

$$F(x,\rho) \le CF(y,r).$$

(iii) For any $0 < \rho \le r$, there exists a constant $C > 0$ depending on ε, n, p, α and α_1 such that the following hold

$$\begin{aligned}
\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{\omega}_1(r) &\le C\breve{\omega}_1(\rho), \\
\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{g}_1(x,r) &\le C\breve{g}_1(x,\rho), \\
\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{h}_1(x,r) &\le C\breve{h}_1(x,\rho).
\end{aligned}$$

Proof. We will only show the proof for g since the other cases are similar. Noting that

$$g_1(x,s) \le \varepsilon^{1-n} g_1(x,t), \quad g_1(x,\varepsilon t) \le \varepsilon^{1-n} g_1(x,s)$$

whenever $\varepsilon t \le s \le t$ and $\hat{g}_1(x,t) \le \varepsilon^{-\alpha_2}\hat{g}_1(x,\varepsilon t)$, assertion (i) follows. Assertion (ii) follows similarly by observing the fact that $\Omega_{\varepsilon^{-i}\rho}(x) \subset \Omega_{\varepsilon^{-i}r}(y)$ whenever $i \ge 0$ since $\Omega_\rho(x) \subset \Omega_r(y)$. It remains to prove assertion (iii). Since $0 < \rho \le r$, there exists an integer $j \ge 0$ such that $\varepsilon^{-j}\rho \le r < \varepsilon^{-j-1}\rho$. Therefore, by part (i),

$$\begin{aligned}
&\left(\frac{\rho}{r}\right)^{\alpha_2} \hat{g}_1(x,r) \le C\varepsilon^{\alpha_2 j}\hat{g}_1(x,\varepsilon^{-j}\rho) \\
&\le C\sum_{j=0}^{\infty}\sum_{i=1}^{\infty} \varepsilon^{\alpha_2(i+j)} g_1(x,\varepsilon^{-i-j}\rho) \\
&= C\sum_{k=1}^{\infty} k\varepsilon^{\alpha_2 k} g_1(x,\varepsilon^{-k}\rho) \le C\breve{g}_1(x,\rho),
\end{aligned}$$

where we used the fact that $k\varepsilon^{\alpha_2 k} \le C\varepsilon^{\alpha_1 k}$ in the last inequality, since $\alpha_1 < \alpha_2$. $\square$

Now we are ready to prove Lemma 2.5.7.

Proof of Lemma 2.5.7. Without loss of generality, we may assume $x = 0$. Note that if $r/16 \le \rho \le r$, then (2.5.30) follows from the definition of ϕ. Hence we only need to consider the case when $0 < \rho < r/16$. We consider the following three cases:

$$r/4 \le \operatorname{dist}(0, \partial\Omega), \quad \operatorname{dist}(0, \partial\Omega) \le 4\rho, \quad 4\rho < \operatorname{dist}(0, \partial\Omega) < r/4.$$

Case 1: $r/4 \le$ dist$(0, \partial\Omega)$. Set $r_1 = r/16$. Since $B_{4r_1} \subset \Omega$, from (2.5.29) we have

$$\begin{aligned}
\phi(0,\rho) &\le C\left(\frac{\rho}{r_1}\right)^{\alpha_2}\phi(0,r_1) + C\hat{h}_1(0,2\rho) \\
&\qquad + C\hat{g}_1(0,2\rho)\left(\|Du\|_{L^\infty(\Omega_{2r_1})} + s\right)^{2-p} + C\hat{\omega}_1(\rho)\left(\|Du\|_{L^\infty(\Omega_{2r_1})} + s\right).
\end{aligned}$$

Thus we can easily get (2.5.30) from Lemma 2.5.8 and the fact that

$$\phi(0, r_1) \le C r^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(\Omega_r)}.$$

Case 2: dist$(0, \partial\Omega) \le 4\rho$. Choose $y_0 \in \partial\Omega$ such that $\operatorname{dist}(0, \partial\Omega) = |y_0|$. Then $B_{10\rho}(y_0) \subset B_{14\rho} \subset B_r$ and from (2.5.28), we have

$$\begin{aligned}
&\phi(0,\rho) \le C\psi(y_0, 5\rho) \\
&\le C\left(\frac{\rho}{r}\right)^{\alpha_2}\psi(y_0, r/2) + C\hat{h}_1(y_0, 10\rho) \\
&\qquad + C\hat{g}_1(y_0,10\rho)\left(\|Du\|_{L^\infty(\Omega_r(y_0))} + s\right)^{2-p} + C\hat{\omega}_1(5\rho)\left(\|Du\|_{L^\infty(\Omega_r(y_0))} + s\right).
\end{aligned}$$

Thus from the fact that $\Omega_r(y_0) \subset \Omega_{2r}$, $\Omega_{r/2}(y_0) \subset \Omega_r$, and $\Omega_{10\rho}(y_0) \subset \Omega_{14\rho}$, we get (2.5.30) by using Lemma 2.5.8.

Case 3: $4\rho <$ dist$(0, \partial\Omega) < r/4$. Set $r_1 = \operatorname{dist}(0, \partial\Omega)/4 > \rho$. Using (2.5.29), we obtain

$$
\begin{aligned}
\phi(0,\rho) \le C\left(\frac{\rho}{r_1}\right)^{\alpha_2}\phi(0,r_1) + C\hat{h}_1(0,2\rho) \\
+ C\hat{g}_1(0,2\rho)\left(\|Du\|_{L^\infty(\Omega_{2r_1})}+s\right)^{2-p} + C\hat{\omega}_1(\rho)\left(\|Du\|_{L^\infty(\Omega_{2r_1})}+s\right).
\end{aligned}
$$

On the other hand, choose $y_0 \in \partial\Omega$ such that $\operatorname{dist}(0,\partial\Omega) = |y_0|$. Therefore, $B_{r_1} \subset B_{5r_1}(y_0)$, $B_r(y_0) \subset B_{2r}$ and from (2.5.28) we have

$$
\begin{aligned}
&\phi(0,r_1) \le C\psi(y_0,5r_1) \\
&\le C\left(\frac{r_1}{r}\right)^{\alpha_2}\psi(y_0,r/2) + C\hat{h}_1(y_0,10r_1) \\
&\quad + C\hat{g}_1(y_0,10r_1)\left(\|Du\|_{L^\infty(\Omega_r(y_0))}+s\right)^{2-p} + C\hat{\omega}_1(5r_1)\left(\|Du\|_{L^\infty(\Omega_r(y_0))}+s\right).
\end{aligned}
$$

Noting that $\Omega_r(y_0) \subset \Omega_{2r}$, $\Omega_{r/2}(y_0) \subset \Omega_r$, and $\Omega_{10r_1}(y_0) \subset \Omega_{14r_1}$, we get (2.5.30) by combining the last two estimates and applying Lemma 2.5.8.

Finally, replacing ρ with $\varepsilon^j\rho$ and summing in j, we get (2.5.31) by using Lemma 2.5.8 and the comparison principle of Riemann integrals. The lemma is proved. □

Remark 2.5.9. We emphasize that Lemma 2.5.7 has a local nature. Indeed, it can be seen from the proof that we only need Dirichlet boundary condition $u = 0$ on $\partial\Omega \cap B_{R/2}(x_0)$ and $C^{1,\mathrm{Dini}}$ regularity of $\partial\Omega \cap B_{R/2}(x_0)$ for these estimates to hold. Therefore, our Lipschitz estimates and modulus of continuity estimates, which will be deduced from Lemma 2.5.7, also have a local nature.

Recall the definition of $\mathbf{q}_{x,r}$ from (2.5.20) and keep (2.5.4) in mind. By following almost the same proof of (2.4.7), we obtain that for any Lebesgue point $x \in \Omega$ of the vector-valued function Du and $\rho \in (0, R_1]$,

$$
|Du(x) - \mathbf{q}_{x,\rho}| \le C\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho), \tag{2.5.32}
$$

where C is a constant depending only on n and γ_0.

Proof of Theorem 2.1.11. We will prove a boundary Lipschitz estimate

$$\|Du\|_{L^\infty(\Omega_{R/16}(x_0))} \le C\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(\Omega_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\||Du|+s\|_{L^{2-p}(\Omega_R(x_0))} \quad (2.5.33)$$

for any $x_0 \in \partial\Omega$ and $R \le R_1$, assuming that

$$\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(B_R(x_0))} < \infty. \quad (2.5.34)$$

Then (2.1.19) follows by a standard covering argument using (2.1.13) and (2.5.33). The proof of (2.5.33) is similar to that of Theorem 2.1.3 so we will only focus on the differences.

Step 1: The case when $u \in C^1(\overline{\Omega_{R/2}(x_0)})$.

With (2.5.31) in place of (2.4.6), using the same iteration technique as in the proof of Theorem 2.1.3, we get the following estimate:

$$\begin{aligned}\|Du\|_{L^\infty(\Omega_{R/4}(x_0))} + s &\le CR^{-n/\gamma_0}\|Du\|_{L^{\gamma_0}(\Omega_{R/2}(x_0))} \\ &+ C\sup_{x\in\Omega_{R/2}(x_0)}\int_0^{\frac{R}{2}}\frac{\breve{h}_1(x,t)}{t}\,dt + C\sup_{x\in\Omega_{R/2}(x_0)}\left(\int_0^{\frac{R}{2}}\frac{\breve{g}_1(x,t)}{t}\,dt\right)^{\frac{1}{p-1}} + Cs.\end{aligned} \quad (2.5.35)$$

Using (2.5.27) and direct computations, we have

$$\begin{aligned}&\int_0^{R/2}\frac{\breve{g}_1(x,t)}{t}\,dt \le C\,\mathbf{I}_1^R(|\mu|)(x) + C\,\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}},\\ &\int_0^{R/2}\frac{\tilde{h}_1(x,t)}{t}\,dt \le C\left(\mathbf{I}_1^R(|\mu|)(x)\right)^{\frac{1}{p-1}} + C\left(\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}}\right)^{\frac{1}{p-1}}.\end{aligned}$$

Therefore, from (2.5.35) and the fact that $\gamma_0 \le 2-p$ (cf. Lemma 2.5.2), we obtain

$$\|Du\|_{L^\infty(\Omega_{R/4}(x_0))} \le C\big\|\mathbf{I}_1^R(|\mu|)\big\|_{L^\infty(\Omega_R(x_0))}^{\frac{1}{p-1}} + CR^{-\frac{n}{2-p}}\||Du|+s\|_{L^{2-p}(\Omega_R(x_0))}. \quad (2.5.36)$$

Step 2: The general case.

We use an approximation argument with the aid of the regularized distance introduced by Lieberman [67]. Here we refer to a modified version in [26]. Let $d(\cdot)$ be the regularized distance defined in [26, Lemma 5.1] ($\psi(\cdot)$ in that paper) and $\Omega^k = \{x \in \Omega : d(x) > 1/k\}$. Then from [26, Lemma 5.1], we know that Ω^k has a smooth boundary and the $C^{1,\text{Dini}}$-properties of $\partial\Omega_k$ are the same as those of $\partial\Omega$ up to some constant independent of k. We take a sequence of standard mollifiers $\{\varphi_k\}$ and mollify μ and a by setting

$$\mu_k(x) = (\mu * \varphi_k)(x), \quad x \in \Omega; \quad a^k(x) = (a * \varphi_k)(x), \quad x \in \Omega^k.$$

We know that $\mu \in W^{-1,p'}(\Omega_R(x_0))$ and therefore

$$\|\mu_k - \mu\|_{W^{-1,p'}(\Omega_R(x_0))} \to 0.$$

Recalling the fact that we have a $C^{1,\text{Dini}}$ coordinate in $\Omega_R(x_0)$ since $R \le R_1$, we can take a sequence of cut-off functions $\zeta_k \in C^\infty(\mathbb{R}^n)$ satisfying $\zeta_k = 1$ in $\Omega^{k/4} \cap B_R(x_0)$, $\zeta_k = 0$ in $(\Omega \backslash \Omega^{k/2}) \cap B_R(x_0)$, and $\|D\zeta_k\|_{L^\infty} \le 16k$.

Next we let $u_k \in u\zeta_k + W_0^{1,p}(\Omega^k \cap B_R(x_0))$ be the unique solution to

$$\begin{cases} -\text{div}\left(a^k(x)(|Du_k|^2 + s^2)^{\frac{p-2}{2}} Du_k\right) = \mu_k & \text{in} \quad \Omega^k \cap B_R(x_0), \\ \qquad\qquad\qquad\qquad\qquad\qquad u_k = u\zeta_k & \text{on} \quad \partial(\Omega^k \cap B_R(x_0)). \end{cases} \tag{2.5.37}$$

Since $u_k = u\zeta_k = 0$ on $B_R(x_0) \cap \partial\Omega^k$, we can always assume $u_k \in u\zeta_k + W_0^{1,p}(\Omega_R(x_0))$ by taking the zero extension of u_k in $(\Omega \backslash \Omega^k) \cap B_R(x_0)$. Since $u = 0$ on $B_R(x_0) \cap \partial\Omega$, by Hardy's inequality, we have

$$\begin{aligned}&\|uD\zeta_k\|_{L^p\left(\Omega_R(x_0)\right)} \le 16\|ku\|_{L^p\left((\Omega\setminus\Omega^{k/4})\cap B_R(x_0)\right)}\\&\le C\|u(x)/d(x)\|_{L^p\left((\Omega\setminus\Omega^{k/4})\cap B_R(x_0)\right)} \le C\|Du\|_{L^p\left((\Omega\setminus\Omega^{k/4})\cap B_R(x_0)\right)} \to 0\end{aligned}$$

as $k \to \infty$. Therefore, we know that

$$\|u - u\zeta_k\|_{W^{1,p}(\Omega_R(x_0))} \to 0. \tag{2.5.38}$$

Thus by choosing $u_k - u\zeta_k$ as a test function in (2.5.37), following the proof of (2.4.18), and using (2.5.38), we can show that $\|Du_k\|_{L^p(\Omega_R(x_0))}$ is uniformly bounded in k. Also, choosing $(u_k - u\zeta_k)1_{\Omega_R(x_0)}$ as a test function in (2.5.1) and (2.5.37), similarly we obtain

$$\int_{\Omega_R(x_0)} |V(Du_k) - V(Du)|^2 \, dx \to 0 \quad \text{as}\, k \to \infty,$$

which again implies

$$Du_k \to Du \quad \text{strongly in } L^p(\Omega_R(x_0)).$$

By the classical boundary regularity theory (see, for instance, [68]), we have $u_k \in C^1(\overline{\Omega^k \cap B_{R/2}(x_0)})$. Note that for sufficiently large k, there exists $x_k \in B_R(x_0) \cap \partial\Omega^k$, such that $|x_k - x_0| \le R/16$. Thus $B_{R/16}(x_0) \subset B_{R/8}(x_k)$, $B_{R/4}(x_k) \subset B_{R/2}(x_0)$ and $B_{R/2}(x_k) \subset B_{3R/4}(x_0)$. Therefore, using (2.5.36) in Step 1 and Remark 2.5.9, we get

$$\begin{aligned}&\|Du_k\|_{L^\infty\left(\Omega^k\cap B_{R/16}(x_0)\right)} \le \|Du_k\|_{L^\infty\left(\Omega^k\cap B_{R/8}(x_k)\right)}\\&\le C\left\|\mathbf{I}_1^{R/2}(|\mu_k|)\right\|^{\frac{1}{p-1}}_{L^\infty\left(\Omega^k\cap B_{R/2}(x_k)\right)} + CR^{-\frac{n}{2-p}}\||Du_k| + s\|_{L^{2-p}\left(\Omega^k\cap B_{R/2}(x_k)\right)}\\&\le C\left\|\mathbf{I}_1^{R/2}(|\mu_k|)\right\|^{\frac{1}{p-1}}_{L^\infty\left(\Omega^k\cap B_{3R/4}(x_0)\right)} + CR^{-\frac{n}{2-p}}\||Du_k| + s\|_{L^{2-p}\left(\Omega^k\cap B_{3R/4}(x_0)\right)}.\end{aligned}$$

By extracting a subsequence and taking the limit as $k \to \infty$, we obtain (2.5.33). $\square$

Proof of Corollary 2.1.12. By testing (2.5.1) with u, following the proof of (2.4.18), we obtain

$$\|Du\|_{L^p(\Omega)} \le C\|\mu\|_{W^{-1,p'}(\mathbb{R}^n)}^{\frac{1}{p-1}} + Cs,$$

where $p' = p/(p-1)$. From [45, Theorem 1], we also have

$$\|\mu\|_{W^{-1,p'}(\mathbb{R}^n)} \le C\Big(\int_{\mathbb{R}^n} \mathbf{W}^1_{1,p}(|\mu|)d|\mu|\Big)^{\frac{p-1}{p}} \le C\big\|\mathbf{I}^1_1(|\mu|)\big\|_{L^\infty(\Omega)}.$$

Therefore, Corollary 2.1.12 follows by combining (2.1.19), Hölder's inequality, and the last two inequalities. $\square$

Now we turn to global modulus of continuity estimates of the gradient. Recall that we fixed an $\varepsilon \in (0, 1/4)$ sufficiently small such that

$$C\varepsilon^{\alpha-\alpha_2} < 1 \quad \text{and} \quad \varepsilon^{\alpha_2} < 1/4$$

for both constants C in (2.3.13) and (2.5.19), where $\alpha \in (0, 1)$ is the same constant as in Theorem 2.2.1, $\alpha_1 \in (0, \alpha)$, and $\alpha_2 = (\alpha_1 + \alpha)/2$. We also took $R \in (0, R_1]$ and defined

$$\begin{aligned}
&\tilde{\omega}_1(t) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\left(\omega_1(\varepsilon^{-i}t)[\varepsilon^{-i}t \le R/2] + \omega_1(R/2)[\varepsilon^{-i}t > R/2]\right),\\
&\tilde{\mathbf{I}}^{\rho}_1(|\mu|)(x) = \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\left(\mathbf{I}_1^{\varepsilon^{-i}\rho}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{I}_1^{R/2}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\right),\\
&\tilde{\mathbf{W}}^{\rho}_{1/p,p}(|\mu|)(x)\\
&= \sum_{i=1}^{\infty} \varepsilon^{\alpha_1 i}\left(\mathbf{W}^{\varepsilon^{-i}\rho}_{1/p,p}(|\mu|)(x)[\varepsilon^{-i}\rho \le R/2] + \mathbf{W}^{R/2}_{1/p,p}(|\mu|)(x)[\varepsilon^{-i}\rho > R/2]\right),
\end{aligned} \tag{2.5.39}$$

where we used the Iverson bracket notation, i.e., $[P] = 1$ if P is true and $[P] = 0$

otherwise, and $\mathbf{I}_1$ and $\mathbf{W}_{1/p,p}$ are the Riesz and Wolff potentials defined in (2.1.9) and (2.1.11), respectively.

Our global modulus of continuity estimate of the gradient is stated as follows.

Theorem 2.5.10. *Assume the conditions of Theorem 2.1.11 and $\alpha_1 \in (0, \alpha)$, where α is the constant in Theorem 2.2.1. Then there exist constants $R_1 = R_1(R_0, \omega_0) \in (0, R_0)$ and $C = C(n, p, \lambda, \alpha_1, \omega, R_0, \omega_0)$, such that for any $x_0 \in \bar{\Omega}$, $R \in (0, R_1]$, and $x, y \in \Omega_{R/4}(x_0)$ being Lebesgue points of the vector-valued function Du, it holds that*

$$
\begin{aligned}
&|Du(x) - Du(y)| \\
&\le C\,\mathbf{M}_1\Big[\Big(\frac{\rho}{R}\Big)^{\alpha_1} + \int_0^\rho \frac{\tilde{\omega}_1(t)}{t}\,dt\Big] + C\,\big\|\tilde{\mathbf{W}}^\rho_{1/p,p}(|\mu|)\big\|_{L^\infty(\Omega_{R/4}(x_0))} \\
&\quad + C\,\mathbf{M}_1^{2-p}\big\|\tilde{\mathbf{I}}^\rho_1(|\mu|)\big\|_{L^\infty(\Omega_{R/4}(x_0))}
\end{aligned}
\tag{2.5.40}
$$

where $\rho = |x-y|$, $\omega_1 = \omega + \omega_0$, $\tilde{\omega}_1$, $\tilde{\mathbf{W}}_{1/p,p}$, and $\tilde{\mathbf{I}}_1$ are defined in (2.5.39), and

$$
\mathbf{M}_1 := R^{-\frac{n}{2-p}}\big\||Du| + s\big\|_{L^{2-p}(\Omega_R(x_0))} + \big\|\mathbf{I}^R_1(|\mu|)\big\|^{\frac{1}{p-1}}_{L^\infty(\Omega_R(x_0))}.
$$

Proof. For any $x, y \in \Omega_{R/4}(x_0)$ being Lebesgue points of Du and $\rho > 0$,

$$
\begin{aligned}
&|Du(x) - Du(y)|^{\gamma_0} \\
&\le |Du(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |Du(y) - \mathbf{q}_{y,2\rho}|^{\gamma_0} + |\mathbf{q}_{x,\rho} - \mathbf{q}_{y,2\rho}|^{\gamma_0} \\
&\le |Du(x) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |Du(y) - \mathbf{q}_{y,2\rho}|^{\gamma_0} + |Du(z) - \mathbf{q}_{x,\rho}|^{\gamma_0} + |Du(z) - \mathbf{q}_{y,2\rho}|^{\gamma_0}.
\end{aligned}
$$

We set $\rho = |x-y|$, take the average over $z \in \Omega_\rho(x)$, and then take the γ_0-th root to

get

$$\begin{aligned}
&|Du(x)-Du(y)|\\
&\le C\,|Du(x)-\mathbf{q}_{x,\rho}|+C\,|Du(y)-\mathbf{q}_{y,2\rho}|+C\phi(x,\rho)+C\phi(y,2\rho)\\
&\le C\sum_{j=0}^{\infty}\phi(x,\varepsilon^j\rho)+C\sum_{j=0}^{\infty}\phi(y,2\varepsilon^j\rho)+C\phi(x,\rho)+C\phi(y,2\rho)\\
&\le C\sup_{y_0\in\Omega_{R/4}(x_0)}\sum_{j=0}^{\infty}\phi(y_0,2\varepsilon^j\rho),
\end{aligned}$$

where we used the fact that $\Omega_\rho(x)\subset\Omega_{2\rho}(y)$ in the first inequality and (2.5.32) in the second inequality.

If $\rho<R/16$, by using (2.5.31) with $R/8$ in place of r and the fact that

$$\Omega_{R/4}(y_0)\subset\Omega_{R/2}(x_0)\quad\forall y_0\in\Omega_{R/4}(x_0),$$

we obtain

$$\begin{aligned}
&|Du(x)-Du(y)|\\
&\le C\left(\frac{\rho}{R}\right)^{\alpha_2}\|Du\|_{L^\infty(\Omega_{R/2}(x_0))}+C\sup_{y_0\in\Omega_{R/4}(x_0)}\int_0^\rho\frac{\breve{h}_1(y_0,t)}{t}\,dt\\
&\quad+C\left(\|Du\|_{L^\infty(\Omega_{R/2}(x_0))}+s\right)^{2-p}\sup_{y_0\in\Omega_{R/4}(x_0)}\int_0^\rho\frac{\breve{g}_1(y_0,t)}{t}\,dt\\
&\quad+C\left(\|Du\|_{L^\infty(\Omega_{R/2}(x_0))}+s\right)\int_0^\rho\frac{\breve{\omega}_1(t)}{t}\,dt.
\end{aligned}\tag{2.5.41}$$

Clearly, (2.5.41) still holds when $\rho\ge R/16$. Using (2.5.27) and similar calculations as in the proof of Theorem 2.4.3, for any $y_0\in\Omega_{R/4}(x_0)$ and $\rho\in(0,R/2)$, we have

$$\int_0^\rho\frac{\breve{g}_1(y_0,t)}{t}\,dt\le\tilde{\mathbf{I}}_1^\rho(|\mu|)(y_0)+C\left(\frac{\rho}{R}\right)^{\alpha_1}\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}},\tag{2.5.42}$$

$$\int_0^\rho\frac{\breve{h}_1(y_0,t)}{t}\,dt\le\tilde{\mathbf{W}}_{1/p,p}^\rho(|\mu|)(y_0)+C\left(\frac{\rho}{R}\right)^{\alpha_1}\left(\frac{|\mu|(B_{R/2}(x_0))}{R^{n-1}}\right)^{\frac{1}{p-1}}.\tag{2.5.43}$$

Using (2.1.19), (2.5.42), and (2.5.43), (2.5.41) implies (2.5.40). The theorem is proved.

□

CHAPTER 3

Gradient estimates for singular parabolic p-Laplace type equations with measure data

We are concerned with gradient estimates for solutions to a class of singular quasilinear parabolic equations with measure data, whose prototype is given by the parabolic p-Laplace equation $u_t - \Delta_p u = \mu$ with $p \in (1, 2)$. The case when $p \in \left(2 - \frac{1}{n+1}, 2\right)$ were studied in [55]. In this chapter, we extend the results in [55] to the open case when $p \in \left(\frac{2n}{n+1}, 2 - \frac{1}{n+1}\right]$ if $n \geq 2$ and $p \in (\frac{5}{4}, \frac{3}{2}]$ if $n = 1$. More specifically, in a more singular range of p as above, we establish pointwise gradient estimates via linear parabolic Riesz potential and gradient continuity results via certain assumptions on parabolic Riesz potential.

3.1 Introduction

In this chapter, we consider the quasilinear parabolic equation with measure data

$$u_t - \operatorname{div}(a(x,t,Du)) = \mu \tag{3.1.1}$$

in a cylindrical domain $\Omega_T = \Omega \times (-T,0) \subset \mathbb{R}^n$, where $\Omega \subset \mathbb{R}^n$ is a bounded domain and $T > 0$. Here and in what follows, the operators "D" and "div" stand for the gradient and divergence with respect to the space variable x. Moreover, μ is a finite signed Radon measure in Ω_T, namely, $|\mu|(\Omega_T) < \infty$. The vector field $a = (a_1, \dots, a_n) : \Omega_T \times \mathbb{R}^n \to \mathbb{R}^n$ is assumed to satisfy the following growth, ellipticity, and continuity conditions: there exist constants $0 < \nu \le L$, $s \ge 0$, and $p > 1$ such that

$$|a(x,t,\xi)| + (s^2 + |\xi|^2)^{1/2} |D_\xi a(x,t,\xi)| \le L(s^2 + |\xi|^2)^{(p-1)/2}, \tag{3.1.2}$$

$$\langle D_\xi a(x,t,\xi)\eta, \eta \rangle \ge \nu (s^2 + |\xi|^2)^{(p-2)/2} |\eta|^2, \tag{3.1.3}$$

and

$$|a(x,t,\xi) - a(x_0,t,\xi)| \le L\, \omega(|x - x_0|)(s^2 + |\xi|^2)^{(p-1)/2} \tag{3.1.4}$$

hold for every $x, x_0 \in \Omega$, $t \in (-T,0)$, and $(\xi,\eta) \in \mathbb{R}^n \times \mathbb{R}^n \setminus \{(0,0)\}$, where $\omega : [0,\infty) \to [0,1]$ is a concave non-decreasing function satisfying

$$\lim_{\rho \to 0^+} \omega(r) = \omega(0) = 0$$

and the Dini condition

$$\int_0^1 \omega(\rho) \, \frac{d\rho}{\rho} < +\infty. \tag{3.1.5}$$

A typical model equation is given by the (possibly nondegenerate) parabolic p-Laplace equation with measure data and $s \geq 0$:

$$u_t - \operatorname{div}\left((|Du|^2 + s^2)^{\frac{p-2}{2}} Du\right) = \mu \quad \text{in } \Omega_T.$$

By a (weak) solution to the equation (3.1.1), we mean a function

$$u \in C^0(-T, 0;\ L^2(\Omega)) \cap L^p(-T, 0;\ W^{1,p}(\Omega))$$

such that the distributional relation

$$-\int_{\Omega_T} u\varphi_t \, dxdt + \int_{\Omega_T} \langle a(x,t,Du), D\varphi \rangle \, dxdt = \int_{\Omega_T} \varphi \, d\mu$$

holds whenever $\varphi \in C_0^\infty(\Omega_T)$ has compact support in Ω_T.

The gradient estimates for the super-quadratic case when $p \geq 2$ were well studied in the literature. See [39, 59, 60] and also [37, 56, 57, 74] for estimates for elliptic problems. However, the corresponding results for the singular case when $p \in (1, 2)$ are still not complete.

In this chapter, we are concerned with only the singular case when $p \in (1, 2)$.

3.1.1 Pointwise gradient estimates

The first gradient potential result for singular parabolic p-Laplace type equations was obtained by Kuusi and Mingione in [55] for the case when $p \in (2 - \frac{1}{n+1}, 2]$ using intrinsic geometry and exit time arguments. More precisely, they first showed that there exists a constant $c = c(n, p, \nu, L, \omega)$ such that if

$$c \fint_{Q_{r_\lambda}^\lambda(x_0,t_0)} (|Du| + s) \, dxdt + c \int_0^{2r_\lambda} \frac{|\mu|(Q_\rho^\lambda(x_0, t_0)}{\rho^{n+1}} \frac{d\rho}{\rho} \leq \lambda$$

for some constant $\lambda > 0$, then $|Du(x_0, t_0)| \leq \lambda$. Here $r_\lambda := \lambda^{(p-2)/2} r$ and

$$Q_\rho^\lambda(x_0, t_0) := B_\rho(x_0) \times (t_0 - \lambda^{2-p}\rho^2, t_0) \tag{3.1.6}$$

is called an intrinsic cylinder for ρ, $\lambda > 0$. They also used the intrinsic Riesz potential result above to establish the following parabolic Riesz potential bound when $p \in (2 - \frac{1}{n+1}, 2]$:

$$\begin{aligned}|Du(x_0, t_0)| &\leq c\,[\mathbf{I}_1^{|\mu|}(x_0, t_0, 2r)]^{2/[(n+1)p-2n]} \\ &\quad + c\Big(\fint_{Q_r(x_0,t_0)} (|Du| + s + 1)\, dxdt\Big)^{2/[2-n(2-p)]}\end{aligned}$$

holds for any solution u to the equation (3.1.1) in the standard parabolic cylinder $Q_{2r}(x_0, t_0) := B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T$. Here

$$\mathbf{I}_1^{|\mu|}(x_0, t_0; r) := \int_0^r \frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{n+1}} \frac{d\rho}{\rho} \tag{3.1.7}$$

is the truncated first-order parabolic Riesz potential. For more notation in parabolic (intrinsic) geometry, see Section 3.2.1 below.

In this chapter, we extend their results to include the case when $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where

$$p^*(n) := \max\Big\{\frac{2n}{n+1}, \frac{3n+2}{2n+2}\Big\} = \begin{cases} \dfrac{5}{4} & \text{when} \quad n = 1, \\ \dfrac{2n}{n+1} & \text{when} \quad n \geq 2. \end{cases} \tag{3.1.8}$$

Note that $p^*(n) < 2 - \frac{1}{n+1}$ holds for every integer $n \geq 1$. Moreover, it is clear that if $p \in (p^*(n), 2 - \frac{1}{n+1}]$, then

$$0 < \max\Big\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}\Big\} < p - \frac{n}{n+1} \leq 1$$

so that we can choose a constant $q \in (0,1)$ satisfying

$$q \in \Big(\max\Big\{ \frac{n+2}{2(n+1)}, \frac{(2-p)n}{2} \Big\}, \, p - \frac{n}{n+1} \Big) \subset (0,1). \tag{3.1.9}$$

Our first main result is stated as follows.

Theorem 3.1.1 (Intrinsic Riesz potential estimate)**.** *Let u be a solution to (3.1.1) with $p \in (p^*(n), 2-\frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). Let $q \in (0,1)$ satisfy (3.1.9). Under the assumptions (3.1.2)–(3.1.5), there exist constants $c \geq 1$ and $R_0 \in (0,1/2]$, both depending only on n, p, ν, L, q, and ω, such that the following holds for a.e. $(x_0,t_0) \in \Omega_T$: If*

$$c \Big(\fint_{Q^{\lambda}_{r_\lambda}(x_0,t_0)} (|Du|+s)^q \, dxdt \Big)^{1/q} + c \int_0^{2r_\lambda} \frac{|\mu|(Q^{\lambda}_{\rho}(x_0,t_0))}{\rho^{n+1}} \frac{d\rho}{\rho} \leq \lambda, \tag{3.1.10}$$

where $\lambda > 0$ is a constant, $r_\lambda := \lambda^{(p-2)/2} r \in (0,R_0]$, $Q^{\lambda}_{2r_\lambda}(x_0,t_0) \subset \Omega_T$, and $Q^{\lambda}_{\rho}(x_0,t_0)$ is the intrinsic cylinder defined in (3.1.6), then it holds that

$$|Du(x_0,t_0)| \leq \lambda.$$

Theorem 3.1.1 implies pointwise gradient estimates in standard parabolic cylinders as in Theorem 3.1.2 below.

Theorem 3.1.2 (Pointwise gradient estimate via parabolic Riesz potential)**.** *Let u be a solution to (3.1.1) with $p \in (p^*(n), 2-\frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). Let $q \in (0,1)$ satisfy (3.1.9). Under the assumptions (3.1.2)–(3.1.5), there exist constants $c \geq 1$ and $R_0 \in (0,1/2]$, both depending only on n, p, ν, L, q, and ω, such that*

$$\begin{aligned} |Du(x_0,t_0)| &\leq c\,[\mathbf{I}_1^{|\mu|}(x_0,t_0,2r)]^{2/[(n+1)p-2n]} \\ &\quad + c \Big(\fint_{Q_r(x_0,t_0)} (|Du|+s+1)^q \, dxdt \Big)^{2q/[2q-n(2-p)]} \end{aligned} \tag{3.1.11}$$

holds for a.e. $(x_0, t_0) \in \Omega_T$ *and every* $Q_{2r}(x_0, t_0) \equiv B_{2r}(x_0) \times (t_0 - 4r^2, t_0) \subset \Omega_T$ *with* $r \in (0, R_0]$*, where* $\mathbf{I}_1^{|\mu|}$ *is the parabolic Riesz potential defined in* (3.1.7)*.*

We remark that a preliminary assumption $p > \frac{2n}{n+2}$ always occurs in regularity theory for parabolic p-Laplace type equations. See for instance [23]. In this chapter, we need the lower bound $p > \frac{2n}{n+2}$ for the applications of reverse Hölder type estimates (Theorem 3.4.2) and Lipschitz estimates (Theorem 3.4.5). Our proofs of the main theorems require a stronger assumption $p > \frac{2n}{n+1}$ when $n \geq 2$. At the time of this writing, it is not clear to us whether this assumption is optimal for these potential estimates to hold. However, we note that the assumption $p > \frac{2n}{n+1}$ is also needed for the existence of fundamental solution to the parabolic p-Laplace equation (the Barenblatt solution; see [10],[87, Chapter 11], [55, Section 1.3]), and the Harnack inequalities (see [23, Chapter 7]).

3.1.2 Gradient continuity results

In [55], the authors proved a sufficient condition for gradient continuity in the case when $p \in (2 - \frac{1}{n+1}, 2)$, namely, the Riesz potential $\mathbf{I}_1^{|\mu|}(x_0, t_0, r) \to 0$ uniformly with respect to (x_0, t_0) when $r \to 0$. We extend that result to the case when $p \in (p^*(n), 2 - \frac{1}{n+1}]$.

Theorem 3.1.3 (Gradient continuity via Riesz potential)**.** *Let* u *be a solution to* (3.1.1) *with* $p \in (p^*(n), 2 - \frac{1}{n+1}]$*, where* $p^*(n)$ *is defined in* (3.1.8)*. Assume that* (3.1.2)–(3.1.5) *are satisfied and that the functions*

$$(x, t) \mapsto \mathbf{I}_1^{|\mu|}(x, t, r) \text{ converge locally uniformly to zero in } \Omega_T \text{ as } r \to 0. \tag{3.1.12}$$

Then Du *is continuous in* Ω_T*.*

Recall the Lorentz space $L^{n+2,1}$ is the collection of measurable functions f such that

$$\int_0^\infty |\{(x,t) : |f(x,t)| \geq h\}|^{\frac{1}{n+2}}\, dh < \infty.$$

Theorem 3.1.3 has the following corollary.

Corollary 3.1.4 (Gradient continuity via Lorentz spaces)**.** *Let u be a solution to (3.1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). Assume that (3.1.2)–(3.1.5) are satisfied and that*

$$\mu \in L^{n+2,1} \text{ holds locally in } \Omega_T. \tag{3.1.13}$$

Then Du is continuous in Ω_T.

A further, actually immediate, corollary of Theorem 3.1.3 concerns measures with certain density properties.

Corollary 3.1.5 (Gradient continuity via density)**.** *Let u be a solution to (3.1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). Assume that (3.1.2)–(3.1.5) are satisfied and that μ satisfies*

$$|\mu|(Q_\rho(x,t)) \leq c_D \rho^{n+1} h(\rho) \tag{3.1.14}$$

for every standard parabolic cylinder $Q_\rho(x,t) = B_\rho(x) \times (t_0 - \rho^2, t_0) \subset\subset \Omega_T$, where c_D is a positive constant and $h : [0,\infty) \to [0,\infty)$ is a function satisfying the Dini condition

$$\int_0^R h(r)\, \frac{dr}{r} < \infty \text{ for some } R > 0. \tag{3.1.15}$$

Then Du is continuous in Ω_T.

We also establish the following measure density criterion to ensure gradient Hölder continuity, which is a parabolic generalization of Lieberman's result in [70]. Recall that in parabolic setting, for any $\beta \in (0, 1)$ and any set $\mathcal{C} \subset \mathbb{R}^{n+1}$, the Hölder space

$C^{0,\beta}(\mathcal{C})$ is the collection of measurable functions f such that

$$\|f\|_{C^{0,\beta}(\mathcal{C})} := \sup_{\mathcal{C}} |f| + \sup_{\substack{(x_1,t_1),(x_2,t_2)\in\mathcal{C} \\ (x_1,t_1)\neq(x_2,t_2)}} \frac{|f(x_1,t_1) - f(x_2,t_2)|}{|(x_1,t_1) - (x_2,t_2)|_{\mathrm{par}}^{\beta}} < \infty, \tag{3.1.16}$$

where

$$|(x_1,t_1) - (x_2,t_2)|_{\mathrm{par}} := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$$

is the parabolic distance between those two points. Moreover, $C^{0,\beta}_{\mathrm{loc}}(\mathcal{C})$ is defined as the collection of measurable functions f such that $f \in C^{0,\beta}(\mathcal{K})$, for every compact set $\mathcal{K} \subset\subset \mathcal{C}$.

Theorem 3.1.6 (Gradient Hölder continuity via Riesz potential)**.** *Let u be a solution to (3.1.1) with $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). Assume that (3.1.2)–(3.1.5) are satisfied and that ω, μ satisfies*

$$\omega(r) \le c_D r^{\delta} \quad \text{and} \quad |\mu|(Q_\rho(x,t)) \le c_D \rho^{n+1+\delta} \tag{3.1.17}$$

for every $r \in (0,1)$ and every standard parabolic cylinder $Q_\rho(x,t) = B_\rho(x) \times (t_0 - \rho^2, t_0) \subset\subset \Omega_T$, where $c_D \ge 1$ and $\delta \in (0,1)$. Then there exists an exponent $\beta \in (0,1)$ depending only on n, p, ν, L, c_D, and δ, such that $Du \in C^{0,\beta}_{loc}(\Omega_T)$.

Let us give a brief description of the proofs. We first prove decay estimates of the L^q-mean oscillation of the gradient for a solution to the homogeneous equation with x-independent nonlinearity

$$v_t - \mathrm{div}(a(x_0,t,Dv)) = 0, \tag{3.1.18}$$

where $q \in (0,1)$ and $x_0 \in \mathbb{R}^n$ is fixed. Here by the L^q-mean oscillation of Dv in a domain $\mathcal{C} \subset \mathbb{R}^{n+1}$, we mean

$$\inf_{\Theta \in \mathbb{R}^n} \left(\fint_{\mathcal{C}} |Dv - \Theta|^q \, dxdt \right)^{1/q}.$$

Our proof of the decay estimates adapts the singular iteration scheme in [55, Section 3] to the L^q setting for $q \in (0,1)$. For the precise definition of the L^q-mean oscillation with $q \in (0,1)$ and some of its properties, see Section 3.2.2. We also refer the reader to [17, 19, 25, 26, 31, 36, 53] for its applications in other problems.

Our proofs of the pointwise gradients estimates and the gradient continuity results are all based on the decay estimates for Dv mentioned above and comparison estimates between the original solution u to (3.1.1) and a solution v to (3.1.18). As a bridge between u and v, we introduce the solution w to the homogeneous equation

$$w_t - \operatorname{div}(a(x,t,Dw)) = 0$$

in a cylinder Q with the boundary condition $w = u$ on $\partial_{\mathrm{par}} Q$. Under appropriate boundary condition on v, we obtain an L^p bound for $Dw - Dv$, which originated from [54, Lemma 4.3]. We also utilize a comparison estimate between u and w in [80, Lemma 3.1], which provides an L^q bound for $Du - Dw$ in terms of μ for some $q \in (0,1)$. By proving a reverse Hölder type inequality for Dw, we establish an L^q estimate for $Du - Dv$ for some $q \in (0,1)$.

With the decay estimates of the L^q-mean oscillation for Dv and L^q estimate for $Du - Dv$ in hand, we then borrow the idea in [25] by estimating the L^q-mean oscillation and adapt the exit time argument and iteration argument used, for instance, in [55, Theorem 1.1] to prove the pointwise gradient estimates. For gradient continuity results, we first prove a uniform decay estimate of the L^q-mean oscillation of Du in Proposition 3.5.1. Then for the gradient Hölder continuity result, we show the decay rate of the L^q-mean oscillation of Du and adapt Campanato's idea of characterizing Hölder continuity to the L^q setting for some $q \in (0,1)$. Finally, for the gradient continuity result, we adapt the "maximal iteration chain" argument introduced in [55, Theorem 1.5] to our L^q setting and apply the uniform decay of the L^q-mean oscillation of Du.

The rest of the chapter is organized as follows. In the next section, we collect basic notation and give the definition and some basic properties of the L^q-mean oscillation for $q \in (0,1)$. In Section 3.3, we prove some decay estimates for the L^q-mean oscillation of the gradients of solutions to the homogeneous equation with x-independent nonlinearities. In Section 3.4, we derive some comparison estimates and give the proofs of Theorems 3.1.1 and 3.1.2. Finally, Section 3.5 is devoted to the gradient continuity results Theorem 3.1.3–Theorem 3.1.6.

3.2 Notation and basic inequalities

3.2.1 Notation

In this chapter, we adapt the same notation as in [55] for comparison purposes. For completeness, we briefly record the notation that will be used throughout this chapter. For any vector $y = (y_1, \dots, y_n) \in \mathbb{R}^n$, we define two different norms

$$|y| := \Big(\sum_{i=1}^n y_i^2\Big)^{1/2} \quad \text{and} \quad \|y\| := \max_{1 \le i \le n} |y_i|.$$

These two norms are equivalent since

$$\|y\| \le |y| \le \sqrt{n}\|y\|, \quad \forall\, y \in \mathbb{R}^n.$$

We use

$$B_r(x_0) := \{x \in \mathbb{R}^n : |x - x_0| < r\}$$

to denote the open Euclidean ball in $\mathbb{R}^n$ with center x_0 and radius r and denote

$$Q_r(x_0, t_0) := B_r(x_0) \times (t_0 - r^2, t_0)$$

as the standard parabolic cylinder with center (x_0, t_0) and radius r. For $\lambda > 0$, we define the intrinsic cylinders as

$$Q_r^\lambda(x_0, t_0) := B_r(x_0) \times (t_0 - \lambda^{2-p} r^2, t_0).$$

Clearly, when $\lambda = 1$, an intrinsic cylinder becomes a standard parabolic cylinder, namely, $Q_r^1(x_0, t_0) = Q_r(x_0, t_0)$. For simplicity, we also denote

$$\delta Q_r^\lambda(x_0, t_0) := Q_{\delta r}^\lambda(x_0, t_0) = B_{\delta r}(x_0) \times (t_0, \lambda^{2-p} \delta^2 r^2, t_0).$$

Namely, the parameter $\delta > 0$ before an intrinsic cylinder $Q_r^\lambda(x_0, r_0)$ should be viewed as a dilation factor of the radius r. We often denote $r_\lambda := \lambda^{(p-2)/2} r$ and therefore

$$Q_{r_\lambda}^\lambda(x_0, t_0) = Q_{\lambda^{(p-2)/2} r}^\lambda(x_0, t_0) = B_{\lambda^{(p-2)/2} r}(x_0) \times (t_0 - r^2, t_0).$$

A useful property is that when $p \in (1, 2)$,

$$Q_{r_{\lambda_2}}^{\lambda_2}(x_0, t_0) \subset Q_{r_{\lambda_1}}^{\lambda_1}(x_0, t_0), \quad \text{if} \quad 0 < \lambda_1 \leq \lambda_2.$$

When there is no confusion and no need to specify the center, we also denote $Q_r^\lambda := Q_r^\lambda(x_0, t_0)$. For any cylindrical domain $\mathcal{C} = D \times (t_1, t_2)$ with $D \subset \mathbb{R}^n$, the parabolic boundary is defined as

$$\partial_{\text{par}}\mathcal{C} := D \times \{t_1\} \cup \partial D \times [t_1, t_2).$$

The parabolic distance between two points is defined as

$$|(x_1, t_1) - (x_2, t_2)|_{\text{par}} := \max\{|x_1 - x_2|, \sqrt{|t_1 - t_2|}\}$$

and the corresponding parabolic distance between two sets is defined as

$$\text{dist}_{\text{par}}(\mathcal{A}_1, \mathcal{A}_2) = \inf\{|(x_1, t_1) - (x_2, t_2)|_{\text{par}} : (x_1, t_1) \in \mathcal{A}_1, (x_2, t_2) \in \mathcal{A}_2\}.$$

Next, for any measurable mapping $g : \mathcal{A} \subset \mathbb{R}^{n+1} \to \mathbb{R}^n$, we denote its integral average as

$$\fint_{\mathcal{A}} g \, dxdt := \frac{1}{|\mathcal{A}|} \int_{\mathcal{A}} g(x,t) \, dxdt$$

and we denote its oscillation as

$$\operatorname*{osc}_{\mathcal{A}} g := \sup_{(x,t), (x_0,t_0) \in \mathcal{A}} |g(x,t) - g(x_0, t_0)|.$$

Finally, throughout the chapter, we denote by c, c', c'' some general constants which may differ from line to line. We also use c_1, c_2, c_3, ... to denote specific constants which may be used later.

3.2.2 Definition of the L^q-mean oscillation and some basic inequalities

For $q \in (0,1)$, we first recall some basic inequalities in one dimension:

$$(a+b)^q \leq a^q + b^q, \quad (a+b)^{1/q} \leq 2^{1/q-1}(a^{1/q} + b^{1/q}) \quad \forall\, a, b > 0. \tag{3.2.1}$$

Therefore, for any Θ_1, $\Theta_2 \in \mathbb{R}^k$, where k is a positive integer, we have

$$|\Theta_1 + \Theta_2|^q \leq (|\Theta_1| + |\Theta_2|)^q \leq |\Theta_1|^q + |\Theta_2|^q. \tag{3.2.2}$$

We now consider a measurable function $F : \mathbb{R}^{n+1} \to \mathbb{R}^k$ for some integer $k \geq 1$. For simplicity, we assume that $F \in L^1_{\text{loc}}(\mathbb{R}^{n+1} : \mathbb{R}^k)$. For any $q \in (0,1)$ and any bounded domain $\mathcal{C} \subset \mathbb{R}^{n+1}$, we define the L^q-mean oscillation of F on $\mathcal{C}$ as

$$\phi_q(F, \mathcal{C}) := \inf_{\Theta \in \mathbb{R}^k} \left(\fint_{\mathcal{C}} |F(x,t) - \Theta|^q \, dxdt \right)^{1/q}.$$

By (3.2.2) and the fact that $F \in L^1(\mathcal{C})$, we know that the function

$$h(\Theta) := \fint_{\mathcal{C}} |F(x,t) - \Theta|^q \, dxdt$$

is a continuous function of $\Theta \in \mathbb{R}^k$ satisfying $\lim_{|\Theta| \to \infty} h(\Theta) = \infty$. Therefore the minimum of $h(\cdot)$ can be attained in $\mathbb{R}^k$ and we choose $\mathbf{m}(F, \mathcal{C}) \in \mathbb{R}^k$ such that

$$\left(\fint_{\mathcal{C}} |F(x,t) - \mathbf{m}(F, \mathcal{C})|^q \, dxdt \right)^{1/q} = \phi_q(F, \mathcal{C}).$$

Next, we prove some useful properties related to the L^q-mean oscillation. By (3.2.2), we have

$$|\mathbf{m}(F, \mathcal{C})|^q \leq |F(x,t) - \mathbf{m}(F, \mathcal{C})|^q + |F(x,t)|^q.$$

By taking the average over $(x,t) \in \mathcal{C}$, taking the q-th root, and using (3.2.1), we obtain

$$|\mathbf{m}(F, \mathcal{C})| \leq 2^{1/q-1} \phi_q(F, \mathcal{C}) + 2^{1/q-1} \left(\fint_{\mathcal{C}} F^q \, dxdt \right)^{1/q} \leq 2^{1/q} \left(\fint_{\mathcal{C}} F^q \, dxdt \right)^{1/q}. \tag{3.2.3}$$

For two bounded domains $\mathcal{C}_1 \subset \mathcal{C}_2 \subset \mathbb{R}^{n+1}$, using the same argument as above, we also have

$$\begin{aligned}|\mathbf{m}(F,\mathcal{C}_1)-\mathbf{m}(F,\mathcal{C}_2)| &\le 2^{1/q-1}\phi_q(F,\mathcal{C}_1)+2^{1/q-1}\Big(\fint_{\mathcal{C}_1}|F-\mathbf{m}(F,\mathcal{C}_2)|^q\,dxdt\Big)^{1/q}\\ &\le 2^{1/q-1}\phi_q(F,\mathcal{C}_1)+2^{1/q-1}\Big(\frac{|\mathcal{C}_2|}{|\mathcal{C}_1|}\Big)^{1/q}\phi_q(F,\mathcal{C}_2).\end{aligned} \tag{3.2.4}$$

3.3 Gradient estimates for homogeneous equations

In this section, we derive decay estimates of the L^q-mean oscillation of gradients of solutions to the homogeneous equations of the type

$$v_t - \operatorname{div}(a_0(t,Dv)) = 0 \tag{3.3.1}$$

in a given cylinder $Q = B \times (t_1,t_2)$, where $a_0 = a_0(\tau,\xi)$ is a vector field independent of x satisfying conditions (3.1.2) and (3.1.3) for some $s \ge 0$, $L \ge \nu > 0$, and $p \in (1,2]$. First, we recall an oscillation estimate given in [55, Theorem 3.2].

Theorem 3.3.1. *Suppose that v is a solution to (3.3.1) in a given cylinder Q under assumptions (3.1.2) and (3.1.3). If $p \in (1,2]$, $\lambda > 0$, and*

$$s + \sup_{Q_r^\lambda}\|Dv\| \le A\lambda$$

holds for a constant $A \ge 1$ and an intrinsic cylinder $Q_r^\lambda \subset Q$, then there exists a constant $\alpha \in (0,1)$ depending only on n, p, ν, L, A, such that

$$|Dv(x_1,t_1) - Dv(x_2,t_2)| \le 4\sqrt{n}A\lambda\Big(\frac{\rho}{r}\Big)^{\alpha}$$

holds for any (x_1,t_1), $(x_2,t_2) \in Q_\rho^\lambda$, where $Q_\rho^\lambda \subset Q_r^\lambda$ is another intrinsic cylinder with *the same center. Moreover, if* $p \in (\frac{2n}{n+2}, 2]$*, then* Dv *is Hölder continuous in* Q*.*

The main goal of this section is to derive the following decay estimate of the L^q-mean oscillation of the gradient with $q \in (0,1)$.

Theorem 3.3.2. *Let* $p \in (1,2]$*. Suppose that* v *is a solution to* (3.3.1) *in an intrinsic cylinder* Q_r^λ *under assumptions* (3.1.2) *and* (3.1.3) *and that* A*,* B*,* q*, and* γ *are constants satisfying* A*,* $B \geq 1$ *and* q*,* $\gamma \in (0,1)$*. Then there exist constants* $\delta_\gamma \in (0,1/2)$ *depending on* n*,* p*,* ν*,* L*,* A*,* B*,* γ*,* q*, and* $\xi \in (0,1/4)$ *depending only on* n*,* p*,* ν*,* L*,* A*,* B*,* γ*, such that if*

$$\lambda \leq \max\left\{\frac{s}{\xi}, B \sup_{\delta_\gamma Q_r^\lambda} \|Dv\|\right\}, \quad s + \sup_{Q_r^\lambda} \|Dv\| \leq A\lambda, \tag{3.3.2}$$

where $\lambda > 0$ *is a constant, then*

$$\phi_q(Dv, \delta_\gamma Q_r^\lambda) \leq \gamma \phi_q(Dv, Q_r^\lambda). \tag{3.3.3}$$

Moreover, there exist constants $\alpha \in (0,1)$ *depending only on* n*,* p*,* ν*,* L*,* A*,* B*,* γ*, but not on* q*, and* $c(A,B) \geq 1$ *depending on* n*,* p*,* ν*,* L*,* A*,* B*,* γ*,* q*, such that*

$$\delta_\gamma = \frac{\gamma^{1/\alpha}}{c(A,B)}.$$

Also, the estimate (3.3.3) *still holds when replacing* δ_γ *with a smaller number.*

In the proof of gradient continuity results in Section 3.5, we need a different version of Theorem 3.3.2 under a stronger condition as follows, which gives a more precise dependence of δ_γ in terms of A, B, and γ, namely,

$$\delta_\gamma = \frac{\gamma^{1/\alpha_2}}{c(A) B^{1/\alpha_1}}$$

for some exponent $\alpha_1 \in (0,1)$ independent of B and some $\alpha_2 \in (0,1)$ independent of

B and q.

Theorem 3.3.3. *Let $p \in (1,2]$. Suppose that v is a solution to (3.3.1) in an intrinsic cylinder Q_r^λ under assumptions (3.1.2) and (3.1.3) and that A, B, q, and γ are constants satisfying A, $B \geq 1$ and q, $\gamma \in (0,1)$. Then there exists a constant $\delta_\gamma \in (0,1/2)$ depending only on n, p, ν, L, A, B, γ, q, such that if*

$$\lambda \leq B \sup_{\delta_\gamma Q_r^\lambda} \|Dv\|, \quad s + \sup_{Q_r^\lambda} \|Dv\| \leq A\lambda,$$

where $\lambda > 0$ is a constant, then

$$\phi_q(Dv, \delta_\gamma Q_r^\lambda) \leq \gamma \phi_q(Dv, Q_r^\lambda). \tag{3.3.4}$$

Moreover, there exist constants $\alpha_1 \in (0,1)$ and $c(A) \geq 1$, both depending only on n, p, ν, L, A, γ, q, but not on B, and $\alpha_2 \in (0,1)$ depending only on n, p, ν, L, A, γ, but not on q and B, such that

$$\delta_\gamma = \frac{\gamma^{1/\alpha_2}}{c(A) B^{1/\alpha_1}}. \tag{3.3.5}$$

Also, the estimate (3.3.4) still holds when replacing δ_γ by a smaller number.

As in [55], we first assume $s > 0$. The next two De Giorgi-Nash-Moser type estimates for Dv can be found in [55] for vector field $a_0(\cdot)$ with no time dependence. However, they still hold here since their proofs only rely on differentiating the equation with respect to the space variable. See [55, Remark 3.6].

Proposition 3.3.4. *Assume $s > 0$ and $\lambda > 0$. Suppose that*

$$s + \sup_{Q_r^\lambda} \|Dv\| \leq A\lambda \tag{3.3.6}$$

holds for some constant $A \geq 1$. There exists a constant $\sigma \in (0,1/2)$ depending only *on n, p, ν, L, A such that if either*

$$|Q_r^\lambda \cap \{D_{x_i}v < \lambda/2\}| \le \sigma |Q_r^\lambda| \tag{3.3.7}$$

or

$$|Q_r^\lambda \cap \{D_{x_i}v > -\lambda/2\}| \le \sigma |Q_r^\lambda| \tag{3.3.8}$$

holds for some $i \in \{1, \dots, n\}$, then

$$|D_{x_i}v| \ge \lambda/4 \quad a.e. \ in \ Q_{r/2}^\lambda.$$

Proposition 3.3.5. *Assume $s > 0$ and $\lambda > 0$. Suppose that (3.3.6) holds and that neither (3.3.7) nor (3.3.8) is satisfied for the constant σ in Proposition 3.3.4. Then there exists a constant $\eta \in (1/2, 1)$ depending only on n, p, ν, L, A, such that*

$$\|Dv\| \le \eta A \lambda \quad a.e. \ in \ Q_{\sigma r/2}^\lambda. \tag{3.3.9}$$

Next, we prove a decay result for the L^q-mean oscillation of gradients of solutions to linear parabolic equations with $q \in (0, 1)$.

Lemma 3.3.6. *Suppose that $\tilde{u} \in L^2(-1, 0; W^{1,2}(B_1(0)))$ is a solution to the following linear parabolic equation*

$$\tilde{u}_t - \operatorname{div}(A(x,t)D\tilde{u}) = 0, \tag{3.3.10}$$

where the matrix $A(x,t)$ has measurable entries and satisfies

$$\nu_0 |\xi|^2 \le \langle A(x,t)\xi, \xi \rangle, \quad |A(x,t)| \le L_0$$

for any $\xi \in \mathbb{R}^n$, where $0 < \nu_0 \le L_0$ are fixed constants. Let $q \in (0, 1)$. Then there exist constants $c_1, c_2 \ge 1$ depending on n, ν_0, L_0, q, and $\beta_0 \in (0, 1)$ depending only on

n, ν_0, L_0, such that

$$\sup_{Q_{1/2}} |\tilde{u}| \leq c_1 \Big(\fint_{Q_1} |\tilde{u}|^q \, dxdt \Big)^{1/q} \tag{3.3.11}$$

and

$$\phi_q(\tilde{u}, Q_\delta) \leq c_2 \delta^{\beta_0} \phi_q(\tilde{u}, Q_1) \tag{3.3.12}$$

hold for any $\delta \in (0,1)$. Here for each $\rho > 0$, $Q_\rho \equiv B_\rho(0) \times (-\rho^2, 0)$ stands for the standard parabolic cylinder centered at 0 with radius ρ.

Proof. First, the estimate (3.3.11) follows from standard local boundedness estimates for nondegenerate parabolic equations (see for example (3.26) in [60, Lemma 3.1]) and a standard interpolation and iteration argument. See also [71, Theorem 6.17]. Next we give the proof of (3.3.12) for $\delta \in (0, 1/4)$. From the classical De Giorgi-Nash-Moser theory for linear parabolic equations (see for example [71, Theorems 6.28-6.29]), there exist constants $c_0 \geq 1$ and $\beta_0 \in (0,1)$, both depending on n, ν_0, L_0, such that $\tilde{u} \in C^{\beta_0}(Q_{1/4})$ and that

$$\|\tilde{u}\|_{C^{\beta_0}(Q_{1/4})} \leq c_0 \sup_{Q_{1/2}} |\tilde{u}|,$$

where $\|\cdot\|_{C^{\beta_0}}$ is the parabolic Hölder norm defined in (3.1.16). Thus, using the last estimate and the assumption that $\delta \in (0, 1/4)$, we have

$$\phi_q(\tilde{u}, Q_\delta) \leq \Big(\fint_{Q_\delta} |\tilde{u} - \tilde{u}(0,0)|^q \, dxdt \Big)^{1/q} \leq (2\delta)^{\beta_0} \|\tilde{u}\|_{C^{\beta_0}(Q_{1/4})} \leq c\delta^{\beta_0} \sup_{Q_{1/2}} |\tilde{u}|, \tag{3.3.13}$$

for some constant $c = c(n, \nu_0, L_0) > 0$. Combining (3.3.13) and (3.3.11), we get

$$\phi_q(\tilde{u}, Q_\delta) \leq c_2 \delta^{\beta_0} \Big(\fint_{Q_1} |\tilde{u}|^q \, dxdt \Big)^{1/q},$$

and therefore (3.3.12) follows by replacing $\tilde{u}$ with $\tilde{u} - \mathbf{m}(\tilde{u}, Q_1)$ since $\tilde{u} - \mathbf{m}(\tilde{u}, Q_1)$ is still a solution to (3.3.10) and $\phi_q(\tilde{u}, Q_\delta) \equiv \phi_q(\tilde{u} - \mathbf{m}(\tilde{u}, Q_1), Q_\delta)$.

Finally, for $\delta \in [1/4, 1)$, the proof of (3.3.12) follows standard manipulations as follows. Recalling the definition of $\mathbf{m}$, we have

$$
\begin{aligned}
&\phi_q(\tilde{u}, Q_\delta) \leq \Big(\fint_{Q_\delta} |\tilde{u} - \mathbf{m}(\tilde{u}, Q_1)|^q \, dxdt \Big)^{1/q} \\
&\leq \Big(\frac{|Q_1|}{|Q_\delta|} \Big)^{1/q} \Big(\fint_{Q_1} |\tilde{u} - \mathbf{m}(\tilde{u}, Q_1)|^q \, dxdt \Big)^{1/q} \leq 4^{(n+2)/q} \phi_q(\tilde{u}, Q_1).
\end{aligned}
$$

The proof is now completed. $\square$

Combining Proposition 3.3.4 and Lemma 3.3.6, we obtain the following result by using a scaling argument.

Proposition 3.3.7. *Suppose that $s > 0$, $\lambda > 0$, and that (3.3.6) holds. There exist constants $\beta \in (0, 1)$ depending on n, p, ν, L, A, and $c_d \geq 1$ depending on n, p, ν, L, A, q, such that if either (3.3.7) or (3.3.8) holds for the constant σ in Proposition 3.3.4 and some $i \in \{1, \ldots, n\}$, then*

$$
\phi_q(Dv, Q^\lambda_{\delta r}) \leq c_d \delta^\beta \phi_q(Dv, Q^\lambda_r). \tag{3.3.14}
$$

Proof. We will only give the proof for $\delta \in (0, 1/2)$ since the proof for $\delta \in [1/2, 1)$ follows by standard manipulations as in Lemma 3.3.6. From Proposition 3.3.4, we have

$$
\lambda/4 \leq \|Dv(x,t)\| \leq s + \|Dv(x,t)\| \leq A\lambda \quad \forall\, (x,t) \in Q^\lambda_{r/2}. \tag{3.3.15}
$$

Then we rescale the solution in the cylinder Q_1, namely,

$$
\tilde{v}(x,t) := \frac{1}{r_1} v(r_1 x, \lambda^{2-p} r_1^2 t), \quad (x,t) \in Q_1, \tag{3.3.16}
$$

where $r_1 = r/2$. Thus $\tilde{v}$ satisfies

$$\lambda^{p-2}\tilde{v}_t - \operatorname{div}\tilde{a}_0(t, D\tilde{v}) = 0, \tag{3.3.17}$$

where

$$\tilde{a}_0(t,z) = a_0(\lambda^{2-p}r_1^2 t, z), \quad t \in (-1,0),\ z \in \mathbb{R}^n.$$

Also, (3.3.15) implies

$$\lambda/4 \leq \|D\tilde{v}(x,t)\| \leq s + \|D\tilde{v}(x,t)\| \leq A\lambda \quad \forall\,(x,t) \in Q_1. \tag{3.3.18}$$

Since $D\tilde{v}$ is bounded, we know that

$$D\tilde{v} \in L^2_{\text{loc}}(-1,0;\, W^{1,2}_{\text{loc}}(B_1,\mathbb{R}^n)) \cap C^0(-1,0;\, L^2_{\text{loc}}(B_1,\mathbb{R}^n)).$$

See [23, Chapter 8, Section 3] for details. Therefore we can differentiate (3.3.17) in x_i-direction for each $i \in \{1,\ldots,n\}$ and get

$$(\tilde{v}_{x_i})_t - \operatorname{div}(A(x,t)D\tilde{v}_{x_i}) = 0, \quad \text{where } A(x,t) := \lambda^{2-p}\partial_z\tilde{a}_0(t, D\tilde{v}(x,t)). \tag{3.3.19}$$

By (3.3.18), the matrix $A(x,t)$ is uniformly elliptic, namely, for any $\eta \in \mathbb{R}^n$

$$\langle A(x,t)\eta,\eta\rangle \geq \nu\lambda^{2-p}(|D\tilde{v}(x,t)|^2 + s^2)^{\frac{p-2}{2}}|\eta|^2 \geq \nu(\sqrt{n}A)^{p-2}|\eta|^2,$$

and

$$|A(x,t)| \leq L\lambda^{2-p}(|D\tilde{v}(x,t)|^2 + s^2)^{\frac{p-2}{2}} \leq 4^{2-p}L.$$

holds whenever $\xi \in \mathbb{R}^n$ for some constant $\Lambda \geq 1$ depending only on n, p, ν, L, A.

Therefore we can apply Lemma 3.3.6 to $\tilde{v}_{x_i}$ and get

$$\phi_q(\tilde{v}_{x_i}, Q_{\delta_0}) \le c\delta_0^\beta \phi_q(\tilde{v}_{x_i}, Q_1) \tag{3.3.20}$$

for any $\delta_0 \in (0,1)$ and $i \in \{1,\dots,n\}$, where $\beta = \beta(n,p,\nu,L,A) \in (0,1)$ and $c = c(n,p,\nu,L,A,q) \ge 1$. Let $\Theta_0 := (\mathbf{m}(\tilde{v}_{x_1}, Q_{\delta_0}),\dots,\mathbf{m}(\tilde{v}_{x_n}, Q_{\delta_0}))$ and $\Theta_1 := (\mathbf{m}(\tilde{v}_{x_1}, Q_1),\dots,\mathbf{m}(\tilde{v}_{x_n}, Q_1))$. Using the triangle inequality and Jensen's inequality, we have

$$\phi_q(D\tilde{v}, Q_{\delta_0}) \le \Big(\fint_{Q_{\delta_0}} |D\tilde{v} - \Theta_0|^q \, dxdt \Big)^{1/q} \le n^{1/q-1} \sum_{i=1}^n \phi_q(\tilde{v}_{x_i}, Q_{\delta_0}). \tag{3.3.21}$$

On the other hand, by the definition of ϕ_q, we have

$$\phi_q(\tilde{v}_{x_i}, Q_1) \le \phi_q(D\tilde{v}, Q_1), \quad \forall\, i \in \{1,\dots,n\}$$

and therefore

$$\sum_{i=1}^n \phi_q(\tilde{v}_{x_i}, Q_1) \le n\phi_q(D\tilde{v}, Q_1). \tag{3.3.22}$$

By (3.3.20), (3.3.21), and (3.3.22), we obtain

$$\phi_q(D\tilde{v}, Q_{\delta_0}) \le c'\delta_0^\beta \phi_q(D\tilde{v}, Q_1) \tag{3.3.23}$$

for some $c' = c'(n,p,\nu,L,A,q) \ge 1$. Rescaling back in v, (3.3.23) becomes

$$\phi_q(Dv, Q_{\delta r}^\lambda) \le c'\delta^\beta \phi_q(Dv, Q_{r/2}^\lambda), \tag{3.3.24}$$

where $\delta = \delta_0/2 \in (0,1/2)$. Moreover, by the definition of ϕ_q, we see that

$$\phi_q(Dv, Q_{r/2}^\lambda) \le \Big(\fint_{Q_{r/2}^\lambda} |Dv - \mathbf{m}(Dv, Q_r^\lambda)|^q \, dxdt \Big)^{1/q} \le 2^{(n+2)/q} \phi_q(Dv, Q_r^\lambda). \tag{3.3.25}$$

Combining (3.3.24) and (3.3.25), we obtain (3.3.14) for any $\delta \in (0, 1/2)$. The proof is now completed. $\square$

Similarly, we have the following result for relatively large s.

Proposition 3.3.8. *Let $\lambda, A > 0$ be constants. Assume that*

$$\sup_{Q_r^\lambda} \|Dv\| \le A\lambda \quad \textit{and} \quad 0 < \xi\lambda \le s \le \xi_1 A\lambda, \quad \textit{where } 0 < \xi \le \xi_1 A. \tag{3.3.26}$$

Then there exist constants $\beta_1 \in (0,1)$ depending on n, p, ν, L, A, ξ, ξ_1, and $\tilde{c}_d \ge 1$ depending on n, p, ν, L, A, ξ, ξ_1, q, such that

$$\phi_q(Dv, Q_{\delta r}^\lambda) \le \tilde{c}_d \delta^{\beta_1} \phi_q(Dv, Q_r^\lambda) \tag{3.3.27}$$

holds for any $\delta \in (0, 1)$.

Proof. First, we rescale the solution in Q_1 as in (3.3.16), but this time with $r_1 = r$. Then the rescaled solution $\tilde{v}$ stills satisfies (3.3.17) in Q_1 and $\tilde{v}_{x_i}$ solves (3.3.19). This time we can use (3.3.26) to get

$$\langle A(x,t)\eta, \eta\rangle \ge \nu\lambda^{2-p}(|D\tilde{v}(x,t)|^2 + s^2)^{\frac{p-2}{2}}|\eta|^2 \ge \nu(n + \xi_1^2)^{\frac{p-2}{2}} A^{p-2}|\eta|^2,$$

and

$$|A(x,t)| \le L\lambda^{2-p}(|D\tilde{v}(x,t)|^2 + s^2)^{\frac{p-2}{2}} \le \xi^{p-2}L.$$

Then, we can proceed exactly as in Proposition 3.3.7 and eventually obtain (3.3.27). $\square$

Now we are ready to give the proofs of Theorems 3.3.2 and 3.3.3.

Proof of Theorem 3.3.2. Our proof closely follows that of [55, Theorem 3.1] using the "singular iteration" argument with Propositions 3.3.7, 3.3.8, and 3.3.5 in place of

[55, Propositions 3.8, 3.9, and 3.11]. More specifically, the case when Proposition 3.3.7 occurs is called the nonsingular alternative and the case when Proposition 3.3.5 occurs is called the singular alternative. The main idea is to construct a chain of intrinsic cylinders where the singular alternative occurs and show that the chain will stop in a finite time because of the occurrence of the nonsingular alternative. As in [55], we first assume $s > 0$ in Steps 1-5 below.

Step 1: Stopping time for the singular iteration and the choice of ξ. Let $\eta_1 := (1+\eta)/2 \in (0,1)$, where η is given in Proposition 3.3.5, and therefore η_1 depends only on n, p, ν, L, and A. Define m as the smallest integer such that

$$\eta_1^m A < \frac{1}{2B}. \tag{3.3.28}$$

Thus we know that m depends on n, p, ν, L, A, B, and we have

$$\frac{\log(2AB)}{-\log(\eta_1)} < m \leq 1 + \frac{\log(2AB)}{-\log(\eta_1)}. \tag{3.3.29}$$

Next we choose

$$\xi := \min\{1/8,\, (\eta_1 - \eta)\eta_1^{2m} A\}, \tag{3.3.30}$$

which also depends on n, p, ν, L, A, and B.

Step 2: The first nonsingular case: $\xi\lambda \leq s$. Proposition 3.3.8 implies that (3.3.3) holds whenever

$$\delta_\gamma \leq (\gamma/\tilde{c}_d)^{1/\beta_1}. \tag{3.3.31}$$

Note that here $\beta_1 \in (0,1)$ depends only on n, p, ν, L, A, B, and $\tilde{c}_d \geq 1$ depends on n, p, ν, L, A, B, and q. From now on, we assume instead

$$s < \xi\lambda \leq (\eta_1 - \eta)\eta_1^{2m} A\lambda, \tag{3.3.32}$$

where the second inequality always holds by the definition of ξ in (3.3.30).

Step 3: The singular iteration. Given a cylinder Q_r^λ, where (3.3.6) holds, then by Propositions 3.3.7 and 3.3.5, we know that at least one of (3.3.14) and (3.3.9) must hold. The case when Proposition 3.3.7 applies is called the nonsingular alternative, while the case when Proposition 3.3.5 applies is called the singular alternative. We now let $\sigma_1 = \sigma/2$, where σ is the constant in Proposition 3.3.4, and therefore σ_1 depends only on n, p, ν, L, and A. We define the sequences $\lambda_0 := \lambda$, $\lambda_{i+1} := \eta_1\lambda_i$ and $R_0 := r$, $R_{i+1} := \sigma_1 R_i$. Exactly as in the proof of [55, Theorem 3.1], we build the singular iteration scheme by induction. Assume that the singular alternative holds in $Q_{R_{i-1}}^{\lambda_{i-1}}$ whenever $i \in \{1, \ldots, j\}$ for some $1 \le j \le 2m$. Arguing exactly as in [55], under the condition (3.3.32) we obtain

$$s + \sup_{Q_{R_j}^{\lambda_j}} \|Dv\| \le A\lambda_j, \tag{3.3.33}$$

which verifies the upper bound (3.3.6) in the cylinder $Q_{R_j}^{\lambda_j}$. Then we can determine whether the singular alternative or the nonsingular alternative occurs in $Q_{R_j}^{\lambda_j}$.

Step 4: The second nonsingular case. Let

$$\delta_\gamma \le \underline{\delta} := (\eta_1^{(2-p)/2}\sigma_1)^{m+1}, \tag{3.3.34}$$

and define

$$\tilde{m} = \min\{k \in \mathbb{N} : \text{ the singular alternative does not hold in } Q_{R_k}^{\lambda_k}\}. \tag{3.3.35}$$

By (3.3.2), (3.3.32), and the definition of m in (3.3.28), arguing exactly as in [55], we have $\tilde{m} \leq m$. Moreover, by (3.3.33) and the second inequality in (3.3.2), it holds that

$$s + \sup_{Q^{\lambda_j}_{R_j}} \|Dv\| \leq A\lambda_j, \quad \text{for } j \in \{0, \ldots, \tilde{m}\}. \tag{3.3.36}$$

Since (3.3.36) in particular holds for $j = \tilde{m}$, we can apply Proposition 3.3.7 in $Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}$ and get

$$\phi_q(Dv, Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}}) \leq c_d \delta^\beta \phi_q(Dv, Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}) \tag{3.3.37}$$

for any $\delta \in (0,1)$, where $c_d \geq 1$ is a constant depending on n, p, ν, L, A, and q. Moreover, using the fact that $\tilde{m} \leq m$, we have

$$\begin{aligned}\phi_q(Dv, Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}) &\leq \Big(\frac{|Q^\lambda_r|}{|Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}|}\Big)^{1/q} \Big(\fint_{Q^\lambda_r} |Dv - \mathbf{m}(Dv, Q^\lambda_r)|^q \, dxdt\Big)^{1/q} \\ &\leq \sigma_1^{-m(n+2)/q} \eta_1^{-m(2-p)/q} \phi_q(Dv, Q^\lambda_r).\end{aligned} \tag{3.3.38}$$

Furthermore, by the definition of $\underline{\delta}$ in (3.3.34) and again the fact that $\tilde{m} \leq m$, we have $Q^\lambda_{\delta\underline{\delta}r} \subset Q^{\lambda_m}_{\delta R_m} \subset Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}}$ and thus

$$\begin{aligned}\phi_q(Dv, Q^\lambda_{\delta\underline{\delta}r}) &\leq \Big(\frac{|Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}}|}{|Q^\lambda_{\delta\underline{\delta}r}|}\Big)^{1/q} \Big(\fint_{Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}}} |Dv - \mathbf{m}(Dv, Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}})|^q \, dxdt\Big)^{1/q} \\ &\leq \underline{\delta}^{-(n+2)/q} \phi_q(Dv, Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}}).\end{aligned} \tag{3.3.39}$$

Combining (3.3.37), (3.3.38), and (3.3.39), we obtain

$$\phi_q(Dv, Q^\lambda_{\delta\underline{\delta}r}) \leq c_d \delta^\beta \Big[\underline{\delta}^{-(n+2)} \sigma_1^{-m(n+2)} \eta_1^{-m(2-p)}\Big]^{1/q} \phi_q(Dv, Q^\lambda_r)$$

for any $\delta \in (0,1)$. Thus (3.3.3) holds for any

$$\delta_\gamma := \delta\underline{\delta}, \quad \text{with} \quad \delta \leq \Big(\frac{\gamma}{c_d}\Big)^{\frac{1}{\beta}} \Big(\underline{\delta}^{n+2} \sigma_1^{m(n+2)} \eta_1^{m(2-p)}\Big)^{\frac{1}{q\beta}}. \tag{3.3.40}$$

Step 5: Determining the constant δ_γ. In view of conditions (3.3.31) and (3.3.40), we can define

$$\delta_\gamma = \underline{\delta}\Big(\underline{\delta}^{n+2}\sigma_1^{m(n+2)}\eta_1^{m(2-p)}\Big)^{\frac{1}{q\beta}}\left(\frac{\gamma}{c_d+\tilde{c}_d}\right)^{\frac{1}{\min\{\beta,\beta_1\}}},$$

where we recall that $\underline{\delta} = (\eta_1^{(2-p)/2}\sigma_1)^{m+1}$. The proof is now completed for the case $s > 0$.

Step 6: The case $s = 0$ via approximation. The approximation scheme introduced in [55, Section 3.3] also works perfectly here. The only differences are that v is in place of w and that instead of choosing $\tilde{\gamma} = 2^{-(n+2)}\gamma$ as in [55, Eq. (3.86)], we now take $\tilde{\gamma} = 2^{-(n+2)/q}\gamma$. We also remark that for a strongly convergent sequence in L^p for some $p \geq 1$, the L^q-mean oscillation of the sequence also converges for any $q \in (0, 1)$. Indeed, by the triangle inequality, we have

$$\begin{aligned}(\phi_q(Dv_\varepsilon, Q_\rho^\lambda))^q &\leq \fint_{Q_\rho^\lambda} |Dv_\varepsilon - \mathbf{m}(Dv, Q_\rho^\lambda)|^q \, dxdt \\ &\leq (\phi_q(Dv, Q_\rho^\lambda))^q + \fint_{Q_\rho^\lambda} |Dv_\varepsilon - Dv|^q \, dxdt.\end{aligned}$$

Switching Dv_ε with Dv, we know that

$$|(\phi_q(Dv_\varepsilon, Q_\rho^\lambda))^q - (\phi_q(Dv, Q_\rho^\lambda))^q| \leq \fint_{Q_\rho^\lambda} |Dv_\varepsilon - Dv|^q \, dxdt.$$

Therefore if $Dv_\varepsilon \to Dv$ strongly in $L^p(Q_\rho^\lambda)$, then by Hölder's inequality and the previous estimate, we have $\phi_q(Dv_\varepsilon, Q_\rho^\lambda) \to \phi_q(Dv, Q_\rho^\lambda)$. □

Proof of Theorem 3.3.3. The proof also closely follow that of [55, Theorem 3.5] and therefore we only give an outline and indicate the necessary modifications. As in the proof of Theorem 3.3.2, we assume $s > 0$ and use the singular iteration. The case $s = 0$ follows by applying the same approximation scheme as in the proof of Theorem 3.3.2. We still define $\tilde{m}$ as in (3.3.35). However, in this time since (3.3.32) is no longer in force, the iteration may stop at $\tilde{m}$ either because the nonsingular alternative occurs or because the upper bound

$$s+\sup_{Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}}\|Dv\|\le A\lambda_{\tilde{m}}$$

does not hold. In the former case, we proceed as in Theorem 3.3.2 and we can take δ_γ as in (3.3.40). In the latter case, exactly as in Step 4 of the proof of [55, Theorem 3.5], we can show that

$$\sup_{Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}}\|Dv\|\le A\lambda_{\tilde{m}}\quad\text{and}\quad(\eta_1-\eta)A\lambda_{\tilde{m}}\le s\le\frac{A}{\eta_1}\lambda_{\tilde{m}}.$$

Applying Proposition 3.3.8 in $Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}}$ with $\xi=(\eta_1-\eta)A$ and $\xi_1=1/\eta_1$, we obtain

$$\phi_q(Dv,Q^{\lambda_{\tilde{m}}}_{\delta R_{\tilde{m}}})\le\tilde{c}'_d\delta^{\beta'_1}\phi_q(Dv,Q^{\lambda_{\tilde{m}}}_{R_{\tilde{m}}})\tag{3.3.41}$$

for any $\delta\in(0,1)$, where the constant $\tilde{c}'_d\ge 1$ depends on n, p, ν, L, A, q and the constant $\beta'_1\in(0,1)$ depends only on n, p, ν, L, A. With the estimate (3.3.41) in place of (3.3.37), we can proceed as in Theorem 3.3.2, Step 4. In this case, (3.3.4) holds for any

$$\delta_\gamma:=\delta\underline{\delta},\quad\text{with}\quad\delta\le\left(\frac{\gamma}{\tilde{c}'_d}\right)^{\frac{1}{\beta'_1}}\left(\underline{\delta}^{n+2}\sigma_1^{m(n+2)}\eta_1^{m(2-p)}\right)^{\frac{1}{q\beta'_1}}.$$

Finally, to ensure the above inequality and (3.3.40), we can choose

$$\delta_\gamma:=\underline{\delta}\Big(\underline{\delta}^{n+2}\sigma_1^{m(n+2)}\eta_1^{m(2-p)}\Big)^{\frac{1}{\min\{\beta,\beta'_1\}q}}\left(\frac{\gamma}{c_d+\tilde{c}'_d}\right)^{\frac{1}{\min\{\beta,\beta'_1\}}},\tag{3.3.42}$$

where $\underline{\delta}=(\eta_1^{(2-p)/2}\sigma_1)^{m+1}$. Since the constants σ_1, η_1, β, β'_1 depend only on n, p, ν, L, A, and the constants c_d, $\tilde{c}'_d$ depend only on n, p, ν, L, A, q, we know that the only constant depending on B in (3.3.42) is m. From the characterization of m described in (3.3.29), we can take δ_γ in the form of (3.3.5) by further decreasing the constant in

(3.3.42). More specifically, we can take

$$\alpha_1 := \frac{\min\{\beta, \beta_1'\}q}{(n+5) + (2n+5)\log(\sigma_1)/\log(\eta_1)}, \quad \alpha_2 := \min\{\beta, \beta_1'\},$$

and

$$c(A) := (2A)^{1/\alpha_1}\eta_1^{p-2}\sigma_1^{-2}\Big(\frac{c_d + \tilde{c}_d'}{\eta_1^{2(n+3)/q}\sigma_1^{3(n+2)/q}}\Big)^{\frac{1}{\alpha_2}}.$$

The proof is now completed. □

3.4 Pointwise gradient estimates

This section is devoted to the proofs of the pointwise gradient estimates. We derive some comparison results in Section 3.4.1 and give the proofs of Theorems 3.1.1 and 3.1.2 in Section 3.4.2.

3.4.1 Comparison results

Let $\rho, \lambda > 0$ and $Q_\rho^\lambda := Q_\rho^\lambda(x_0, t_0) \subset \Omega_T$ be a parabolic cylinder. We consider the unique solution $w \in C^0([t_0-\lambda^{2-p}\rho^2, t_0);\ L^2(B_\rho(x_0)))\cap L^p(t_0-\lambda^{2-p}\rho^2, t_0;\ W^{1,p}(B_\rho(x_0)))$ to the Cauchy-Dirichlet problem:

$$\begin{cases} w_t - \operatorname{div}(a(x,t,Dw)) = 0 & \text{in} \quad Q_\rho^\lambda, \\ \qquad\qquad\qquad\quad w = u & \text{on} \quad \partial_{\mathrm{par}} Q_\rho^\lambda. \end{cases} \tag{3.4.1}$$

We have the following comparison estimate between u and w from [80, Lemma 3.1].

Lemma 3.4.1. *Let w be a solution to (3.4.1) under the assumptions (3.1.2)–(3.1.5) and assume that $p \in (\frac{3n+2}{2n+2}, 2 - \frac{1}{n+1}]$. Then there exists a constant $c_3 = c_3(n, p, \nu, L, q) \geq 1$*

such that

$$\left(\fint_{Q_\rho^\lambda} |Du - Dw|^q \, dxdt\right)^{1/q}$$
$$\leq c_3 \left[\frac{|\mu|(Q_\rho^\lambda)}{|Q_\rho^\lambda|^{\frac{n+1}{n+2}}}\right]^{\frac{n+2}{(n+1)p-n}} + c_3 \left[\frac{|\mu|(Q_\rho^\lambda)}{|Q_\rho^\lambda|^{\frac{n+1}{n+2}}}\right] \left(\fint_{Q_\rho^\lambda} (|Du| + s)^q \, dxdt\right)^{\frac{(2-p)(n+1)}{q(n+2)}}$$

for any constant q such that $\frac{n+2}{2(n+1)} < q < p - \frac{n}{n+1} \leq 1$.

We remark that only the case $s = 0$ was considered in [80, Lemma 3.2], but their proof also works in the case $s > 0$ with $V_s(\xi) := (|\xi|^2 + s^2)^{\frac{p-2}{4}}\xi$ in place of $V(\xi) := |\xi|^{\frac{p-2}{2}}\xi$ for $\xi \in \mathbb{R}^n$.

We also have a reverse Hölder type inequality for Dw.

Theorem 3.4.2. *Let $\lambda > 0$ and w be a solution to (3.4.1) under the assumptions (3.1.2) and (3.1.3) and assume that* $\max\{1, \frac{2n}{n+2}\} < p < 2$. *Then there exists a constant $c_4 = c_4(n, p, \nu, L, q) \geq 1$ such that*

$$\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt \leq c_4\lambda^p + c_4 s^p + c_4 \lambda^{\frac{n(p-2)(p-q)}{n(p-2)+2q}} \left(\fint_{Q_\rho^\lambda} (|Dw| + s)^q \, dxdt\right)^{\frac{n(p-2)+2p}{n(p-2)+2q}} \tag{3.4.2}$$

holds for every $q \in (\frac{n(2-p)}{2}, \frac{2n}{n+2})$ when $n \geq 2$, and that

$$\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt \leq c_4\lambda^p + c_4 s^p + c_4 \lambda^{\frac{(p-2)(p-q)}{p+q-2}} \left(\fint_{Q_\rho^\lambda} (|Dw| + s)^q \, dxdt\right)^{\frac{2p-2}{p+q-2}} \tag{3.4.3}$$

holds for every $q \in (2 - p, 1)$ when $n = 1$.

Proof. This type of estimates can be deduced from higher integrability results as in [11, Theorem 1] and standard interpolation and iteration arguments as in [41, Remark 6.12]. See also [51], [80, Lemma 3.3]. For completeness, we give a direct proof below. First, we note that it suffices to show (3.4.2) and (3.4.3) for the special case when

$\lambda = \rho = 1$, $(x_0, t_0) = 0$ by using a standard rescaling argument. Indeed, we can define

$$w_1(x,t) := (\lambda\rho)^{-1} w(x_0 + \rho x, t_0 + \lambda^{2-p}\rho^2 t)$$

and

$$a_1(x,t,\xi) := \lambda^{1-p} a(x_0 + \rho x, t_0 + \lambda^{2-p}\rho^2 t, \lambda\xi).$$

Then w_1 solves

$$\partial_t w_1 - \operatorname{div}(a_1(x,t,Dw_1)) = 0 \quad \text{in} \quad Q_1^1(0) \equiv B_1(0) \times (-1, 0)$$

and a_1 satisfies the assumptions (3.1.2) and (3.1.3) with $s_1 := \lambda^{-1}s$ in place of s. We can see that if (3.4.2) (or (3.4.3)) holds for w_1 in $Q_1^1(0)$ with $s_1 := \lambda^{-1}s$ in place of s, then by rescaling back to w, (3.4.2) (or (3.4.3)) also holds for w in Q_ρ^λ.

Next, we prove (3.4.2) and (3.4.3) for the case when $\lambda = \rho = 1$, $(x_0, t_0) = 0$. For simplicity, in this proof, we denote $B := B_1(0)$, $Q := Q_1^1(0) \equiv B_1(0) \times (-1, 0)$, and $\frac{1}{2}Q := \frac{1}{2}Q_1^1(0) \equiv B_{\frac{1}{2}}(0) \times (-\frac{1}{4}, 0)$. First we take $\zeta_1 \in C_0^\infty(B)$ such that $0 \le \zeta_1 \le 1$ in B, $\zeta_1 = 1$ in $\frac{1}{2}B$ and $|D\zeta_1| \le 4$ in B. We then define $\zeta := \zeta_1 / (\int_B \zeta_1(x)\,dx)$ and therefore $\int_B \zeta(x)\,dx = 1$. Now we define $\bar{w}(t) := \int_B w(x,t)\zeta(x)\,dx$. If $n \ge 2$, it follows from the Sobolev-Poincáre inequality that

$$\int_B |w(x,t) - \bar{w}(t)|^2\,dx \le c\Big(\int_B |Dw(x,t)|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n}{n+2}} \tag{3.4.4}$$

holds for some constant $c = c(n)$. On the other hand, if $n = 1$, then we have

$$\int_B |w(x,t) - \bar{w}(t)|^2\,dx \le |B| \sup_B |w - \bar{w}(t)|^2 \le c\Big(\int_B |Dw(x,t)|\,dx\Big)^2. \tag{3.4.5}$$

We now proceed with the case when $n \ge 2$ and we will indicate necessary modifications for the case when $n=1$ at the end of our proof. By (3.4.1), we know that

$$\frac{d}{dt}\bar{w}(t) = -\int_B a(x,t,Dw)D\zeta\,dx := g(t)$$

holds in distribution sense and therefore the equation (3.4.1) also reads

$$(w-\bar{w})_t - \operatorname{div}(a(x,t,Dw)) + g(t) = 0 \quad \text{in} \quad Q. \tag{3.4.6}$$

Note that by (3.1.2), we have

$$\int_{-1}^{0} |g(t)|dt \le c\int_Q (|Dw|^2+s^2)^{\frac{p-1}{2}}\,dxdt \tag{3.4.7}$$

for some constant $c = c(n,p,L)$. Next, we choose a nondecreasing smooth function $\eta : \mathbb{R} \to [0,1]$ satisfying $\eta \equiv 1$ on $[-1/4,\infty)$, $\eta \equiv 0$ on $(-\infty,-1]$ and $|\eta'| \le 2$ in $\mathbb{R}$. We then take φ as the product of ζ and η, namely, $\varphi(x,t) = \zeta(x)\eta(t)$ for every $(x,t) \in Q$. We also choose

$$\tilde{p} := \max\left\{\frac{2p}{2-p}, \frac{n+2}{n}\right\}.$$

Now formally we test the equation (3.4.6) with $\psi := (w-\bar{w})\varphi^{\tilde{p}}1_{(-\infty,\tau]}(t)$ for $\tau \in (-1,0)$ and get

$$\begin{aligned}
0 &= \frac{1}{2}\int_B (w(x,\tau)-\bar{w}(\tau))^2(\varphi(x,\tau))^{\tilde{p}}\,dx - \frac{\tilde{p}}{2}\int_{B\times(-1,\tau]}(w-\bar{w})^2\varphi^{\tilde{p}-1}\varphi_t\,dxdt \\
&\quad + \tilde{p}\int_{B\times(-1,\tau]} a(x,t,Dw)(w-\bar{w})\varphi^{\tilde{p}-1}D\varphi\,dxdt \\
&\quad + \int_{B\times(-1,\tau]} a(x,t,Dw)Dw\,\varphi^{\tilde{p}}\,dxdt + \int_{B\times(-1,\tau]}(w-\bar{w})\varphi^{\tilde{p}}g(t)\,dxdt \\
&:= \mathrm{I} + \mathrm{II} + \mathrm{III} + \mathrm{IV} + \mathrm{V}.
\end{aligned} \tag{3.4.8}$$

Note that the computations above can be justified in a standard way using Steklov averages as in [23]. Then we estimate the terms in (3.4.8). By Hölder's inequality, Young's inequality with conjugate exponents $(\tilde{p},\ \tilde{p}/(\tilde{p}-1))$, and (3.4.4), we have

$$
\begin{aligned}
|\mathrm{II}| &\le \frac{\tilde{p}}{2}\int_{-1}^{\tau}\Big(\int_B (w-\bar{w})^2\varphi^{\tilde{p}}\,dx\Big)^{\frac{\tilde{p}-1}{\tilde{p}}}\Big(\int_B (w-\bar{w})^2\varphi_t^{\tilde{p}}\,dx\Big)^{\frac{1}{\tilde{p}}}dt \\
&\le \frac{\tilde{p}}{2}\sup_{t\in(-1,0)}\Big(\int_B (w-\bar{w}(t))^2\varphi^{\tilde{p}}\,dx\Big)^{\frac{\tilde{p}-1}{\tilde{p}}}\int_{-1}^{0}\Big(\int_B (w-\bar{w})^2\varphi_t^{\tilde{p}}\,dx\Big)^{\frac{1}{\tilde{p}}}dt \\
&\le \frac{1}{8}\sup_{t\in(-1,0)}\int_B (w-\bar{w}(t))^2\varphi^{\tilde{p}}\,dx + c\Big(\int_{-1}^{0}\Big(\int_B (w-\bar{w})^2\,dx\Big)^{\frac{1}{\tilde{p}}}dt\Big)^{\tilde{p}} \\
&\le \frac{1}{8}\sup_{t\in(-1,0)}\int_B (w-\bar{w}(t))^2\varphi^{\tilde{p}}\,dx + c\Big(\int_{-1}^{0}\Big(\int_B |Dw|^{\frac{2n}{n+2}}\,dx\Big)^{\frac{n+2}{n\tilde{p}}}dt\Big)^{\tilde{p}} \\
&\le \frac{1}{8}\sup_{t\in(-1,0)}\int_B (w-\bar{w}(t))^2\varphi^{\tilde{p}}\,dx + c\Big(\int_Q |Dw|^{\frac{2n}{n+2}}\,dxdt\Big)^{\frac{n+2}{n}}
\end{aligned}
\tag{3.4.9}
$$

for some constant $c=c(n,p)$. Here we also used the fact that $\frac{n+2}{n\tilde{p}}\le 1$ in the last line.

By (3.1.2), Young's inequality with exponents $(p/(p-1),\ p)$ and $(2/p,\ 2/(2-p))$, and the fact that $\frac{2}{p}(\tilde{p}-p)\ge\tilde{p}$, we have

$$
\begin{aligned}
|\mathrm{III}| &\le \tilde{p}L\int_{B\times(-1,\tau]}(s^2+|Dw|^2)^{\frac{p-1}{2}}|w-\bar{w}|\varphi^{\tilde{p}-1}|D\varphi|\,dxdt \\
&\le \frac{\nu}{2}\int_{B\times(-1,\tau]}(s^2+|Dw|^2)^{\frac{p}{2}}\varphi^{\tilde{p}}\,dxdt + c\int_{B\times(-1,\tau]}\varphi^{\tilde{p}-p}|w-\bar{w}|^p|D\varphi|^p\,dxdt \\
&\le \frac{\nu}{2}\int_{B\times(-1,\tau]}(s^2+|Dw|^2)^{\frac{p}{2}}\varphi^{\tilde{p}}\,dxdt + \frac{1}{8}\int_{B\times(-1,\tau]}|w-\bar{w}|^2\varphi^{\tilde{p}}\,dxdt \\
&\quad + c\int_{B\times(-1,\tau]}|D\varphi|^{\frac{2p}{2-p}}\,dxdt \\
&\le \frac{\nu}{2}\int_{B\times(-1,\tau]}|Dw|^p\varphi^{\tilde{p}}\,dxdt + \frac{1}{8}\sup_{t\in(-1,0)}\int_B (w-\bar{w}(t))^2\varphi^{\tilde{p}}\,dx + c(1+s^p)
\end{aligned}
\tag{3.4.10}
$$

for some constant $c=c(n,p,\nu,L)$. Here ν is the constant in (3.1.3).

By (3.1.3), we know that $a(x,t,\xi)\xi\ge\nu(s^2+|\xi|^2)^{\frac{p-2}{2}}|\xi|^2\ge\nu(\xi^p-s^p)$ for every $\xi\in\mathbb{R}^n$ and therefore

$$
\mathrm{IV}\ge\nu\int_{B\times(-1,\tau]}|Dw|^p\varphi^{\tilde{p}}\,dxdt - cs^p
\tag{3.4.11}
$$

for some constant $c = c(n)$.

Finally, by Hölder's inequality, Young's inequality with conjugate exponents 2 and 2, and the estimate (3.4.7), we obtain

$$
\begin{aligned}
|\mathrm{V}| &\le \int_{-1}^{\tau} \Big(\int_B |w-\bar{w}|^2 \varphi^{\tilde{p}}\, dx \Big)^{\frac{1}{2}} \Big(\int_B |g(t)|^2 \varphi^{\tilde{p}}\, dx \Big)^{\frac{1}{2}} dt \\
&\le c \sup_{t\in(-1,0)} \Big(\int_B (w-\bar{w}(t))^2 \varphi^{\tilde{p}}\, dx \Big)^{\frac{1}{2}} \int_{-1}^{\tau} |g(t)| dt \\
&\le \frac{1}{8} \sup_{t\in(-1,0)} \int_B (w-\bar{w}(t))^2 \varphi^{\tilde{p}}\, dx + c\Big(\int_Q |Dw|^{p-1}\, dxdt \Big)^2 + c(1+s^p)
\end{aligned} \tag{3.4.12}
$$

for some constant $c = c(n, p, L)$. Here we also used the fact that $2(p-1) \le p$ and therefore $s^{2(p-1)} \le 1 + s^p$ in the last inequality. Using the estimates (3.4.9)–(3.4.12) in (3.4.8), we get

$$
\begin{aligned}
&\frac{1}{2} \int_B (w(x,\tau) - \bar{w}(\tau))^2 (\varphi(x,\tau))^{\tilde{p}}\, dx + \frac{\nu}{2} \int_{B\times(-1,\tau]} |Dw|^p \varphi^{\tilde{p}}\, dxdt \\
&\le \frac{3}{8} \sup_{t\in(-1,0)} \int_B (w-\bar{w}(t))^2 \varphi^{\tilde{p}}\, dx + c \Big(\int_Q |Dw|^{\frac{2n}{n+2}}\, dxdt \Big)^{\frac{n+2}{n}} \\
&+ c \Big(\int_Q |Dw|^{p-1}\, dxdt \Big)^2 + c(1+s^p)
\end{aligned}
$$

for another constant $c = c(n, p, \nu, L)$. By taking the supremum over $\tau \in (-1, 0)$, we have

$$
\begin{aligned}
&\frac{1}{8} \sup_{t\in(-1,0)} \int_B (w-\bar{w}(t))^2 \varphi^{\tilde{p}}\, dx \\
&\le c \Big(\int_Q |Dw|^{\frac{2n}{n+2}}\, dxdt \Big)^{\frac{n+2}{n}} + c \Big(\int_Q |Dw|^{p-1}\, dxdt \Big)^2 + c(1+s^p),
\end{aligned}
$$

and therefore

$$
\begin{aligned}
&\int_{\frac{1}{2}Q} |Dw|^p \, dxdt \le c \int_{B\times(-1,0)} |Dw|^p \varphi^{\tilde{p}} \, dxdt \\
&\le c \Big(\int_Q |Dw|^{\frac{2n}{n+2}} \, dxdt \Big)^{\frac{n+2}{n}} + c \Big(\int_Q |Dw|^{p-1} \, dxdt \Big)^2 + c(1+s^p)
\end{aligned}
\tag{3.4.13}
$$

for some constant $c = c(n,p,\nu,L)$. Next, we interpolate the terms in (3.4.13). We recall that we assume $p \in (\frac{2n}{n+2}, 2)$ and $q \in (\frac{n(2-p)}{2}, \frac{2n}{n+2})$ for $n \ge 2$. Since $q < \frac{2n}{n+2} < p$, by Hölder's inequality, we have

$$
\Big(\int_Q |Dw|^{\frac{2n}{n+2}} \, dxdt \Big)^{\frac{n+2}{n}} \le \Big(\int_Q |Dw|^p \, dxdt \Big)^{\alpha \frac{n+2}{n}} \Big(\int_Q |Dw|^q \, dxdt \Big)^{(1-\alpha)\frac{n+2}{n}}, \tag{3.4.14}
$$

where $\alpha \in (0,1)$ is a constant such that $\alpha p + (1-\alpha) q = \frac{2n}{n+2}$. Namely,

$$
\alpha = (\frac{2n}{n+2} - q)/(p-q).
$$

Since $q > \frac{n(2-p)}{2}$, we know that $\alpha \frac{n+2}{n} < 1$. Therefore by (3.4.14) and Young's inequality with exponents $1/(\alpha \frac{n+2}{n})$ and $1/(1-\alpha\frac{n+2}{n})$, we obtain

$$
\Big(\int_Q |Dw|^{\frac{2n}{n+2}} \, dxdt \Big)^{\frac{n+2}{n}} \le \varepsilon \int_Q |Dw|^p \, dxdt + c_\varepsilon \Big(\int_Q |Dw|^q \, dxdt \Big)^{\frac{n(p-2)+2p}{n(p-2)+2q}} \tag{3.4.15}
$$

for any $\varepsilon \in (0,1)$, where c_ε is a constant depending only on ε, n, p, and q. We then estimate the term $\Big(\int_Q |Dw|^{p-1} \, dxdt \Big)^2$. Since $p - 1 < 1 \le \frac{2n}{n+2}$, by Hölder's inequality, we have

$$
\begin{aligned}
&\Big(\int_Q |Dw|^{p-1} \, dxdt \Big)^2 \le c \Big(\int_Q |Dw|^{\frac{2n}{n+2}} \, dxdt \Big)^{\frac{(p-1)(n+2)}{n}} \\
&\le c \Big[1 + \Big(\int_Q |Dw|^{\frac{2n}{n+2}} \, dxdt \Big)^{\frac{n+2}{n}} \Big]
\end{aligned}
\tag{3.4.16}
$$

for some constant $c = c(n,p,q)$. Therefore, it follows by using (3.4.15) and (3.4.16) in

(3.4.13), and a standard iteration argument that

$$\fint_{\frac{1}{2}Q}(|Dw|+s)^p\,dxdt \le c(1+s^p)+c\Big(\fint_Q(|Dw|+s)^q\,dxdt\Big)^{\frac{n(p-2)+2p}{n(p-2)+2q}}$$

holds for every $q \in (\frac{(2-p)n}{2}, \frac{2n}{n+2})$ and some constant $c = c(n,p,\nu,L,q)$. Note that the last estimate is exactly (3.4.2) in the case when $\lambda = \rho = 1$, $(x_0,t_0) = 0$. The proof for the case when $n \ge 2$ is now completed. Finally, we briefly indicate the modifications needed for the case when $n = 1$. In this case, we test the equation (3.4.6) with $(w-\bar{w})\varphi^{\tilde{p}'}1_{(-\infty,\tau]}(t)$ for $\tau \in (-1,0)$, where

$$\tilde{p}' := \max\big\{\frac{2p}{2-p}, 2\big\}.$$

Arguing exactly as in the case when $n \ge 2$, with the estimate (3.4.5) in place of (3.4.4), we obtain

$$\int_{\frac{1}{2}Q}|Dw|^p\,dxdt \le c\Big(\int_Q|Dw|\,dxdt\Big)^2 + c\Big(\int_Q|Dw|^{p-1}\,dxdt\Big)^2 + c(1+s^p). \quad (3.4.17)$$

Finally, it follows from the last estimate and similar interpolation and iteration arguments as in the case when $n \ge 2$ that

$$\fint_{\frac{1}{2}Q}(|Dw|+s)^p\,dxdt \le c(1+s^p)+c\Big(\fint_Q(|Dw|+s)^q\,dxdt\Big)^{\frac{2p-2}{p+q-2}}$$

holds for every $q \in (2-p,1)$ and some constant $c = c(p,\nu,L,q)$. This proves (3.4.3) in the case when $\lambda = \rho = 1$ and $(x_0,t_0) = 0$. The Theorem is now proved. $\square$

Remark 3.4.3. It can be seen from the proof of Theorem 3.4.2 that when $\max\{1, \frac{2n}{n+2}\} < p < 2$, reverse Hölder type estimates actually hold for every $q > \frac{n(2-p)}{2}$ if $n \ge 2$ and for every $q > 2-p$ if $n = 1$. Indeed, by (3.4.13), (3.4.17), Hölder's inequality, and similar rescaling arguments, there exists a constant $c = c(n, p, \nu, L, q) \geq 1$ such that

$$\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt \leq c\lambda^p + cs^p + c\lambda^{p-2}\Big(\fint_{Q_\rho^\lambda} (|Dw| + s)^q \, dxdt\Big)^{\frac{2}{q}}$$

holds for every $q \geq \frac{2n}{n+2}$ if $n \geq 2$, and for every $q \geq 1$ if $n = 1$.

Now we consider the unique solution $v \in C^0([t_0 - \lambda^{2-p}\rho^2, t_0); L^2(B_\rho(x_0))) \cap L^p(t_0 - \lambda^{2-p}\rho^2, t_0; W^{1,p}(B_\rho(x_0)))$ to the Cauchy-Dirichlet problem:

$$\begin{cases} v_t - \operatorname{div}(a(x_0, t, Dv)) = 0 \quad \text{in} \quad \dfrac{1}{2}Q_\rho^\lambda, \\ \qquad\qquad\qquad v = w \quad \text{on} \quad \partial_{\mathrm{par}}\big(\dfrac{1}{2}Q_\rho^\lambda\big). \end{cases} \tag{3.4.18}$$

We have the following comparison result between w and v.

Lemma 3.4.4. *Let w be a solution to (3.4.1) and v be a solution to (3.4.18) under the assumptions (3.1.2)–(3.1.5). Assume that $p \in (1, 2)$. Then there exists a constant $c_5 = c_5(n, p, \nu, L) \geq 1$, such that*

$$\fint_{\frac{1}{2}Q_\rho^\lambda} |Dw - Dv|^p \, dxdt \leq c_5\big[\omega(\rho)\big]^p \fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt. \tag{3.4.19}$$

Proof. By [54, Remark 4.1], we know that

$$\fint_{\frac{1}{2}Q_\rho^\lambda} |V_s(Dw) - V_s(Dv)|^2 \, dxdt \leq c\big[\omega(\rho/2)\big]^2 \fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt, \tag{3.4.20}$$

where $V_s(\xi) := (|\xi|^2 + s^2)^{\frac{p-2}{4}}\xi$ for $\xi \in \mathbb{R}^n$ and $c > 0$ is a constant depending only on n, p, ν, and L. Also similar to [54, Eq. (4.11)], we have the following inequality

$$|Dw - Dv| \leq c(|Dw| + s)^{(2-p)/2}|V_s(Dw) - V_s(Dv)| + c|V_s(Dv) - V_s(Dw)|^{2/p},$$

and therefore by Young's inequality with exponents $2/(2 - p)$ and $2/p$, and the fact that $\omega(\rho) \in [0,1]$ for every $\rho > 0$, we obtain

$$|Dw - Dv|^p \leq c\big[\omega(\rho)\big]^p (|Dw| + s)^p + c\big[\omega(\rho)\big]^{p-2} |V_s(Dw) - V_s(Dv)|^2. \qquad (3.4.21)$$

Thus (3.4.19) follows from (3.4.20), (3.4.21) and the fact that ω is a nondecreasing function. $\square$

We also have a Lipschitz estimate for v from [23, Chapter 8, Theorem 5.2'].

Theorem 3.4.5. *Let v be a solution to (3.4.18) under the assumptions (3.1.2)–(3.1.5). Assume that $\max\{1, \frac{2n}{n+2}\} < p < 2$ and that $q \in (\frac{(2-p)n}{2}, p]$. Then there exists a constant $c_6 = c_6(n, p, \nu, L, q) \geq 1$, such that*

$$\sup_{\frac{1}{4}Q_\rho^\lambda} \|Dv\| \leq c_6(\lambda + s) + c_6 \lambda^{\frac{n(p-2)}{n(p-2)+2q}} \Big(\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dv| + s)^q \, dxdt \Big)^{\frac{2}{n(p-2)+2q}}.$$

In the rest of this section, we always assume

$$p \in \big(p^*(n), 2 - \frac{1}{n+1}\big] \quad \text{and} \quad q \in \big(\max\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}\}, p - \frac{n}{n+1}\big),$$

where $p^*(n)$ is defined in (3.1.8) so that all of the assumptions in Lemma 3.4.1–Theorem 3.4.5 are satisfied. In particular, we have $q \in (\frac{1}{2}, 1)$ and therefore

$$(a+b)^q \leq a^q + b^q, \quad (a+b)^{1/q} \leq 2a^{1/q} + 2b^{1/q}, \quad \forall\, a, b > 0. \qquad (3.4.22)$$

Lemma 3.4.6. *Assume that*

$$\Big(\fint_{Q_\rho^\lambda} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} \leq \lambda, \quad \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \leq \lambda. \qquad (3.4.23)$$

Then there exists a constant $c_7 = c_7(n, p, \nu, L, q) \geq 1$, such that

$$\Big(\fint_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dxdt \Big)^{\frac{1}{q}} \leq c_7 \, \omega(\rho)\lambda + c_7 \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big]. \tag{3.4.24}$$

Proof. First, by (3.4.23) and Lemma 3.4.1, we can argue as in the proof of [55, Corollary 4.4] and get

$$\fint_{Q_\rho^\lambda} |Du - Dw|^q \, dxdt \leq c \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big]^q. \tag{3.4.25}$$

Indeed, we know that

$$\frac{|\mu|(Q_\rho^\lambda)}{|Q_\rho^\lambda|^{\frac{n+1}{n+2}}} = |B_1|^{-\frac{n+1}{n+2}} \lambda^{\frac{(n+1)(p-2)}{n+2}} \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big], \tag{3.4.26}$$

where $|B_1|$ is the volume of a unit ball in $\mathbb{R}^n$. Moreover, by (3.4.26) and the second inequality in (3.4.23), we also have

$$\begin{aligned} \Bigg[\frac{|\mu|(Q_\rho^\lambda)}{|Q_\rho^\lambda|^{\frac{n+1}{n+2}}} \Bigg]^{\frac{n+2}{(n+1)p-n}} &= |B_1|^{-\frac{n+1}{(n+1)p-n}} \lambda^{\frac{(n+1)(p-2)}{(n+1)p-n}} \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big]^{1+\frac{(n+1)(2-p)}{(n+1)p-n}} \\ &\leq |B_1|^{-\frac{n+1}{(n+1)p-n}} \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big]. \end{aligned} \tag{3.4.27}$$

The estimate (3.4.25) now follows by applying Lemma 3.4.1 and using (3.4.27), (3.4.26), and the first inequality in (3.4.23).

Next, using (3.4.23), (3.4.25), and the triangle inequality, we have

$$\fint_{Q_\rho^\lambda} (|Dw| + s)^q \, dxdt \leq c\lambda^q. \tag{3.4.28}$$

From (3.4.23) we also have $s \leq \lambda$, which together with (3.4.28) and Theorem 3.4.2 implies

$$\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dw| + s)^p \, dxdt \leq c\lambda^p.$$

Thus, Lemma 3.4.4, the last inequality, and Hölder's inequality imply

$$\fint_{\frac{1}{2}Q_\rho^\lambda} |Dw - Dv|^q \, dxdt \le c\,\omega(\rho)^q \lambda^q. \tag{3.4.29}$$

The estimate (3.4.24) now follows using (3.4.25), (3.4.29) and the triangle inequality.

□

Lemma 3.4.7. *Let $\delta, \theta \in (0, 1/2)$. Assume that*

$$\Big(\fint_{Q_\rho^\lambda} (|Du| + s)^q \, dxdt\Big)^{\frac{1}{q}} \le \lambda, \quad \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \le \frac{\delta^{\frac{n+2}{q}} \theta^{\frac{1}{q}}}{2c_7} \lambda, \quad \omega(\rho) \le \frac{\delta^{\frac{n+2}{q}} \theta^{\frac{1}{q}}}{2c_7}, \tag{3.4.30}$$

where c_7 is the same constant as in Lemma 3.4.6. Then there exists a constant $c_8 = c_8(n, p, \nu, L, q) \ge 1$, such that

$$s + \sup_{\frac{1}{4}Q_\rho^\lambda} \|Dv\| \le c_8 \lambda.$$

Moreover, it holds that

$$\fint_{\delta Q_\rho^\lambda} |Du|^q \, dxdt - \theta \lambda^q \le \fint_{\delta Q_\rho^\lambda} |Dv|^q \, dxdt \le \sqrt{n}\big(\sup_{\delta Q_\rho^\lambda} \|Dv\|\big)^q.$$

Proof. Using (3.4.30) and Lemma 3.4.6, we have

$$\fint_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dxdt \le \delta^{n+2} \theta \lambda^q. \tag{3.4.31}$$

This together with the first inequality in (3.4.30) and the triangle inequality implies

$$\fint_{\frac{1}{2}Q_\rho^\lambda} (|Dv| + s)^q \, dxdt \le \fint_{\frac{1}{2}Q_\rho^\lambda} (|Du| + s)^q \, dxdt + \fint_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dxdt \le (2^{n+2} + 1)\lambda^q.$$

Thus Theorem 3.4.5 and the last inequality imply

$$s + \sup_{\frac{1}{4}Q_\rho^\lambda} \|Dv\| \leq c_8 \lambda$$

for some constant $c_8 = c_8(n, p, \nu, L, q) \geq 1$. Moreover, using (3.4.31) and the triangle inequality, we also have

$$\begin{aligned}
\fint_{\delta Q_\rho^\lambda} |Du|^q \, dxdt &\leq \fint_{\delta Q_\rho^\lambda} |Dv|^q \, dxdt + \fint_{\delta Q_\rho^\lambda} |Du - Dv|^q \, dxdt \\
&\leq \fint_{\delta Q_\rho^\lambda} |Dv|^q \, dxdt + (2\delta)^{-(n+2)} \fint_{\frac{1}{2}Q_\rho^\lambda} |Du - Dv|^q \, dxdt \\
&\leq \fint_{\delta Q_\rho^\lambda} |Dv|^q \, dxdt + \theta \lambda^q.
\end{aligned}$$

The lemma is proved. □

Lemma 3.4.8. *Let $\delta \in (0, 1/4)$ and $\varepsilon \in (0, 1]$. Suppose that the bounds*

$$\Big(\fint_{Q_\rho^\lambda} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} \leq \lambda, \qquad \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \leq \lambda$$

hold, and that Dv satisfies

$$\phi_q(Dv, \delta Q_\rho^\lambda) \leq \frac{\varepsilon}{2^{4(n+3)}} \phi_q(Dv, \frac{1}{4} Q_\rho^\lambda). \tag{3.4.32}$$

Then

$$\phi_q(Du, \delta Q_\rho^\lambda) \leq \frac{\varepsilon}{4} \phi_q(Du, Q_\rho^\lambda) + c_7 \delta^{-(n+2)/q} \Big[\frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \Big] + c_7 \delta^{-(n+2)/q} \omega(\rho) \lambda, \tag{3.4.33}$$

where $c_7 = c_7(n, p, \nu, L, q)$ is the same constant as in Lemma 3.4.6.

Proof. Recalling the definition of ϕ_q and using (3.4.32), (3.4.22), and the triangle inequality, we obtain

$$
\begin{aligned}
\phi_q(Du,\delta Q^\lambda_\rho) &\le \Big(\fint_{\delta Q^\lambda_\rho} |Du-\mathbf{m}(Dv,\delta Q^\lambda_\rho)|^q\,dxdt\Big)^{1/q}\\
&\le 2\phi_q(Dv,\delta Q^\lambda_\rho)+2\Big(\fint_{\delta Q^\lambda_\rho}|Du-Dv|^q\,dxdt\Big)^{1/q}\\
&\le \varepsilon 2^{-(4n+11)}\phi_q(Dv,\frac{1}{4}Q^\lambda_\rho)+2(2\delta)^{-(n+2)/q}\Big(\fint_{\frac{1}{2}Q^\lambda_\rho}|Du-Dv|^q\,dxdt\Big)^{1/q}.
\end{aligned}
\tag{3.4.34}
$$

Again using the definition of ϕ_q and the triangle inequality, we have

$$
\begin{aligned}
\phi_q(Dv,\frac{1}{4}Q^\lambda_\rho) &\le 2\phi_q(Du,\frac{1}{4}Q^\lambda_\rho)+2\Big(\fint_{\frac{1}{4}Q^\lambda_\rho}|Du-Dv|^q\,dxdt\Big)^{1/q}\\
&\le 2\cdot 4^{(n+2)/q}\phi_q(Du,Q^\lambda_\rho)+2\cdot 2^{(n+2)/q}\Big(\fint_{\frac{1}{2}Q^\lambda_\rho}|Du-Dv|^q\,dxdt\Big)^{1/q}.
\end{aligned}
\tag{3.4.35}
$$

Note that $q\in(1/2,1)$. Therefore, the estimate (3.4.33) follows by combining (3.4.34) and (3.4.35) and applying Lemma 3.4.6. □

3.4.2 Proof of pointwise gradient estimates.

Proof of Theorem 3.1.1. First, for $\lambda>0$, we define the Lebesgue set

$$
\mathcal{L}_\lambda := \Big\{(x_0,t_0)\in\Omega_T:\ \lim_{\rho\to 0}\fint_{Q^\lambda_\rho(x_0,t_0)}|Du-Du(x_0,t_0)|\,dxdt=0\Big\}. \tag{3.4.36}
$$

Direct calculations imply that $\mathcal{L}:=\mathcal{L}_1=\mathcal{L}_\lambda$ for all $\lambda>0$. Moreover, by the Lebesgue differentiation theorem, $\Omega_T\backslash\mathcal{L}$ has zero lebesgue measure. We will prove Theorem 3.1.1 for every $(x_0,t_0)\in\mathcal{L}$.

Step 1: Choices of constants and basic inequalities. First, we take the constant $c_8=c_8(n,p,\nu,L,q)$ in Lemma 3.4.7 and define

$$
A:=c_8,\quad B:=2560000n,\quad \gamma:=2^{-4(n+3)}. \tag{3.4.37}
$$

We fix the constants δ_γ and α in Theorem 3.3.3 with the choices of A, B, γ in (3.4.37). We now define

$$\delta_1 = \delta_\gamma/4, \quad r_i = \delta_1^i r_\lambda, \quad Q_i = Q_{r_i}^\lambda$$

for any integer $i \geq 0$. Next, we choose k as the smallest integer larger than or equal to 2 such that

$$4\sqrt{n} A \delta_1^{(k-1)\alpha} \leq \frac{\delta_1^{(n+2)/q}}{1600}, \tag{3.4.38}$$

and we define

$$H_1 := 400\delta_1^{-(n+2)/q}, \quad H_2 := 5120000 c_7 \delta_1^{-(k+2)(n+2)/q}, \quad c := \max\{H_1, H_2\}, \tag{3.4.39}$$

where $c_7 = c_7(n, p, \nu, L, q)$ is the constant in Lemma 3.4.6. We then choose $R_0 = R_0(n, p, \nu, L, q) \in (0, \frac{1}{2}]$ to be the largest number in $(0, \frac{1}{2}]$ such that

$$\int_0^{2R_0} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\delta_1^{(k+2)(n+2)/q}}{5120000 c_7}. \tag{3.4.40}$$

The constants c and R_0 defined in (3.4.39) and (3.4.40) are the same constants we choose in the statement of Theorem 3.1.1. The choice of H_1 in (3.4.39) and (3.1.10) imply that

$$\begin{aligned} &\Big(\fint_{Q_0} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} + \delta_1^{-(n+2)/q} \phi_q(Du, Q_0) \\ &\leq (1 + \delta_1^{-(n+2)/q}) \Big(\fint_{Q_{r_\lambda}^\lambda} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} \leq \frac{\lambda}{100} \end{aligned} \tag{3.4.41}$$

and that

$$s \leq \frac{\lambda}{400}. \tag{3.4.42}$$

Since $\delta_1 \in (0, 1/4)$, by using the comparison principle for the Riemann integrals, we have

$$\int_0^{2r_\lambda} \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \frac{d\rho}{\rho} \geq \sum_{i=0}^{\infty} \int_{r_i}^{2r_i} \frac{|\mu|(Q_\rho^\lambda)}{\rho^{n+1}} \frac{d\rho}{\rho}$$
$$\geq \frac{1}{n+1}(1-\frac{1}{2^{n+1}}) \sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} \geq \delta_1^{n+2} \sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}}. \tag{3.4.43}$$

The estimate (3.4.43) together with the choice of H_2 in (3.4.39) and (3.1.10) imply that

$$\sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} \leq \frac{\delta_1^{(k+1)(n+2)/q}}{5120000c_7}\lambda. \tag{3.4.44}$$

Similarly, since we have $r_\lambda \leq R_0$, the inequality (3.4.40) implies that

$$\sum_{i=0}^{\infty} \omega(r_i) \leq \delta_1^{-(n+2)} \int_0^{2r_\lambda} \omega(\rho) \frac{d\rho}{\rho} \leq \frac{\delta_1^{(k+1)(n+2)/q}}{5120000c_7}. \tag{3.4.45}$$

Step 2: Exit time argument.

For any integer $i \geq 0$, we define

$$C_i := \Big(\fint_{Q_i} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} + \delta_1^{-(n+2)/q} \phi_q(Du, Q_i).$$

Thus $C_0 \leq \lambda/100$ by (3.4.41). We can now assume without loss of generality that there exists a exit time $i_e \geq 0$, satisfying

$$C_{i_e} \leq \frac{\lambda}{100}, \qquad C_j > \frac{\lambda}{100} \quad \forall j > i_e.$$

In fact, if this is not true, we can always find a increasing subsequence j_i such that $C_{j_i} \leq \lambda/100$ for every integer $i \geq 0$ and therefore

$$|Du(x_0, t_0)| \leq \limsup_{i \to \infty} \Big(\fint_{Q_{j_i}} |Du|^q \, dxdt \Big)^{1/q} \leq \frac{\lambda}{100},$$

since $(x_0, t_0) \in \mathcal{L}$ is a Lebesgue point.

Step 3: Estimates after the exit time. The core of the proof is the following iteration lemma.

Lemma 3.4.9. *If $i \geq i_e$ and*

$$\Big(\fint_{Q_i} (|Du| + s)^q \, dxdt \Big)^{\frac{1}{q}} \leq \lambda, \tag{3.4.46}$$

then

$$\phi_q(Du, Q_{i+1}) \leq \frac{1}{4}\phi_q(Du, Q_i) + c_7 \delta_1^{-(n+2)/q} \Big[\frac{|\mu|(Q_i)}{r_i^{n+1}} \Big] + c_7 \delta_1^{-(n+2)/q} \omega(r_i)\lambda. \tag{3.4.47}$$

Proof. Let w and v be the solutions to (3.4.1) and (3.4.18) respectively, with $\rho = r_i$. We then take $\rho = r_i$ and $\theta = 1/1600$ in Lemma 3.4.7. By (3.4.46), (3.4.44) and (3.4.45), we can apply Lemma 3.4.7 both with $\delta = \delta_1$ and with $\delta = \delta_1^k$. We obtain

$$s + \sup_{Q_{i+1}} \|Dv\| \leq s + \sup_{\frac{1}{4}Q_i} \|Dv\| \leq c_8 \lambda = A\lambda \tag{3.4.48}$$

and

$$\fint_{Q_{i+k}} |Du|^q \, dxdt - \frac{\lambda^q}{1600} \leq \fint_{Q_{i+k}} |Dv|^q \, dxdt \leq \sqrt{n} \big(\sup_{Q_{i+k}} \|Dv\| \big)^q. \tag{3.4.49}$$

Next, by Theorem 3.3.1 with $\rho = r_{i+k}$ and $r = r_{i+1}$, (3.4.48), and (3.4.38), we have

$$\operatorname*{osc}_{Q_{i+k}} Dv \leq 4\sqrt{n} A \delta_1^{(k-1)\alpha} \lambda \leq \frac{\delta_1^{(n+2)/q}}{1600} \lambda.$$

Again by (3.4.46), (3.4.44) and (3.4.45), we can apply Lemma 3.4.6 with $\rho = r_i$ and

get

$$\Big(\fint_{Q_{i+k}} |Du - Dv|^q \,dxdt\Big)^{\frac{1}{q}} \le \Big(\frac{|\frac{1}{2}Q_i|}{|Q_{i+k}|}\Big)^{\frac{1}{q}}\Big(\fint_{\frac{1}{2}Q_i} |Du - Dv|^q \,dxdt\Big)^{\frac{1}{q}}$$
$$\le \frac{c_7}{2}\delta_1^{-k(n+2)/q}\omega(r_i)\lambda + \frac{c_7}{2}\delta_1^{-k(n+2)/q}\Big[\frac{|\mu|(Q_i)}{r_i^{n+1}}\Big]$$
$$\le \frac{\delta_1^{(n+2)/q}}{1600}\lambda.$$

Therefore, from the above two inequalities,

$$\phi_q(Du, Q_{i+k}) \le 2^{\frac{1}{q}-1}\phi_q(Dv, Q_{i+k}) + 2^{\frac{1}{q}-1}\Big(\fint_{Q_{i+k}} |Du - Dv|^q \,dxdt\Big)^{\frac{1}{q}}$$
$$\le 2 \operatorname*{osc}_{Q_{i+k}} Dv + 2\Big(\fint_{Q_{i+k}} |Du - Dv|^q \,dxdt\Big)^{\frac{1}{q}}$$
$$\le \frac{\delta_1^{(n+2)/q}}{400}\lambda.$$

The last estimate and (3.4.42) imply that

$$C_{i+k} = \Big(\fint_{Q_{i+k}} (|Du| + s)^q \,dxdt\Big)^{\frac{1}{q}} + \delta_1^{-(n+2)/q}\phi_q(Du, Q_{i+k})$$
$$\le 2\Big(\fint_{Q_{i+k}} |Du|^q \,dxdt\Big)^{\frac{1}{q}} + 2s + \frac{\lambda}{400}$$
$$\le 2\Big(\fint_{Q_{i+k}} |Du|^q \,dxdt\Big)^{\frac{1}{q}} + \frac{3\lambda}{400}.$$

Therefore, by (3.4.49) and the fact that $C_{i+k} > \lambda/100$, we have

$$\sup_{Q_{i+1}} \|Dv\| \ge \sup_{Q_{i+k}} \|Dv\| \ge \Big(\frac{1}{1600\sqrt{n}}\Big)^{\frac{1}{q}}\lambda \ge \frac{\lambda}{2560000n} = \frac{\lambda}{B}. \tag{3.4.50}$$

By (3.4.48) and (3.4.50), we can apply Theorem 3.3.3 with $Q_r^\lambda = \frac{1}{4}Q_i$ and constants A, B, γ chosen in (3.4.37) and get

$$\phi_q(Dv, Q_{i+1}) = \phi_q(Dv, \frac{\delta_\gamma}{4}Q_i) \le 2^{-4(n+3)}\phi_q(Dv, \frac{1}{4}Q_i).$$

Finally, using (3.4.46), (3.4.44), and the last inequality, we can apply Lemma 3.4.8 with $\varepsilon = 1$ to obtain (3.4.47). $\square$

Step 4: Iteration and conclusion. For any integer $j \geq 0$, we denote

$$\Phi_j := \phi_q(Du, Q_j), \quad \mathbf{m}_j := \mathbf{m}(Du, Q_j).$$

Since we have

$$(|Du(x,t)| + s)^q \leq |Du(x,t) - \mathbf{m}_j|^q + (|\mathbf{m}_j| + s)^q,$$

by taking the average over $(x,t) \in Q_j$ and then taking the q-th root, we obtain

$$\left(\fint_{Q_j} (|Du| + s)^q \, dxdt \right)^{\frac{1}{q}} \leq 2\Phi_j + 2|\mathbf{m}_j| + 2s. \tag{3.4.51}$$

Here we also used the fact that $q \in (1/2, 1)$. Moreover, by (3.2.3), we have

$$|\mathbf{m}_j| \leq 2\Phi_j + 2\left(\fint_{Q_j} |Du(x,t)|^q \, dxdt \right)^{1/q} \leq 2C_j. \tag{3.4.52}$$

Using (3.2.4), we also have

$$|\mathbf{m}_{j+1} - \mathbf{m}_j| \leq 2\Phi_{j+1} + 2\delta_1^{-(n+2)/q}\Phi_j. \tag{3.4.53}$$

We now prove by induction that

$$\Phi_j + |\mathbf{m}_j| + s \leq \frac{\lambda}{2} \tag{3.4.54}$$

holds for any $j \geq i_e$. First, by the definition of exit time i_e and (3.4.52), we know that

$$C_{i_e} \leq \frac{\lambda}{100}, \quad |\mathbf{m}_{i_e}| \leq \frac{\lambda}{50},$$

and thus

$$\Phi_{i_e} + |\mathbf{m}_{i_e}| + s \leq \frac{\lambda}{2}.$$

Assume that (3.4.54) holds for any $j \in \{i_e, \dots, i\}$. Then by (3.4.51) we have

$$\left(\fint_{Q_j} (|Du| + s)^q \, dxdt \right)^{\frac{1}{q}} \leq \lambda$$

for any $j \in \{i_e, \dots, i\}$. Therefore we can apply Lemma 3.4.9 to get

$$\Phi_{j+1} \leq \frac{1}{4}\Phi_j + c_7 \delta_1^{-(n+2)/q} \Big[\frac{|\mu|(Q_i)}{r_j^{n+1}} \Big] + c_7 \delta_1^{-(n+2)/q} \omega(r_j)\lambda \tag{3.4.55}$$

for any $j \in \{i_e, \dots, i\}$. Thus, using (3.4.54) with $j = i$, (3.4.55), (3.4.44) and (3.4.45), we have

$$\Phi_{i+1} \leq \frac{\lambda}{8} + c_7 \delta_1^{-(n+2)/q} \Big[\frac{|\mu|(Q_i)}{r_i^{n+1}} \Big] + c_7 \delta_1^{-(n+2)/q} \omega(r_i)\lambda \leq \frac{\lambda}{4}. \tag{3.4.56}$$

Summing up (3.4.55) in $j \in \{i_e, \dots, i\}$, we also have

$$\sum_{j=i_e}^{i+1} \Phi_j \leq \Phi_{i_e} + \frac{1}{4} \sum_{j=i_e}^{i} \Phi_j + c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^{i} \Big[\frac{|\mu|(Q_i)}{r_j^{n+1}} \Big] + c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^{i} \omega(r_j)\lambda$$

and thus

$$\sum_{j=i_e}^{i+1} \Phi_j \leq \frac{4}{3}\Phi_{i_e} + \frac{4}{3} c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^{i} \Big[\frac{|\mu|(Q_j)}{r_j^{n+1}} \Big] + \frac{4}{3} c_7 \delta_1^{-(n+2)/q} \sum_{j=i_e}^{i} \omega(r_j)\lambda. \tag{3.4.57}$$

Using (3.4.53), (3.4.57), (3.4.44), and (3.4.45), we obtain

$$\begin{aligned}
|\mathbf{m}_{i+1} - \mathbf{m}_{i_e}| &\le \sum_{j=i_e}^{i} |\mathbf{m}_{j+1} - \mathbf{m}_j| \le 4\delta_1^{-(n+2)/q} \sum_{j=i_e}^{i+1} \Phi_j \\
&\le 8\delta_1^{-(n+2)/q}\Phi_{i_e} + 8c_7\delta_1^{-2(n+2)/q} \sum_{j=i_e}^{i} \left[\frac{|\mu|(Q_j)}{r_j^{n+1}}\right] + 8c_7\delta_1^{-2(n+2)/q} \sum_{j=i_e}^{i} \omega(r_j)\lambda \\
&\le 8\delta_1^{-(n+2)/q}\Phi_{i_e} + \frac{\lambda}{100}.
\end{aligned}$$

Therefore it follows from (3.4.52) and the previous inequality that

$$|\mathbf{m}_{i+1}| \le |\mathbf{m}_{i_e}| + 8\delta_1^{-(n+2)/q}\Phi_{i_e} + \frac{\lambda}{100} \le 10C_{i_e} + \frac{\lambda}{100} \le \frac{\lambda}{8}. \tag{3.4.58}$$

By (3.4.42), (3.4.56) and (3.4.58), we obtain

$$\Phi_{i+1} + |\mathbf{m}_{i+1}| + s \le \frac{\lambda}{4} + \frac{\lambda}{8} + \frac{\lambda}{400} \le \frac{\lambda}{2},$$

which completes the induction. Since we have

$$|\mathbf{m}_j - Du(x_0,t_0)|^q \le |Du(x,t) - \mathbf{m}_j|^q + |Du(x,t) - Du(x_0,t_0)|^q,$$

by taking the average over $(x,t) \in Q_j$ and then taking the q-th root, we obtain

$$\begin{aligned}
|\mathbf{m}_j - Du(x_0,t_0)| &\le 2\phi_q(Du, Q_j) + 2\Big(\fint_{Q_j} |Du - Du(x_0,t_0)|^q \, dxdt\Big)^{1/q} \\
&\le 4\Big(\fint_{Q_j} |Du - Du(x_0,t_0)|^q \, dxdt\Big)^{1/q}.
\end{aligned} \tag{3.4.59}$$

Since $(x_0,t_0) \in \mathcal{L}$ is a Lebesgue point, using (3.4.59) and (3.4.54), we obtain

$$|Du(x_0,t_0)| = \lim_{j\to\infty} |\mathbf{m}_j| \le \frac{\lambda}{2}.$$

The proof of Theorem 3.1.1 is completed. $\square$

Proof of Theorem 3.1.2. Without loss of generality, we assume that $\mathbf{I}_1^{|\mu|}(x_0, t_0, 2r) < \infty$. We consider the function

$$
\begin{aligned}
h(\lambda) &:= \lambda - c\Big(\fint_{Q^{\lambda}_{r_\lambda}(x_0,t_0)} (|Du| + s + 1)^q \, dxdt\Big)^{1/q} - c\int_0^{2r_\lambda} \frac{|\mu|(Q^{\lambda}_{\rho}(x_0,t_0))}{\rho^{n+1}} \frac{d\rho}{\rho} \\
&= \lambda - c\lambda^{\frac{n(2-p)}{2q}} A(\lambda) - c\lambda^{\frac{(n+1)(2-p)}{2}} B(\lambda),
\end{aligned}
$$

where $r_\lambda = \lambda^{(p-2)/2} r$ and

$$
A(\lambda) := \frac{1}{|Q_r(x_0,t_0)|^{1/q}} \Big(\int_{Q^{\lambda}_{r_\lambda}} (|Du| + s + 1)^q \, dxdt\Big)^{1/q}
$$

and

$$
B(\lambda) := \int_0^{2r} \frac{|\mu|(Q^{\lambda}_{\lambda^{(p-2)/2}\rho}(x_0,t_0))}{\rho^{n+1}} \frac{d\rho}{\rho}.
$$

Here c is the same constant as in Theorem 3.1.1. Since $p \in (1,2)$, we have $Q^{\lambda_2}_{\lambda_2^{(p-2)/2}\rho} \subset Q^{\lambda_1}_{\lambda_1^{(p-2)/2}\rho}$ for every $\lambda_2 > \lambda_1 > 0$ and $\rho > 0$ and, therefore, A and B are nonincreasing functions of λ. Moreover, the functions A, B, h are well defined for $\lambda \in [1,\infty)$ since $Q^{\lambda}_{2\lambda^{(p-2)/2}r}(x_0,t_0) \subset Q_{2r}(x_0,t_0) \subset \Omega$ for any $\lambda \in [1,\infty)$. Clearly, h is a continuous function on $[1,\infty)$ and $h(1) \le 0$ since $c \ge 1$ and $A(1) \ge 1$. On the other hand, since $q > \frac{n(2-p)}{2}$ and $p > \frac{2n}{n+1}$, we have

$$
\lim_{\lambda\to\infty} h(\lambda) \ge \lim_{\lambda\to\infty} \Big(\lambda - c\lambda^{\frac{n(2-p)}{2q}} A(1) - c\lambda^{\frac{(n+1)(2-p)}{2}} B(1)\Big) = \infty.
$$

Thus there exists some $\lambda \ge 1$ such that $h(\lambda) = 0$ and therefore (3.1.10) holds for such λ. Since $\lambda \ge 1$ and $r \in (0, R_0]$, we have $r_\lambda \equiv \lambda^{(p-2)/2} r \in (0, R_0]$. Applying Theorem 3.1.1 and using the fact that $h(\lambda) = 0$, we obtain

$$
\lambda + |Du(x_0,t_0)| \le 2\lambda = 2c\lambda^{\frac{n(2-p)}{2q}} A(\lambda) + 2c\lambda^{\frac{(n+1)(2-p)}{2}} B(\lambda). \tag{3.4.60}
$$

Using the fact that A, B are nonincreasing functions and Young's inequality with conjugate exponents $\left(\frac{2q}{n(2-p)}, \frac{2q}{2q-n(2-p)}\right)$ and $\left(\frac{2}{(n+1)(2-p)}, \frac{2}{(n+1)p-2n}\right)$, we obtain

$$2c\lambda^{\frac{n(2-p)}{2q}} A(\lambda) \leq 2c\lambda^{\frac{n(2-p)}{2q}} A(1) \leq \frac{\lambda}{4} + c' \left[A(1)\right]^{\frac{2q}{2q-n(2-p)}}$$

and

$$2c\lambda^{\frac{(n+1)(2-p)}{2}} B(\lambda) \leq 2c\lambda^{\frac{(n+1)(2-p)}{2}} B(1) \leq \frac{\lambda}{4} + c' \left[B(1)\right]^{\frac{2}{(n+1)p-2n}}. \tag{3.4.61}$$

Therefore (3.1.11) follows by using the last two inequalities and (3.4.60). $\square$

3.5 Gradient continuity results

3.5.1 Preliminary choices of constants and the geometry

In this section, we always assume $p \in (p^*(n), 2 - \frac{1}{n+1}]$, where $p^*(n)$ is defined in (3.1.8). We also choose $q := q(n,p) \in (1/2, 1)$ as a fixed constant depending only on n and p. For instance, we can take

$$q := \frac{1}{2}\Big(\max\{\frac{n+2}{2(n+1)}, \frac{(2-p)n}{2}\} + p - \frac{n}{n+1}\Big) \in (\frac{1}{2}, 1).$$

The choices of geometry in this section are essentially the same as in [55, Section 5.1]. For completeness, we shall still briefly report the choices. First, we fix an open cylinder $\tilde{Q} \subset\subset \Omega_T$ and take another cylinder $\tilde{Q}'$ such that $\tilde{Q} \subset\subset \tilde{Q}' \subset\subset \Omega_T$. Let $\bar{R}_1 := \mathrm{dist}_{\mathrm{par}}(\tilde{Q}, \partial_{\mathrm{par}}\tilde{Q}') > 0$. Under the assumptions of Theorem 3.1.3 or Theorem 3.1.6, it always holds that the Riesz potential $\mathbf{I}_1^{|\mu|}(x,t;r)$ is locally bounded in Ω_T for some $r > 0$. Therefore we can apply Theorem 3.1.2 to obtain that Du is locally bounded in Ω_T and that in particular, Du is bounded in $\tilde{Q}'$. Thus we can choose

$$M := 1 + s + \sup_{\tilde{Q}'} |Du| < \infty.$$

Let

$$\lambda_M := M \quad \text{and} \quad R_1 := \frac{1}{4}\lambda_M^{(p-2)/2}\bar{R}_1. \tag{3.5.1}$$

Then we have $Q_r^{\lambda_M}(x_0,t_0) \subset \tilde{Q}'$ whenever $(x_0,t_0) \in \tilde{Q}$ and $r \in (0, R_1]$, and therefore

$$s + \sup_{Q_r^{\lambda_M}(x_0,t_0)} |Du| \le \lambda_M, \quad \forall r \in (0, R_1]. \tag{3.5.2}$$

By (3.5.1) and (3.5.2), we have

$$\left(\fint_{Q_\rho^{\lambda_M}(x_0,t_0)} (|Du| + s)^q \, dxdt \right)^{1/q} \le M \equiv \lambda_M \tag{3.5.3}$$

for any $(x_0,t_0) \in \tilde{Q}$ and any $\rho \in (0, R_1]$.

3.5.2 Proof of gradient continuity results

First, we prove the following proposition.

Proposition 3.5.1. *Let $\varepsilon \in (0,1]$. Assume that the Riesz potential $\mathbf{I}_1^{|\mu|}(x,t;r)$ is locally bounded in Ω_T for some $r > 0$ and that*

$$\lim_{r \to 0} \frac{|\mu|(Q_r(x,t))}{r^{n+1}} = 0 \quad \textit{locally uniformly in } (x,t) \in \Omega_T. \tag{3.5.4}$$

Then there exists constants $\alpha = \alpha(n,p,\nu,L) \in (0,1)$, $c_9 = c_9(n,p,\nu,L) \ge 1$, and $R_\varepsilon = R_\varepsilon(n,p,\nu,L,M,\mu,\varepsilon) \in (0,R_1)$ such that

$$\phi_q(Du, Q_\rho^{\lambda_M}(x_0,t_0)) < \varepsilon\lambda_M \tag{3.5.5}$$

holds for every $(x_0,t_0) \in \tilde{Q}$ and every $\rho \in (0, \rho_\varepsilon]$, where

$$\rho_\varepsilon = \frac{\varepsilon^{2/\alpha}}{c_9} R_\varepsilon.$$

More specifically, the constant R_ε is determined in (3.5.8)–(3.5.9) *below.*

Proof. First we take

$$\lambda = \lambda_M, \quad A = c_8, \quad B = \frac{400n}{\varepsilon}, \quad \gamma = \frac{\varepsilon}{2^{4(n+3)}}, \tag{3.5.6}$$

where $c_8 = c_8(n, p, \nu, L)$ is the same constant as in Lemma 3.4.7. Then we choose $\delta_\gamma = \delta_\gamma(n, p, \nu, L, \varepsilon) \in (0, 1/2)$ as in Theorem 3.3.3 with the choices of A, B, γ in (3.5.6) and we set $\delta_1 = \delta_\gamma/4$. Then by (3.3.5), we have

$$\delta_1 = \frac{\varepsilon^{2/\alpha}}{c_9} \tag{3.5.7}$$

for some constants $\alpha \in (0, 1)$ and $c_9 \geq 1$ both depending on n, p, ν, L. Next, we take $R_\varepsilon \in (0, R_1)$ such that

$$\omega(R_\varepsilon) \leq \frac{\delta_1^{\frac{n+2}{q}} \varepsilon}{800 c_7} \tag{3.5.8}$$

and that

$$\sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\leq\lambda_M^{(2-p)/2} R_\varepsilon} \frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{n+1}} \leq \frac{\delta_1^{\frac{n+2}{q}} \varepsilon}{800 c_7 \lambda_M^{(n+1)(2-p)/2}}, \tag{3.5.9}$$

where $c_7 = c_7(n, p, \nu, L)$ is the same constant as in Lemma 3.4.6. Thus we have

$$\begin{aligned} \sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\leq R_\varepsilon} \frac{|\mu|(Q_\rho^{\lambda_M}(x_0, t_0))}{\rho^{n+1}} &\leq \sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\leq R_\varepsilon} \frac{|\mu|(Q_{\lambda_M^{(2-p)/2}\rho}(x_0, t_0))}{\rho^{n+1}} \\ &\leq \frac{\delta_1^{\frac{n+2}{q}} \varepsilon}{800 c_7} \leq \frac{\delta_1^{\frac{n+2}{q}} \varepsilon}{800 c_7} \lambda_M. \end{aligned} \tag{3.5.10}$$

For $i \in \mathbb{N}$, we define

$$Q_i := Q_{r_i}^{\lambda_M}(x_0, t_0), \quad r_i := \delta_1^i r, \quad r \in (\delta_1 R_\varepsilon, R_\varepsilon].$$

We will prove that for every $i \geq 1$, it holds that

$$\phi_q(Du, Q_i) < \varepsilon\lambda_M. \tag{3.5.11}$$

Let $i \geq 1$. We consider two different cases. First, suppose that

$$\Big(\fint_{Q_i} |Du|^q \, dxdt\Big)^{1/q} < \frac{\varepsilon}{10}\lambda_M. \tag{3.5.12}$$

In this case, the definition of ϕ_q implies that

$$\phi_q(Du, Q_i) \leq \Big(\fint_{Q_i} |Du|^q \, dxdt\Big)^{1/q} < \frac{\varepsilon}{10}\lambda_M$$

and therefore (3.5.11) holds. On the other hand, suppose that (3.5.12) does not hold. Then we have

$$\Big(\fint_{Q_i} |Du|^q \, dxdt\Big)^{1/q} \geq \frac{\varepsilon}{10}\lambda_M. \tag{3.5.13}$$

Let w and v be the solutions defined in (3.4.1) and (3.4.18) respectively, with the choices $\rho = r_{i-1}$ and $\lambda = \lambda_M$. By (3.5.3), (3.5.8), and (3.5.10), we can apply Lemma 3.4.7 with the parameters $\rho = r_{i-1}$, $\lambda = \lambda_M$, $\delta = \delta_1$, and $\theta = \varepsilon^q/20$. Thus using (3.5.13) we obtain

$$\frac{\lambda_M}{B} = \frac{\varepsilon\lambda_M}{400n} \leq \sup_{Q_i} \|Dv\|, \quad s + \sup_{\frac{1}{4}Q_{i-1}} \|Dv\| \leq c_8\lambda_M = A\lambda_M.$$

Recalling that $\delta_1 = \delta_\gamma/4$ and applying Theorem 3.3.3 in $\frac{1}{4}Q_{i-1} = \frac{1}{4}Q^{\lambda_M}_{r_{i-1}}$, we have

$$\phi_q(Dv, Q_i) = \phi(Dv, \frac{\delta_\gamma}{4}Q_{i-1}) \leq \frac{\varepsilon}{2^{4(n+3)}}\phi_q(Dv, \frac{1}{4}Q_{i-1}). \tag{3.5.14}$$

By (3.5.3), (3.5.10), and (3.5.14), we can apply Lemma 3.4.8 and get

$$\begin{aligned}\phi_q(Du, Q_i) &\le \frac{\varepsilon}{4}\phi_q(Du, Q_{i-1}) + c_7\delta_1^{-(n+2)/q}\Big[\frac{|\mu|(Q_{i-1}}{r_{i-1}^{n+1}}\Big] + c_7\delta_1^{-(n+2)/q}\omega(r_{i-1})\lambda_M \\ &\le \frac{\varepsilon}{4}\lambda_M + \frac{\varepsilon}{400}\lambda_M < \varepsilon\lambda_M.\end{aligned}$$

The proof of (3.5.11) is completed. Now we take $\rho_\varepsilon = \delta_1 R_\varepsilon$, where δ_1 has the form in (3.5.7). Since for any $\rho \in (0, \rho_\varepsilon]$, there exists $r \in (\delta_1 R_\varepsilon, R_\varepsilon]$ and an integer $k \ge 1$ such that $\rho = \delta_1^k r$, (3.5.5) follows directly from (3.5.11). □

A corollary of Proposition 3.5.1 is Theorem 3.1.6.

Proof of Theorem 3.1.6. We are now able to determine the exact form of R_ε in Proposition 3.5.1 for any $\varepsilon \in (0,1)$ thanks to the assumption (3.1.17). By (3.5.7), to verify (3.5.8) and (3.5.9), we need to show that

$$\omega(R_\varepsilon) \le \frac{\varepsilon^{\frac{2(n+2)}{q\alpha}+1}}{800c_7c_9^{(n+2)/q}},$$

and that

$$\sup_{(x_0,t_0)\in\tilde{Q}}\ \sup_{0<\rho\le\lambda_M^{(2-p)/2}R_\varepsilon} \frac{|\mu|(Q_\rho(x_0,t_0))}{\rho^{n+1}} \le \frac{\varepsilon^{\frac{2(n+2)}{q\alpha}+1}}{800c_7c_9^{(n+2)/q}\lambda_M^{(n+1)(2-p)/2}}.$$

Thus using (3.1.17), it is sufficient to take $R_\varepsilon \in (0, R_1)$ such that

$$R_\varepsilon \le \Big(\frac{\varepsilon^{\frac{2(n+2)}{q\alpha}+1}}{800c_Dc_7c_9^{(n+2)/q}\lambda_M^{(n+2)(2-p)/2}}\Big)^{1/\delta} := \frac{\varepsilon^{1/\theta}}{c_{10}},$$

where $\theta = \theta(n, p, \nu, L, \delta) \in (0,1)$ and $c_{10} = c_{10}(n, p, \nu, L, \delta, c_D, M) \ge 1$. Now we take

$$R_\varepsilon = \min\{1, R_1\}\frac{\varepsilon^{1/\theta}}{c_{10}} \quad \text{and} \quad \rho_\varepsilon = \min\{1, R_1\}\frac{\varepsilon^{2/\alpha+1/\theta}}{c_9c_{10}}.$$

Thus we can apply Lemma 3.5.1 and obtain

$$\phi_q(Du, Q^{\lambda_M}_{\rho_\varepsilon}(x_0,t_0)) < \varepsilon\lambda_M.$$

Since ε is an arbitrary number in $(0,1)$, the last inequality implies that

$$\phi_q(Du, Q^{\lambda_M}_{\rho}(x_0,t_0)) < c\rho^\beta \tag{3.5.15}$$

holds for every $(x_0,t_0) \in \tilde{Q}$ and every $\rho \in (0, R_2]$, where $R_2 := \min\{1, R_1\}/(c_9c_{10}) \in (0,1)$, $c>0$ is a constant depending on n, p, ν, L, δ, c_D, M, and R_1, and

$$\beta := \frac{1}{\frac{2}{\alpha}+\frac{1}{\theta}} \in (0,1)$$

depends only on n, p, ν, L, and δ.

We are ready to prove $Du \in C^{0,\beta}(\tilde{Q})$ using (3.5.15). Let (x_1,t_1), $(x_2,t_2) \in \tilde{Q} \cap \mathcal{L}$, such that $\rho := |(x_1,t_1)-(x_2,t_2)|_{\mathrm{par}} \le R_2/2$. Here $\mathcal{L} \equiv \mathcal{L}_{\lambda_M}$ is the set of Lebesgue points of Du defined in (3.4.36). Without loss of generality, we assume that $t_1 \le t_2$. Arguing exactly as in (3.4.59), we know that

$$\lim_{r\to 0} |\mathbf{m}(Q^{\lambda_M}_r(x_1,t_1) - Du(x_1,t_1)| = 0. \tag{3.5.16}$$

Using (3.2.4), (3.5.16), (3.5.15), and the fact that $q \in (1/2, 1)$, we obtain

$$\begin{aligned}
&|\mathbf{m}(Du, Q^{\lambda_M}_{\rho}(x_1,t_1)) - Du(x_1,t_1)| \\
&\le \sum_{j=0}^{\infty} |\mathbf{m}(Du, Q^{\lambda_M}_{2^{-j}\rho}(x_1,t_1)) - \mathbf{m}(Du, Q^{\lambda_M}_{2^{-(j+1)}\rho}(x_1,t_1))| \\
&\le 2^{2n+6}\sum_{j=0}^{\infty} \phi_q(Du, Q^{\lambda_M}_{2^{-j}\rho}(x_1,t_1)) \le c\,2^{2n+6}\sum_{j=0}^{\infty}(2^{-j}\rho)^\beta \le c'\rho^\beta.
\end{aligned} \tag{3.5.17}$$

Similarly, we have

$$|\mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2)) - Du(x_2, t_2)| \le c'' \rho^{\beta}. \tag{3.5.18}$$

By (3.5.17), (3.5.18) and the triangle inequality, it holds that

$$\begin{aligned}
&|Du(x_1, t_1) - Du(x_2, t_2)| \\
&\le |Du(x_1, t_1) - \mathbf{m}(Du, Q_{\rho}^{\lambda_M}(x_1, t_1))| + |\mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2)) - Du(x_2, t_2)| \\
&\quad + |\mathbf{m}(Du, Q_{\rho}^{\lambda_M}(x_1, t_1)) - \mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2))| \\
&\le c\rho^{\beta} + |\mathbf{m}(Du, Q_{\rho}^{\lambda_M}(x_1, t_1)) - \mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2))|.
\end{aligned} \tag{3.5.19}$$

Recalling the definition of parabolic distance and the assumption that $t_1 \le t_2$, we know that $Q_{\rho}^{\lambda_M}(x_1, t_1) \subset Q_{2\rho}^{\lambda_M}(x_2, t_2)$ and therefore by (3.2.4), (3.5.15) and the fact that $q \in (\frac{1}{2}, 1)$, we obtain

$$\begin{aligned}
&|\mathbf{m}(Du, Q_{\rho}^{\lambda_M}(x_1, t_1)) - \mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2))| \\
&\le 2\phi_q(Du, Q_{\rho}^{\lambda_M}(x_1, t_1)) + 2^{2n+5}\phi_q(Du, Q_{2\rho}^{\lambda_M}(x_2, t_2)) \le c\rho^{\beta}.
\end{aligned} \tag{3.5.20}$$

Combining (3.5.19), (3.5.20) and using the fact that $\rho = |(x_1, t_1) - (x_2, t_2)|_{\mathrm{par}}$, we have

$$|Du(x_1, t_1) - Du(x_2, t_2)| \le c\rho^{\beta} = c\,|(x_1, t_1) - (x_2, t_2)|_{\mathrm{par}}^{\beta},$$

where c is a constant depending only on n, p, ν, L, δ, c_D, M, and R_1. Recall that by Theorem 3.1.2, Du is bounded in $\tilde{Q}$. Therefore the last estimate implies that $Du \in C^{0,\beta}(\tilde{Q})$ since $|\tilde{Q} \backslash (\mathcal{L} \cap \tilde{Q})| = 0$. The proof is now completed. □

Next, we turn to the proofs of Theorem 3.1.3 and its corollaries. We start with the following proposition.

Proposition 3.5.2. *Let $\varepsilon \in (0, 1]$. Under the same assumptions as in Theorem 3.1.3,* *there exists a radius* $R'_\varepsilon \in (0, R_1)$ *depending only on* n, p, ν, L, δ, c_D, M, *and* R_1, *such that*

$$|\mathbf{m}(Du, Q^{\lambda_M}_{\rho_1}(x_0,t_0)) - \mathbf{m}(Du, Q^{\lambda_M}_{\rho_2}(x_0,t_0))| \leq 3\varepsilon\lambda_M \tag{3.5.21}$$

holds for every ρ_1, $\rho_2 \in (0, R'_\varepsilon]$ *and* $(x_0,t_0) \in \tilde{Q}$.

Proof. We still use an exit time argument similar to the proof of [55, Theorem 1.6].

Step 1: Choices of constants. First, we take

$$\lambda = \lambda_M, \quad A = c_8, \quad B = \frac{400n}{\varepsilon}, \quad \gamma = 2^{-4(n+3)}, \tag{3.5.22}$$

where $c_8 = c_8(n,p,\nu,L)$ is the same constant as in Lemma 3.4.7. Then we choose $\delta_\gamma = \delta_\gamma(n,p,\nu,L,\varepsilon) \in (0,1/2)$ as in Theorem 3.3.3 with the choices of A, B, γ in (3.5.22) and we set $\delta_1 = \delta_\gamma/4$.

Next, We take $R'_\varepsilon \in (0,R_1)$ depending only on n, p, ν, L, δ, c_D, M, and R_1, such that

$$\int_0^{2R'_\varepsilon} \omega(\rho)\frac{d\rho}{\rho} \leq \frac{\delta_1^{\frac{4(n+2)}{q}}\varepsilon}{800c_7}, \tag{3.5.23}$$

$$\omega(R'_\varepsilon) \leq \frac{\delta_1^{\frac{n+2}{q}}\varepsilon}{800c_7}, \tag{3.5.24}$$

$$\sup_{(x_0,t_0)\in\tilde{Q}} \int_0^{2\lambda_M^{(2-p)/2}R'_\varepsilon} \frac{|\mu|(Q_\rho(x_0,t_0))}{\rho^{n+1}}\frac{d\rho}{\rho} \leq \frac{\delta_1^{\frac{4(n+2)}{q}}\varepsilon}{800c_7\lambda_M^{(n+1)(2-p)/2}}, \tag{3.5.25}$$

$$\sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\leq\lambda_M^{(2-p)/2}R'_\varepsilon} \frac{|\mu|(Q_\rho(x_0,t_0))}{\rho^{n+1}} \leq \frac{\delta_1^{\frac{n+2}{q}}\varepsilon}{800c_7\lambda_M^{(n+1)(2-p)/2}}, \tag{3.5.26}$$

and

$$\sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\leq R'_\varepsilon} \phi_q(Du, Q^{\lambda_M}_\rho(x_0,t_0)) \leq \frac{\delta_1^{\frac{2(n+2)}{q}}\varepsilon}{800}, \tag{3.5.27}$$

where $c_7 = c_7(n, p, \nu, L)$ is the same constant as in Lemma 3.4.6. Let us briefly explain why we can choose such an R'_ε. First, (3.5.23) and (3.5.24) are possible since ω is a nondecreasing function satisfying the Dini condition (3.1.5). Moreover, (3.5.25) and (3.5.26) are possible by using the assumption (3.1.12). Finally, (3.5.27) is possible by Proposition 3.5.1, which is applicable since the assumption (3.1.12) directly implies (3.5.4).

Arguing exactly as in (3.5.10), the bound (3.5.26) implies that

$$\sup_{(x_0,t_0)\in\tilde{Q}} \sup_{0<\rho\le R'_\varepsilon} \frac{|\mu|(Q_\rho^{\lambda_M}(x_0,t_0))}{\rho^{n+1}} \le \frac{\delta_1^{\frac{n+2}{q}}\varepsilon}{800c_7}\lambda_M. \tag{3.5.28}$$

Now we fix $(x_0, t_0) \in \tilde{Q}$ and define a sequence of shrinking intrinsic cylinders for $i \in \mathbb{N}$, namely,

$$Q_i := Q_{r_i}^{\lambda_M}(x_0, t_0), \quad r_i := \delta_1^i R'_\varepsilon.$$

Step 2: The iteration step. We have the following lemma.

Lemma 3.5.3. *Assume that*

$$\Big(\fint_{Q_{i+1}} |Du|^q \, dxdt\Big)^{1/q} \ge \frac{\varepsilon}{10}\lambda_M. \tag{3.5.29}$$

Then we have

$$\phi_q(Du, Q_{i+1}) \le \frac{1}{4}\phi_q(Du, Q_i) + c_7\delta_1^{-(n+2)/q}\Big[\frac{|\mu|(Q_i}{r_i^{n+1}}\Big] + c_7\delta_1^{-(n+2)/q}\omega(r_i)\lambda_M. \tag{3.5.30}$$

Proof. Let w and v be the solutions defined in (3.4.1) and (3.4.18) respectively, with the choices $\rho = r_i$ and $\lambda = \lambda_M$. By (3.5.3) and (3.5.28), we can apply Lemma 3.4.7 with choices of parameters $\rho = r_i$, $\lambda = \lambda_M$, $\delta = \delta_1$, and $\theta = \varepsilon^q/20$. Thus using (3.5.29) we obtain

$$\frac{\lambda_M}{B} = \frac{\varepsilon\lambda_M}{400n} \leq \sup_{Q_{i+1}} \|Dv\|, \quad s + \sup_{\frac{1}{4}Q_i} \|Dv\| \leq c_8\lambda_M = A\lambda_M.$$

Recalling that $\delta_1 = \delta_\gamma/4$ and applying Theorem 3.3.3 in $\frac{1}{4}Q_i = \frac{1}{4}Q_{r_i}^{\lambda_M}$, we have

$$\phi_q(Dv, Q_{i+1}) = \phi(Dv, \frac{\delta_\gamma}{4}Q_i) \leq 2^{-4(n+3)}\phi_q(Dv, \frac{1}{4}Q_i). \tag{3.5.31}$$

By (3.5.3), (3.5.24), (3.5.28), and (3.5.31), we can apply Lemma 3.4.8 (with $\varepsilon = 1$) and get (3.5.30). □

Step 3: Exit time argument. The main result we want to prove is as follows.

Lemma 3.5.4. *It holds that*

$$|\mathbf{m}(Du, Q_j) - \mathbf{m}(Du, Q_k)| < \varepsilon\lambda_M, \quad \forall\, 0 \leq j \leq k.$$

Proof. For simplicity, we still denote $\mathbf{m}_i := \mathbf{m}(Du, Q_i)$ and $\Phi_i := \phi_q(Du, Q_i)$ for $i \geq 0$. We denote the set

$$L := \Big\{ i \in \mathbb{N} : \Big(\fint_{Q_i} |Du|^q \, dxdt \Big)^{1/q} < \frac{\varepsilon}{10}\lambda_M \Big\}. \tag{3.5.32}$$

We can assume $0 \leq j < k$ and there are two different cases:

$$L \cap \{j+1, \ldots, k\} = \emptyset, \quad \text{or} \quad L \cap \{j+1, \ldots, k\} \neq \emptyset.$$

Case 1: $L \cap \{j+1, \ldots, k\} = \emptyset$. By the definition of the set L in (3.5.32), we can apply Lemma 3.5.3 for $i \in \{j, \ldots, k-1\}$ and obtain

$$\Phi_{i+1} \leq \frac{1}{4}\Phi_i + c_7\delta_1^{-(n+2)/q}\Big[\frac{|\mu|(Q_i)}{r_i^{n+1}}\Big] + c_7\delta_1^{-(n+2)/q}\omega(r_i)\lambda_M. \tag{3.5.33}$$

Summing up (3.5.33) for $i \in \{j, \dots, k-1\}$, and using standard manipulations as in the proof of (3.4.57), we have

$$\begin{aligned}\sum_{i=j}^{k} \Phi_i &\leq 2\Phi_j + 2c_7\delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \frac{|\mu|(Q_i)}{r_i^{n+1}} + 2c_7\delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \omega(r_i)\lambda_M \\ &\leq \frac{\delta_1^{4(n+2)/q}\varepsilon}{400} + 2c_7\delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \frac{|\mu|(Q_i)}{r_i^{n+1}} + 2c_7\delta_1^{-(n+2)/q} \sum_{i=j}^{k-1} \omega(r_i)\lambda_M.\end{aligned} \tag{3.5.34}$$

Here we also used (3.5.27) in the last inequality. Using (3.4.43) (with $\lambda = 1$), (3.5.25), and the fact that $Q_i \subset Q_{\lambda_M^{(2-p)/2} r_i}(x_0, t_0)$, we have

$$\begin{aligned}\sum_{i=0}^{\infty} \frac{|\mu|(Q_i)}{r_i^{n+1}} &\leq \sum_{i=0}^{\infty} \frac{|\mu|\big(Q_{\lambda_M^{(2-p)/2} r_i}(x_0, t_0)\big)}{r_i^{n+1}} \\ &\leq \lambda_M^{(n+1)(2-p)/2} \delta_1^{-(n+2)} \int_0^{2\lambda_M^{(2-p)/2} R'_\varepsilon} \frac{|\mu|(Q_\rho(x_0, t_0))}{\rho^{n+1}} \frac{d\rho}{\rho} \\ &\leq \frac{\delta_1^{3(n+2)/q}\varepsilon}{800c_7}.\end{aligned} \tag{3.5.35}$$

Similarly, from (3.5.23) we have

$$\sum_{i=0}^{\infty} \omega(r_i)\lambda_M \leq \frac{\delta_1^{3(n+2)/q}\varepsilon}{800c_7}. \tag{3.5.36}$$

Combining (3.5.34), (3.5.35), and (3.5.36) we obtain

$$\sum_{i=j}^{k} \Phi_i \leq \frac{\delta_1^{2(n+2)/q}\varepsilon}{100}.$$

By (3.2.4), the triangle inequality, and the previous inequality, it holds that

$$|\mathbf{m}_k - \mathbf{m}_j| \leq \sum_{i=j}^{k-1} |\mathbf{m}_{i+1} - \mathbf{m}_i| \leq 4\delta_1^{-(n+2)/q} \sum_{i=j}^{k} \Phi_i \leq \frac{\varepsilon\lambda_M}{25}.$$

Case 2: $L \cap \{j+1, \dots, k\} \neq \emptyset$. We prove in this case that

$$|\mathbf{m}_j| < \frac{\varepsilon\lambda_M}{2} \quad \text{and} \quad |\mathbf{m}_k| < \frac{\varepsilon\lambda_M}{2}.$$

We only give the proof of the former inequality and the proof for the latter is similar. By the assumption that $L \cap \{j+1, \dots, k\} \neq \emptyset$, we can define $j' := \min\{l \in L : l \geq j+1\}$ and we have $j' \in L$. Thus by (3.2.3) and the fact that $q \in (1/2, 1)$, we obtain

$$|\mathbf{m}_{j'}| \leq 4\Big(\fint_{Q_j} |Du(x,t)|^q \, dxdt \Big)^{1/q} < \frac{2\varepsilon\lambda_M}{5}. \tag{3.5.37}$$

There are two possibilities, namely, $j' = j+1$ or $j' > j+1$. First, we assume $j' = j+1$. Using (3.2.4) and (3.5.27), we have

$$|\mathbf{m}_j - \mathbf{m}_{j+1}| \leq 2\Phi_{j+1} + 2\delta_1^{-(n+2)/q}\Phi_j \leq \frac{\varepsilon}{200} \leq \frac{\varepsilon\lambda_M}{200}. \tag{3.5.38}$$

Therefore, by the triangle inequality, (3.5.37), and (3.5.38),

$$|\mathbf{m}_j| \leq |\mathbf{m}_j - \mathbf{m}_{j+1}| + |\mathbf{m}_{j'}| < \frac{\varepsilon\lambda_M}{25} + \frac{2\varepsilon\lambda_M}{5} < \frac{\varepsilon\lambda_M}{2}.$$

Otherwise, we have $j' > j+1$. Then by the definition of j', we know that $L \cap \{j+1, \dots, j'-1\} = \emptyset$. Therefore we can apply Lemma 3.5.3 for $i \in \{j, \dots, j'-2\}$. From now on, we can argue exactly as in Case 1 to get

$$|\mathbf{m}_j - \mathbf{m}_{j'-1}| \leq \frac{\varepsilon\lambda_M}{25}. \tag{3.5.39}$$

Again using (3.2.4) and (3.5.27), we have

$$|\mathbf{m}_{j'-1} - \mathbf{m}_{j'}| \leq 2\Phi_{j'} + 2\delta_1^{-(n+2)/q}\Phi_{j'-1} \leq \frac{\varepsilon}{200} \leq \frac{\varepsilon\lambda_M}{200}. \tag{3.5.40}$$

Therefore, by the triangle inequality, (3.5.37), (3.5.39), and (3.5.40),

$$|\mathbf{m}_j| \le |\mathbf{m}_j - \mathbf{m}_{j'-1}| + |\mathbf{m}_{j'-1} - \mathbf{m}_{j'}| + |\mathbf{m}_{j'}| < \frac{\varepsilon\lambda_M}{25} + \frac{\varepsilon\lambda_M}{200} + \frac{2\varepsilon\lambda_M}{5} < \frac{\varepsilon\lambda_M}{2}.$$

The proof of the lemma is now completed. □

Step 4: Conclusion. For any $\rho_1,\ \rho_2 \in (0, R'_\varepsilon]$, there exists two integers $j,\ k \ge 0$ such that

$$\delta_1^{j+1} R'_\varepsilon < \rho_1 \le \delta_1^j R'_\varepsilon \quad \text{and} \quad \delta_1^{k+1} R'_\varepsilon < \rho_2 \le \delta_1^k R'_\varepsilon.$$

By (3.2.4), we have

$$\begin{aligned}
&|\mathbf{m}(Du, Q_{\rho_1}^{\lambda_M}(x_0, t_0)) - \mathbf{m}(Du, Q_j)| \\
&\le 2\phi_q(Du, Q_{\rho_1}^{\lambda_M}(x_0, t_0)) + 2\Big(\frac{|Q_j|}{|Q_{\rho_1}^{\lambda_M}(x_0, t_0)|}\Big)^{1/q} \phi_q(Du, Q_j) \le \frac{\varepsilon}{200}.
\end{aligned}$$

Here we used (3.5.27) in the last line. Similarly, it holds that

$$|\mathbf{m}(Du, Q_{\rho_2}^{\lambda_M}(x_0, t_0)) - \mathbf{m}(Du, Q_k)| \le \frac{\varepsilon}{200}.$$

Thus, the estimate (3.5.21) follows by using Lemma 3.5.4, the triangle inequality, and the last two inequalities. The proposition is proved. □

Proof of Theorem 3.1.3. For any $(x_0, t_0) \in \tilde{Q} \cap \mathcal{L}$, where $\mathcal{L} \equiv \mathcal{L}_{\lambda_M}$ is the set of Lebesgue points of Du defined in (3.4.36), arguing exactly as in (3.4.59), we know that

$$\lim_{r\to 0} |\mathbf{m}(Q_r^{\lambda_M}(x_0, t_0) - Du(x_0, t_0)| = 0.$$

Hence by Proposition 3.5.2,

$$\lim_{r\to 0} |\mathbf{m}(Q_r^{\lambda_M}(x_0, t_0) - Du(x_0, t_0)| = 0 \quad \text{uniformly in } (x_0, t_0) \in \tilde{Q} \cap \mathcal{L}. \tag{3.5.41}$$

Let (x_1,t_1), $(x_2,t_2) \in \tilde{Q}\cap\mathcal{L}$ and $\rho := |(x_1,t_1)-(x_2,t_2)|_{\text{par}}$. Without loss of generality, we assume that $t_1 \le t_2$. Therefore $Q_\rho^{\lambda_M}(x_1,t_1) \subset Q_{2\rho}^{\lambda_M}(x_2,t_2)$ and by (3.2.4) and the fact that $q \in (\frac{1}{2},1)$, we have

$$
\begin{aligned}
&|\mathbf{m}(Du, Q_\rho^{\lambda_M}(x_1,t_1)) - \mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2))| \\
&\le 2\phi_q(Du, Q_\rho^{\lambda_M}(x_1,t_1)) + 2^{2n+5}\phi_q(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2)).
\end{aligned}
$$

By the triangle inequality and the previous inequality, we obtain

$$
\begin{aligned}
&|Du(x_1,t_1) - Du(x_2,t_2)| \\
&\le |Du(x_1,t_1) - \mathbf{m}(Du, Q_\rho^{\lambda_M}(x_1,t_1))| + |\mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2)) - Du(x_2,t_2)| \\
&\quad + |\mathbf{m}(Du, Q_\rho^{\lambda_M}(x_1,t_1)) - \mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2))|. \\
&\le |Du(x_1,t_1) - \mathbf{m}(Du, Q_\rho^{\lambda_M}(x_1,t_1))| + |\mathbf{m}(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2)) - Du(x_2,t_2)| \\
&\quad + 2\phi_q(Du, Q_\rho^{\lambda_M}(x_1,t_1)) + 2^{2n+5}\phi_q(Du, Q_{2\rho}^{\lambda_M}(x_2,t_2)).
\end{aligned}
\tag{3.5.42}
$$

Using (3.5.41), (3.5.42), and Proposition 3.5.1, it follows that for (x_1,t_1), $(x_2,t_2) \in \tilde{Q}\cap\mathcal{L}$,

$$
|Du(x_1,t_1) - Du(x_2,t_2)| \to 0 \quad \text{when } \rho \equiv |(x_1,t_1)-(x_2,t_2)|_{\text{par}} \to 0.
$$

Since $|\tilde{Q}\backslash\mathcal{L}| = 0$, we conclude that Du is continuous in $\tilde{Q}$. □

Proof of Corollary 3.1.4. By [55, Lemma 2.1], we know that the assumption (3.1.13) implies (3.1.12). Therefore, Corollary 3.1.4 is a direct consequence of Theorem 3.1.3. □

Proof of Corollary 3.1.5. Corollary 3.1.5 follows directly from Theorem 3.1.3 since the assumptions (3.1.14) and (3.1.15) directly imply (3.1.12). □

CHAPTER 4

The insulated conductivity problem with p-Laplacian

We study the insulated conductivity problem with closely spaced insulators embedded in a homogeneous matrix where the current-electric field relation is the power law $J = |E|^{p-2}E$. The gradient of solutions may blow up as ε, the distance between insulators, approaches to 0. We prove an upper bound of the gradient to be of order $\varepsilon^{-\alpha}$, where $\alpha = 1/2$ when $p \in (1, n+1]$ and any $\alpha > n/(2(p-1))$ when $p > n+1$. We provide examples to show that this exponent is almost optimal in 2D. Additionally in dimensions $n \geq 3$, for any $p > 1$, we prove another upper bound of order $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$, and show that $\beta \nearrow 1/2$ as $n \to \infty$.

4.1 Introduction and Main results

We investigate the phenomenon of electric field concentration in high-contrast composites. Such a phenomenon can occur when two approaching inclusions possess material properties that differ significantly from the background matrix (see e.g. [16, 49, 73]). The study of this area originated from [6], where the problem with inclusions closely located in a linear background medium was studied numerically. In this chapter, we study the scenario in which the inclusions are insulators, and the background matrix follows the current-electric field relation described by the power law:

$$J = \sigma |E|^{p-2} E, \quad p > 1, \tag{4.1.1}$$

where J, E, and σ denote current, electric field, and conductivity, respectively. Physically, such power law can occur in various materials, including dielectrics, plastic moulding, plasticity phenomena, viscous flows in glaciology, electro-rheological and thermo-rheological fluids. See e.g., [5, 40, 46, 61, 81, 82], the second paragraph of [14] and the references therein.

Let us describe the mathematical setup: let $n \geq 2$, $\Omega \subset \mathbb{R}^n$ be a bounded domain with $C^{1,1}$ boundary containing two $C^{1,1}$ open sets $\mathcal{D}_1$ and $\mathcal{D}_2$ with $\operatorname{dist}(\mathcal{D}_1 \cup \mathcal{D}_2, \partial\Omega) > c > 0$. Let

$$\varepsilon := \operatorname{dist}(\mathcal{D}_1, \mathcal{D}_2),$$

$\widetilde{\Omega} := \Omega \setminus \overline{(\mathcal{D}_1 \cup \mathcal{D}_2)}$, and $\sigma = \mathbb{1}_{\widetilde{\Omega}}$. The voltage potential u satisfies the following p-Laplace equation with $p > 1$:

$$\begin{cases} -\operatorname{div}(|Du|^{p-2} Du) = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \partial\mathcal{D}_i, \ i = 1, 2, \\ u = \varphi & \text{on } \partial\Omega, \end{cases} \tag{4.1.2}$$

where $\varphi \in C^{1,1}(\partial\Omega)$ is given, and $\nu = (\nu_1, \ldots, \nu_n)$ denotes the inner normal vector on $\partial\mathcal{D}_1 \cup \partial\mathcal{D}_2$.

Our goal is to quantitatively analyze the concentration of the electric field $E = -Du$ between the inclusions, and this is a challenging problem even in the linear case when $p = 2$. While the optimal blow-up rate for the linear case in two dimensions was captured about two decades ago in [3,4], the optimal rate in dimensions $n \geq 3$ was only recently identified in [28,29]. This optimal rate is linked to the first non-zero eigenvalue of an elliptic operator on $\mathbb{S}^{n-2}$, which is determined by the principal curvatures of the inclusions. This phenomenon is completely different from the perfect conductivity problem, where the optimal blow-up rates do not depend on the curvatures of the inclusions (see [3, 4, 8]). For other earlier work on the linear insulated conductivity problem, we refer the reader to [9, 65, 66, 88, 91]. In the case when the current-electric field relation is given by (4.1.1), Gorb and Novikov in [43] and Ciraolo and Sciammetta in [21] studied the field concentration when $\mathcal{D}_1$ and $\mathcal{D}_2$ are perfect conductors. They proved that for $n \geq 2$,

$$\|\nabla u\|_{L^\infty(\widetilde{\Omega})} \leq \begin{cases} C\varepsilon^{-\frac{n-1}{2(p-1)}} & p > \dfrac{n+1}{2}, \\ C\varepsilon^{-1}|\log \varepsilon|^{\frac{1}{1-p}} & p = \dfrac{n+1}{2}, \\ C\varepsilon^{-1} & p < \dfrac{n+1}{2}. \end{cases}$$

These bounds were shown to be optimal in their respective papers. However, the phenomenon of electric field concentration between insulators has not been studied before.

Before stating our main results, let us introduce some notation. We denote $x = (x', x_n)$, where $x' \in \mathbb{R}^{n-1}$. By choosing a coordinate system properly, we can assume that near the origin, the part of $\partial\mathcal{D}_1$ and $\partial\mathcal{D}_2$, denoted by Γ_+ and Γ_-, are respectively the graphs of two $C^{1,1}$ functions in terms of x'. That is

$$\Gamma_+ = \left\{x_n = \frac{\varepsilon}{2} + h_1(x'),\ |x'| < 1\right\}, \quad \Gamma_- = \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'),\ |x'| < 1\right\},$$

where h_1 and h_2 are $C^{1,1}$ functions satisfying

$$h_1(0') = h_2(0') = 0, \quad D_{x'}h_1(0') = D_{x'}h_2(0') = 0, \tag{4.1.3}$$

$$c_1|x'|^2 \le h_1(x') - h_2(x') \quad \text{for } \ 0 < |x'| < 1, \tag{4.1.4}$$

$$\|h_1\|_{C^{1,1}} \le c_2, \quad \|h_2\|_{C^{1,1}} \le c_2, \tag{4.1.5}$$

with some positive constants c_1, c_2. For $x_0 \in \widetilde{\Omega}$, $0 < r \le 1$, we denote

$$\Omega_{x_0,r} := \left\{(x', x_n) \in \widetilde{\Omega} : \ -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'),\ |x' - x_0'| < r\right\},$$

and $\Omega_r := \Omega_{0,r}$. We use $B_r(x_0)$ to denote the open ball of radius r centered at x_0 and we set

$$B_r = B_r(0), \quad \Omega_r(x_0) = \widetilde{\Omega} \cap B_r(x_0).$$

By classical $C^{1,\alpha}$ estimates for the p-Laplace equation and the maximum principle (see e.g. [68,86]), the solution $u \in W^{1,p}(\widetilde{\Omega})$ of (4.1.2) satisfies

$$\|u\|_{L^\infty(\widetilde{\Omega})} + \|u\|_{C^{1,\alpha}(\widetilde{\Omega}\setminus\Omega_{1/2})} \le C\|\varphi\|_{C^{1,1}(\partial\Omega)} \tag{4.1.6}$$

for some constants $\alpha = \alpha(n,p) \in (0,1)$ and $C > 0$ independent of ε, φ, and u. As such, we focus on the following problem near the origin:

$$\begin{cases} -\operatorname{div}(|Du|^{p-2}Du) = 0 & \text{in } \Omega_1, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } \Gamma_+ \cup \Gamma_-. \end{cases} \tag{4.1.7}$$

For any domain $\mathcal{D}$, we denote the oscillation of u in $\mathcal{D}$ by

$$\operatorname*{osc}_{\mathcal{D}} u := \sup_{\mathcal{D}} u - \inf_{\mathcal{D}} u.$$

Our first main result is the following pointwise gradient estimate of order $\varepsilon^{-1/2}$ for any $p > 1$ and $n \ge 2$.

Theorem 4.1.1. *Let h_1, h_2 be $C^{1,1}$ functions satisfying (4.1.3)-(4.1.5), $p > 1$, $n \ge 2$,*

$\varepsilon \in (0,1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (4.1.7). Then there exists a constant $C > 0$ depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/2}$ and $\eta = \frac{1}{4}(\varepsilon + |x'|^2)^{\frac{1}{2}}$,

$$|Du(x)| \leq C(\varepsilon + |x'|^2)^{-\frac{1}{2}} \underset{\Omega_{x,\eta}}{osc}\, u. \tag{4.1.8}$$

When $n \geq 3$, we improve the upper bound in Theorem 4.1.1 to the order of $\varepsilon^{-1/2+\beta}$ for some $\beta > 0$.

Theorem 4.1.2. *Let h_1, h_2 be $C^{1,1}$ functions satisfying (4.1.3)-(4.1.5), $p > 1$, $n \geq 3$, $\varepsilon \in (0,1)$, and $u \in W^{1,p}(\Omega_1)$ be a solution of (4.1.7). Then there exist positive constants C and β depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/2}$,*

$$|Du(x)| \leq C(\varepsilon + |x'|^2)^{-1/2+\beta} \underset{\Omega_1}{osc}\, u. \tag{4.1.9}$$

When $p > n+1$, we derive a more explicit upper bound, under an additional assumption that h_1 and h_2 are C^2 strictly convex and strictly concave, respectively. That is, for some positive constants κ_1 and κ_2,

$$\kappa_1 I_{n-1} \leq D^2 h_1(x') \leq \kappa_2 I_{n-1}, \quad \kappa_1 I_{n-1} \leq -D^2 h_2(x') \leq \kappa_2 I_{n-1} \quad \text{for} \ \ 0 \leq |x'| < 1. \tag{4.1.10}$$

Our pointwise gradient estimate of order $-\frac{n+2\delta}{2(p-1)}$, for any $\delta > 0$ when $p > n+1$, is as follows.

Theorem 4.1.3. *Let $n \geq 2$, $p > n+1$, h_1, h_2 be C^2 functions satisfying (4.1.3) and (4.1.10), and let $u \in W^{1,p}(\Omega_1)$ be a solution of (4.1.7). Then for any $\delta > 0$ and $\varepsilon \in (0,1)$, we have*

$$|Du(x)| \leq C(\varepsilon + |x'|^2)^{-\frac{n+2\delta}{2(p-1)}} \underset{\Omega_1}{osc}\, u \quad \text{for} \ \ x \in \Omega_{1/2}, \tag{4.1.11}$$

where C is a positive constant depending on n, p, δ, κ_1, κ_2, and the modulus of continuity for $D^2h_1(x')$ and $D^2h_2(x')$ at $x'=0$.

Furthermore, we show that when $n=2$, the blow-up exponents $-1/2$ for $p\leq 3$ and $-1/(p-1)$ for $p>3$ obtained in Theorems 4.1.1 and 4.1.3 are critical in the sense that

Theorem 4.1.4. *For $n=2$, $p>1$, $\varepsilon\in(0,1)$, let $\Omega=B_5$, and $\mathcal{D}_1,\mathcal{D}_2$ be the unit balls center at $(0,1+\varepsilon/2)$ and $(0,-1-\varepsilon/2)$, respectively. Let $\varphi=x_1$ and $u\in W^{1,p}(\widetilde{\Omega})$ be the solution of (4.1.2). Then for any $\delta>0$, there exists a positive constant C depending only on p and δ, such that when $p\in(1,3]$,*

$$\|Du\|_{L^\infty(\widetilde{\Omega}\cap B_{8\sqrt{\varepsilon/\delta}})}\geq \frac{1}{C}\varepsilon^{\frac{-1+\delta}{2}},$$

and when $p>3$,

$$\|Du\|_{L^\infty(\widetilde{\Omega}\cap B_{8\sqrt{\varepsilon/\delta}})}\geq \frac{1}{C}\varepsilon^{\frac{-1+\delta}{p-1}}.$$

Finally, we also establish a blow-up rate of $\varepsilon^{-1/2+\beta}$ for the gradient, for any $p>1$ and sufficiently large n, with more explicit constant $\beta\in[0,1/2)$. For this, we impose a further assumption that h_1 and h_2 are $C^{2,\mathrm{Dini}}$ strictly convex and strictly concave respectively, satisfying (4.1.10) for some positive constants κ_1 and κ_2.

Theorem 4.1.5. *Let h_1, h_2 be $C^{2,Dini}$ functions satisfying (4.1.3) and (4.1.10), $p>1$, $\beta\in[0,1/2)$, $\varepsilon\in(0,1)$, and $u\in W^{1,p}(\Omega_1)$ be a solution of (4.1.7). If n, p, and β satisfy either*

$$p\geq 2,\quad n\geq \frac{5(p-1)}{2}\left(\frac{p+1-2\beta(p-1)}{2}+\frac{\kappa_2}{(1-2\beta)\kappa_1}\right)+1, \tag{4.1.12}$$

or

$$1<p<2,\quad n\geq \frac{5}{2}\left(\frac{3-2\beta}{2}+\frac{\kappa_2}{(1-2\beta)\kappa_1}\right)+3-p, \tag{4.1.13}$$

then there exist a positive constant C depending only on n, p, β, κ_1, and κ_2, such that

$$|Du(x)| \le C\|u\|_{L^\infty(\Omega_1)}(\varepsilon + |x'|^2)^{-\frac{1}{2}+\beta} \quad \textit{for} \;\; x \in \Omega_{1/2}. \tag{4.1.14}$$

Remark 4.1.6. By (4.1.12) and (4.1.13), when $n \to \infty$, β can be chosen arbitrarily close to $1/2$. In view of (4.1.14), the singularity of Du diminishes as the dimension n increases. We also note that by refining the inequalities in the proof of Theorem 4.1.5 further, it is possible to improve the lower bounds of n in both (4.1.12) and (4.1.13). However, we have decided not to pursue this in the current chapter.

The rest of the chapter is organized as follows. In the next section, we give the proof of Theorem 4.1.1 using mean oscillation estimates. In Section 4.3, we demonstrate the proof of Theorem 4.1.2 by utilizing a delicate change of variables, an extension argument, and the Krylov-Safonov Harnack inequality. Sections 4.4 and 4.5 are devoted to the proofs of Theorems 4.1.3 and 4.1.4, respectively, for which we construct suitable sub- and super-solutions. Finaally, in Section 4.6, we employ a Bernstein type argument to prove Theorem 4.1.5. Here we use the fact that for any $q \ge p$, $|Du|^q$ is a subsolution to the normalized p-Laplace equation, as originally observed by Uhlenbeck [85].

4.2 Mean oscillation estimates

In this section, we give the proof of Theorem 4.1.1 using mean oscillation estimates. We fix a point $x_0 \in \Omega_{1/2}$ and prove (4.1.8) at $x = x_0$. Note that we can always assume $\varepsilon + |x_0'|^2 \le c$, where $c = c(n, p, c_1, c_2) > 0$ could be any sufficiently small constant depending only on n, p, c_1, and c_2. Otherwise, by classical estimates (see [68, 86]), (4.1.8) directly follows. Next, we derive some mean oscillation estimates of Du on a ball $B_r(x_0)$ for different radii r.

4.2.1 Mean oscillation estimates for small r

We recall a classical interior mean oscillation estimate when $B_r(x_0) \subset \Omega_1$. Estimates of this type, with different exponents involved, were developed in [24, 38, 69].

Lemma 4.2.1. *Let $u \in W^{1,p}(\Omega_1)$ be a solution to (4.1.7). There exist constants $C > 1$ and $\alpha \in (0, 1)$ depending only on n and p, such that $u \in C^{1,\alpha}(\Omega_1)$ and for every $B_r(x_0) \subset \Omega_1$ and $\rho \in (0, r]$, we have*

$$\left(\fint_{B_\rho(x_0)} |Du - (Du)_{B_\rho(x_0)}|^p \right)^{\frac{1}{p}} \leq C \left(\frac{\rho}{r} \right)^\alpha \left(\fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^p \right)^{\frac{1}{p}}.$$

We denote

$$\phi(x_0, r) = \left(\fint_{B_r(x_0)} |Du - (Du)_{B_r(x_0)}|^p \right)^{\frac{1}{p}}.$$

Then Lemma 4.2.1 also implies

Corollary 4.2.2. *Under the assumptions of Lemma 4.2.1, there exists a constant $\mu_1 \in (0, 1)$ depending only on n and p, such that for any $\mu \in (0, \mu_1]$, $B_r(x_0) \subset \Omega_1$, and $K \in \mathbb{N}$, it holds that*

$$\sum_{k=0}^{K+1} \phi(x_0, \mu^k r) \leq 2\, \phi(x_0, r). \tag{4.2.1}$$

Proof. We take $\mu_1 \in (0, 1)$ such that $C\mu_1^\alpha = \frac{1}{2}$, where C and α are the same constants as in Lemma 4.2.1. Replacing r with $\mu^k r$ and setting $\rho = \mu^{k+1} r$, we get

$$\phi(x_0, \mu^{k+1} r) \leq \frac{1}{2} \phi(x_0, \mu^k r).$$

Summing the above inequality over $k = 0, 1, \ldots, K$, we obtain (4.2.1). $\square$

4.2.2 Mean oscillation estimates for intermediate r

Next, we consider the case when $B_r(x_0)$ intersects with only one of Γ_+ and Γ_-. In this case, we choose $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ such that $\operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$ and we derive mean oscillation estimates around $\hat{x}_0$. Note that we can assume $\varepsilon + c_2|x_0'|^2 \le 1/4$ and thus by (4.1.5) and the triangle inequality, $|\hat{x}_0'| \le 3/4$. Without loss of generality, we assume $\hat{x}_0 \in \Gamma_-$. Then by (4.1.4) and (4.1.5), there exists a constant $c = c(n, c_1, c_2) \in (0, 1/4)$, such that $B(\hat{x}_0, r) \cap \Gamma_+ = \emptyset$ for any $r \in (0, c(\varepsilon + |\hat{x}_0'|^2))$.

We first choose a coordinate $y = (y', y_n)$ such that $y(\hat{x}_0) = 0$, the direction of axis y_n is the upper normal vector at $\hat{x}_0 \in \Gamma_-$, and $\Omega_{R_0}(\hat{x}_0) = \{y \in B_{R_0} : y_n > \chi(y')\}$, where $R_0 = c_3(\varepsilon + |\hat{x}_0'|^2) \in (0, 1/4)$ for some constant $c_3 = c_3(n, c_1, c_2) \in (0, 1/8)$ and $\chi : \{y' \in \mathbb{R}^{n-1} : |y'| < R_0\} \to \mathbb{R}$ is a $C^{1,1}$ function in the coordinate system depending on $\hat{x}_0$ such that

$$\chi(0') = 0, \quad D_{y'}\chi(0') = 0, \quad \|\chi\|_{C^{1,1}} \le C\|h_2\|_{C^{1,1}}. \tag{4.2.2}$$

Then we let

$$z = \Lambda(y) = (y',\, y_n - \chi(y')).$$

Since Γ_- is $C^{1,1}$, by (4.2.2) there exist constants $C = C(n, c_1, c_2)$, $c_4 = c_4(n, c_1, c_2) \in (0, c_3) \subset (0, 1/8)$, and $R_1 = c_4(\varepsilon + |\hat{x}_0'|^2)$ such that

$$|D_{y'}\chi(y')| \le C\,|y'| \le 1/2 \quad \text{if} \quad |y'| \le 2R_1, \tag{4.2.3}$$

$$\Omega_{r/2}(\hat{x}_0) \subset \Lambda^{-1}(B_r^+) \subset \Omega_{2r}(\hat{x}_0) \quad \forall\, r \in (0, 2R_1], \tag{4.2.4}$$

and thus

$$|D\Lambda(y) - I_n| \le C\,|y'| \le 1/2 \quad \text{if} \quad |y'| \le 2R_1, \tag{4.2.5}$$

Therefore, there exist positive constants $c(n)$ and $c'(n)$ depending only on n, such that for any $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$ and $0 < r \leq c_4(\varepsilon + |\hat{x}_0'|^2)$,

$$c(n)r^n \leq |\Omega_r(\hat{x}_0)| \leq c'(n)r^n. \tag{4.2.6}$$

Note that

$$\det(D\Lambda) \equiv 1. \tag{4.2.7}$$

Then $u_1(z) := u(\Lambda^{-1}(z))$ satisfies the following equation with conormal boundary condition

$$\begin{cases} -\operatorname{div}_z \left(|A^T D_z u_1|^{p-2} A A^T D_z u_1\right) = 0 \quad \text{in } B^+_{R_1}, \\ \left(|A^T D_z u_1|^{p-2} A A^T D_z u_1\right)_n = 0 \quad \text{on } B_{R_1} \cap \partial\mathbb{R}^n_+, \end{cases} \tag{4.2.8}$$

where we denote

$$A := A(z) := (a_{ij}(z)) := D\Lambda(\Lambda^{-1}(z)).$$

Next we extend u_1 and the coefficient matrix A to the whole ball B_{R_1}. We take the even extension of u_1, a_{nn}, and $a_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $z_n = 0$, and take the odd extension of a_{in} and $a_{ni}, i = 1, 2, \ldots, n-1$, with respect to $z_n = 0$. We still denote these functions by u_1 and A after the extension. Because of the conormal boundary condition, u_1 satisfies

$$-\operatorname{div}_z \left(\mathbf{A}(z, D_z u_1)\right) = 0 \quad \text{in } B_{R_1}, \tag{4.2.9}$$

where the nonlinear operator $\mathbf{A}$ is defined as

$$\mathbf{A}(z, \xi) = |A^T \xi|^{p-2} A A^T \xi \qquad \text{for} \quad z \in B_{R_1},\ \xi \in \mathbb{R}^n.$$

Lemma 4.2.3. *There exists a constant $C = C(n, p, c_1, c_2) > 0$, such that for any*

$z \in B_{R_1}$ *and* $\xi \in \mathbb{R}^n$,

$$|\mathbf{A}(z,\xi) - |\xi|^{p-2}\xi| \leq C\,|z'|\,|\xi|^{p-1}. \tag{4.2.10}$$

Proof. We recall a well-known inequality (see [44, Lemma 2.1]): for any $p > 1$ and $\xi_1,\, \xi_2 \in \mathbb{R}^n$, it holds that

$$c^{-1}\big(|\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}} \leq \frac{\big||\xi_2|^{p-2}\xi_2 - |\xi_1|^{p-2}\xi_1\big|}{|\xi_2 - \xi_1|} \leq c\,\big(|\xi_1|^2 + |\xi_2|^2\big)^{\frac{p-2}{2}}, \tag{4.2.11}$$

where $c = c(n,p) > 1$ is a positive constant. Using (4.2.11), (4.2.5), and the triangle inequality, we obtain

$$\begin{aligned}
|\mathbf{A}(z,\xi) - |\xi|^{p-2}\xi| &\leq \big||A^T\xi|^{p-2}(A - I_n)A^T\xi\big| + \big||A^T\xi|^{p-2}A^T\xi - |\xi|^{p-2}\xi\big| \\
&\leq |A - I_n|\,|A^T\xi|^{p-1} + c\left(|A^T\xi|^2 + |\xi|^2\right)^{\frac{p-2}{2}}|A^T\xi - \xi| \leq C\,|z'|\,|\xi|^{p-1}.
\end{aligned}$$

Thus the proof of (4.2.10) is completed. $\square$

Assume that $r \in (0, R_1]$. We let $v_1 \in u_1 + W_0^{1,p}(B_r)$ be the unique solution to

$$\begin{cases} -\operatorname{div}_z(|D_z v_1|^{p-2}D_z v_1) = 0 & \text{in } B_r, \\ v_1 = u_1 & \text{on } \partial B_r. \end{cases} \tag{4.2.12}$$

By testing (4.2.12) and (4.2.9) with $v_1 - u_1$ and using (4.2.10), we have the comparison estimate

$$\fint_{B_r} |D_z u_1 - D_z v_1|^p \leq C r^{\min\{2,p\}} \fint_{B_r} |D_z u_1|^p, \tag{4.2.13}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. For detailed proof of (4.2.13), see [38, Eq. (4.35)] when $p \in (1,2)$ and [39, Lemma 3.4] when $p \geq 2$.

Applying Lemma 4.2.1 and the comparison estimate (4.2.13), we have

Lemma 4.2.4. *Suppose that $u_1 \in W^{1,p}(B^+_{R_1})$ is a solution to (4.2.8). Then for any $\mu \in (0,1)$ and $r \in (0, R_1]$, we have*

$$\left(\fint_{B^+_{\mu r}} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_{\mu r}}|^p + |D_{z_n}u_1|^p \right)^{1/p}$$
$$\leq C\mu^\alpha \left(\fint_{B^+_r} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_r}|^p + |D_{z_n}u_1|^p \right)^{1/p} + C_\mu r^{\theta_p} \left(\fint_{B^+_r} |D_z u_1|^p \right)^{1/p}, \tag{4.2.14}$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 4.2.1, C_μ is a constant depending on μ, n, p, c_1, c_2 and C is a constant depending on n, p, c_1, c_2.

Proof. By Lemma 4.2.1, (4.2.13), and the triangle inequality, we have

$$\begin{aligned}
&\left(\fint_{B_{\mu r}} |D_z u_1 - (D_z u_1)_{B_{\mu r}}|^p \right)^{1/p} \\
&\leq C \left(\fint_{B_{\mu r}} |D_z v_1 - (D_z v_1)_{B_{\mu r}}|^p \right)^{1/p} + C \left(\fint_{B_{\mu r}} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\
&\leq C\mu^\alpha \left(\fint_{B_r} |D_z v_1 - (D_z v_1)_{B_r}|^p \right)^{1/p} + C\mu^{-\frac{n}{p}} \left(\fint_{B_r} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\
&\leq C\mu^\alpha \left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p \right)^{1/p} + C\mu^{-\frac{n}{p}} \left(\fint_{B_r} |D_z u_1 - D_z v_1|^p \right)^{1/p} \\
&\leq C\mu^\alpha \left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p \right)^{1/p} + C_\mu r^{\theta_p} \left(\fint_{B_r} |D_z u_1|^p \right)^{1/p}.
\end{aligned} \tag{4.2.15}$$

Since u_1 is even in z_n, (4.2.15) directly implies (4.2.14). The proof is completed. □

We now define

$$\psi(\hat{x}_0, r) = \left(\fint_{\Omega_r(\hat{x}_0)} |D_{y'}u - (D_{y'}u)_{\Omega_r(\hat{x}_0)}|^p + |D_{y_n}u|^p \right)^{1/p}. \tag{4.2.16}$$

Let $\mu \in (0,1)$ and $r \in (0, R_1/2]$ be constants. By using change of variables, (4.2.3), (4.2.4), (4.2.7), and the triangle inequality, we have

$$
\begin{aligned}
&\left(\fint_{B^+_{\mu r}} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_{\mu r}}|^p + |D_{z_n}u_1|^p\right)^{1/p} \\
&= \left(\fint_{\Lambda^{-1}(B^+_{\mu r})} |D_{y'}u + D_{y_n}u\, D_{y'}\chi - (D_{y'}u + D_{y_n}u\, D_{y'}\chi)_{\Lambda^{-1}(B^+_{\mu r})}|^p + |D_{y_n}u|^p\right)^{1/p} \\
&\geq C\left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |D_{y'}u - (D_{y'}u)_{\Omega_{\mu r/2}(x_0)}|^p + |D_{y_n}u|^p\right)^{1/p} \\
&\quad - C'\left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |D_{y_n}u\; D_{y'}\chi|^p\right)^{1/p} \\
&\geq C\psi(\hat{x}_0, \mu r/2) - C'\mu r\left(\fint_{\Omega_{\mu r/2}(\hat{x}_0)} |Du|^p\right)^{1/p},
\end{aligned}
\tag{4.2.17}
$$

where C and C' are positive constants depending on n, p, c_1, and c_2. Similarly,

$$
\begin{aligned}
&\left(\fint_{B^+_r} |D_{z'}u_1 - (D_{z'}u_1)_{B^+_r}|^p + |D_{z_n}u_1|^p\right)^{1/p} \\
&\leq C''\psi(\hat{x}_0, 2r) + C''r\left(\fint_{\Omega_{2r}(\hat{x}_0)} |Du|^p\right)^{1/p},
\end{aligned}
\tag{4.2.18}
$$

where C'' is a positive constant depending only on n, p, c_1, and c_2. Therefore, by using (4.2.17), (4.2.18), and (4.2.6), (4.2.14) implies that

$$
\psi(\hat{x}_0, \mu r/2) \leq C\mu^\alpha \psi(\hat{x}_0, 2r) + C_\mu r^{\theta_p}\left(\fint_{\Omega_{2r}(\hat{x}_0)} |Du|^p\right)^{1/p}.
$$

By replacing $\mu/4$ and $2r$ with μ and r respectively, we obtain

$$
\psi(\hat{x}_0, \mu r) \leq C\mu^\alpha \psi(\hat{x}_0, r) + C_\mu r^{\theta_p}\left(\fint_{\Omega_r(\hat{x}_0)} |Du|^p\right)^{1/p}
$$

for $\mu \in (0, 1/4)$ and $r \in (0, R_1]$, where we recall that $R_1 = c_4(\varepsilon + |\hat{x}'_0|^2)$. Note that the same argument above also holds when $\hat{x}_0 \in \Gamma_+$. Therefore, using the same argument as in Corollary 4.2.2, we have

Lemma 4.2.5. *Suppose that u is a solution to (4.1.2) and $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \le 3/4\}$. Then there exist constants $c_4 \in (0,1)$ and $C > 0$, both depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0, 1/4)$ and $r \in (0,\, c_4(\varepsilon + |\hat{x}_0'|^2)]$, it holds that*

$$\psi(\hat{x}_0, \mu r) \le C\mu^\alpha \psi(\hat{x}_0, r) + C_\mu r^{\theta_p} \left(\fint_{\Omega_r(\hat{x}_0)} |Du|^p \right)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 4.2.1, and ψ is defined in (4.2.16). Moreover, there exist constants $\mu_2 = \mu_2(n, p, c_1, c_2) \in (0, 1/4)$ and $C_\mu' = C_\mu'(n, p, c_1, c_2, \mu) > 0$, such that for any $\mu \in (0, \mu_2]$ and $K \in \mathbb{N}$, it holds that

$$\sum_{k=0}^{K+1} \psi(\hat{x}_0, \mu^k r) \le 2\, \psi(\hat{x}_0, r) + C_\mu' \sum_{k=0}^{K} (\mu^k r)^{\theta_p} \left(\fint_{\Omega_{\mu^k r}(\hat{x}_0)} |Du|^p \right)^{1/p}.$$

4.2.3 Mean oscillation estimates for large r

Finally, we consider the case when $B_r(x_0)$ could potentially intersects with both Γ_+ and Γ_-. In this case, we assume $x_0 \in \Omega_{1/2}$ and $\frac{c_4}{12}(\varepsilon + |x_0'|^2) \le r \le c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$, where c_4 is the same constant as in Lemma 4.2.5 and c_5 is a constant which will be determined later. We define the map $\mathcal{Z} = \tilde{\Lambda}(x)$ by

$$\begin{cases} \mathcal{Z}' = x' - x_0', \\ \mathcal{Z}_n = (h_1(x_0') - h_2(x_0') + \varepsilon)\Big(\dfrac{x_n - h_2(x') + \varepsilon/2}{h_1(x') - h_2(x') + \varepsilon} - \dfrac{1}{2} \Big). \end{cases}$$

Thus $\tilde{\Lambda}$ is invertible in $\Omega_{x_0,1/2}$,

$$Q_{1/2} := \tilde{\Lambda}(\Omega_{x_0,1/2}) = \big\{ (\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < \frac{1}{2},\, |\mathcal{Z}_n| < \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon) \big\},$$

and

$$\begin{aligned}\tilde{\Gamma}_{\pm} &:= \tilde{\Lambda}(\Gamma_{\pm} \cap \{x \in \mathbb{R}^n : |x' - x_0'| < \frac{1}{2}\}) \\ &= \big\{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n :\ |\mathcal{Z}'| < \frac{1}{2},\ \mathcal{Z}_n = \pm\frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon)\big\}.\end{aligned}$$

Then $u_2(\mathcal{Z}) := u(\tilde{\Lambda}^{-1}(\mathcal{Z}))$ satisfies the following equation with homogeneous conormal boundary condition

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}}\big(|B^T D_{\mathcal{Z}} u_2|^{p-2}(\det(B))^{-1} B B^T D_{\mathcal{Z}} u_2\big) = 0 & \text{in } Q_{1/2} \\ \big(|B^T D_{\mathcal{Z}} u_2|^{p-2}(\det(B))^{-1} B B^T\, D_{\mathcal{Z}} u_2\big)_n = 0 & \text{on } \tilde{\Gamma}_{\pm}, \end{cases}$$

where we denote

$$B := B(\mathcal{Z}) := (b_{ij}(\mathcal{Z})) := D\tilde{\Lambda}(\tilde{\Lambda}^{-1}(\mathcal{Z})).$$

For $\mathcal{Z} \in Q_{1/2}$, let $x = \tilde{\Lambda}^{-1}(\mathcal{Z})$. Then

$$b_{ii}(\mathcal{Z}) = 1 \quad \text{for } i \in \{1, 2, \ldots, n-1\},$$

$$b_{ij}(\mathcal{Z}) = 0 \quad \text{for } i \neq j,\ i \in \{1, 2, \ldots, n-1\},\ j \in \{1, 2, \ldots, n\},$$

$$\begin{aligned}b_{nj}(\mathcal{Z}) = &\frac{h_1(x_0') - h_2(x_0') + \varepsilon}{(h_1(x') - h_2(x') + \varepsilon)^2} \\ &\cdot \big[D_{x_j} h_2(x')\big(x_n - h_1(x') - \frac{\varepsilon}{2}\big) - D_{x_j} h_1(x')\big(x_n - h_2(x') + \frac{\varepsilon}{2}\big)\big]\end{aligned}$$

for $j \in \{1, 2, \ldots, n-1\}$, and

$$b_{nn}(\mathcal{Z}) = \frac{h_1(x_0') - h_2(x_0') + \varepsilon}{h_1(x') - h_2(x') + \varepsilon}.$$

Therefore,

$$\det(B(\mathcal{Z})) = b_{nn}(\mathcal{Z}) = \frac{h_1(x_0') - h_2(x_0') + \varepsilon}{h_1(x_0' + \mathcal{Z}') - h_2(x_0' + \mathcal{Z}') + \varepsilon}$$

is a function independent of $\mathcal{Z}_n$. Assume

$$\frac{c_4}{12}(\varepsilon + |x_0'|^2) \le r \le \frac{1}{4}(\varepsilon + |x_0'|^2)^{\frac{1}{2}} \le \frac{1}{2} \tag{4.2.19}$$

and let $\mathcal{Z}_0 = \tilde{\Lambda}(x_0)$. Then for any $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \le r$ and $x = \tilde{\Lambda}^{-1}(\mathcal{Z})$, by the triangle inequality, we have

$$|x'| \le r + |x_0'| \le \big(1 + \sqrt{12/c_4}\big) r^{\frac{1}{2}} \quad \text{and} \quad |x'|^2 \ge \frac{1}{2}|x_0'|^2 - r^2 \ge \frac{1}{4}(|x_0'|^2 - \varepsilon).$$

Thus, using (4.1.3), (4.1.4), and (4.1.5), we infer that for $j = 1, 2, \ldots, n-1$ and some constant $C > 0$ depending only on n, p, c_1, and c_2,

$$\begin{aligned} |b_{nj}(\mathcal{Z})| &\le 2c_2 \frac{|x'|(h_1(x_0') - h_2(x_0') + \varepsilon)}{h_1(x') - h_2(x') + \varepsilon} \le 2c_2 \frac{|x'|(2c_2|x_0'|^2 + \varepsilon)}{c_1|x'|^2 + \varepsilon} \\ &\le C|x'| \le Cr^{\frac{1}{2}} \le \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} &|b_{nn}(\mathcal{Z}) - 1| = \Big| \frac{\int_0^1 \frac{d}{dt}\big(h_1(tx' + (1-t)x_0') - h_2(tx' + (1-t)x_0')\big)\,dt}{h_1(x') - h_2(x') + \varepsilon} \Big| \\ &\le 2c_2 \frac{(|x'| + |x_0'|)|x' - x_0'|}{c_1|x'|^2 + \varepsilon} \le \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}, \end{aligned}$$

and similarly,

$$\big|\big(\det(B(\mathcal{Z}))\big)^{-1} - 1\big| = \big|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\big| \le \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}. \tag{4.2.20}$$

Therefore, when (4.2.19) holds and $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \le r$, we have for some constant $C = C(n, p, c_1, c_2) > 0$,

$$|B(\mathcal{Z}) - I_n| \leq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}. \tag{4.2.21}$$

In particular, there exists $c_5 = c_5(n, p, c_1, c_2) \in (0, 1/4)$, such that for any $\frac{c_4}{12}(\varepsilon+|x_0'|^2) \leq r \leq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$ and $\mathcal{Z} \in Q_{1/2}$ with $|\mathcal{Z}'| \leq r$, it also holds that

$$|B(\mathcal{Z}) - I_n| \leq 1/2 \quad \text{and} \quad \left|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\right| \leq 1/2. \tag{4.2.22}$$

Note that we can always assume $\varepsilon + |x_0'|^2$ to be sufficiently small so that $c_4(\varepsilon + |x_0'|^2) \leq c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$. Next we extend u_2 and B to the whole cylinder $\mathcal{C}_{1/2} := \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1/2\}$. We take the even extension of u_2, b_{nn}, and $b_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon)$, and take the odd extension of b_{in} and $b_{ni}, i = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = \frac{1}{2}(h_1(x_0') - h_2(x_0') + \varepsilon)$. Then we take the periodic extension in $\mathcal{Z}_n$ axis, so that the period is equal to $2(h_1(x_0') - h_2(x_0') + \varepsilon)$. We still denote these functions by u_2 and B after the extension. Then because of the conormal boundary condition, u_2 satisfies

$$-\operatorname{div}_{\mathcal{Z}} \big(\mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}} u_2)\big) = 0 \quad \text{in } \mathcal{C}_{1/2}, \tag{4.2.23}$$

where the nonlinear operator $\mathbf{B}$ is defined as

$$\mathbf{B}(\mathcal{Z}, \xi) = d(\mathcal{Z}')|B^T\xi|^{p-2}BB^T\xi \qquad \text{for} \quad \mathcal{Z} \in \mathcal{C}_{1/2},\ \xi \in \mathbb{R}^n,$$

and

$$d(\mathcal{Z}') := \big(b_{nn}(\mathcal{Z})\big)^{-1} = \frac{h_1(\mathcal{Z}' + x_0') - h_2(\mathcal{Z}' + x_0') + \varepsilon}{h_1(x_0') - h_2(x_0') + \varepsilon}.$$

Similar to (4.2.10), using (4.2.20), (4.2.21), (4.2.22), and (4.2.11), we obtain that for any $r \in \left[\frac{c_4}{12}(\varepsilon + |x_0'|^2), c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}\right]$, $\mathcal{Z} \in B_r(\mathcal{Z}_0)$, and $\xi \in \mathbb{R}^n$,

$$|\mathbf{B}(\mathcal{Z},\xi) - |\xi|^{p-2}\xi| \leq \frac{Cr}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}|\xi|^{p-1}, \tag{4.2.24}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. Now we let $v_2 \in u_2 + W_0^{1,p}(B_r(\mathcal{Z}_0))$ be the unique solution to

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}}\left(|D_{\mathcal{Z}}v_2|^{p-2}D_{\mathcal{Z}}v_2\right) = 0 & \text{in } B_r(\mathcal{Z}_0), \\ \qquad\qquad\qquad\qquad v_2 = u_2 & \text{on } \partial B_r(\mathcal{Z}_0). \end{cases}$$

Using (4.2.24), similar to (4.2.13), we have the following comparison estimate

$$\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2 - D_{\mathcal{Z}}v_2|^p \leq C\Big(\frac{r}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}\Big)^{\min\{2,p\}} \fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2|^p, \tag{4.2.25}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2.

We define

$$\tilde{\phi}(x_0, r) = \left(\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2 - (D_{\mathcal{Z}}u_2)_{B_r(\mathcal{Z}_0)}|^p\right)^{1/p}. \tag{4.2.26}$$

Then following the same proof as that of Lemma 4.2.5 with (4.2.25) in place of (4.2.13), we have

Lemma 4.2.6. *Suppose that $x_0 \in \Omega_{1/2}$ and u_2 is a solution to (4.2.23). Then there exist constants $c_5 \in (0, 1/4)$ and $C > 0$, both depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0,1)$ and $r \in \left[\frac{c_4}{12}(\varepsilon + |x_0'|^2),\ c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}\right]$, it holds that*

$$\tilde{\phi}(x_0, \mu r) \leq C\mu^\alpha \tilde{\phi}(x_0, r) + C_\mu\Big(\frac{r}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}\Big)^{\theta_p}\left(\fint_{B_r(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2|^p\right)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 4.2.1, c_4 is the same *constant as in Lemma 4.2.5, and $\tilde{\phi}$ is defined in (4.2.26). Moreover, there exist constants $\mu_3 = \mu_3(n, p, c_1, c_2) \in (0, 1)$ and $C'_\mu = C'_\mu(n, p, c_1, c_2, \mu) > 0$, such that for any $\mu \in (0, \mu_3]$ and k_1, $k_2 \in \mathbb{N}$ satisfying $\frac{c_4}{12}(\varepsilon + |x'_0|^2) \le \mu^{k_2} r \le \mu^{k_1} r \le c_5(\varepsilon + |x'_0|^2)^{\frac{1}{2}}$, it holds that*

$$\sum_{k=k_1}^{k_2+1} \tilde{\phi}(x_0, \mu^k r) \le 2\,\tilde{\phi}(x_0, \mu^{k_1} r) + C'_\mu \sum_{k=k_1}^{k_2} \Big(\frac{\mu^k r}{(\varepsilon + |x'_0|^2)^{\frac{1}{2}}} \Big)^{\theta_p} \left(\fint_{B_{\mu^k r}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p \right)^{\frac{1}{p}}.$$

4.2.4 Proof of Theorem 4.1.1

Now we are ready to prove Theorem 4.1.1.

Proof of Theorem 4.1.1. We prove the theorem at any point $x_0 \in \Omega_{1/2}$.

Step 1: Notation and choices of constants. We choose $\mu = \frac{1}{6}\min\{\mu_1, \mu_2, \mu_3\}$, where μ_1, μ_2, and μ_3 are the same constants as in Corollary 4.2.2, Lemma 4.2.5, and Lemma 4.2.6. We define $r_j = \frac{c_5}{2}\mu^j(\varepsilon + |x'_0|^2)^{\frac{1}{2}}$ and let j_1, j_2 be the integers such that

$$r_{j_1} \ge \frac{c_4}{6}(\varepsilon + |x'_0|^2), \quad r_{j_1+1} < \frac{c_4}{6}(\varepsilon + |x'_0|^2),$$

and

$$r_{j_2} \ge \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-), \quad r_{j_2+1} < \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-),$$

where c_4 and c_5 are the same constants as in Lemma 4.2.5 and Lemma 4.2.6. Note that we can assume $\varepsilon + |x'_0|^2$ to be sufficiently small so that $\frac{c_5}{2}(\varepsilon + |x'_0|^2)^{\frac{1}{2}} > \frac{c_4}{6}(\varepsilon + |x'_0|^2)$ and thus $j_1 \ge 0$.

We denote

$$\phi_j = \begin{cases} \left(\fint_{B_{r_j}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2 - (D_{\mathcal{Z}} u_2)_{B_{r_j}(\mathcal{Z}_0)}|^p \right)^{1/p} & \text{if} \quad 0 \le j \le j_1, \\ \left(\fint_{\Omega_{r_j}(x_0)} |Du - (Du)_{\Omega_{r_j}(x_0)}|^p \right)^{1/p} & \text{if} \quad j \ge j_1 + 1, \end{cases}$$

$$T_j = \begin{cases} \left(\fint_{B_{r_j}(\mathcal{Z}_0)} |D_{\mathcal{Z}} u_2|^p \right)^{1/p} & \text{if} \quad 0 \le j \le j_1, \\ \left(\fint_{\Omega_{3r_j}(x_0)} |Du|^p \right)^{1/p} & \text{if} \quad j \ge j_1 + 1, \end{cases}$$

and

$$\mathbf{m}_j = \begin{cases} (D_{\mathcal{Z}} u_2)_{B_{r_j}(\mathcal{Z}_0)} & \text{if} \quad 0 \le j \le j_1, \\ (Du)_{\Omega_{r_j}(x_0)} & \text{if} \quad j \ge j_1 + 1, \end{cases}$$

where u_2, $\mathcal{Z}_0$, and the coordinate $\mathcal{Z}$ are defined in Section 4.2.3. In the following proof, we use C, C' to denote positive constants depending only on n, p, c_1, and c_2, which may differ from line to line.

Step 2: Preliminary estimates and iterations. Next, we derive some preliminary estimates. First, we show that there exists a constant $c = c(n) > 0$, such that for any $j \ge j_1 + 1$,

$$|\Omega_{r_j}(x_0)| \ge c\, r_j^n. \tag{4.2.27}$$

If $r_j \le 2\,\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$, then $B_{\frac{1}{2}r_j}(x_0) \subset \Omega_{r_j}(x_0)$ and (4.2.27) clearly holds. Otherwise, assume $r_j > 2\,\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$. Then we choose $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ such that $\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$, and thus $\Omega_{\frac{1}{2}r_j}(\hat{x}_0) \subset \Omega_{r_j}(x_0)$. Note that we can assume $\varepsilon + c_2|x_0'|^2 \le 1/4$ and thus by (4.1.5) and the triangle inequality, $|\hat{x}_0'| \le 3/4$. Since

$j \geq j_1 + 1$, by the triangle inequality again, we know that $|x_0'| \leq |\hat{x}_0'| + r_j/2$ and

$$r_j < \frac{c_4}{6}(\varepsilon + |x_0'|^2) \leq \frac{c_4}{6}\big(\varepsilon + 2|\hat{x}_0'|^2 + \frac{1}{2}r_j^2\big).$$

Since $c_4 \in (0,1)$ and $r_j \in (0,1)$, we get

$$r_j < \frac{c_4}{2}(\varepsilon + |\hat{x}_0'|^2).$$

By using (4.2.6), we have

$$|\Omega_{r_j}(x_0)| \geq |\Omega_{\frac{1}{2}r_j}(\hat{x}_0)| \geq c\, r_j^n.$$

Thus, (4.2.27) holds for every $j \geq j_1 + 1$.

By (4.2.27) and Hölder's inequality, for any $j \in \mathbb{N}$, we have

$$|\mathbf{m}_j| \leq C T_j. \tag{4.2.28}$$

Note that since $\mu \leq 1/6$, $\Omega_{3r_{j+1}}(x_0) \subset \Omega_{\frac{1}{2}r_j}(x_0)$ and thus by (4.2.27), (4.2.22) and the definition of j_1,

$$T_{j_1+1} \leq C \left(\fint_{\Omega_{\frac{1}{2}r_{j_1}}(x_0)} |Du|^p \right)^{1/p} \leq C\, T_{j_1}.$$

Therefore, there exists $c_6 = c_6(n, p, c_1, c_2) > 0$, such that for any $j \in \mathbb{N}$,

$$T_{j+1} \leq c_6\, T_j. \tag{4.2.29}$$

By (4.2.29) and the triangle inequality, for any $j \leq j_1$, we have

$$T_{j+1} \leq c_6\, T_j \leq C\, |\mathbf{m}_j| + C\, \phi_j.$$

For $j \geq j_1 + 1$, since $\mu \leq 1/6$, $\Omega_{3r_{j+1}}(x_0) \subset \Omega_{r_j}(x_0)$ and thus by (4.2.27) and the triangle inequality, we have

$$T_{j+1} \leq C \left(\fint_{\Omega_{r_j}(x_0)} |Du|^p \right)^{1/p} \leq C\,|\mathbf{m}_j| + C\,\phi_j.$$

Therefore, there exists $c_7 = c_7(n, p, c_1, c_2) > 0$, such that for any $j \in \mathbb{N}$,

$$T_{j+1} \leq c_7\,|\mathbf{m}_j| + c_7\,\phi_j. \tag{4.2.30}$$

For any $0 \leq k \leq j_1$, since

$$|\mathbf{m}_k - \mathbf{m}_{k-1}|^p \leq 2^{p-1}|\mathbf{m}_k - D_{\mathcal{Z}} u_2(\mathcal{Z})|^p + 2^{p-1}|D_{\mathcal{Z}} u_2(\mathcal{Z}) - \mathbf{m}_{k-1}|^p,$$

by taking the average over $\mathcal{Z} \in B_{r_k}(\mathcal{Z}_0)$ and then taking the p-th root, we obtain

$$|\mathbf{m}_k - \mathbf{m}_{k-1}| \leq C\phi_k + C\phi_{k-1}.$$

Then by iterating, we get

$$|\mathbf{m}_j - \mathbf{m}_{j_0}| \leq C \sum_{k=j_0}^{j} \phi_k, \tag{4.2.31}$$

for any integers j_0, j satisfying $0 \leq j_0 \leq j \leq j_1$. By (4.2.31), (4.2.28), and the triangle inequality, we have

$$|\mathbf{m}_j| \leq C\,T_{j_0} + C \sum_{k=j_0}^{j} \phi_k, \tag{4.2.32}$$

for $j_0 \le j \le j_1$. Similarly, for any integers j, l satisfying $j_1+1 \le l \le j$, we also have

$$|\mathbf{m}_j - \mathbf{m}_l| \le C \sum_{k=l}^{j} \phi_k,$$

and

$$|\mathbf{m}_j| \le C\, T_l + C \sum_{k=l}^{j} \phi_k. \tag{4.2.33}$$

For $j \in \{j_0, \dots, j_1\}$, from Lemma 4.2.6 and (4.2.32), we know that

$$|\mathbf{m}_j| + \sum_{k=j_0}^{j} \phi_k \le C\, T_{j_0} + C \sum_{k=j_0}^{j} \Big(\frac{r_k}{(\varepsilon + |x_0'|^2)^{\frac{1}{2}}}\Big)^{\theta_p} T_k \le C\, T_{j_0} + C \sum_{k=j_0}^{j} \mu^{k\theta_p} T_k. \tag{4.2.34}$$

For $j \in \{j_1+1, \dots, j_2\}$, we have $r_j \ge \mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-)$. Choose $\hat{x}_0 \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \le 3/4\}$ such that $\mathrm{dist}(x_0, \Gamma_+ \cup \Gamma_-) = |\hat{x}_0 - x_0|$, and thus $\Omega_{r_j}(x_0) \subset \Omega_{2r_j}(\hat{x}_0) \subset \Omega_{3r_j}(x_0)$. Then

$$r_j < \frac{c_4}{6}(\varepsilon + |x_0'|^2) \le \frac{c_4}{6}(\varepsilon + 2|\hat{x}_0'|^2 + 2r_j^2),$$

which also implies

$$2r_j < c_4(\varepsilon + |\hat{x}_0'|^2) \tag{4.2.35}$$

since $c_4 \in (0,1)$ and $r_j \in (0,1)$.

By (4.2.35), we can apply Lemma 4.2.5 at $\hat{x}_0 \in \Gamma_+ \cup \Gamma_-$ and use (4.2.27) and (4.2.6) to obtain

$$\begin{aligned} \sum_{k=j_1+1}^{j} \phi_k &\le C \sum_{k=j_1+1}^{j} \psi(\hat{x}_0, 2r_k) \le C\, Y_{j_1+1} + C \sum_{k=j_1+1}^{j} r_k^{\theta_p} Y_k \\ &\le C\, T_{j_1+1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k, \end{aligned} \tag{4.2.36}$$

where

$$Y_j := \left(\fint_{\Omega_{2r_j}(\hat{x}_0)} |Du|^p \right)^{1/p}.$$

Moreover, from (4.2.33), (4.2.30), and (4.2.36) we also know that

$$\begin{aligned} |\mathbf{m}_j| + \sum_{k=j_1+1}^{j} \phi_j &\le C\, T_{j_1+1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \\ &\le C\, |\mathbf{m}_{j_1}| + C\, \phi_{j_1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \end{aligned} \tag{4.2.37}$$

holds for any $j \in \{j_1 + 1, \dots, j_2\}$.

For $j \ge j_2 + 1$, from Corollary 4.2.2, (4.2.33), and (4.2.30) we have

$$|\mathbf{m}_j| + \sum_{k=j_2+1}^{j} \phi_k \le C\, T_{j_2+1} \le C\, |\mathbf{m}_{j_2}| + C\, \phi_{j_2}. \tag{4.2.38}$$

Combining (4.2.37) and (4.2.38), we know that

$$|\mathbf{m}_j| + \sum_{k=j_1+1}^{j} \phi_j \le C\, |\mathbf{m}_{j_1}| + C\, \phi_{j_1} + C \sum_{k=j_1+1}^{j} \mu^{k\theta_p} T_k \tag{4.2.39}$$

holds for any $j \ge j_1 + 1$. Note that (4.2.39) also holds if $j_2 \le j_1$ since in that case $r_j \le \operatorname{dist}(x_0, \Gamma_+ \cup \Gamma_-)$ for any $j \ge j_1 + 1$ and thus we can directly use Corollary 4.2.2 and (4.2.33) to get (4.2.39).

Moreover, combining (4.2.39) and (4.2.34), we know that

$$|\mathbf{m}_j| + \sum_{k=j_0}^{j} \phi_k \le C\, T_{j_0} + C \sum_{k=j_0}^{j} \mu^{k\theta_p} T_k \tag{4.2.40}$$

holds for any $0 \le j_0 \le j_1$ and $j \ge j_0$.

Step 3: A stopping time argument. We choose $j_0 = j_0(n, p, c_1, c_2) \in \mathbb{N}$ sufficiently large such that

$$(c_7+1)\,C\sum_{k=j_0}^{\infty}\mu^{k\theta_p}\le\frac{1}{10},\tag{4.2.41}$$

where c_7 is the constant in (4.2.30) and C is the constant in (4.2.40). Note that we can assume $\varepsilon+|x_0'|^2$ to be sufficiently small so that $\frac{c_5}{2}\mu^{j_0}(\varepsilon+|x_0'|^2)^{\frac{1}{2}}>\frac{c_4}{6}(\varepsilon+|x_0'|^2)$ and thus $j_1\ge j_0$. Now we show that

$$|Du(x_0)|\le C\,T_{j_0}.\tag{4.2.42}$$

We consider the following three possibilities.

Case 1: If $|Du(x_0)|\le T_{j_0}$, then (4.2.42) directly follows.

Case 2: If $T_j<|Du(x_0)|$, $\forall j_0\le j\le j_3$, and $|Du(x_0)|\le T_{j_3+1}$, then by (4.2.30), we have

$$|Du(x_0)|\le T_{j_3+1}\le c_7\,|\mathbf{m}_{j_3}|+c_7\,\phi_{j_3}.\tag{4.2.43}$$

Now applying (4.2.40) with $j=j_3$, from (4.2.43) and (4.2.41), we get

$$|Du(x_0)|\le C'\,T_{j_0}+C'\sum_{k=j_0}^{j_3}\mu^{k\theta_p}|Du(x_0)|\le C'\,T_{j_0}+\frac{1}{10}|Du(x_0)|,$$

where $C'=c_7\,C$, C is the constant in (4.2.40). The last inequality directly implies (4.2.42) as desired.

Case 3: If $T_j<|Du(x_0)|$ for every $j\ge j_0$, then from (4.2.40), we infer that for any $j\ge j_0$,

$$|\mathbf{m}_j|\le C\,T_{j_0}+C\sum_{k=j_0}^{j}\mu^{k\theta_p}|Du(x_0)|\le C\,T_{j_0}+\frac{1}{10}|Du(x_0)|.$$

Here we used (4.2.41) in the last inequality. Letting $j\to\infty$ and using the fact that

$u \in C^1(\Omega_1)$, we get

$$|Du(x_0)| \le C\, T_{j_0} + \frac{1}{10}|Du(x_0)|,$$

which directly implies (4.2.42). The proof of the inequality (4.2.42) is completed.

Step 4: Caccioppoli inequality and conclusion. Let $\lambda \in \mathbb{R}$ and ζ be a nonnegative smooth function satisfying $\zeta = 1$ in $B_{r_{j_0}}(\mathcal{Z}_0)$, $|D_{\mathcal{Z}}\zeta| \le 2r_{j_0}^{-1}$, and $\zeta = 0$ outside $B_{2r_{j_0}}(\mathcal{Z}_0)$. Since $2r_{j_0} \le c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$, using $\zeta^p(u_2 - \lambda)$ as a test function in (4.2.23), by (4.2.22), Young's inequality, we obtain

$$\begin{aligned}
&\frac{1}{2^{p+1}}\int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \zeta^p|D_{\mathcal{Z}}u_2|^p \le \int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \langle \mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2), \zeta^p D_{\mathcal{Z}}u_2\rangle \\
&= -p\int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \langle \mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2), \zeta^{p-1}(u_2-\lambda)D_{\mathcal{Z}}\zeta\rangle \\
&\le p2^{p+2}r_{j_0}^{-1}\int_{B_{2r_{j_0}}(\mathcal{Z}_0)} \zeta^{p-1}|D_{\mathcal{Z}}u_2|^{p-1}|u_2-\lambda| \\
&\le \frac{1}{2^{p+2}}\int_{B_{2r_{j_0}}} \zeta^p|D_{\mathcal{Z}}u_2|^p + c(p)\, r_{j_0}^{-p}\int_{B_{2r_{j_0}}(\mathcal{Z}_0)} |u_2-\lambda|.
\end{aligned}$$

Therefore, we have the following Caccioppoli inequality

$$\int_{B_{r_{j_0}}(\mathcal{Z}_0)} |D_{\mathcal{Z}}u_2|^p \le c(p)\, r_{j_0}^{-p}\int_{B_{2r_{j_0}}(\mathcal{Z}_0)} |u_2-\lambda|^p, \tag{4.2.44}$$

where λ is an arbitrary constant and $c(p)$ is a positive constant depending only on p. Since $2r_{j_0} \le c_5(\varepsilon + |x_0'|^2)^{\frac{1}{2}} \le \frac{1}{4}(\varepsilon + |x_0'|^2)^{\frac{1}{2}} \le 1/2$, by choosing $\lambda = (u_2)_{B_{2r_{j_0}}(\mathcal{Z}_0)}$ in (4.2.44), and using (4.2.42), we obtain the pointwise blow-up estimate

$$|Du(x_0)| \le C(\varepsilon + |x_0'|^2)^{-\frac{1}{2}} \operatorname*{osc}_{\Omega_{x_0,\eta}} u,$$

where $\eta = \frac{1}{4}(\varepsilon + |x_0'|^2)^{\frac{1}{2}}$. $\square$

4.3 Improved gradient estimates

In this section, we utilize a similar approach of flattening the boundaries and extending the equation, as described in [66], to derive an improved gradient estimate for (4.1.7) in dimensions $n \geq 3$. However, in contrast to [66], since our equation is degenerate, we need to exploit the nondivergence form of the normalized equation. Consequently, the argument of flattening the boundaries becomes much more intricate, and unlike in [66], where the De Giorgi-Nash-Moser Harnack inequality is applied, we use the Krylov-Safonov Harnack inequality for nondivergence form equations. Furthermore, there are additional first-order terms that require control over the size of the coefficients.

To prove Theorem 4.1.2, for $\eta > 0$, we consider the approximating equation

$$\begin{cases} -\operatorname{div}\left((\eta + |Du_\eta|^2)^{\frac{p-2}{2}} Du_\eta\right) = 0 & \text{in } \widetilde{\Omega}, \\ \dfrac{\partial u_\eta}{\partial \nu} = 0 & \text{on } \partial D_i,\ i = 1, 2, \\ u_\eta = \varphi & \text{on } \partial \Omega. \end{cases} \tag{4.3.1}$$

Since $\|u_\eta\|_{C^{1,\alpha}(\Omega_1)}$ is bounded independent of η, it suffices to prove (4.1.9) for u_η. Therefore, we will focus on (4.3.1) throughout the rest of this section, and denote $u = u_\eta$ for simplicity. Note that u satisfies the normalized p-Laplace equation

$$a^{ij} D_{ij} u = 0 \quad \text{in } \Omega_1,$$

where

$$a^{ij} = \delta_{ij} + (p-2)(\eta + |Du|^2)^{-1} D_i u D_j u \tag{4.3.2}$$

satisfies

$$
\begin{aligned}
&(p-1)|\xi|^2 \le a^{ij}\xi_i\xi_j \le |\xi|^2, \quad \forall \xi \in \mathbb{R}^n \quad \text{when} \;\; 1<p<2,\\
&|\xi|^2 \le a^{ij}\xi_i\xi_j \le (p-1)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n \quad \text{when} \;\; p \ge 2.
\end{aligned} \tag{4.3.3}
$$

For a small r_0 independent of ε, we only need to show (4.1.9) in Ω_{r_0}, as $|Du|$ is bounded in $\Omega_{1/2} \setminus \Omega_{r_0}$ independent of ε. For any $x \in \Omega_{r_0}$, we estimate $|Du(x)|$ as follows: First we consider the equation in $\Omega_{2r} \setminus \Omega_{r/4}$ for $r \in (\sqrt{\varepsilon}, r_0]$, we perform a suitable change of variables that maps the domain to a flat "annular cylinder". After the change of variables, u will satisfy a second-order uniformly elliptic equation in non-divergence form, and the Neumann boundary condition on the upper and lower boundaries of the domain. Then we obtain a Harnack inequality through the Krylov-Safonov theorem. Together with the maximum principle, this gives the oscillation of u in Ω_r for $r \in (\sqrt{\varepsilon}, r_0]$ with a decay rate $r^{2\beta}$ for some positive ε-independent β. Then the desired estimate on $|Du(x)|$ follows from the decay rate of $\operatorname{osc}_{2(\varepsilon+|x'|^2)^{1/2}} u$ and Theorem 4.1.1.

Let $r \in (\sqrt{\varepsilon}, r_0]$, where r_0 is an ε-independent constant to be determined later. We define

$$
\tilde{h}_i(x') := \begin{cases} h_i(x') & \text{when} \;\; |x'| \le 2r_0, \\ 0 & \text{when} \;\; |x'| > 2r_0 \end{cases}
$$

for $i = 1, 2$. We denote

$$
Q_{s,t} := \{ y = (y', y_n) \in \mathbb{R}^n \;\big|\; |y'| < s, |y_n| < t \},
$$

and for $y \in Q_{2r,r^2} \setminus Q_{r/4,r^2}$, we define the map $x = \Phi(y)$ by

$$\begin{cases} x' = y' - g(y), \\ x_n = \dfrac{1}{2}\Big[\dfrac{y_n}{r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) + \tilde{h}_1(y') + \tilde{h}_2(y')\Big], \end{cases} \tag{4.3.4}$$

where

$$g(y) = (y_n - r^2)(y_n + r^2)(\Theta y_n + \Xi), \tag{4.3.5}$$

$$\begin{cases} \Theta = \dfrac{1}{8r^6}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1^{\mu}(y') + \tilde{h}_2^{\mu}(y')], \\ \Xi = \dfrac{1}{8r^4}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')]D_{y'}[\tilde{h}_1^{\mu}(y') - \tilde{h}_2^{\mu}(y')], \end{cases}$$

$\tilde{h}_i^{\mu}$ is a mollification of $\tilde{h}_i$ given by

$$\tilde{h}_i^{\mu}(y') := \int_{\mathbb{R}^{n-1}} \tilde{h}_i(y' - \mu z')\varphi(z')\, dz', \tag{4.3.6}$$

φ is a positive smooth function with unit integral supported in $B_1 \subset \mathbb{R}^{n-1}$, and

$$\mu = \frac{r^4 - y_n^2}{r} \geq 0.$$

Here we briefly explain the motivation for defining the map Φ as above: to ensure that $y' = x'$ on $\{y_n = \pm r^2\}$, which is $g\big|_{y_n = \pm r^2} = 0$, we setup the ansatz (4.3.5) for g. Next, we want the function $v(y) := u(\Phi(y))$ to satisfy the Neumann boundary condition on $\{y_n = \pm r^2\}$, which leads to (see details in Lemma 4.3.1)

$$\begin{cases} \Big(- D_{y_n} g, \dfrac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_1, 1) \quad \text{on} \quad \{y_n = r^2\}, \\ \Big(- D_{y_n} g, \dfrac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_2, 1) \quad \text{on} \quad \{y_n = -r^2\}. \end{cases}$$

Using the ansatz (4.3.5) and solving for Θ and Ξ, we have

$$\begin{cases} \Theta = \dfrac{1}{8r^6}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')] D_{y'}[\tilde{h}_1(y') + \tilde{h}_2(y')], \\ \Xi = \dfrac{1}{8r^4}[\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')] D_{y'}[\tilde{h}_1(y') - \tilde{h}_2(y')]. \end{cases}$$

Note that the equation of v involves second-order derivatives of Φ, and hence involves third-order derivatives of $\tilde{h}_1$ and $\tilde{h}_2$. However, $\tilde{h}_1$ and $\tilde{h}_2$ are only $C^{1,1}$, so we introduce the mollification (4.3.6) to overcome the lack of regularities. Here μ is chosen so that $\tilde{h}_i^\mu = \tilde{h}_i$ on $\{y_n = \pm r^2\}$, and the coefficients of the equation of v have the desired estimates.

Throughout this section, unless specify otherwise, we use C to denote positive constants that could be different from line to line, and depend only on n, p, c_1, and c_2, where c_1 and c_2 are defined in (4.1.4) and (4.1.5), respectively.

Lemma 4.3.1. *There exists an $r_0 > 0$ independent of ε, such that when $r \in (\sqrt{\varepsilon}, r_0]$ and Φ is given as (4.3.4), then:*

(a) There exists a positive constant C independent of ε and r, such that

$$\frac{I}{C} \le D\Phi(y) \le CI, \quad y \in Q_{2r,r^2} \setminus Q_{r/4,r^2},$$

and hence Φ is invertible.

(b)

$$Q_{1.9r,r^2} \setminus Q_{0.35r,r^2} \subset \Phi^{-1}(\Omega_{2r} \setminus \Omega_{r/4}),$$

and

$$\Omega_r \setminus \Omega_{r/2} \subset \Phi(Q_{1.1r,r^2} \setminus Q_{0.4r,r^2}).$$

(c) Let $u \in W^{1,p}(\Omega_{2r} \setminus \Omega_{r/4})$ be a solution of

$$\begin{cases} -\operatorname{div}\Big((\eta + |Du|^2)^{\frac{p-2}{2}} Du\Big) = 0 & \text{in } \Omega_{2r} \setminus \Omega_{r/4}, \\ \dfrac{\partial u}{\partial \nu} = 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega_{2r} \setminus \Omega_{r/4}} \end{cases} \tag{4.3.7}$$

for some $\eta > 0$, and $v(y) = u(\Phi(y))$. Then v satisfies an elliptic equation

$$\begin{cases} \tilde{a}^{ij} D_{ij} v(y) + \tilde{b}_i D_i v(y) = 0 & \text{in } Q_{1.9r,r^2} \setminus Q_{0.35r,r^2}, \\ \dfrac{\partial v}{\partial \nu}(y) = 0 & \text{on } \{y_n = \pm r^2\}, \end{cases} \tag{4.3.8}$$

with

$$\frac{I}{C} \le \tilde{a} \le CI, \quad |\tilde{b}| \le \frac{C}{r}.$$

Proof. By (4.1.3), we have

$$|D^k_{y'} \tilde{h}_i(y')| \le Cr^{2-k}, \quad |D^k_{y'} \tilde{h}^\mu_i(y')| \le Cr^{2-k} \quad \text{for } \ i = 1, 2 \ \text{ and } \ k = 0, 1, 2. \tag{4.3.9}$$

Therefore, when $y \in Q_{2r,r^2} \setminus Q_{r/4,r^2}$,

$$|D_{y'} x' - I_{(n-1)\times(n-1)}| = |D_{y'} g(y)| \le Cr^2.$$

$$\begin{aligned} D_{y_n} x' &= -D_{y_n} g(y) \\ &= -2y_n(\Theta y_n + \Xi) - (y_n^2 - r^4)[(D_{y_n}\Theta) y_n + \Theta + D_{y_n}\Xi], \end{aligned}$$

and

$$|\Theta| \le Cr^{-3}, \quad |\Xi| \le Cr^{-1}.$$

By (4.3.6), we have for $y \in Q_{2r,r^2} \setminus Q_{r/4,r^2}$,

$$
\begin{aligned}
|D_{y_n} D_{y'} \tilde{h}_i^{\mu}(y')| =& \left| -\frac{\partial \mu}{\partial y_n} \int_{\mathbb{R}^{n-1}} D_{y'}^2 \tilde{h}_i(y' - \mu z') z' \varphi(z') \, dz' \right| \\
=& \left| \frac{2y_n}{r} \int_{\mathbb{R}^{n-1}} D_{y'}^2 \tilde{h}_i(y' - \mu z') z' \varphi(z') \, dz' \right| \leq Cr \quad \text{for} \;\; i = 1, 2.
\end{aligned}
$$

Therefore,

$$
|D_{y_n} \Theta| \leq C r^{-3}, \quad |D_{y_n} \Xi| \leq C r^{-1},
$$

and hence

$$
\begin{aligned}
|D_{y_n} x'| \leq& |2y_n(\Theta y_n + \Xi)| + |(y_n^2 - r^4)[(D_{y_n}\Theta) y_n + \Theta + D_{y_n}\Xi]| \\
\leq& Cr + Cr^4[r^{-1} + r^{-3} + r^{-1}] \leq Cr.
\end{aligned}
$$

By (4.3.9),

$$
|D_{y'} x_n| = \frac{1}{2} \left| \frac{y_n}{r^2} (D_{y'} \tilde{h}_1(y') - D_{y'} \tilde{h}_2(y')) + D_{y'} \tilde{h}_1(y') + D_{y'} \tilde{h}_2(y') \right| \leq Cr.
$$

And lastly,

$$
D_{y_n} x_n = \frac{1}{2r^2} (\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')).
$$

By (4.1.4),

$$
\frac{1}{C} \leq D_{y_n} x_n \leq C.
$$

Then (a) follows by shrinking r_0 to be sufficiently small.

Since $g(y) = 0$ when $y = \pm r^2$, Φ maps the upper and lower boundaries of $Q_{2r,r^2} \setminus Q_{r/4,r^2}$ onto the upper and lower boundaries of $\Omega_{2r} \setminus \Omega_{r/4}$, respectively. Then (b) simply follows from the fact that $|g(y)| \leq Cr^3$, and we can shrink r_0 so that $|g(y)| \leq r/10$.

To verify (c), note that u is smooth from the classical elliptic theory. We compute by the chain rule,

$$\begin{aligned} D_{x_k}u(x) =& D_{y_i}v(y)D_{x_k}y_i, \\ D_{x_kx_l}u(x) =& D_{y_i}D_{y_j}v(y)D_{x_k}y_iD_{x_l}y_j + D_{y_i}v(y)D_{x_kx_l}y_i. \end{aligned}$$

Recall that $u(x)$ satisfies the equation

$$a^{kl}D_{x_kx_l}u(x) = 0,$$

where the matrix a is given by (4.3.2). If we define

$$\tilde{a}^{ij} := a^{kl}D_{x_k}y_iD_{x_l}y_j, \quad \tilde{b}^i := a^{kl}D_{x_kx_l}y_i,$$

then $v(y)$ satisfies

$$\tilde{a}^{ij}D_{ij}v(y) + \tilde{b}_iD_iv(y) = 0.$$

Next, we show that v satisfies the Neumann boundary condition on $\{y_n = \pm r^2\}$. We will show the boundary condition on $\{y_n = r^2\}$, as the one on $\{y_n = -r^2\}$ follows similarly. By the chain rule,

$$D_yv(y) \cdot e_n = D_xu(x) \cdot D_y\Phi e_n,$$

where $e_n := (0, \ldots, 0, 1)$. Therefore, it suffices to show that

$$D_y\Phi e_n = \Big(- D_{y_n}g, \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y'))\Big) \parallel (-D_{x'}\tilde{h}_1, 1) \quad \text{on} \ \ \{y_n = r^2\}. \tag{4.3.10}$$

Note that when $y_n = r^2$, we have $g(y) = 0$, $y' = x'$, $\mu = 0$, and $\tilde{h}_1^\mu = \tilde{h}_1$. Therefore,

$$\begin{aligned}
D_{y_n} g =& (y_n + r^2)(\Theta y_n + \Xi)\Big|_{y_n = r^2} \\
=& \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) D_{y'} \tilde{h}_1^\mu(y') \\
=& \frac{1}{2r^2}(\varepsilon + \tilde{h}_1(y') - \tilde{h}_2(y')) D_{x'} \tilde{h}_1(x').
\end{aligned}$$

This implies (4.3.10).

Finally, we show that the coefficients $\tilde{a}$ and $\tilde{b}$ satisfy the desired estimates. From part (a), we know that

$$\frac{I}{C} \leq D_x y = D_x \Phi^{-1}(x) \leq CI,$$

which together with (4.3.3) implies that

$$\frac{I}{C} \leq \tilde{a} \leq CI.$$

To estimate $\tilde{b}$, we differentiate $\partial y_i / \partial x_k \cdot \partial x_k / \partial y_j = \delta_{ij}$ in x_l. Note that by chain rule, we have

$$\frac{\partial^2 y_i}{\partial x_k \partial x_l} \frac{\partial x_k}{\partial y_j} + \frac{\partial y_i}{\partial x_k} \frac{\partial y_m}{\partial x_l} \frac{\partial^2 x_k}{\partial y_j \partial y_m} = 0.$$

Since $I/C \leq D_x y \leq CI$ and $I/C \leq D_y x \leq CI$, it suffices to estimate $D_y^2 x$, which is $D_y^2 \Phi(y)$. It is easy to see that

$$\left| \frac{\partial^2 x_n}{\partial y^2} \right| \leq \frac{C}{r}.$$

To estimate $\partial^2 x' / \partial y^2$, the key terms are

$$D_{y'}^3 \tilde{h}_i^\mu(y'), \quad D_{y_n} D_{y'}^2 \tilde{h}_i^\mu(y'), \quad D_{y_n}^2 D_{y'} \tilde{h}_i^\mu(y'), \quad i = 1, 2.$$

By (4.3.6) and integration by parts, we have

$$
\begin{aligned}
D_{y'}\tilde{h}_i^{\mu}(y') =& \int_{\mathbb{R}^{n-1}} D_{y'}\tilde{h}_i(y'-\mu z')\varphi(z')\,dz' \\
=& -\frac{1}{\mu}\int_{\mathbb{R}^{n-1}} D_{z'}\tilde{h}_i(y'-\mu z')\varphi(z')\,dz' \\
=& \frac{1}{\mu}\int_{\mathbb{R}^{n-1}} \tilde{h}_i(y'-\mu z')D_{z'}\varphi(z')\,dz'.
\end{aligned}
$$

Then

$$
|D^3_{y'}\tilde{h}_i^{\mu}(y')| \leq \frac{C\|h_i\|_{C^{1,1}}}{\mu} \leq \frac{Cr}{r^4-y_n^2}.
$$

Similarly,

$$
\begin{aligned}
D_{y_n}\tilde{h}_i^{\mu}(y') =& \frac{2y_n}{r}\int_{\mathbb{R}^{n-1}} D_{y'}\tilde{h}_i(y'-\mu z')\cdot z'\varphi(z')\,dz' \\
=& \frac{2y_n}{\mu r}\int_{\mathbb{R}^{n-1}} \tilde{h}_i(y'-\mu z')D_{z'}\cdot(z'\varphi(z'))\,dz',
\end{aligned} \tag{4.3.11}
$$

so

$$
|D_{y_n}D^2_{y'}\tilde{h}_i^{\mu}(y')| = \left|\frac{2y_n}{\mu r}\int_{\mathbb{R}^{n-1}} D^2_{y'}\tilde{h}_i(y'-\mu z')D_{z'}\cdot(z'\varphi(z'))\,dz'\right| \leq \frac{Cr^2}{r^4-y_n^2}.
$$

Differentiating the first line of (4.3.11) in y_n, we have

$$
\begin{aligned}
D^2_{y_n}\tilde{h}_i^{\mu}(y') =& \frac{2}{r}\int_{\mathbb{R}^{n-1}} \sum_{k=1}^{n-1} D_{y_k}\tilde{h}_i(y'-\mu z')z_k\varphi(z')\,dz' \\
&+ \frac{4y_n^2}{r^2}\int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^{n-1} D_{y_k y_l}\tilde{h}_i(y'-\mu z')z_k z_l\varphi(z')\,dz' \\
=& \frac{2}{r}\int_{\mathbb{R}^{n-1}} \sum_{k=1}^{n-1} D_{y_k}\tilde{h}_i(y'-\mu z')z_k\varphi(z')\,dz' \\
&+ \frac{4y_n^2}{\mu r^2}\int_{\mathbb{R}^{n-1}} \sum_{k,l=1}^{n-1} D_{y_l}\tilde{h}_i(y'-\mu z')D_{z_k}\big(z_k z_l\varphi(z')\big)\,dz'.
\end{aligned}
$$

Therefore,

$$\begin{aligned}|D_{y_n}^2 D_{y'}\tilde{h}_i^{\mu}(y')| \leq & \frac{2}{r}\int_{\mathbb{R}^{n-1}} |D_{y'}^2\tilde{h}_i(y'-\mu z')||z'\varphi(z')|\,dz' \\ & + \frac{4y_n^2}{\mu r^2}\int_{\mathbb{R}^{n-1}} |D_{y'}^2\tilde{h}_i(y'-\mu z')||D_{z'}(z'\otimes z'\varphi(z'))|\,dz' \\ \leq & \frac{Cr^3}{r^4-y_n^2}.\end{aligned}$$

By these estimates above and straightforward computations, we have

$$|D_y^2\Phi(y)| \leq Cr^{-1},$$

which implies $|\tilde{b}| \leq Cr^{-1}$. □

Lemma 4.3.2. *Let r_0 be as in Lemma 4.3.1, and let $r \in (\sqrt{\varepsilon}, r_0]$. If $u \in W^{1,p}(\Omega_{2r} \setminus \Omega_{r/4})$ is a nonnegative solution of (4.3.7) for some $\eta > 0$, then,*

$$\sup_{\Omega_r\setminus\Omega_{r/2}} u \leq C \inf_{\Omega_r\setminus\Omega_{r/2}} u, \tag{4.3.12}$$

for some constant $C > 0$ depending only on n, p, c_1, and c_2, but independent of ε, η, r, and u.

Proof. We take the change of variable $y = \Phi^{-1}(x)$, where Φ is given as (4.3.4). Let $v(y) = u(x)$. By Lemma 4.3.1 (c), v satisfies the equation (4.3.8).

For $i, j = 1, 2, \ldots, n-1$, we take the even extension of $\tilde{a}^{ij}$, $\tilde{a}^{nn}$, $\tilde{b}^i$, and v with respect to $y_n = r^2$, and take odd extension of $\tilde{a}^{in}$, $\tilde{a}^{ni}$, and $\tilde{b}^n$ with respect to $y_n = r^2$. Then we take the periodic extension (so that the period is equal to $4r^2$). We still denote them by $\tilde{a}$, $\tilde{b}$, and v after the extension. Then v satisfies

$$\tilde{a}^{ij}D_{ij}v(y) + \tilde{b}_i D_i v(y) = 0 \quad \text{in } Q_{1.9r,2r} \setminus Q_{0.35r,2r}.$$

Setting $\bar{a}^{ij}(y) = \tilde{a}^{ij}(ry)$, $\bar{b}^i(y) = r\tilde{b}^i(ry)$, and $\bar{v}(y) = v(ry)$, we see that $\bar{v}$ satisfies

$$\bar{a}^{ij} D_{ij}\bar{v}(y) + \bar{b}_i D_i \bar{v}(y) = 0 \quad \text{in } Q_{1.9,2} \setminus Q_{0.35,2},$$

with

$$\frac{I}{C} \le \bar{a} \le CI, \quad |\bar{b}| \le C.$$

Since $Q_{1.9,2} \setminus Q_{0.35,2}$ is connected when $n \ge 3$, by the Krylov-Safonov theorem (see Section 4.2 of [52]), we have

$$\sup_{Q_{1.1,1} \setminus Q_{0.4,1}} \bar{v} \le C \inf_{Q_{1.1,1} \setminus Q_{0.4,1}} \bar{v}.$$

This implies

$$\sup_{Q_{1.1r,r^2} \setminus Q_{0.4r,r^2}} v \le C \inf_{Q_{1.1r,r^2} \setminus Q_{0.4r,r^2}} v.$$

Finally, (4.3.12) follows by reverting the changes of variables and Lemma 4.3.1 (b). $\square$

The following estimate on the oscillation of u is a direct consequence of Lemma 4.3.2.

Corollary 4.3.3. *For $n \ge 3$, let $u \in W^{1,p}(\Omega_1)$ be a solution of (4.3.1) for some $\eta > 0$. Then there exist positive constants C and β, depending only on n, p, c_1, and c_2, such that*

$$\underset{\Omega_r}{osc}\, u \le C r^\beta \underset{\Omega_1}{osc}\, u, \quad \forall\, r \in (\sqrt{\varepsilon}, 1/2). \tag{4.3.13}$$

Proof. It suffices to prove (4.3.13) for $r \in (\sqrt{\varepsilon}, r_0]$, where r_0 is the same as in Lemma 4.3.1. Let $\sqrt{\varepsilon} < r \le r_0$ and $v = u - \inf_{\Omega_{2r}} u$. Then $v \ge 0$ in Ω_{2r}. By Lemma 4.3.2, we have

$$\sup_{\Omega_r \setminus \Omega_{r/2}} v \le C_1 \inf_{\Omega_r \setminus \Omega_{r/2}} v,$$

where $C_1 > 1$ is a constant independent of r. Since v satisfies equation (4.1.7), by the maximum principle,

$$\sup_{\Omega_r \setminus \Omega_{r/2}} v = \sup_{\Omega_r} v, \qquad \inf_{\Omega_r \setminus \Omega_{r/2}} v = \inf_{\Omega_r} v.$$

Therefore,

$$\sup_{\Omega_r} v \le C_1 \inf_{\Omega_r} v,$$

which implies

$$\sup_{\Omega_r} u \le C_1 \inf_{\Omega_r} u - (C_1 - 1) \inf_{\Omega_{2r}} u.$$

Adding the above inequality with

$$(C_1 - 1) \sup_{\Omega_r} u \le (C_1 - 1) \sup_{\Omega_{2r}} u,$$

and dividing both sides by C_1, we have

$$\operatorname*{osc}_{\Omega_r} u \le \frac{C_1 - 1}{C_1} \operatorname*{osc}_{\Omega_{2r}} u.$$

Finally, (4.3.13) follows from iterating the inequality above. □

Now we are ready to prove Theorem 4.1.2.

Proof of Theorem 4.1.2. It suffices to show (4.1.9) for $x \in \Omega_{1/8}$ and $\varepsilon \in (0, 1/32)$. By Corollary 4.3.3, there exist positive constants C and β, depending only on n, p, c_1, and c_2, such that

$$\operatorname*{osc}_{\Omega_{8\eta}} u \le C(\varepsilon + |x'|^2)^\beta \operatorname*{osc}_{\Omega_1} u,$$

where $\eta=\frac{1}{4}(\varepsilon+|x'|^2)^{\frac{1}{2}}$. Then by Theorem 4.1.1, we have

$$\begin{aligned}|Du(x)|\leq & C(\varepsilon+|x'|^2)^{-\frac{1}{2}}\underset{\Omega_{x,\eta}}{\operatorname{osc}}\ u\\ \leq & C(\varepsilon+|x'|^2)^{-\frac{1}{2}}\underset{\Omega_{8\eta}}{\operatorname{osc}}\ u\\ \leq & C(\varepsilon+|x'|^2)^{-\frac{1}{2}+\beta}\underset{\Omega_{1}}{\operatorname{osc}}\ u.\end{aligned}$$

The theorem is proved. □

4.4 The $p>n+1$ case

In this section, we establish a more explicit gradient estimate for the equation (4.1.7) when $p>n+1$, with a blow-up rate of order $\varepsilon^{-\alpha}$ for any $\alpha>\frac{n}{2(p-1)}$. Throughout this section, in addition to (4.1.3) and (4.1.4), we need to further assume that h_1 and h_2 are C^2 strictly convex and strictly concave functions respectively, satisfying (4.1.10). Let ν denote the normal vector on $\Gamma_{\pm}$, pointing upwards and downwards respectively.

To obtain the improved gradient estimate, in the following lemma, we construct a supersolution to show that the oscillation of u enjoys a better decay rate. Then the desired gradient estimate (4.1.11)follows by using Theorem 4.1.1.

Lemma 4.4.1. *Let $n\geq 2$, $p>n+1$, $\Gamma_+,\Gamma_-,h_1,h_2$ be as above. For any $\delta\in(0,p-n-1)$, let $v(x)=(|x'|^2+(2+\delta)x_n^2)^{\gamma/2}$. Then for any $\gamma\in(0,\frac{p-n-1-\delta}{p-1})$, there exists a constant $\mu\in(0,1/2)$ depending only on n, p, δ, γ, κ_1, κ_2, and the modulus of continuity for $D^2h_1(x')$ and $D^2h_2(x')$ at $x'=0$, such that for any $\varepsilon\in(0,\mu^2/\kappa_2)$,*

$$\begin{cases}-\operatorname{div}(|Dv|^{p-2}Dv)>0 & \text{in } \Omega_{\mu/\kappa_2}\setminus\Omega_{\varepsilon/\mu},\\ \dfrac{\partial v}{\partial\nu}>0 & \text{on } (\Gamma_+\cup\Gamma_-)\cap\overline{\Omega}_{\mu/\kappa_2}.\end{cases}$$

Proof. We denote $R(x)=(|x'|^2+(2+\delta)x_n^2)^{1/2}$, so that $v(x)=R(x)^\gamma$. Using Taylor expansion up to order 2, from (4.1.3) we know that for any $|x'| < 1$.

$$\begin{aligned} h_1(x') &= \Big\langle \int_0^1 (1-t) D^2 h_1(tx') dt \cdot x', x' \Big\rangle, \\ h_2(x') &= \Big\langle \int_0^1 (1-t) D^2 h_2(tx') dt \cdot x', x' \Big\rangle. \end{aligned} \tag{4.4.1}$$

By (4.4.1) and (4.1.10), we have

$$\frac{\kappa_1}{2}|x'|^2 \le h_1(x') \le \frac{\kappa_2}{2}|x'|^2, \quad \frac{\kappa_1}{2}|x'|^2 \le h_2(x') \le \frac{\kappa_2}{2}|x'|^2 \quad \text{for} \quad |x'| < 1. \tag{4.4.2}$$

We also note that for any $|x'| < 1$,

$$\big\langle Dh_1(x'), x' \big\rangle = \Big\langle \int_0^1 \frac{d}{dt} Dh_1(tx') dt, x' \Big\rangle = \Big\langle \int_0^1 D^2 h_1(tx') dt \cdot x', x' \Big\rangle, \tag{4.4.3}$$

and similarly

$$\big\langle Dh_2(x'), x' \big\rangle = \Big\langle \int_0^1 D^2 h_2(tx') dt \cdot x', x' \Big\rangle.$$

Since h_1 and h_2 are C^2, for any $\delta \in (0, p-n-1)$, there is a sufficiently small $r_0 \in (0, 1/2)$ depending only on n, δ, κ_1, and the modulus of continuity for $D^2 h_1(x')$ and $D^2 h_2(x')$ at $x' = 0$, such that

$$|D^2 h_1(x') - D^2 h_1(0)| \le \frac{\kappa_1 \delta}{8 + 2\delta}, \quad |D^2 h_2(x') - D^2 h_2(0)| \le \frac{\kappa_1 \delta}{8 + 2\delta} \quad \text{for} \quad |x'| \le r_0. \tag{4.4.4}$$

Thus by (4.4.1), (4.4.3), (4.4.4), (4.1.10), and the triangle inequality, we obtain

$$\begin{aligned}
&(2+\delta)h_1(x') - \big\langle Dh_1(x'), x'\big\rangle \\
&= \Big\langle \int_0^1 (2+\delta)(1-t)D^2h_1(tx')dt \cdot x', x'\Big\rangle - \Big\langle \int_0^1 D^2h_1(tx')dt \cdot x', x'\Big\rangle \\
&\geq \Big\langle \int_0^1 (2+\delta)(1-t)D^2h_1(0)dt \cdot x', x'\Big\rangle - \frac{2+\delta}{2}\frac{\kappa_1\delta}{8+2\delta}|x'|^2 \\
&\quad - \Big\langle \int_0^1 D^2h_1(0)dt \cdot x', x'\Big\rangle - \frac{\kappa_1\delta}{8+2\delta}|x'|^2 \\
&= \frac{\delta}{2}\langle D^2h_1(0)\cdot x', x'\rangle - \frac{\kappa_1\delta}{4}|x'|^2 \geq \frac{\kappa_1\delta}{4}|x'|^2 \geq 0.
\end{aligned} \tag{4.4.5}$$

By direct computations, we have

$$\begin{aligned}
Dv &= \gamma R^{\gamma-2}(x', (2+\delta)x_n), \\
D_{ii}v &= \gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}x_i^2, \quad \text{for } i \in \{1,2,\ldots,n-1\}, \\
D_{ij}v &= \gamma(\gamma-2)R^{\gamma-4}x_ix_j, \quad \text{for } i \neq j,\ i,j \in \{1,2,\ldots,n-1\}, \\
D_{in}v &= \gamma(\gamma-2)R^{\gamma-4}(2+\delta)x_ix_n, \quad \text{for } i \in \{1,2,\ldots,n-1\}, \\
D_{nn}v &= \gamma R^{\gamma-2}(2+\delta) + \gamma(\gamma-2)R^{\gamma-4}(2+\delta)^2x_n^2.
\end{aligned}$$

On $\Gamma_+ \cap \overline{\Omega}_{r_0}$,

$$\nu = \frac{1}{\sqrt{1+|Dh_1(x')|^2}}(-Dh_1(x'), 1).$$

Then by (4.4.5),

$$\begin{aligned}
\frac{\partial v}{\partial \nu} =& \frac{\gamma R^{\gamma-2}}{\sqrt{1+(Dh_1(x'))^2}}\Big[- \langle Dh_1(x'), x'\rangle + (2+\delta)(\frac{\varepsilon}{2} + h_1(x'))\Big] \\
\geq& \frac{\gamma R^{\gamma-2}}{\sqrt{1+(Dh_1(x'))^2}}\Big[\frac{\kappa_1\delta}{4}|x'|^2 + (1+\frac{\delta}{2})\varepsilon\Big] > 0.
\end{aligned}$$

A similar computation shows that $\frac{\partial v}{\partial \nu} > 0$ on $\Gamma_- \cap \overline{\Omega}_{r_0}$.

Next, we compute in $\Omega_1 \setminus \{0\}$,

$$
\begin{aligned}
&\operatorname{div}(|Dv|^{p-2}Dv)|Dv|^{4-p} = |Dv|^2\Delta v + (p-2)D_ivD_jvD_{ij}v \\
&= \gamma^2 R^{2\gamma-4}\Big(|x'|^2 + (2+\delta)^2x_n^2\Big)\Big((n+1+\delta)\gamma R^{\gamma-2} \\
&\qquad + \gamma(\gamma-2)R^{\gamma-4}(|x'|^2 + (2+\delta)^2x_n^2)\Big) \\
&\quad + (p-2)\gamma^2R^{2\gamma-4}|x'|^2\Big(\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}|x'|^2\Big) \\
&\quad + 2(p-2)(2+\delta)^2\gamma^2R^{2\gamma-4}\gamma(\gamma-2)R^{\gamma-4}|x'|^2x_n^2 \\
&\quad + (p-2)\gamma^2R^{2\gamma-4}(2+\delta)^2x_n^2\Big(\gamma R^{\gamma-2}(2+\delta) + \gamma(\gamma-2)R^{\gamma-4}(2+\delta)^2x_n^2\Big) \\
&= \gamma^3R^{3\gamma-8}\Big\{|x'|^4\big[n+\delta+(p-1)(\gamma-1)\big] \\
&\qquad + |x'|^2x_n^2\big[(2+\delta)(n+\delta+p-1) \\
&\qquad\quad + (2+\delta)^2(n+1+\delta+(p-2)(2+\delta)+2(\gamma-2)(p-1))\big] \\
&\qquad + x_n^4\big[(2+\delta)^3(n+1+\delta+(p-2)(2+\delta)+(\gamma-2)(p-1)(2+\delta))\big]\Big\}.
\end{aligned}
$$

Thus $\operatorname{div}(|Dv|^{p-2}Dv)|Dv|^{4-p}\gamma^{-3}R^{8-3\gamma}$ is a 4th order homogeneous polynomial of $|x'|$ and x_n. Since $\gamma \in (0, \frac{p-n-1-\delta}{p-1})$, we have

$$n+\delta+(p-1)(\gamma-1) < 0.$$

Therefore there exists a sufficiently small $\mu_0 \in (0,1/2)$ depending only on n, p, γ, and δ, such that if $|x_n| \leq \mu_0|x'|$ and $x \neq 0$, we have

$$\operatorname{div}(|Dv|^{p-2}Dv) > 0.$$

We then take $\mu = \min\{\mu_0, \kappa_2 r_0\}$, so that $\mu/\kappa_2 \leq r_0$ and thus $\frac{\partial v}{\partial \nu} > 0 \quad \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{\mu/\kappa_2}$. Note that when $x \in \Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}$, by (4.4.2) we have

$$|x_n| \leq \frac{\varepsilon}{2} + \frac{\kappa_2}{2}|x'|^2 \leq \mu|x'| \leq \mu_0|x'|.$$

This concludes the proof. $\square$

Proof of Theorem 4.1.3. It suffices to show (4.1.11) for $\delta \in (0,(p-n-1)/2)$, $x \in \Omega_{\mu/2\kappa_2}$, and $\varepsilon \in (0,\mu^2/(\kappa_2+\kappa_2^2))$, where $\mu \in (0,1/2)$ is defined in Lemma 4.4.1 with $\gamma = \frac{p-1-n-2\delta}{p-1} \in (0,1)$. Without loss of generality, we may assume that $u(0)=0$ and $\operatorname{osc}_{\Omega_1} u = 1$. By Theorem 4.1.1,

$$|u(x))| \le C\sqrt{\varepsilon} \quad \text{for} \quad x \in \overline{\Omega}_{\varepsilon/\mu}. \tag{4.4.6}$$

Let v be the function defined in Lemma 4.4.1 with $\gamma = \frac{p-1-n-2\delta}{p-1}$ and $v_1 = v + \sqrt{\varepsilon}$. Note that $u \le Cv_1$ and $-u \le Cv_1$ on $(\{|x'| = \varepsilon/\mu\} \cup \{|x'| = \mu/\kappa_2\}) \cap \overline{\Omega}_1$ for some ε-independent constant C. By the comparison principle, we have $|u| \le Cv$ in $\Omega_{\mu/\kappa_2} \backslash \Omega_{\varepsilon/\mu}$. In particular, we have

$$|u(x)| \le C(\varepsilon^2 + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}} + C\varepsilon^{1/2} \quad \text{for} \quad x \in \Omega_{\mu/\kappa_2} \setminus \Omega_{\varepsilon/\mu}. \tag{4.4.7}$$

Since $\frac{p-1-n-2\delta}{2(p-1)} \in (0,1/2)$, by combining (4.4.6) and (4.4.7), we obtain

$$|u(x)| \le C(\varepsilon + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}} \quad \text{for} \quad x \in \Omega_{\mu/\kappa_2}.$$

This implies that for any $x \in \Omega_{\mu/2\kappa_2}$ and $\varepsilon \in (0,\mu^2/(\kappa_2+\kappa_2^2))$,

$$\operatorname*{osc}_{\Omega_{x,\eta}} u \le C(\varepsilon + |x'|^2)^{\frac{p-1-n-2\delta}{2(p-1)}}, \tag{4.4.8}$$

where $\eta = \frac{1}{4}(\varepsilon + |x'|^2)^{1/2}$, and C is a constant depending only on n, p, δ, κ_1, κ_2, and the modulus of continuity for $D^2h_1(x')$ and $D^2h_2(x')$ at $x'=0$. Then (4.1.11) follows from (4.4.8) and Theorem 4.1.1. $\square$

4.5 A two dimensional example

In this section, we provide an example showing that the estimates (4.1.8) and (4.1.11) are close to optimal in 2D. In the following and throughout this section, we set our domain $\Omega = B_5 \subset \mathbb{R}^2$, $\mathcal{D}_1$ and $\mathcal{D}_2$ to be the unit balls centered at $(0, 1+\varepsilon/2)$ and $(0, -1-\varepsilon/2)$, respectively. That is,

$$\Gamma_+ = \left\{x_2 = \frac{\varepsilon}{2} + 1 - \sqrt{1-x_1^2}\right\},\ \Gamma_- = \left\{x_2 = -\frac{\varepsilon}{2} - 1 + \sqrt{1-x_1^2}\right\},\ x_1 \in (-1,1). \tag{4.5.1}$$

Lemma 4.5.1. *Let $n=2$ and Γ_+, Γ_- be as (4.5.1). For any $\delta \in (0,1/2)$, $\varepsilon \in (0, \delta/10)$, and $\gamma > \max\{\frac{p-3+\delta}{p-1}, 0\}$, there exists a constant $r_0 \in (0,1/2)$ depending only on p and δ, such that the function*

$$w(x) := \left[\left(x_1^2 + (2-\delta)x_2^2\right)^{\frac{\gamma}{2}} - (4\sqrt{\varepsilon/\delta})^\gamma\right]_+$$

satisfies

$$\begin{cases} -\operatorname{div}(|Dw|^{p-2}Dw) \le 0 & \text{in } \Omega_{r_0}, \\ \dfrac{\partial w}{\partial \nu} \le 0 & \text{on } (\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0}. \end{cases} \tag{4.5.2}$$

Proof. We denote $R = R(x) = \left(x_1^2 + (2-\delta)x_2^2\right)^{\frac{1}{2}}$ and $v(x) = R(x)^\gamma$. Then

$$Dv = \gamma R^{\gamma-2}(x_1, (2-\delta)x_2).$$

On Γ_+, the upward normal vector $\nu = (-x_1, 1+\varepsilon/2 - x_2)$. Therefore,

$$\begin{aligned} \frac{\partial v}{\partial \nu} =& \gamma R^{\gamma-2}\Big[-x_1^2 + (2-\delta)x_2\Big(1+\frac{\varepsilon}{2} - x_2\Big)\Big] \\ =& \gamma R^{\gamma-2}\Big[\Big(x_2 - 1 - \frac{\varepsilon}{2}\Big)^2 - 1 + (2-\delta)x_2\Big(1+\frac{\varepsilon}{2}-x_2\Big)\Big] \\ =& \gamma R^{\gamma-2}\Big[-(1-\delta)x_2^2 - \delta x_2\Big(1+\frac{\varepsilon}{2}\Big) + \varepsilon + \frac{\varepsilon^2}{4}\Big]. \end{aligned}$$

One can see that $\partial v/\partial\nu < 0$ if $x_2 > \varepsilon/\delta$. Since $x_2 = \frac{\varepsilon}{2} + 1 - \sqrt{1 - x_1^2}$, $|x_1| > 2\sqrt{\varepsilon/\delta}$ implies $x_2 > \varepsilon/\delta$. Note that $w = 0$ on Γ_+ when $|x_1| \le 2\sqrt{\varepsilon/\delta}$, therefore $\partial w/\partial\nu \le 0$ on Γ_+. The fact that $\partial w/\partial\nu \le 0$ on Γ_- follows from a similar argument.

Next, following a similar computation as Lemma 4.4.1, we have in the region $\{x_1^2 + (2-\delta)x_2^2 > 16\varepsilon/\delta\} \cap \Omega_1$,

$$\begin{aligned}
&\operatorname{div}(|Dw|^{p-2}Dw)|Dw|^{4-p} \\
&= \gamma^2 R^{2\gamma-4}\Big(x_1^2 + (2-\delta)^2 x_2^2\Big)\Big((3-\delta)\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}(x_1^2 + (2-\delta)^2 x_2^2)\Big) \\
&\quad + (p-2)\gamma^2 R^{2\gamma-4} x_1^2\Big(\gamma R^{\gamma-2} + \gamma(\gamma-2)R^{\gamma-4}x_1^2\Big) \\
&\quad + (p-2)\gamma^2 R^{2\gamma-4}(2-\delta)^2 x_2^2\Big((2-\delta)\gamma R^{\gamma-2} + (2-\delta)^2\gamma(\gamma-2)R^{\gamma-4}x_2^2\Big) \\
&\quad + 2(2+\delta)^2(p-2)\gamma^2 R^{2\gamma-4}\gamma(\gamma-2)R^{\gamma-4}x_1^2 x_2^2.
\end{aligned}$$

Note that in $\{x_1^2 + (2-\delta)x_2^2 > 16\varepsilon/\delta\} \cap \Omega_1$, $|x_1| > 2\sqrt{\varepsilon/\delta}$. Therefore,

$$|x_2| \le \frac{\varepsilon}{2} + 1 - \sqrt{1 - x_1^2} \le \frac{\delta}{8}x_1^2 + 1 - \sqrt{1 - x_1^2} \le \Big(\frac{\delta}{8} + 1\Big)x_1^2,$$

and thus

$$R^2 - (2-\delta)\Big(\frac{\delta}{8} + 1\Big)^2 R^4 \le x_1^2 \le R^2.$$

Hence for small $R > 0$, the leading term of $\operatorname{div}(|Dw|^{p-2}Dw)|Dw|^{4-p}$ is given by

$$\gamma^2 R^{2\gamma-2}\Big((3-\delta)\gamma + \gamma(\gamma-2)\Big)R^{\gamma-2} + (p-2)\gamma^2 R^{2\gamma-2}\gamma(\gamma-1)R^{\gamma-2}.$$

If we set

$$(3-\delta) + (\gamma-2) + (p-2)(\gamma-1) > 0 \quad \text{and} \quad \gamma > 0,$$

which implies

$$\gamma > \max\Big\{\frac{p-3+\delta}{p-1}, 0\Big\},$$

then there exists a sufficient small $r_0 \in (0, 1/2)$, depending only on p and δ, such that $\mathrm{div}(|Dw|^{p-2}Dw) \geq 0$ in Ω_{r_0}. $\square$

Proof of Theorem 4.1.4. We only need to prove the theorem for any $\delta \in (0, 1/2)$ and $\varepsilon \in (0, r_0^2\delta/64)$, where r_0 is the constant stated in Lemma 4.5.1. By symmetry and the maximum principle, we have $u(0, x_2) = 0$ for $|x_2| < \varepsilon/2$ and $u(x) > 0$ when $x_1 > 0$.

Next, we use interior Harnack inequality to show that there exists a positive constant C_0 depending only on p and δ such that $u \geq C_0$ on $\widetilde{\Omega} \cap \{x_1 = r_0\}$. Since $u(x) = x_1$ on ∂B_5, by (4.1.6), there exists $r_1 \in (4, 5)$ such that $u(r_1, 0) \geq 4$ and $5 - r_1$ is bounded from below by a positive constant depending only on p. Let $\mu = \min\{r_0^2/10, (5 - r_1)/10\}$ and $N = \lfloor \frac{r_1 - r_0}{\mu} \rfloor$. We then define a chain of balls $\mathcal{B}_k = B_\mu(p_k)$, where $k = 0, 1, \ldots, N$ and $p_k = (r_0 + k\mu, 0)$. Since $B_{2\mu}(p_k) \subset \widetilde{\Omega} \cap \{x_1 > 0\}$, by the Harnack inequality (see e.g. [84]), there exists a constant $C_0' > 0$ depending only on p, such that

$$\max_{\mathcal{B}_k} u \leq C_0' \min_{\mathcal{B}_k} u$$

holds for any $k = 0, 1, \ldots, N$. Since $(r_1, 0) \in \mathcal{B}_N$, by iteration, we have

$$u(r_0, 0) \geq (C_0')^{-N-1} u(r_1, 0) \geq 4(C_0')^{-N-1}. \tag{4.5.3}$$

Now we set $x_0 = (r_0, 0)$ and perform the flattening and extension of u to u_2 in $\mathcal{C}_{1/2} = \{(\mathcal{Z}_1, \mathcal{Z}_2) \in \mathbb{R}^2 : |\mathcal{Z}_1| < 1/2\}$ as in Section 4.2.3. Since u_2 is a nonnegative solution to (4.2.23) when $x_1 > 0$, by the Harnack inequality and a similar iteration argument as above, we obtain that

$$u_2(0, \mathcal{Z}_2) \geq C_0'' u_2(0, 0) \tag{4.5.4}$$

holds for any $\mathcal{Z}_2 \in \big(-1 + \sqrt{1 - r_0^2} - \varepsilon/2, 1 - \sqrt{1 - r_0^2} + \varepsilon/2\big)$, where $C_0'' > 0$ is a constant depending only on p and δ. In the original coordinate, (4.5.4) directly implies that

$$u(x) \geq C_0'' u(r_0, 0) \tag{4.5.5}$$

for any $x \in \widetilde{\Omega} \cap \{x_1 = r_0\}$. Combining (4.5.3) and (4.5.5), we get

$$u \geq C_0 \quad \text{on} \quad \widetilde{\Omega} \cap \{x_1 = r_0\},$$

where $C_0 = 4C_0''(C_0')^{-N-1}$ is a positive constant depending only on p and δ.

Let w be the function defined in Lemma 4.5.1 with

$$\gamma = \begin{cases} \delta & \text{when} \quad 1 < p \leq 3, \\ \dfrac{p-3+2\delta}{p-1} & \text{when} \quad p > 3. \end{cases}$$

By the comparison principle, there exists a positive constant C depending only on p and δ, such that $u \geq \frac{1}{C} w$ in $\Omega_{r_0} \cap \{x_1 > 0\}$. In particular, since $8\sqrt{\varepsilon/\delta} < r_0$, we have

$$u(8\sqrt{\varepsilon/\delta}, 0) \geq \frac{1}{C} \varepsilon^{\frac{\gamma}{2}}.$$

The desired lower bounds on Du follow from the mean value theorem since $u(0) = 0$. $\square$

Remark 4.5.2. When $\varepsilon = 0$ and $p > 5$, it can be shown that $w(x) := (x_1^2 + 2x_2^2)^{\gamma/2}$ is also a subsolution satisfying (4.5.2) for $\gamma = (p-3)/(p-1)$ and some absolute constant $r_0 \in (0, 1)$. Therefore, a similar argument as in the proof of Theorem 4.1.4 gives $u(x_1, 0) \geq c_0\, x_1^{(p-3)/(p-1)}$ for some constant $c_0 = c_0(p) > 0$ and any $x_1 \in (0, r_0)$.

Thus, for any $r \in (0,1)$, there exists $x_1 \in (0,r)$ such that

$$D_1 u(x_1, 0) \geq c\, x_1^{-2/(p-1)},$$

where $c > 0$ is a constant depending only on p.

4.6 Bernstein type argument

In this section, we adapt the Bernstein type argument used in [88] (see also [20] and [21]) to prove improved gradient estimates for (4.1.7) in high dimensions. As mentioned before, our proof also relies on the fact that for any $q \geq p$, $|Du|^q$ is a subsolution to the normalized p-Laplace equation, which was originally observed by Uhlenbeck [85]. In addition to (4.1.3) and (4.1.4), we need to further assume that h_1 and h_2 are $C^{2,\mathrm{Dini}}$ (so that u is C^2 at the points where $Du \neq 0$, see [30, Theorem 2.4]), strictly convex and strictly concave respectively, satisfying (4.1.10).

Let ν denote the normal vector on $\Gamma_{\pm}$, pointing upwards and downwards respectively. We have the following lemma.

Lemma 4.6.1. *Let $\Gamma_+, \Gamma_-, h_1, h_2$ be as above, $s \geq 2$. If u is twice differentiable and $D_\nu u = 0$ on $\Gamma_+ \cup \Gamma_-$, then at any point $x_0 \in \Gamma_+ \cup \Gamma_-$,*

$$s\kappa_1 |Du(x_0)|^s \leq D_\nu |Du(x_0)|^s \leq s\kappa_2 |Du(x_0)|^s. \tag{4.6.1}$$

Proof. We only prove (4.6.1) at $x_0 \in \Gamma_+$. By a rotation, we may assume that $x_0' = 0$ and $D_{x'} h_1(x_0') = 0$. The normal vector ν on Γ_+ is given by

$$\nu = \frac{1}{\sqrt{1 + |D_{x'} h_1|^2}} (-D_1 h_1, \cdots, -D_{n-1} h_1, 1). \tag{4.6.2}$$

Then $D_\nu u = 0$ is equivalent to

$$\sum_{j=1}^{n-1} D_j u D_j h_1 - D_n u = 0.$$

Applying D_i to the equation above for $i = 1, \ldots, n-1$, we have at x_0,

$$\sum_{j=1}^{n-1} D_j u D_{ij} h_1 - D_{in} u = 0. \tag{4.6.3}$$

By direct computation, at x_0,

$$\begin{aligned} D_\nu |Du|^s &= s|Du|^{s-2} \sum_{i=1}^{n} D_i u D_{in} u \\ &= s|Du|^{s-2} \sum_{i=1}^{n-1} \sum_{j=1}^{n-1} D_i u D_j u D_{ij} h_1, \end{aligned}$$

where in the second line, we used (4.6.3) and $D_n u(x_0) = 0$. Then (4.6.1) follows from (4.1.10). □

Proof of Theorem 4.1.5. For convenience, we let $\gamma = 2\beta \in [0, 1)$.

Case 1: For $p \geq 2$, we consider the quantity

$$F = Q^{\frac{p - p\gamma}{2}} |Du|^p,$$

where

$$Q = \frac{\varepsilon}{\kappa_1} + |x'|^2 - \frac{5\kappa_2}{2(1-\gamma)\kappa_1} x_n^2. \tag{4.6.4}$$

We will show by contradiction that F does not achieve its maximum on $(\Gamma_+ \cup \Gamma_-) \cap \overline{\Omega}_{r_0}$ or in Ω_{r_0} for some suitable r_0 which is independent of ε. Therefore, F can only achieve maximum on

$$\{|x'| = r_0\} \cap \Omega_1,$$

and (4.1.14) follows.

If $F^{q/2}$ achieves it maximum at a point x_0, we may assume that $Du(x_0) \neq 0$. First we show that $x_0 \notin \Gamma_+ \cap \overline{\Omega}_{r_0}$. A similar argument applies to $\Gamma_- \cap \overline{\Omega}_{r_0}$. On Γ_+, the normal vector ν is given by (4.6.2). At x_0, by (4.6.1) with $s = p$,

$$\begin{aligned} D_\nu F =& \frac{p-p\gamma}{2} Q^{\frac{p-p\gamma}{2}-1} D_\nu Q |Du|^p + Q^{\frac{p-p\gamma}{2}} D_\nu |Du|^p \\ \leq& - \frac{(p-p\gamma)Q^{\frac{p-p\gamma}{2}-1}}{\sqrt{1+|D_{x'}h_1|^2}} \Big[\sum_{j=1}^{n-1} D_j h_1 x_j + \frac{5\kappa_2}{2(1-\gamma)\kappa_1}(\varepsilon/2 + h_1) \Big] |Du|^p \\ &+ p\kappa_2 Q^{\frac{p-p\gamma}{2}} |Du|^p. \end{aligned}$$

We choose r_0 small enough such that

$$\frac{-1}{\sqrt{1+|D_{x'}h_1|^2}} \leq \frac{-1}{\sqrt{1+|\kappa_2 x'|^2}} \leq -\frac{4}{5} \quad \text{for} \quad |x'| < r_0.$$

By (4.1.3) and (4.1.10), we have

$$\sum_{j=1}^{n-1} D_j h_1 x_j \geq \kappa_1 |x'|^2 \quad \text{and} \quad h_1 \geq \frac{1}{2}\kappa_1 |x'|^2.$$

Therefore,

$$\begin{aligned} D_\nu F \leq& Q^{\frac{p-p\gamma}{2}-1} |Du|^p \bigg(-\frac{4}{5}(p-p\gamma)\Big[\kappa_1|x'|^2 + \frac{5\kappa_2}{4(1-\gamma)\kappa_1}(\varepsilon + \kappa_1|x'|^2)\Big] \\ &+ p\kappa_2\Big(\frac{\varepsilon}{\kappa_1} + |x'|^2 - \frac{5\kappa_2}{2(1-\gamma)\kappa_1} x_n^2\Big)\bigg) \\ =& Q^{\frac{p-p\gamma}{2}-1} |Du|^p \Big(-\frac{4}{5}(p-p\gamma)\kappa_1|x'|^2 - \frac{5p\kappa_2^2}{2(1-\gamma)\kappa_1} x_n^2 \Big) < 0. \end{aligned}$$

Hence F does not achieve its maximum on $\Gamma_+ \cap \overline{\Omega}_{r_0}$. Next, we will show that $x_0 \notin \Omega_{r_0}$ by assuming otherwise and showing that $a^{ij} D_{ij} F(x_0) > 0$, where

$$a^{ij}(x) = \delta_{ij} + (p-2)|Du|^{-2} D_i u D_j u \tag{4.6.5}$$

is symmetric. Note that

$$|\xi|^2 \le a^{ij}\xi_i\xi_j \le (p-1)|\xi|^2, \quad \forall \xi \in \mathbb{R}^n, \tag{4.6.6}$$

and if u is a solution of (4.1.7), then $a^{ij}D_{ij}u = 0$. Since $Du(x_0) \neq 0$. By the continuity of Du, $Du \neq 0$ in a neighborhood of x_0, and hence a^{ij} is well defined in the neighborhood. By direction computations,

$$\begin{aligned} &a^{ij}D_{ij}F \\ &= Q^{\frac{p-p\gamma}{2}}(a^{ij}D_{ij}|Du|^p) + |Du|^p(a^{ij}D_{ij}Q^{\frac{p-p\gamma}{2}}) + 2a^{ij}(D_i|Du|^p)(D_jQ^{\frac{p-p\gamma}{2}}). \end{aligned} \tag{4.6.7}$$

Next, we estimate the three terms on the right-hand side above. First,

$$\begin{aligned} &a^{ij}D_{ij}Q^{\frac{p-p\gamma}{2}} \\ &= a^{ij}\Big[\frac{p-p\gamma}{2}Q^{\frac{p-p\gamma}{2}-1}D_{ij}Q + \frac{p-p\gamma}{2}\Big(\frac{p-p\gamma}{2}-1\Big)Q^{\frac{p-p\gamma}{2}-2}D_iQD_jQ\Big]. \end{aligned} \tag{4.6.8}$$

Since $a^{ij}D_{ij}u = 0$, applying D_k gives

$$a^{ij}D_{ijk}u + D_ka^{ij}D_{ij}u = 0.$$

Since

$$D_ka^{ij} = (p-2)\left[\frac{D_{ik}uD_ju + D_iuD_{jk}u}{|Du|^2} - 2\frac{D_iuD_juD_{kl}uD_lu}{|Du|^4}\right],$$

we have

$$\begin{aligned} a^{ij}D_{ijk}uD_ku &= -D_ka^{ij}D_{ij}uD_ku \\ &= -2(p-2)|D|Du||^2 + 2(p-2)|Du|^{-4}|\Delta_\infty u|^2, \end{aligned} \tag{4.6.9}$$

where $\Delta_\infty u := D_i u D_j u D_{ij} u$. By (4.6.5) and (4.6.9), we have

$$\begin{aligned}
a^{ij} D_{ij} |Du|^p =& p a^{ij} \Big[(p-2)|Du|^{p-4} D_{ik} u D_k u D_{jl} u D_l u \\
&+ |Du|^{p-2} D_{ik} u D_{jk} u + |Du|^{p-2} D_k u D_{ikj} u \Big] \\
=& p|Du|^{p-4} \Big[(p-2)|Du|^2 |D|Du||^2 + (p-2)^2 |Du|^{-2} |\Delta_\infty u|^2 \\
&+ |Du|^2 |D^2 u|^2 + (p-2)|Du|^2 |D|Du||^2 - 2(p-2)|Du|^2 |D|Du||^2 \\
&+ 2(p-2)|Du|^{-2} |\Delta_\infty u|^2 \Big] \\
=& p|Du|^{p-4} \Big[p(p-2)|Du|^{-2} |\Delta_\infty u|^2 + |Du|^2 |D^2 u|^2 \Big]. \qquad (4.6.10)
\end{aligned}$$

Note that at the point x_0, for any $i = 1, 2, \ldots, n$,

$$0 = D_i F = (D_i Q^{\frac{p-p\gamma}{2}})|Du|^p + Q^{\frac{p-p\gamma}{2}} (D_i |Du|^p). \qquad (4.6.11)$$

We split the last term on the right-hand side of (4.6.7) into

$$\frac{2p-1}{p} a^{ij} (D_i |Du|^p)(D_j Q^{\frac{p-p\gamma}{2}}) + \frac{1}{p} a^{ij} (D_i |Du|^p)(D_j Q^{\frac{p-p\gamma}{2}}).$$

For the first term on the right-hand side above, we use (4.6.11) to substitute $D_i |Du|^p$, and for the second term, we substitute $D_jQ^{\frac{p-p\gamma}{2}}$. Then,

$$\begin{aligned}
&2a^{ij}(D_i|Du|^p)(D_jQ^{\frac{p-p\gamma}{2}})\\
&= -\frac{2p-1}{p}Q^{-\frac{p-p\gamma}{2}}|Du|^p a^{ij}(D_iQ^{\frac{p-p\gamma}{2}})(D_jQ^{\frac{p-p\gamma}{2}})\\
&\quad -\frac{1}{p}Q^{\frac{p-p\gamma}{2}}|Du|^{-p}a^{ij}(D_i|Du|^p)(D_j|Du|^p)\\
&= -\frac{2p-1}{p}\frac{(p-p\gamma)^2}{4}Q^{\frac{p-p\gamma}{2}-2}|Du|^p a^{ij}D_iQD_jQ\\
&\quad - pQ^{\frac{p-p\gamma}{2}}|Du|^{p-4}a^{ij}D_{ik}uD_kuD_{jl}uD_lu\\
&= -\frac{2p-1}{p}\frac{(p-p\gamma)^2}{4}Q^{\frac{p-p\gamma}{2}-2}|Du|^p a^{ij}D_iQD_jQ\\
&\quad - pQ^{\frac{p-p\gamma}{2}}|Du|^{p-4}(|Du|^2|D|Du||^2+(p-2)|Du|^{-2}|\Delta_\infty u|^2),
\end{aligned} \tag{4.6.12}$$

where we used (4.6.5) in the last equality. Therefore, by (4.6.7), (4.6.8), (4.6.10), and (4.6.12),

$$\begin{aligned}
a^{ij}D_{ij}F =& pQ^{\frac{p-p\gamma}{2}}|Du|^{p-4}\Big[(p-1)(p-2)|Du|^{-2}|\Delta_\infty u|^2+|Du|^2|D^2u|^2\\
&-|Du|^2|D|Du||^2\Big]+\frac{p-p\gamma}{2}Q^{\frac{p-p\gamma}{2}-1}|Du|^p a^{ij}D_{ij}Q\\
&-\Big[\frac{2p-1}{p}\frac{(p-p\gamma)^2}{4}-\frac{p-p\gamma}{2}\Big(\frac{p-p\gamma}{2}-1\Big)\Big]Q^{\frac{p-p\gamma}{2}-2}|Du|^p a^{ij}D_iQD_jQ.
\end{aligned}$$

Note that

$$|Du|^2|D|Du||^2 \le |Du|^2|D^2u|^2.$$

It remains to show

$$Qa^{ij}D_{ij}Q > \Big[\frac{2p-1}{p}\frac{p-p\gamma}{2}-\Big(\frac{p-p\gamma}{2}-1\Big)\Big]a^{ij}D_iQD_jQ. \tag{4.6.13}$$

Recall that Q is given in (4.6.4). Then

$$DQ = \Big(2x_1,\ldots,2x_{n-1},-\frac{5\kappa_2}{(1-\gamma)\kappa_1}x_n\Big), \tag{4.6.14}$$

and

$$
\begin{aligned}
a^{ij}D_{ij}Q =& 2n-2-\frac{5\kappa_2}{(1-\gamma)\kappa_1} \\
&+(p-2)|Du|^{-2}\Big(2|D_{x'}u|^2-\frac{5\kappa_2}{(1-\gamma)\kappa_1}|D_nu|^2\Big) \\
\geq & 2n-2-(p-1)\frac{5\kappa_2}{(1-\gamma)\kappa_1}.
\end{aligned}
$$

By (4.6.6) and shrinking r_0 if necessary, we have

$$
a^{ij}D_iQD_jQ \leq 4(p-1)\Big(|x'|^2+\frac{25\kappa_2^2}{4(1-\gamma)^2\kappa_1^2}x_n^2\Big) < 5(p-1)Q.
$$

In order to show (4.6.13), we only require

$$
2n-2-(p-1)\frac{5\kappa_2}{(1-\gamma)\kappa_1} \geq 5(p-1)\Big[\frac{2p-1}{p}\frac{p-p\gamma}{2}-\Big(\frac{p-p\gamma}{2}-1\Big)\Big],
$$

which is equivalent to (4.1.12) since $\gamma=2\beta$. This concludes the proof for the case when $p \geq 2$.

Case 2: For $p\in(1,2)$, we consider the quantity

$$
G = Q^{1-\gamma}|Du|^2,
$$

where Q is given in (4.6.4). From the computation of $D_\nu F$ with $p=2$, one can see that G does not attain its maximum on $(\Gamma_+\cup\Gamma_-)\cap\overline{\Omega}_{r_0}$. Next, we assume that G achieves its maximum at $x_0\in\Omega_{r_0}$. By the computations as in (4.6.7) and (4.6.8) with $p=2$, we have

$$
a^{ij}D_{ij}G = Q^{1-\gamma}(a^{ij}D_{ij}|Du|^2)+|Du|^2(a^{ij}D_{ij}Q^{1-\gamma})+2a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}), \tag{4.6.15}
$$

where a^{ij} is given in (4.6.5) with

$$(p-1)|\xi|^2 \le a^{ij}\xi_i\xi_j \le |\xi|^2, \quad \forall \xi \in \mathbb{R}^n. \tag{4.6.16}$$

Next, we estimate the three terms on the right-hand side of (4.6.15). First,

$$a^{ij}D_{ij}Q^{1-\gamma} = a^{ij}\Big[(1-\gamma)Q^{-\gamma}D_{ij}Q - \gamma(1-\gamma)Q^{-1-\gamma}D_iQD_jQ\Big]. \tag{4.6.17}$$

By (4.6.5) and (4.6.9), we have

$$\begin{aligned}
a^{ij}D_{ij}|Du|^2 =& a^{ij}\Big[2D_{ijk}uD_ku + 2D_{jk}uD_{ik}u\Big] \\
=& 4(2-p)|D|Du||^2 - 4(2-p)|Du|^{-4}|\Delta_\infty u|^2 \\
& + 2|D^2u|^2 - 2(2-p)|D|Du||^2 \\
=& 2(2-p)|D|Du||^2 - 4(2-p)|Du|^{-4}|\Delta_\infty u|^2 + 2|D^2u|^2.
\end{aligned} \tag{4.6.18}$$

Note that at the point x_0, for any $i = 1, 2, \ldots, n$,

$$0 = D_iG = (D_iQ^{1-\gamma})|Du|^2 + Q^{1-\gamma}(D_i|Du|^2). \tag{4.6.19}$$

As before, we split the last term on the right-hand side of (4.6.15) as

$$\frac{1}{2}a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}) + \frac{3}{2}a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}).$$

For the first term on the right-hand side, we use (4.6.19) to substitute $D_jQ^{1-\gamma}$, and for the second term, we substitute $D_i|Du|^2$. Then by (4.6.5),

$$\begin{aligned}
2a^{ij}(D_i|Du|^2)(D_jQ^{1-\gamma}) =& -\frac{1}{2}a^{ij}(D_i|Du|^2)(D_j|Du|^2)Q^{1-\gamma}|Du|^{-2} \\
& -\frac{3}{2}a^{ij}(D_iQ^{1-\gamma})(D_jQ^{1-\gamma})|Du|^2Q^{-1+\gamma} \\
=& -2Q^{1-\gamma}|Du|^{-2}a^{ij}D_{ik}uD_kuD_{jl}uD_lu \\
& -\frac{3}{2}(1-\gamma)^2|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ \\
=& -2Q^{1-\gamma}|D|Du||^2 + 2(2-p)Q^{1-\gamma}|Du|^{-4}|\Delta_\infty u|^2 \\
& -\frac{3}{2}(1-\gamma)^2|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ. \qquad (4.6.20)
\end{aligned}$$

Therefore, by (4.6.15), (4.6.17), (4.6.18), and (4.6.20),

$$\begin{aligned}
a^{ij}D_{ij}G =& Q^{1-\gamma}\Big[(2(2-p)-2)|D|Du||^2 - 2(2-p)|Du|^{-4}|\Delta_\infty u|^2 + 2|D^2u|^2\Big] \\
& + (1-\gamma)Q^{-\gamma}|Du|^2a^{ij}D_{ij}Q \\
& - \Big[\gamma(1-\gamma) + \frac{3}{2}(1-\gamma)^2\Big]|Du|^2Q^{-1-\gamma}a^{ij}D_iQD_jQ.
\end{aligned}$$

Since

$$|Du|^{-4}|\Delta_\infty u|^2 \le |D|Du||^2 \le |D^2u|^2,$$

it remains to show

$$Qa^{ij}D_{ij}Q > \Big[\gamma + \frac{3}{2}(1-\gamma)\Big]a^{ij}D_iQD_jQ. \qquad (4.6.21)$$

By (4.6.14), we have

$$\begin{aligned}
a^{ij}D_{ij}Q =& 2n - 2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1} \\
& - (2-p)|Du|^{-2}\Big(2|D_{x'}u|^2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1}|D_nu|^2\Big) \\
\ge& 2n - 2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1} - 2(2-p).
\end{aligned}$$

By (4.6.16) and shrinking r_0 if necessary, we have

$$a^{ij}D_iQD_jQ \leq 4\Big(|x'|^2 + \frac{25\kappa_2^2}{4(1-\gamma)^2\kappa_1^2}x_n^2\Big) < 5Q.$$

In order to show (4.6.21), we only require

$$2n - 2 - \frac{5\kappa_2}{(1-\gamma)\kappa_1} - 2(2-p) \geq 5\Big[\gamma + \frac{3}{2}(1-\gamma)\Big],$$

which is equivalent to (4.1.13) since $\gamma = 2\beta$. This concludes the proof for the case when $p \in (1, 2)$. $\square$

CHAPTER 5

Asymptotics of the solution to the perfect conductivity problem with p-Laplacian

We study the perfect conductivity problem with closely spaced perfect conductors embedded in a homogeneous matrix where the current-electric field relation is the power law $J = \sigma|E|^{p-2}E$. The gradient of solutions may be arbitrarily large as ε, the distance between inclusions, approaches to 0. To characterize this singular behavior of the gradient in the narrow region between two inclusions, we capture the leading order term of the gradient. This is the first gradient asymptotics result on the nonlinear perfect conductivity problem.

5.1 Introduction and Main results

Our study is instigated by the damage analysis in the fiber composite materials [6]. Particularly, when fibers are closely packed and in high-contrast to the background matrix in terms of material properties, the electric field could be amplified by the composite micro-structure. In this chapter, we investigate the specific scenario in which the fiber inclusions are perfect conductors, and the background matrix follows the current-electric field relation described by the power law:

$$J = \sigma |E|^{p-2} E, \quad p > 1, \tag{5.1.1}$$

where J, E, and σ represent current, electric field, and conductivity, respectively.

Before stating our results, let us describe the mathematical setup: let $\Omega \subset \mathbb{R}^n$ be a bounded domain with C^2 boundary, and let $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ be two C^2 open sets with $\operatorname{diam}(\mathcal{D}_i^0) > c > 0$ and $\operatorname{dist}(\mathcal{D}_1^0 \cup \mathcal{D}_2^0, \partial\Omega) > c > 0$, touching at the origin with the inner normal direction of $\partial\mathcal{D}_1^0$ being the positive x_n-axis. We write the variable x as (x', x_n), where $x' \in \mathbb{R}^{n-1}$. For $\varepsilon > 0$, translating $\mathcal{D}_1^0$ and $\mathcal{D}_2^0$ by $\varepsilon/2$ along the x_n-axis, we obtain

$$\mathcal{D}_1^\varepsilon := \mathcal{D}_1^0 + (0', \varepsilon/2) \quad \text{and} \quad \mathcal{D}_2^\varepsilon := \mathcal{D}_2^0 - (0', \varepsilon/2).$$

We denote $\widetilde{\Omega}^\varepsilon := \Omega \setminus \overline{(\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon)}$.

The perfect conductivity problem incorporating the power law (5.1.1) can be modeled by the following p-Laplace equation with $p > 1$:

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2} Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ u_\varepsilon = U_i^\varepsilon & \text{on } \overline{\mathcal{D}_i^\varepsilon},\ i = 1, 2, \\ \displaystyle\int_{\partial\mathcal{D}_i^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu = 0, & i = 1, 2, \\ u_\varepsilon = \varphi & \text{on } \partial\Omega, \end{cases} \tag{5.1.2}$$

where $\varphi \in C^2(\partial\Omega)$ is given, $\nu = (\nu_1, \ldots, \nu_n)$ denotes the outer normal vector on $\partial\mathcal{D}_1^\varepsilon \cup \partial\mathcal{D}_2^\varepsilon$ (pointing away from $\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon$), and U_1^ε, U_2^ε are two constants determined by (5.1.2)$_3$. Here and throughout the chapter, we adopt the notation

$$Du_\varepsilon \cdot \nu(x) := \lim_{t\to 0_+} \frac{u_\varepsilon(x + t\,\nu(x)) - u_\varepsilon(x)}{t} \quad \text{and} \quad Du_\varepsilon(x) := [Du_\varepsilon \cdot \nu(x)]\nu(x)$$

for $x \in \partial\mathcal{D}_i^\varepsilon$, $i = 1, 2$.

The solution $u_\varepsilon \in W^{1,p}(\Omega)$ can be viewed as the unique function which has the minimal energy in appropriate function space: $I_p[u] = \min_{v\in\mathscr{A}^\varepsilon} I_p[v]$, where

$$\begin{aligned} I_p[v] &:= \int_\Omega |Dv|^p, \quad v \in \mathscr{A}^\varepsilon, \\ \mathscr{A}^\varepsilon &:= \{v \in W^{1,p}(\Omega) : Dv \equiv 0 \ \text{ in } \ \mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon, v = \varphi \ \text{ on } \ \partial\Omega\}. \end{aligned} \tag{5.1.3}$$

We refer the reader to the Appendix of [8] for the derivation of (5.1.2) and its equivalence with (5.1.3). Although this derivation specifically addresses the case when $p = 2$, the argument can be readily applied to $p > 1$ with slight modifications.

The perfect conductivity problem (5.1.2) with $p = 2$ has undergone thorough studies. It was proved by Ammari et al. in [4] and [3] that, when $\mathcal{D}_1$ and $\mathcal{D}_2$ are disks of comparable radii in $\mathbb{R}^2$, the blow-up rate of the gradient of the solution is $\varepsilon^{-1/2}$ as ε goes to zero; Yun in [89, 90] generalized the above mentioned result for two strictly convex inclusions in $\mathbb{R}^2$. These gradient estimates in dimension $n = 2$ were localized and extended to higher dimensions by Bao, Li, and Yin in [8]:

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}^\varepsilon)} \le \begin{cases} C\varepsilon^{-1/2}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 2, \\ C|\varepsilon \ln \varepsilon|^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n = 3, \\ C\varepsilon^{-1}\|\varphi\|_{C^2(\partial\Omega)} & \text{when } n \ge 4. \end{cases}$$

These bounds were shown to be optimal in the paper and they are independent of the shape of inclusions, as long as the inclusions are relatively strictly convex. Moreover, numerous studies have been conducted into characterizing the asymptotic behavior of

Du_ε, which are significant in practical applications. For further works on the linear perfect conductivity problem, see e.g. [1, 2, 12, 13, 18, 27, 34, 42, 47, 48, 50, 62–64, 72] and the references therein.

The study on the nonlinear perfect conductivity problem (5.1.2) is less comprehensive. The only results were given by Gorb and Novikov [43] and Ciraolo and Sciammetta [21]. They proved that for $n \geq 2$,

$$\|Du_\varepsilon\|_{L^\infty(\widetilde{\Omega}^\varepsilon)} \leq \begin{cases} C\varepsilon^{-\frac{n-1}{2(p-1)}} & \text{when } p > \dfrac{n+1}{2}, \\ C\varepsilon^{-1}|\ln\varepsilon|^{\frac{1}{1-p}} & \text{when } p = \dfrac{n+1}{2}, \\ C\varepsilon^{-1} & \text{when } 1 < p < \dfrac{n+1}{2}. \end{cases}$$

These bounds were shown to be optimal in their respective papers. In this chapter, we give a more precise characterization of the gradient by capturing its leading order term in the asymptotics expansion.

It is noteworthy that, for the linear case in dimension two, solutions to the perfect conductivity problem and the insulated conductivity problem, representing the two extremes of conductivity, are harmonic conjugate to each other as shown in [4]. Therefore, the behavior of their gradients is essentially identical due to the Cauchy-Riemann equation. The optimal blow-up exponent in dimension two for the insulated conductivity problem with p-Laplacian was identified in Chapter 4. It turns out that the gradient behaves significantly different from that of the solution to (5.1.2) in dimension two. This showcases an intriguing feature of the nonlinear conductivity problem. For more results on the linear insulated conductivity problem, we refer to [9, 28, 29, 65, 66, 88, 91].

To study the asymptotic behavior of u_ε, the solution to (5.1.2), it is important to study the limiting problem (5.1.3) with $\varepsilon = 0$. We will show that the minimizing problem (5.1.3) with $\varepsilon = 0$ is equivalent to

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2}Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_0 & \text{on } \overline{\mathcal{D}_1^0} \cup \overline{\mathcal{D}_2^0}, \\ \displaystyle\int_{\partial\mathcal{D}_1^0 \cup \partial\mathcal{D}_2^0} |Du_0|^{p-2}Du_0 \cdot \nu = 0, & \\ u_0 = \varphi & \text{on } \partial\Omega \end{cases} \tag{5.1.4}$$

for a constant U_0 when $p \geq (n+1)/2$, and is equivalent to

$$\begin{cases} -\operatorname{div}(|Du_0|^{p-2}Du_0) = 0 & \text{in } \widetilde{\Omega}^0, \\ u_0 = U_i & \text{on } \overline{\mathcal{D}_i^0} \setminus \{0\},\ i = 1, 2, \\ \displaystyle\int_{\partial\mathcal{D}_i^0} |Du_0|^{p-2}Du_0 \cdot \nu = 0, & i = 1, 2, \\ u_0 = \varphi & \text{on } \partial\Omega \end{cases} \tag{5.1.5}$$

for constants U_1 and U_2 when $p < (n+1)/2$. We would like to clarify a misunderstanding in the papers [20, 21]. In [20, 21], the authors implicitly claimed that the minimizing problem (5.1.3) with $\varepsilon = 0$ is universally equivalent to (5.1.4), which is not the case. We will justify this in Theorem 5.2.4. We emphasize that while the minimizer u_0 of (5.1.3) with $\varepsilon = 0$ always takes the same value in $\overline{\mathcal{D}_1^0}$ and $\overline{\mathcal{D}_2^0}$ when $p \geq (n+1)/2$, it may take different values when $1 < p < (n+1)/2$. On the other hand, the flux along $\partial\mathcal{D}_1^0$, denoted by

$$\mathcal{F} := \int_{\partial\mathcal{D}_1^0} |Du_0|^{p-2}Du_0 \cdot \nu, \tag{5.1.6}$$

might not be zero when $p \geq (n+1)/2$, but it must be zero when $p < (n+1)/2$.

By the regularity of $\mathcal{D}_1^\varepsilon$ and $\mathcal{D}_2^\varepsilon$, we can assume that near the origin, the part of $\partial\mathcal{D}_1^\varepsilon$ and $\partial\mathcal{D}_2^\varepsilon$, denoted by Γ_+^ε and Γ_-^ε, are respectively the graphs of two C^2 functions in terms of x'. That is,

$$\Gamma_+^\varepsilon = \left\{x_n = \frac{\varepsilon}{2} + h_1(x'),\ |x'| < 1\right\}, \quad \Gamma_-^\varepsilon = \left\{x_n = -\frac{\varepsilon}{2} + h_2(x'),\ |x'| < 1\right\},$$

where h_1 and h_2 are relatively convex C^2 functions satisfying

$$h_1(0') = h_2(0') = 0, \quad D_{x'}h_1(0') = D_{x'}h_2(0') = 0, \tag{5.1.7}$$

$$c_1|x'|^2 \leq h_1(x') - h_2(x') \quad \text{for} \ \ 0 < |x'| < 1, \tag{5.1.8}$$

and

$$\|h_1\|_{C^2} \leq c_2, \quad \|h_2\|_{C^2} \leq c_2, \tag{5.1.9}$$

with some positive constants c_1, c_2. For $x_0 \in \widetilde{\Omega}^\varepsilon$, $0 < r < 1 - |x_0'|$, we denote

$$\Omega_{x_0,r}^\varepsilon := \left\{(x', x_n) \in \widetilde{\Omega}^\varepsilon \ \Big| \ -\frac{\varepsilon}{2} + h_2(x') < x_n < \frac{\varepsilon}{2} + h_1(x'),\ |x' - x_0'| < r\right\},$$

and $\Omega_r^\varepsilon := \Omega_{0,r}^\varepsilon$. We also denote

$$\Gamma_{+,r}^\varepsilon := \Gamma_+^\varepsilon \cap \overline{\Omega}_r^\varepsilon, \quad \Gamma_{-,r}^\varepsilon := \Gamma_-^\varepsilon \cap \overline{\Omega}_r^\varepsilon.$$

We use $B_r(x_0)$ to denote the open ball of radius r centered at x_0 and we set

$$B_r := B_r(0), \quad \Omega_r^\varepsilon(x_0) := \widetilde{\Omega}^\varepsilon \cap B_r(x_0).$$

Throughout this chapter, we denote

$$\underline{\delta}(x) := \varepsilon + |x'|^2 \tag{5.1.10}$$

and

$$\delta(x) := \varepsilon + h_1(x') - h_2(x'). \tag{5.1.11}$$

By (5.1.7)–(5.1.9), it can be easily seen that

$$\min\{c_1, 1\}\underline{\delta}(x) \leq \delta(x) \leq \max\{c_2, 1\}\underline{\delta}(x), \quad \text{for} \quad x \in \Omega_1.$$

We denote

$$\Theta(\varepsilon) := \begin{cases} \varepsilon^{\frac{2p-n-1}{2(p-1)}}, & p > \dfrac{n+1}{2}, \\ |\ln \varepsilon|^{-\frac{1}{p-1}}, & p = \dfrac{n+1}{2}, \\ 1, & 1 < p < \dfrac{n+1}{2}, \end{cases} \tag{5.1.12}$$

and

$$K = \begin{cases} \dfrac{\det\big(D^2_{x'}(h_1 - h_2)(0')\big)^{\frac{1}{2}}\Gamma(p-1)}{(2\pi)^{\frac{n-1}{2}}\Gamma\big(p - \frac{n+1}{2}\big)}, & \text{when} \quad p > \dfrac{n+1}{2}, \\ \dfrac{\det\big(D^2_{x'}(h_1 - h_2)(0')\big)^{\frac{1}{2}}\Gamma\big(\frac{n-1}{2}\big)}{(2\pi)^{\frac{n-1}{2}}}, & \text{when} \quad p = \dfrac{n+1}{2}, \end{cases} \tag{5.1.13}$$

where $\Gamma(z) := \int_0^\infty t^{z-1}e^{-t}$ is the Gamma function defined for $z > 0$.

Our main result is the following asymptotic expansion of $Du_\varepsilon(x)$ for sufficiently small ε and x'.

Theorem 5.1.1. *Let h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $p > 1$, $n \geq 2$, $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (5.1.2), u_0 be the minimizer of (5.1.3) with $\varepsilon = 0$, $\mathcal{F}$ be given in (5.1.6), U_1, U_2 be the constants in $(5.1.5)_2$, $\delta(x)$ be defined in (5.1.11), $\Theta(\varepsilon)$ be given in (5.1.12), and K be defined in (5.1.13). Then there exist constants $\beta \in (0, 1)$ depending only on n and p, and $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that the following holds:*

(i) If $p \geq (n+1)/2$, for $\varepsilon \in (0,1)$ and $x \in \Omega^{\varepsilon}_{1/4}$, we have

$$Du_{\varepsilon}(x) = \big(0', \delta(x)^{-1}\Theta(\varepsilon)(\mathrm{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)} + f_0(\varepsilon))\big) + \mathbf{f}_1(x,\varepsilon), \quad (5.1.14)$$

where $f_0 : \mathbb{R} \to \mathbb{R}$ is a function of ε and $\mathbf{f}_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function of x and ε, such that

$$\lim_{\varepsilon \to 0} f_0(\varepsilon) = 0,$$

$$|\mathbf{f}_1(x,\varepsilon)| \leq C_1\Big(\delta(x)^{\beta/2-1}\Theta(\varepsilon)(|K\mathcal{F}|^{1/(p-1)} + |f_0(\varepsilon)|) + \|\varphi\|_{L^{\infty}(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big). \quad (5.1.15)$$

(ii) If $1 < p < (n+1)/2$, for $\varepsilon \in (0,1)$ and $x \in \Omega^{\varepsilon}_{1/4}$, we have

$$Du_{\varepsilon}(x) = \big(0', \delta(x)^{-1}(U_1 - U_2 + g_0(\varepsilon))\big) + \mathbf{g}_1(x,\varepsilon),$$

where $g_0 : \mathbb{R} \to \mathbb{R}$ is a function of ε and $\mathbf{g}_1 : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ is a function of x and ε, such that

$$\lim_{\varepsilon \to 0} g_0(\varepsilon) = 0,$$

$$|\mathbf{g}_1(x,\varepsilon)| \leq C_1\Big(\delta(x)^{\beta/2-1}(|U_1 - U_2| + |g_0(\varepsilon)|) + \|\varphi\|_{L^{\infty}(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big). \quad (5.1.16)$$

As a consequence of the asymptotic expansion in Theorem 5.1.1, we provide a pointwise upper bound of Du_{ε}.

Remark 5.1.2. Under the hypotheses of Theorem 5.1.1, there exist constants $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that for sufficiently small $\varepsilon > 0$, and any $x \in \Omega^{\varepsilon}_{1/4}$, we have

$$|Du_{\varepsilon}(x)| \leq C_1 \|\varphi\|_{L^{\infty}(\partial\Omega)} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + |x'|^2} + e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big). \quad (5.1.17)$$

In fact, when $1 < p < (n+1)/2$, (5.1.17) follows directly from Proposition 5.2.2.

When $p \geq (n+1)/2$, since $u_0 = U_0$ on $\overline{\mathcal{D}_1^0} \cup \overline{\mathcal{D}_2^0}$, we have $|\mathcal{F}| \leq C\|\varphi\|_{L^\infty(\partial\Omega)}^{p-1}$ and thus (5.1.17) follows from (5.1.14). Indeed, one can see the boundedness of Du_0 on $\Gamma_{+,1/2}^0$ by Lemma 5.2.1, and on $\partial\mathcal{D}_1^0 \setminus \Gamma_{+,1/2}^0$ by classical gradient estimates (see e.g. [68]).

Another direct consequence of Theorem 5.1.1 is the following pointwise positive lower bound of $D_n u_\varepsilon$ near the origin, provided the coefficient of the leading order term in the asymptotic expansion is positive.

Remark 5.1.3. Under the hypotheses of Theorem 5.1.1, if either

$$p \geq \frac{n+1}{2} \quad \text{and} \quad \mathcal{F} > 0 \tag{5.1.18}$$

or

$$1 < p < \frac{n+1}{2} \quad \text{and} \quad U_1 > U_2 \tag{5.1.19}$$

holds, then there exist constants $\kappa_1, \kappa_2 \in (0, 1/4)$, $\gamma > 0$ depending only on n, p, c_1, c_2, $\|\varphi\|_{L^\infty(\partial\Omega)}$, $\mathcal{F}$ (when $p \geq (n+1)/2$), and $U_1 - U_2$ (when $p < (n+1)/2$), such that for sufficiently small $\varepsilon > 0$, and any $x \in \Omega_{1/4}^\varepsilon$ satisfying

$$\begin{cases} |x'| \leq |\ln\varepsilon|^{-\gamma}, & \text{when} \quad p > \dfrac{n+1}{2}, \\ |x'| \leq \kappa_1(\ln|\ln\varepsilon|)^{-1}, & \text{when} \quad p = \dfrac{n+1}{2}, \\ |x'| \leq \kappa_2, & \text{when} \quad 1 < p < \dfrac{n+1}{2}, \end{cases}$$

we have

$$\begin{cases} D_n u_\varepsilon(x) \geq \dfrac{1}{2}\delta(x)^{-1}\Theta(\varepsilon)(K\mathcal{F})^{1/(p-1)}, & \text{when} \quad p \geq \dfrac{n+1}{2}, \\ D_n u_\varepsilon(x) \geq \dfrac{1}{2}\delta(x)^{-1}(U_1 - U_2), & \text{when} \quad 1 < p < \dfrac{n+1}{2}. \end{cases} \tag{5.1.20}$$

Indeed, when $p \geq (n+1)/2$, (5.1.20) follows directly from Theorem 5.1.1 by setting

$$\begin{cases} |f_0(\varepsilon)| \leq \dfrac{1}{6}(K\mathcal{F})^{1/(p-1)}, \\ C_1\delta(x)^{\beta/2} \leq \dfrac{1}{7}, \\ C_1\|\varphi\|_{L^\infty(\partial\Omega)}e^{-\frac{C_2}{\sqrt{\varepsilon+|x'|}}} \leq \dfrac{1}{6}\delta(x)^{-1}\Theta(\varepsilon)(K\mathcal{F})^{1/(p-1)}, \end{cases}$$

and the case when $p \in (1,(n+1)/2)$ follows similarly.

Next, we provide a concrete example whose coefficient of the leading order term in the asymptotic expansion is positive.

Proposition 5.1.4. *Let $\Omega = B_5 \subset \mathbb{R}^n$, $\mathcal{D}_1 = B_2(0',2)$, $\mathcal{D}_2 = B_2(0',-2)$, $\varphi = x_n$, and u_0 be the minimizer of (5.1.3) with $\varepsilon = 0$. Then either (5.1.18) or (5.1.19) is satisfied.*

Our proof of the main result Theorem 5.1.1 relies on the following C^β bound of the gradient, which may be of independent interest.

Proposition 5.1.5. *Let h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $p > 1$, $n \geq 2$, $\varepsilon \in (0,1)$, and $u_\varepsilon \in W^{1,p}(\Omega)$ be a solution of (5.1.2). Then there exist constants $\beta \in (0,1)$ depending only on n and p, and $C > 0$ depending only on n, p, c_1, and c_2, such that for any $x \in \Omega_{1/4}$ and $\underline{\delta}(x) = \varepsilon + |x'|^2$, it holds that*

$$[Du_\varepsilon]_{C^\beta(\Omega^\varepsilon_{x,\sqrt{\underline{\delta}(x)}/4})} \leq C\underline{\delta}(x)^{-\beta/2}\|Du_\varepsilon\|_{L^\infty(\Omega^\varepsilon_{x,\sqrt{\underline{\delta}(x)}/2})}. \tag{5.1.21}$$

Remark 5.1.6. It can be seen from the proof in Section 5.3 that Proposition 5.1.5

holds as long as $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ is a solution of

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ u_\varepsilon = U_1^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_\varepsilon = U_2^\varepsilon & \text{on } \Gamma_-^\varepsilon, \end{cases}$$

for some arbitrary constants U_1^ε and U_2^ε. Moreover, the same estimates also hold for any solution $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ of

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ \dfrac{\partial u_\varepsilon}{\partial \nu} = 0 & \text{on } \Gamma_\pm^\varepsilon, \end{cases}$$

which might be useful for obtaining sharper blow-up estimates for the insulated conductivity problem with p-Laplacian (see [33]).

We briefly describe the steps of proving Theorem 5.1.1. First, we establish a pointwise upper bound of the gradient in terms of $U_1^\varepsilon - U_2^\varepsilon$ for arbitrary given U_1^ε and U_2^ε (Proposition 5.2.2). Then we use mean oscillation estimates to prove a $C^{1,\beta}$ estimate (Proposition 5.1.5). Note that Proposition 5.1.5 implies a power gain of order $\delta^{\beta/2}$ for the oscillation of the gradient in the x_n direction. Because of this power gain and Proposition 5.2.2, we then derive an asymptotic expansion of Du_ε in terms of $U_1^\varepsilon - U_2^\varepsilon$ (Proposition 5.4.1). When $p \geq (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converge to 0 as $\varepsilon \to 0$. In this case, we use the flux conditions to derive the convergence rate for $U_1^\varepsilon - U_2^\varepsilon$ (Theorem 5.4.4). When $p < (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converges to $U_1 - U_2$ (Theorem 5.2.5). Finally, Theorem 5.1.1 follows from putting all the ingredients above together.

We would like to point out that weaker versions of Proposition 5.4.1 were proved in [21, 43]. They derived an asymptotic expansion of Du_ε only on the upper and lower boundaries $\Gamma_\pm^\varepsilon$ by constructing suitable barrier functions and using comparison principle. However, in our opinion, it appears that there is a gap in their proofs. It is nontrivial to see that the normal derivative is bounded in the case when $U_1^\varepsilon - U_2^\varepsilon$ is small as lack of control of the oscillation of the solution in the x' direction (see [21, p.6174] and [43, p.740]). This gap can be filled by Proposition 5.2.2 of this chapter. We would like to remark that our argument in Proposition 5.4.1 is more robust in the sense that the proof does not rely on the fundamental solution of the p-Laplace equation or the maximum principle. In fact, the only place this chapter involves the maximum principle is Proposition 5.2.2. See also Remark 5.2.3. If one can give an alternative proof of (5.2.7) in Proposition 5.2.2 without using the maximum principle, then our results can be extended to nonlinear systems of p-Laplace type.

The rest of the chapter is organized as follows: In Section 5.2, we provide some preliminary estimates and results. In Section 5.3, we use mean oscillation estimates to prove Proposition 5.1.5. A convergence rate of $U_1^\varepsilon - U_2^\varepsilon$ when $p \geq (n+1)/2$ is provided in Section 5.4. Finally, the proofs of Theorem 5.1.1 and Proposition 5.1.4 are given in Section 5.5.

5.2 Preliminaries

In this section, we provide some preliminary results.

Lemma 5.2.1. *Let h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $p > 1$, $n \geq 2$, $\varepsilon \in [0,1)$, and $v \in W^{1,p}(\Omega_1^\varepsilon)$ be a solution of*

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv) = 0 & \text{in } \Omega_1^\varepsilon, \\ v = 0 & \text{on } \Gamma_\pm^\varepsilon. \end{cases} \tag{5.2.1}$$

Then there exist constants $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that

$$|v(x)| + |Dv(x)| \leq C_1 e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}} \|v\|_{L^p(\Omega_1^\varepsilon \setminus \Omega_{1/2}^\varepsilon)} \quad \text{for } x \in \overline{\Omega_{1/2}^\varepsilon}. \tag{5.2.2}$$

Proof. The proof of this lemma essentially follows that of [7, Theorem 1.1], with some modification. For simplicity, we omit the superscript ε in the proof. Without loss of generality, we may assume $\varepsilon \in [0, 1/256)$ and $|x'| < 1/16$ since otherwise (5.2.2) follows from classical estimates for the p-Laplace equation (see e.g. [68]). For any $0 < t < s < 1$, let $\eta = \eta(x')$ be a cutoff function such that $\eta = 1$ in Ω_t, $\eta = 0$ in $\Omega_1 \setminus \Omega_s$, and $|D\eta| \le C(s-t)^{-1}$. Multiplying $v\eta^p$ on both sides of (5.2.1) and integrating by parts, we have

$$\int_{\Omega_1} |Dv|^p \eta^p + p|Dv|^{p-2} Dv \cdot D\eta v \eta^{p-1} = 0.$$

By Young's inequality,

$$\int_{\Omega_1} |Dv|^p \eta^p \le p \int_{\Omega_1} |Dv|^{p-1} \eta^{p-1} |v||D\eta| \le \frac{1}{2} \int_{\Omega_1} |Dv|^p \eta^p + C \int_{\Omega_1} |v|^p |D\eta|^p.$$

Therefore,

$$\int_{\Omega_t} |Dv|^p \le \frac{C}{(s-t)^p} \int_{\Omega_s \setminus \Omega_t} |v|^p.$$

Since $v = 0$ on Γ_-, by the Poincaré inequality in the x_n direction, we have

$$\int_{\Omega_s \setminus \Omega_t} |v|^p \le C(\varepsilon + s^2)^p \int_{\Omega_s \setminus \Omega_t} |Du|^p.$$

Therefore,

$$\int_{\Omega_t} |Dv|^p \le C^* \left(\frac{\varepsilon + s^2}{s-t} \right)^p \int_{\Omega_s \setminus \Omega_t} |Dv|^p. \tag{5.2.3}$$

Let $t_0 = r \in (\sqrt{\varepsilon}, 1/2)$ and $t_j = (1 - jr)r$ for $j \in \mathbb{N}$ such that $j \le 1/r$. Taking $s = t_j, t = t_{j+1}$ in (5.2.3), we have

$$\int_{\Omega_{t_{j+1}}} |Dv|^p \le 2^p C^* \int_{\Omega_{t_j} \setminus \Omega_{t_{j+1}}} |Dv|^p.$$

Adding both sides by $2^pC^* \int_{\Omega_{t_{j+1}}} |Dv|^p$ and dividing both sides by $1+2^pC^*$, we have

$$\int_{\Omega_{t_{j+1}}} |Dv|^p \leq \frac{2^pC^*}{1+2^pC^*} \int_{\Omega_{t_j}} |Dv|^p.$$

Let $k = \lfloor \frac{1}{2r} \rfloor$ and iterate the above inequality k times. We have

$$\int_{\Omega_{r/2}} |Dv|^p \leq \left(\frac{2^pC^*}{1+2^pC^*}\right)^k \int_{\Omega_r} |Dv|^p \leq C\mu^{\frac{1}{r}} \int_{\Omega_1 \setminus \Omega_{1/2}} |v|^p, \tag{5.2.4}$$

where $\mu \in (0,1)$ and C are constants depending only on n, p, c_1, and c_2. Now we take $r = 4(\sqrt{\varepsilon} + |x'|)$ and (5.2.2) follows from classical estimates for the p-Laplace equation in $\Omega_{x',\varepsilon+|x'|^2}$ and (5.2.4). □

Next, we derive a pointwise upper bound of the gradient in terms of $U_1^\varepsilon - U_2^\varepsilon$.

Proposition 5.2.2. *Let h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $p > 1$, $n \geq 2$, $\varepsilon \in [0,1)$, U_1^ε, U_2^ε be arbitrary constants, and $u_\varepsilon \in W^{1,p}(\Omega_1^\varepsilon)$ be a solution of*

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \Omega_1^\varepsilon, \\ u_\varepsilon = U_1^\varepsilon & \text{on } \Gamma_+^\varepsilon, \\ u_\varepsilon = U_2^\varepsilon & \text{on } \Gamma_-^\varepsilon. \end{cases}$$

Then there exist constants $C_1, C_2 > 0$ depending only on n, p, c_1, and c_2, such that for $x \in \Omega_{1/4}^\varepsilon$, it holds that

$$|Du_\varepsilon(x)| \leq C_1\Big(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\varepsilon + |x'|^2} + \|u_\varepsilon\|_{L^\infty(\Omega_1^\varepsilon \setminus \Omega_{1/2}^\varepsilon)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big). \tag{5.2.5}$$

Moreover, if $\varepsilon \in (0,1)$, $u_\varepsilon \in W^{1,p}(\Omega)$ is the solution to (5.1.2) and U_1^ε, and U_2^ε are the same constants in (5.1.2), we have

$$\inf_{\partial\Omega} \varphi \leq U_1^\varepsilon,\, U_2^\varepsilon \leq \sup_{\partial\Omega} \varphi, \tag{5.2.6}$$

and for $x \in \Omega^{\varepsilon}_{1/4}$,

$$|Du_{\varepsilon}(x)| \leq C_1\Big(\frac{|U_1^{\varepsilon}-U_2^{\varepsilon}|}{\varepsilon+|x'|^2}+\|\varphi\|_{L^{\infty}(\partial\Omega)}e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big). \tag{5.2.7}$$

Proof. We first give the proof of (5.2.5). Take a point $x_0 \in \Omega^{\varepsilon}_{1/4}$. In order to estimate the gradient at x_0, we first estimate the oscillation of u_{ε} in $\Omega^{\varepsilon}_{x_0,\underline{\delta}(x_0)/8}$, where $\underline{\delta}(x_0)=\varepsilon+|x_0'|^2$. Without loss of generality, we may assume that $U_1^{\varepsilon} \geq U_2^{\varepsilon}$. Let v be the solution to

$$\begin{cases} -\operatorname{div}(|Dv|^{p-2}Dv)=0 & \text{in } \Omega_1^{\varepsilon}, \\ v=0 & \text{on } \Gamma^{\varepsilon}_{\pm}, \\ v=u_{\varepsilon}-U_1^{\varepsilon} & \text{on } \partial\Omega_1^{\varepsilon}\cap\{x\in\mathbb{R}^n:|x'|=1\}. \end{cases}$$

By Lemma 5.2.1,

$$|v(x)| \leq C_1 e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\|u_{\varepsilon}\|_{L^{\infty}(\Omega_1^{\varepsilon}\setminus\Omega^{\varepsilon}_{1/2})} \quad \text{for } \ x\in\Omega^{\varepsilon}_{1/2}.$$

Since $v \geq u_{\varepsilon}-U_1^{\varepsilon}$ on $\partial\Omega_1^{\varepsilon}$, by the comparison principle, we have

$$u_{\varepsilon}(x)-U_1^{\varepsilon} \leq v(x) \quad \text{in } \ \Omega_1^{\varepsilon}.$$

Similarly, let w be the solution to

$$\begin{cases} -\operatorname{div}(|Dw|^{p-2}Dw)=0 & \text{in } \Omega_1^{\varepsilon}, \\ w=0 & \text{on } \Gamma^{\varepsilon}_{\pm}, \\ w=u_{\varepsilon}-U_2^{\varepsilon} & \text{on } \partial\Omega_1^{\varepsilon}\cap\{x\in\mathbb{R}^n:|x'|=1\}. \end{cases}$$

We have

$$|w(x)| \leq C_1 e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\|u_{\varepsilon}\|_{L^{\infty}(\Omega_1^{\varepsilon}\setminus\Omega^{\varepsilon}_{1/2})} \quad \text{for } \ x\in\Omega^{\varepsilon}_{1/2},$$

and

$$u_\varepsilon(x) - U_2^\varepsilon \geq w(x) \quad \text{in} \;\; \Omega_1^\varepsilon.$$

Therefore,

$$\underset{\Omega^\varepsilon_{x_0,\underline{\delta}(x_0)/8}}{\operatorname{osc}} u_\varepsilon \leq |U_1^\varepsilon - U_2^\varepsilon| + C_1 \|u_\varepsilon\|_{L^\infty(\Omega_1^\varepsilon \setminus \Omega_{1/2}^\varepsilon)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x_0'|}}.$$

Then the gradient estimate (5.2.5) follows from classical boundary and interior estimates for the p-Laplace equation (see e.g. [68]).

Next we prove (5.2.6). Indeed, if $U_i^\varepsilon = \max\{U_1^\varepsilon, U_2^\varepsilon\} > \sup_{\partial\Omega} \varphi$, by the maximum principle and the Hopf lemma (see [86, Theorem 5]), $Du_\varepsilon \cdot \nu > 0$ on $\partial\mathcal{D}_i^\varepsilon$, which violates $(5.1.2)_3$.

Finally, (5.2.7) follows directly from (5.2.5), (5.2.6), and the maximum principle.

□

Remark 5.2.3. For systems of p-Laplace type, instead of (5.2.6), one can still show that

$$|U_i^\varepsilon| \leq C \|\varphi\|_{C^1(\partial\Omega)}, \quad i = 1, 2$$

holds for some ε-independent constant C, by using classical boundary and interior gradient estimates away from the neck region $\Omega_{1/2}^\varepsilon$. However, it is not clear to us whether (5.2.7) (or a weaker version of it) is still true.

In the following, we justify the equivalence between the minimizing problem (5.1.3) with $\varepsilon = 0$ and the equations (5.1.4)-(5.1.5).

Theorem 5.2.4. *u_0 is the minimizer of (5.1.3) with $\varepsilon = 0$ if and only if $u_0 \in W^{1,p}(\Omega)$ satisfies (5.1.4) when $p \geq (n+1)/2$ and satisfies (5.1.5) when $p < (n+1)/2$.*

Proof. First, we prove that (5.1.4) has at most one solution $u \in W^{1,p}(\Omega)$. The same conclusion applies to (5.1.5). Let $u_1, u_2 \in W^{1,p}(\Omega)$ be two solutions of (5.1.4).

Multiplying the equation by $u_1 - u_2$ and integrating by parts, we have for $j = 1, 2$,

$$\begin{aligned}
0 &= \int_{\widetilde{\Omega}^0} |Du_j|^{p-2} Du_j \cdot D(u_1 - u_2)\, dx - \int_{\partial\Omega} |Du_j|^{p-2} Du_j \cdot \nu (u_1 - u_2)\, dS \\
&\quad - \sum_{i=1}^{2} \int_{\partial D_i^0} |Du_j|^{p-2} Du_j \cdot \nu (u_1 - u_2)\, dS \\
&= \int_{\widetilde{\Omega}^0} |Du_j|^{p-2} Du_j \cdot D(u_1 - u_2)\, dx.
\end{aligned}$$

Therefore,

$$\begin{aligned}
0 &= \int_{\widetilde{\Omega}^0} \Big(|Du_1|^{p-2} Du_1 - |Du_2|^{p-2} Du_2 \Big) \cdot D(u_1 - u_2)\, dx \\
&\geq \frac{\min\{1, p-1\}}{2^{p-2}} \int_{\widetilde{\Omega}^0} (|Du_1| + |Du_2|)^{p-2} |Du_1 - Du_2|^2\, dx.
\end{aligned}$$

This implies $u_1 \equiv u_2$. It is straightforward to see that the minimizer of (5.1.3) with $\varepsilon = 0$ is unique, due to the convexity of I_p and $\mathscr{A}^0$. It suffices to show that the minimizer u_0 satisfies (5.1.4) when $p \geq (n+1)/2$ and satisfies (5.1.5) when $p < (n+1)/2$. We show this by taking different test function $v \in \mathscr{A}^0$ in the equation

$$0 = \frac{d}{dt} I_p[u_0 + tv]\Big|_{t=0}. \tag{5.2.8}$$

First we take $v \in C_c^\infty(\widetilde{\Omega}^0)$. Then (5.2.8) reads as

$$0 = \int_{\widetilde{\Omega}^0} |Du_0|^{p-2} Du_0 \cdot Dv\, dx.$$

This implies

$$-\operatorname{div}(|Du_0|^{p-2} Du_0) = 0 \quad \text{in } \widetilde{\Omega}^0.$$

Next, we take $v \in C_c^\infty(\Omega)$ such that $v = 1$ in $\overline{\mathcal{D}_1^0 \cup \mathcal{D}_2^0}$. From (5.2.8) and integration by parts, we have

$$\begin{aligned}0 &= \int_{\widetilde{\Omega}^0} |Du_0|^{p-2} Du_0 \cdot Dv \, dx \\ &= - \int_{\widetilde{\Omega}^0} \operatorname{div}(|Du_0|^{p-2} Du_0) v \, dx + \sum_{i=1}^{2} \int_{\partial \mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu \, dS \\ &= \sum_{i=1}^{2} \int_{\partial \mathcal{D}_i^0} |Du_0|^{p-2} Du_0 \cdot \nu \, dS.\end{aligned}$$

For the case when $p \geq (n+1)/2$, it remains to show that u_0 equals to the same constant on $\overline{\mathcal{D}_1^0}$ and $\overline{\mathcal{D}_2^0}$. Assume that $u_0 = U_1$ in $\overline{\mathcal{D}_1^0}$ and $u_0 = U_2$ in $\overline{\mathcal{D}_2^0}$ with $U_1 \neq U_2$. Then by the fundamental theorem of calculus,

$$U_1 - U_2 = \int_{h_2(x')}^{h_1(x')} D_n u_0(x) \, dx_n.$$

Taking the absolute value and raising to the power of p on the both sides, by Hölder's inequality, (5.1.7), and (5.1.9), we have

$$|U_1 - U_2|^p \leq C |x'|^{2(p-1)} \int_{h_2(x')}^{h_1(x')} |D_n u_0(x)|^p \, dx_n.$$

This implies

$$\int_{|x'|<1/2} \frac{|U_1 - U_2|^p}{|x'|^{2(p-1)}} \, dx' \leq C \int_{\Omega_{1/2}^0} |Du_0|^p \, dx.$$

The left-hand-side diverges since $p \geq (n+1)/2$ and $U_1 \neq U_2$, which leads to a contradiction. Therefore, $U_1 = U_2$.

For the case when $p < (n+1)/2$, we need to show the flux on each of $\partial \mathcal{D}_i^0$ vanishes. We will only show the flux vanishes on $\partial \mathcal{D}_1^0$, as a similar argument applies to $\partial \mathcal{D}_2^0$. Let v be a function compactly supported in Ω such that $v = 1$ in $\overline{\mathcal{D}_1^0}$, $v = 0$ in $\overline{\mathcal{D}_2^0}$,

$$v(x', x_n) = \left(\frac{2x_n - (h_1(x') + h_2(x'))}{h_1(x') - h_2(x')} \right)_+, \quad x \in \Omega_{1/2}^0,$$

and v is smooth in $\widetilde{\Omega}^0 \setminus \Omega^0_{1/2}$. Then $v \in \mathscr{A}^0$. Indeed, we only need to verify

$$\int_{\Omega^0_{1/2}} |Dv|^p \, dx \leq C \int_{|x'|<1/2} \int_{\frac{h_1(x')+h_2(x')}{2}}^{h_1(x')} \frac{1}{|x'|^{2p}} \, dx_n dx' \leq C \int_0^{1/2} r^{n-2p} \leq C$$

since $n - 2p > -1$. Taking this v in (5.2.8) and integrating by parts, we have

$$\int_{\partial \mathcal{D}^0_1} |Du_0|^{p-2} Du_0 \cdot \nu \, dS = 0.$$

The theorem is proved. $\square$

Next, we show that u_ε converges to u_0 in the following sense.

Theorem 5.2.5. *Let $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (5.1.2), and $u_0 \in W^{1,p}(\Omega)$ be the minimizer of (5.1.3) with $\varepsilon = 0$. Then as $\varepsilon \to 0$, $u_\varepsilon \rightharpoonup u_0$ weakly in $W^{1,p}(\Omega)$, and $u_\varepsilon \to u_0$ strongly in $C^{1,\beta}(K)$ for some $\beta > 0$ and any*

$$K \subset\subset \Omega \setminus \Big(\bigcup_{0<\varepsilon\leq\varepsilon_0} (\mathcal{D}^\varepsilon_1 \cup \mathcal{D}^\varepsilon_2) \cup \{0\} \Big) \quad \textit{with } \varepsilon_0 > 0.$$

As a consequence, as $\varepsilon \to 0$, $U^\varepsilon_1 \to U_1$ and $U^\varepsilon_2 \to U_2$ when $p < (n+1)/2$, and $U^\varepsilon_1, U^\varepsilon_2 \to U_0$ when $p \geq (n+1)/2$, where U_0, U_1, U_2 are the constants in (5.1.4) and (5.1.5).

Proof. First we take an arbitrary function $w \in W^{1,p}(\Omega)$ such that $w = \varphi$ on $\partial\Omega$ and $Dw = 0$ in $\mathcal{B}$, where $\mathcal{B} \subset \Omega$ is an open set containing $\cup_{0\leq\varepsilon\leq c} \overline{\mathcal{D}^\varepsilon_1 \cup \mathcal{D}^\varepsilon_2}$, where $0 < c < \mathrm{dist}(\mathcal{D}^0_1 \cup \mathcal{D}^0_2, \partial\Omega)$. Therefore, $w \in \mathscr{A}^\varepsilon$ for all $\varepsilon \in [0, c]$, where $\mathscr{A}^\varepsilon$ is the admissible set defined in (5.1.3). By Theorem 5.2.4,

$$\|Du_\varepsilon\|_{L^p(\Omega)} \leq \|Dw\|_{L^p(\Omega)}.$$

This together with the Poincaré inequality implies that $\|u_\varepsilon\|_{W^{1,p}(\Omega)}$ is bounded uniformly in ε. Then there exists a subsequence $\{\varepsilon_j\}_{j\in\mathbb{N}}$ and a function $u_* \in W^{1,p}(\Omega)$, such that $u_{\varepsilon_j} \rightharpoonup u_*$ weakly in $W^{1,p}(\Omega)$, and $u_{\varepsilon_j} \to u_*$ strongly in $L^p(\Omega)$, as $j \to \infty$. From (5.2.6), we know that $U_1^{\varepsilon_j}$ and $U_2^{\varepsilon_j}$, the values of u_{ε_j} in $\mathcal{D}_1^{\varepsilon_j}$ and $\mathcal{D}_2^{\varepsilon_j}$, are uniformly bounded. Then there exists a subsequence of $\{\varepsilon_j\}_{j\in\mathbb{N}}$, still denoted by $\{\varepsilon_j\}_{j\in\mathbb{N}}$, so that $U_i^{\varepsilon_j} \to U_i$ for some constants U_i, $i = 1, 2$. Therefore, for any $x \in \mathcal{D}_i^0$, we have

$$u_*(x) = \lim_{j\to\infty} U_i^{\varepsilon_j} = U_i, \quad i = 1, 2.$$

On the other hand, for any $k, l \in \mathbb{Z}_+$, we denote

$$\mathcal{K}_{k,l} := \Omega \setminus \Big(\bigcup_{0<\varepsilon\leq 1/k} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup B_{1/l}(0) \Big).$$

By the classical $C^{1,\alpha}$ estimate, when $\varepsilon_j \leq 1/k$, we have

$$\|u_{\varepsilon_j}\|_{C^{1,\alpha}(\overline{\mathcal{K}}_{k,l})} \leq C(k, l).$$

This implies that there is a subsequence that converges in $C^{1,\beta}(\overline{\mathcal{K}}_{k,l})$ for any $\beta < \alpha$. We can apply the Cantor diagonal argument to select a subsequence, still denoted by $\{\varepsilon_j\}_{j\in\mathbb{N}}$, such that

$$u_{\varepsilon_j} \to u_{**} \ \text{ in } \ C^{1,\beta}(K) \quad \text{as} \ \ j \to \infty \tag{5.2.9}$$

for any $K \subset\subset \Omega \setminus \Big(\bigcup_{0<\varepsilon\leq\varepsilon_0} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup \{0\} \Big)$ with $\varepsilon_0 > 0$ and some $C^{1,\beta}$ function u_{**}. Therefore, u_{**} is a weak solution to the p-Laplace equation in $\widetilde{\Omega}^0$. Since $u_\varepsilon \to u_*$ strongly in $L^p(\Omega)$, we have $u_* \equiv u_{**}$. It remains to show that $u_* \equiv u_0$, which implies the convergence of $\{u_\varepsilon\}$.

When $p \geq (n + 1)/2$, by the same argument as in the proof of Theorem 5.2.4, we know that $U_1 = U_2$. It remains to prove that

$$\int_{\partial \mathcal{D}_1^0 \cup \partial \mathcal{D}_2^0} |Du_*|^{p-2} Du_* \cdot \nu = 0. \tag{5.2.10}$$

Let Ω' be an open set such that $\mathcal{D}_1^0 \cup \mathcal{D}_2^0 \subset\subset \Omega' \subset\subset \Omega$. Since u_{ε_j} is the solution to (5.1.2), when j is sufficiently large, by integration by parts in $\Omega' \setminus \overline{\mathcal{D}_1^{\varepsilon_j} \cup \mathcal{D}_2^{\varepsilon_j}}$, we have

$$\int_{\partial \Omega'} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0.$$

Therefore, by (5.2.9), we also have

$$\int_{\partial \Omega'} |Du_*|^{p-2} Du_* \cdot \nu = 0. \tag{5.2.11}$$

Since u_* is a weak solution to the p-Laplace equation in $\widetilde{\Omega}^0$, by integration by parts again in $\Omega' \setminus \overline{\mathcal{D}_1^0 \cup \mathcal{D}_2^0}$, (5.2.11) directly implies (5.2.10).

When $p < (n+1)/2$, it remains to prove that

$$\int_{\partial \mathcal{D}_i^0} |Du_*|^{p-2} Du_* \cdot \nu = 0, \quad i = 1, 2.$$

We will prove it only for $i = 1$. Fix a small $s \in (0, 1/2)$, we take a smooth surface η so that $\mathcal{D}_1^\varepsilon$ is surrounded by $\Gamma^0_{-,s} \cup \eta$. See Figure 5.1.

Since $\int_{\partial \mathcal{D}_1^{\varepsilon_j}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0$, by integration by parts, we have

$$-\int_{\Gamma^0_{-,s}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu + \int_\eta |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu = 0.$$

Note that the minus sign appears because ν on $\Gamma^0_{-,s}$ is pointing upwards, while ν on η

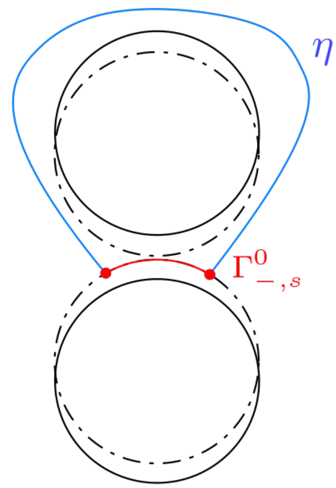

Figure 5.1: An illustration of $\Gamma^0_{-,s} \cup \eta$

is pointing away from $\mathcal{D}_1^\varepsilon$. By (5.2.7), we have $|Du_{\varepsilon_j}(x)| \le C(\varepsilon_j + |x'|^2)^{-1}$. Therefore,

$$\left| \int_{\Gamma^0_{-,s}} |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu \right| \le C \int_{|x'|<s} \frac{1}{|x'|^{2p-2}}\, dx' \le C s^{n-2p+1},$$

where we used $2p-2 < n-1$. By (5.2.9), we know that

$$\int_\eta |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu \to \int_\eta |Du_*|^{p-2} Du_* \cdot \nu \quad \text{as} \;\; j \to \infty.$$

Therefore,

$$\left| \int_\eta |Du_*|^{p-2} Du_* \cdot \nu \right| \le C s^{n-2p+1}.$$

Similarly by (5.2.5), we have $|Du_*(x)| \le C|x'|^{-2}$ and

$$\left| \int_{\Gamma^0_{-,s}} |Du_*|^{p-2} Du_* \cdot \nu \right| \le C s^{n-2p+1}.$$

By integration by parts, we have

$$\begin{aligned} \left| \int_{\mathcal{D}_1^0} |Du_*|^{p-2} Du_* \cdot \nu \right| &= \left| \int_\eta |Du_*|^{p-2} Du_* \cdot \nu - \int_{\Gamma^0_{-,s}} |Du_*|^{p-2} Du_* \cdot \nu \right| \\ &\le C s^{n-2p+1}. \end{aligned}$$

Sending $s \to 0$ and using $p < (n+1)/2$, we have

$$\int_{\mathcal{D}_1^0} |Du_*|^{p-2} Du_* \cdot \nu = 0.$$

Finally, by the uniqueness of solution to (5.1.4) and (5.1.5), we can conclude that $u_* \equiv u_0$, and the full sequence u_ε converges to u_0 in the corresponding topology. $\square$

5.3 Mean oscillation estimates

In this section, we give the proof of Proposition 5.1.5 using mean oscillation estimates. Throughout this section, unless otherwise specified, we use C to denote positive constants depending only on n, p, c_1, and c_2, which could differ from line to line. Here c_1 and c_2 are the same constants in (5.1.8) and (5.1.9), respectively. For simplicity, we denote $u := u_\varepsilon$ and we omit the superscript ε throughout this section when there is no confusion.

First, we fix a point $\bar{x} \in \Omega_{1/2}$ and derive some mean oscillation estimates of Du on a ball intersecting Ω_1, namely $\Omega_r(\bar{x})$, for different radii r.

5.3.1 Mean oscillation estimates for small r

We recall a classical interior mean oscillation estimate when $B_r(\bar{x}) \subset \Omega_1$. Estimates of this type, with different exponents involved, were developed in [24, 38, 69].

Lemma 5.3.1. *Let $u \in W^{1,p}(\Omega_1)$ be a solution to (5.1.2). There exist constants $C > 1$ and $\alpha \in (0,1)$ depending only on n and p, such that $u \in C^{1,\alpha}(\Omega_1)$ and for every $B_r(\bar{x}) \subset \Omega_1$ and $\rho \in (0, r]$, we have*

$$\phi(\bar{x}, \rho) \le C \left(\frac{\rho}{r}\right)^{\alpha} \phi(\bar{x}, r),$$

where we denote

$$\phi(\bar{x}, r) = \left(\fint_{\Omega_r(\bar{x})} |Du - (Du)_{\Omega_r(\bar{x})}|^p \right)^{\frac{1}{p}}. \tag{5.3.1}$$

5.3.2 Mean oscillation estimates for intermediate r

Next, we consider the case when $B_r(\bar{x})$ intersects with only one of Γ_+ and Γ_-. In this case, we derive mean oscillation estimates around any $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$.

Without loss of generality, let $\hat{x} \in \Gamma_- \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$. Then by (5.1.8) and (5.1.9), there exists a constant $c = c(n, c_1, c_2) \in (0, \min\{c_1/4, 1/4\})$, such that $B(\hat{x}, r) \cap \Gamma_+ = \emptyset$ for any $r \in (0, c\underline{\delta}(\hat{x}))$. Here we recall (5.1.10). We first choose a coordinate system $y = (y', y_n)$ such that $y(\hat{x}) = 0$, the direction of axis y_n is the normal vector at $\hat{x} \in \Gamma_-$ pointing upwards. Note that the coordinate y is a rotation (plus a transition) of the coordinate x, namely $y = T(x - \hat{x})$ for some rotation matrix $T \in \mathbb{R}^{n \times n}$, which maps the normal vector of Γ_- at $\hat{x}$ pointing upward, namely $\nu = (-Dh_2(\hat{x}'), 1)/\sqrt{1 + |Dh_2(\hat{x}')|^2}$ in the x-coordinate, to the unit vector $\mathbf{e}_{y_n} = (0, \ldots, 0, 1)$ in y-coordinate. Therefore, by (5.1.7) and (5.1.9), there exists a constant $C = C(n, c_2) > 0$, such that

$$|T - I_n| \leq C|\hat{x}'| \quad \text{and} \quad |T^{-1} - I_n| \leq C|\hat{x}'|. \tag{5.3.2}$$

Thus there exists a constant $c_3 = c_3(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\})$ such that $\Omega_{R_0}(\hat{x}) = \{\hat{x} + T^{-1}y : y \in B_{R_0} : y_n > \chi(y')\}$, where $R_0 = c_3\underline{\delta}(\hat{x}) \in (0, 1/4)$ and $\chi : \{y' \in \mathbb{R}^{n-1} : |y'| < R_0\} \to \mathbb{R}$ is a C^2 function in the y-coordinate system such that

$$\chi(0') = 0, \quad D_{y'}\chi(0') = 0, \quad \|\chi\|_{C^2} \leq C\|h_2\|_{C^2}, \tag{5.3.3}$$

for some constant $C = C(n) > 0$. Then we let

$$z = \Lambda(y) = (y',\, y_n - \chi(y')).$$

Since Γ_- is C^2, by (5.3.3) there exist constants

$$c_4 = c_4(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\}), \tag{5.3.4}$$

$C = C(n, c_1, c_2) > 0$, and $R_1 = c_4\underline{\delta}(\hat{x})$ such that

$$|D_{y'}\chi(y')| \le C\,|y'| \le 1/2 \quad \text{if} \quad |y'| \le 2R_1,$$

$$\Omega_{r/2}(\hat{x}) \subset \Lambda^{-1}(B_r^+) \subset \Omega_{2r}(\hat{x}) \quad \forall\, r \in (0, 2R_1],$$

and thus

$$|D\Lambda(y) - I_n| \le C\,|y'| \le 1/2 \quad \text{if} \quad |y'| \le 2R_1. \tag{5.3.5}$$

Therefore, there exist positive constants $c(n)$ and $c'(n)$ depending only on n, such that for any $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \le 3/4\}$ and $0 < r \le c_4\underline{\delta}(\hat{x})$,

$$c(n)r^n \le |\Omega_r(\hat{x})| \le c'(n)r^n. \tag{5.3.6}$$

Note that

$$\det(D\Lambda) \equiv 1.$$

Then $u_1(z) := u(\Lambda^{-1}(z))$ satisfies the following equation with constant Dirichlet boundary condition

$$
\begin{cases} -\operatorname{div}_z \left(|A^T D_z u_1|^{p-2} A A^T D_z u_1 \right) = 0 & \text{in } B_{R_1}^+, \\ \qquad\qquad\qquad\qquad\qquad\quad u_1 = U_2^\varepsilon & \text{on } B_{R_1} \cap \partial \mathbb{R}_+^n, \end{cases} \tag{5.3.7}
$$

where we denote

$$
A := A(z) := (a_{ij}(z)) := D\Lambda(\Lambda^{-1}(z)).
$$

Next we extend the equation to the whole ball B_{R_1}. We take the even extension of a_{nn} and $a_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $z_n = 0$, and take the odd extension of a_{in} and $a_{ni}, i = 1, 2, \ldots, n-1$, with respect to $z_n = 0$. Then we reflect u_1 with respect to $z_n = 0$. Namely, we define $u_1(z) = 2\, U_2^\varepsilon - u_1(z', -z_n)$ for $z \in B_{R_1}^-$. We still denote these functions by u_1 and A after the extension. Because of the Dirichlet boundary condition, it is easily seen that u_1 satisfies

$$
-\operatorname{div}_z \left(\mathbf{A}(z, D_z u_1) \right) = \ 0 \quad \text{in } B_{R_1}, \tag{5.3.8}
$$

where the nonlinear operator $\mathbf{A}$ is defined as

$$
\mathbf{A}(z, \xi) = |A^T \xi|^{p-2} A A^T \xi \qquad \text{for } z \in B_{R_1},\ \xi \in \mathbb{R}^n.
$$

By (5.3.5), similar to [33, Lemma 2.3], there exists a constant $C = C(n, p, c_1, c_2) > 0$, such that for any $z \in B_{R_1}$ and $\xi \in \mathbb{R}^n$,

$$
|\mathbf{A}(z, \xi) - |\xi|^{p-2} \xi| \le C\, |z'|\, |\xi|^{p-1}. \tag{5.3.9}
$$

Assume that $r \in (0, R_1]$. We let $v_1 \in u_1 + W_0^{1,p}(B_r)$ be the unique solution to

$$\begin{cases} -\operatorname{div}_z(|D_z v_1|^{p-2} D_z v_1) = 0 & \text{in } B_r, \\ v_1 = u_1 & \text{on } \partial B_r. \end{cases} \tag{5.3.10}$$

By testing (5.3.10) and (5.3.8) with $v_1 - u_1$ and using (5.3.9), we have the comparison estimate

$$\fint_{B_r} |D_z u_1 - D_z v_1|^p \le C r^{\min\{2,p\}} \fint_{B_r} |D_z u_1|^p, \tag{5.3.11}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. For detailed proof of (5.3.11), see [38, Eq. (4.35)] when $p \in (1, 2)$ and [39, Lemma 3.4] when $p \ge 2$.

Applying Lemma 5.3.1 and the comparison estimate (5.3.11), we have

Lemma 5.3.2. *Suppose that $u_1 \in W^{1,p}(B^+_{R_1})$ is a solution to (5.3.7). Then for any $\mu \in (0, 1)$ and $r \in (0, R_1]$, we have*

$$\begin{aligned} &\left(\fint_{B^+_{\mu r}} |D_{z'} u_1|^p + |D_{z_n} u_1 - (D_{z_n} u_1)_{B^+_{\mu r}}|^p \right)^{1/p} \\ &\le C\mu^\alpha \left(\fint_{B^+_r} |D_{z'} u_1|^p + |D_{z_n} u_1 - (D_{z_n} u_1)_{B^+_r}|^p \right)^{1/p} + C_\mu r^{\theta_p} \left(\fint_{B^+_r} |D_z u_1|^p \right)^{1/p}, \end{aligned} \tag{5.3.12}$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 5.3.1, C_μ is a constant depending on μ, n, p, c_1, and c_2, and C is a constant depending on n, p, c_1, and c_2.

Proof. By Lemma 5.3.1, (5.3.11), and the triangle inequality, we have

$$\begin{aligned}
&\left(\fint_{B_{\mu r}} |D_z u_1 - (D_z u_1)_{B_{\mu r}}|^p\right)^{1/p} \\
&\le C\left(\fint_{B_{\mu r}} |D_z v_1 - (D_z v_1)_{B_{\mu r}}|^p\right)^{1/p} + C\left(\fint_{B_{\mu r}} |D_z u_1 - D_z v_1|^p\right)^{1/p} \\
&\le C\mu^{\alpha}\left(\fint_{B_r} |D_z v_1 - (D_z v_1)_{B_r}|^p\right)^{1/p} + C\mu^{-\frac{n}{p}}\left(\fint_{B_r} |D_z u_1 - D_z v_1|^p\right)^{1/p} \\
&\le C\mu^{\alpha}\left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p\right)^{1/p} + C\mu^{-\frac{n}{p}}\left(\fint_{B_r} |D_z u_1 - D_z v_1|^p\right)^{1/p} \\
&\le C\mu^{\alpha}\left(\fint_{B_r} |D_z u_1 - (D_z u_1)_{B_r}|^p\right)^{1/p} + C_\mu r^{\theta_p}\left(\fint_{B_r} |D_z u_1|^p\right)^{1/p}.
\end{aligned} \tag{5.3.13}$$

Since u_1 is even in z_n, (5.3.13) directly implies (5.3.12). The proof is completed. $\square$

We now define

$$\psi(\hat{x}, r) = \left(\fint_{\Omega_r(\hat{x})} |D_{y'} u|^p + |D_{y_n} u - (D_{y_n} u)_{\Omega_r(\hat{x})}|^p\right)^{1/p}. \tag{5.3.14}$$

Following a similar argument as in the proof of [33, Lemma 2.5], we have

Lemma 5.3.3. *Suppose that u is a solution to (5.1.2) and $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \le 3/4\}$. Then there exist constants $C > 0$ depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0, 1/4)$ and $r \in (0,\, c_4\underline{\delta}(\hat{x})]$, it holds that*

$$\psi(\hat{x}, \mu r) \le C\mu^{\alpha}\psi(\hat{x}, r) + C_\mu r^{\theta_p}\left(\fint_{\Omega_r(\hat{x})} |Du|^p\right)^{1/p}, \tag{5.3.15}$$

where $\theta_p = \min\{1, 2/p\}$, $\alpha \in (0,1)$ is the same constant as in Lemma 5.3.1, $c_4 = c_4(n, c_1, c_2) \in (0, \min\{c_1/8, 1/8\})$ is the same constant as in (5.3.4) and ψ is defined in (5.3.14).

By iteration, Lemma 5.3.3 also implies

Corollary 5.3.4. *Let u, $\hat{x}$, c_4, α, and θ_p be as in Lemma 5.3.3 and $\alpha_1 \in (0, \min\{\alpha, \theta_p\})$. Then there exists a constant $C > 0$ depending only on n, p, c_1, c_2, and α_1, such that for any $0 < \rho \le r \le c_4 \underline{\delta}(\hat{x})$, it holds that*

$$\psi(\hat{x}, \rho) \le C \left(\frac{\rho}{r}\right)^{\alpha_1} \psi(\hat{x}, r) + C\, \rho^{\alpha_1} \|Du\|_{L^\infty(\Omega_r(\hat{x}))}.$$

Proof. We choose $\mu = \mu(n, p, c_1, c_2, \alpha_1) \in (0, 1/6)$ sufficiently small such that $C\mu^{\alpha - \alpha_1} < 1$, where C is the same constant in (5.3.15). Then Lemma 5.3.3 implies that for any $r \in (0,\ c_4 \underline{\delta}(\hat{x})]$, we have

$$\psi(\hat{x}, \mu r) \le \mu^{\alpha_1} \psi(\hat{x}, r) + C_{\alpha_1} r^{\theta_p} \|Du\|_{L^\infty(\Omega_r(\hat{x}))}, \tag{5.3.16}$$

where $C_{\alpha_1} > 0$ is a constant depending only on n, p, c_1, c_2, and α_1. By iteration, from (5.3.16) we get

$$\begin{aligned} \psi(\hat{x}, \mu^j r) &\le \mu^{\alpha_1 j} \psi(\hat{x}, r) + C_{\alpha_1} \sum_{i=1}^{j} \mu^{\alpha_1 (i-1)} (\mu^{j-i} r)^{\theta_p} \|Du\|_{L^\infty(\Omega_r(\hat{x}))} \\ &= \mu^{\alpha_1 j} \psi(\hat{x}, r) + C_{\alpha_1} \sum_{i=1}^{j} \mu^{\alpha_1 (j-1)} \mu^{(j-i)(\theta_p - \alpha_1)} r^{\theta_p} \|Du\|_{L^\infty(\Omega_r(\hat{x}))} \\ &\le \mu^{\alpha_1 j} \psi(\hat{x}, r) + C_{\alpha_1} (\mu^j r)^{\alpha_1} \|Du\|_{L^\infty(\Omega_r(\hat{x}))}. \end{aligned} \tag{5.3.17}$$

Here in the last inequality we used the facts that $\alpha_1 < \theta_p$ and $r \in (0, 1)$.

Now for any $0 < \rho \le r \le c_4 \underline{\delta}(\hat{x})$, let j be the integer such that $\mu^{j+1} < \rho/r \le \mu^j$. Then by (5.3.17) with $\mu^{-j}\rho$ in place of r, we get

$$\begin{aligned} \psi(\hat{x}, \rho) &\le \mu^{\alpha_1 j} \psi(\hat{x}, \mu^{-j}\rho) + C_{\alpha_1}\, \rho^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\mu^{-j}\rho}(\hat{x}))} \\ &\le C_{\alpha_1} \left(\frac{\rho}{r}\right)^{\alpha_1} \psi(\hat{x}, r) + C_{\alpha_1}\, \rho^{\alpha_1} \|Du\|_{L^\infty(\Omega_r(\hat{x}))}, \end{aligned}$$

where $C_{\alpha_1} > 0$ is a constant depending only on n, p, c_1, c_2, and α_1. The proof is completed. $\square$

5.3.3 Mean oscillation estimates for large r

Finally, we consider the case when $B_r(\bar{x})$ could potentially intersects with both Γ_+ and Γ_-. In this case, we fix $\bar{x} \in \Omega_{1/2}$ and assume $\frac{c_4}{216}\underline{\delta}(\bar{x}) \le r \le c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$, where $\underline{\delta}(\bar{x})$ is defined in (5.1.10), c_4 is the same constant as in (5.3.4), and c_5 is a constant which will be determined later. We define the map $\mathcal{Z} = \tilde{\Lambda}_{\bar{x}}(x)$ by

$$\begin{cases} \mathcal{Z}' = x', \\ \mathcal{Z}_n = (h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big(\dfrac{x_n - h_2(x') + \varepsilon/2}{h_1(x') - h_2(x') + \varepsilon} - \dfrac{1}{2}\Big). \end{cases} \tag{5.3.18}$$

Thus $\tilde{\Lambda}_{\bar{x}}$ is invertible in Ω_1,

$$Q_1 := \tilde{\Lambda}_{\bar{x}}(\Omega_1) = \Big\{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1, |\mathcal{Z}_n| < \frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big\},$$

and

$$\tilde{\Gamma}_\pm := \tilde{\Lambda}_{\bar{x}}(\Gamma_\pm) = \Big\{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1, \mathcal{Z}_n = \pm\frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)\Big\}.$$

Then $u_2(\mathcal{Z}) := u(\tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z}))$ satisfies the following equation with constant Dirichlet boundary conditions

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}}\left(|B^T D_{\mathcal{Z}} u_2|^{p-2}(\det(B))^{-1} B B^T D_{\mathcal{Z}} u_2\right) = 0 \quad \text{in } Q_1, \\ u_2 = U_1^\varepsilon \quad \text{on } \tilde{\Gamma}_+, \quad u_2 = U_2^\varepsilon \quad \text{on } \tilde{\Gamma}_-, \end{cases}$$

where we denote

$$B := B_{\bar{x}} := B(\mathcal{Z}) := (b_{ij}(\mathcal{Z})) := D\tilde{\Lambda}_{\bar{x}}(\tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})).$$

For $\mathcal{Z} \in Q_1$, let $x = \tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})$. Then

$$b_{ii}(\mathcal{Z}) = 1 \quad \text{for } i \in \{1, 2, \dots, n-1\},$$

$$b_{ij}(\mathcal{Z}) = 0 \quad \text{for } i \in \{1, 2, \dots, n-1\},\ j \in \{1, 2, \dots, n\},\ i \neq j,$$

$$\begin{aligned} b_{nj}(\mathcal{Z}) = {} & \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{(h_1(x') - h_2(x') + \varepsilon)^2} \\ & \cdot \big[D_{x_j} h_2(x')\big(x_n - h_1(x') - \frac{\varepsilon}{2}\big) - D_{x_j} h_1(x')\big(x_n - h_2(x') + \frac{\varepsilon}{2}\big)\big] \end{aligned}$$

for $j \in \{1, 2, \dots, n-1\}$, and

$$b_{nn}(\mathcal{Z}) = \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{h_1(x') - h_2(x') + \varepsilon}.$$

Therefore,

$$\det(B(\mathcal{Z})) = b_{nn}(\mathcal{Z}) = \frac{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}{h_1(\mathcal{Z}') - h_2(\mathcal{Z}') + \varepsilon}$$

is a function independent of $\mathcal{Z}_n$. Assume

$$\frac{c_4}{216} \underline{\delta}(\bar{x}) \leq r \leq \frac{1}{8} \underline{\delta}(\bar{x})^{\frac{1}{2}} \tag{5.3.19}$$

and let $\bar{\mathcal{Z}} = \tilde{\Lambda}_{\bar{x}}(\bar{x})$. When $\mathcal{Z} \in \mathbb{R}^n$ satisfies $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \leq r$, by the triangle inequality and (5.3.19), we always have

$$|Z'| \leq |\bar{\mathcal{Z}}'| + r \leq |\bar{x}'| + r < 1. \tag{5.3.20}$$

Then for any $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le r$ and $x = \tilde{\Lambda}_{\bar{x}}^{-1}(\mathcal{Z})$, by the triangle inequality and (5.3.19), we have

$$|x'| \le r + |\bar{x}'| \le \big(1 + \sqrt{216/c_4}\big) r^{\frac{1}{2}} \quad \text{and} \quad |x'|^2 \ge \frac{1}{2}|\bar{x}'|^2 - r^2 \ge \frac{1}{4}(|\bar{x}'|^2 - \varepsilon).$$

Thus, using (5.1.7), (5.1.8), and (5.1.9), we infer that for $j = 1, 2, \ldots, n-1$ and some constants $C > 0$ depending only on n, p, c_1, and c_2,

$$\begin{aligned} |b_{nj}(\mathcal{Z})| &\le 2c_2 \frac{|x'|(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon)}{h_1(x') - h_2(x') + \varepsilon} \le 2c_2 \frac{|x'|(2c_2|\bar{x}'|^2 + \varepsilon)}{c_1|x'|^2 + \varepsilon} \\ &\le C|x'| \le C r^{\frac{1}{2}} \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}, \end{aligned}$$

$$\begin{aligned} |b_{nn}(\mathcal{Z}) - 1| &= \left| \frac{\int_0^1 \frac{d}{dt}\big(h_1(tx' + (1-t)\bar{x}') - h_2(tx' + (1-t)\bar{x}')\big)dt}{h_1(x') - h_2(x') + \varepsilon} \right| \\ &\le 2c_2 \frac{(|x'| + |\bar{x}'|)|x' - \bar{x}'|}{c_1|x'|^2 + \varepsilon} \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}, \end{aligned} \tag{5.3.21}$$

and similarly,

$$\Big|\big(\det(B(\mathcal{Z}))\big)^{-1} - 1\Big| = \Big|\big(b_{nn}(\mathcal{Z})\big)^{-1} - 1\Big| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}. \tag{5.3.22}$$

Therefore, when (5.3.19) holds and $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le r$, we have for some constant $C = C(n, p, c_1, c_2) > 0$,

$$|B(\mathcal{Z}) - I_n| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}. \tag{5.3.23}$$

In particular, there exists a constant

$$c_5 = c_5(n, p, c_1, c_2) \in (0, 1/8), \tag{5.3.24}$$

such that if $\frac{c_4}{216}\underline{\delta}(\bar{x}) \le c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$ and $\mathcal{Z} \in Q_1$ with $|\mathcal{Z}' - \bar{\mathcal{Z}}'| \le c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$, it also holds that

$$|B(\mathcal{Z}) - I_n| \le 1/2 \quad \text{and} \quad \left|(\det(B(\mathcal{Z})))^{-1} - 1\right| \le 1/2. \tag{5.3.25}$$

Next we extend u_2 and B to the whole cylinder $\mathcal{C}_1 := \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1\}$. We denote $H := h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon$. We take the even extension of b_{nn}, and $b_{ij}, i, j = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = H/2$, and take the odd extension of b_{in} and $b_{ni}, i = 1, 2, \ldots, n-1$, with respect to $\mathcal{Z}_n = H/2$. Then we take the periodic extension of B in the $\mathcal{Z}_n$ axis, so that the period is equal to $2H$. Then we inductively reflect u_2 with respect to $\pm(k + \frac{1}{2})H$ for $k \in \mathbb{N}$. Namely, for $\mathcal{Z} \in \mathcal{C}_1$ and $k \in \mathbb{Z}$, we define

$$u_2(\mathcal{Z}) := \begin{cases} 2kU_1^\varepsilon - 2kU_2^\varepsilon + u_2(\mathcal{Z}_n - 2kH), & \text{if } |\mathcal{Z}_n - 2kH| \le \dfrac{H}{2}, \\ (2k+2)U_1^\varepsilon - 2kU_2^\varepsilon - u_2((2k+1)H - \mathcal{Z}_n), & \text{if } |\mathcal{Z}_n - (2k+1)H| \le \dfrac{H}{2}. \end{cases}$$

Then because of the Dirichlet boundary conditions, it is easily seen that u_2 satisfies

$$-\operatorname{div}_{\mathcal{Z}}\left(\mathbf{B}(\mathcal{Z}, D_{\mathcal{Z}}u_2)\right) = 0 \quad \text{in } \mathcal{C}_1, \tag{5.3.26}$$

where the nonlinear operator $\mathbf{B}$ is defined as

$$\mathbf{B}(\mathcal{Z}, \xi) = (\det(B(\mathcal{Z})))^{-1}|B^T\xi|^{p-2}BB^T\xi \qquad \text{for } \mathcal{Z} \in \mathcal{C}_1,\, \xi \in \mathbb{R}^n,$$

and

$$(\det(B(\mathcal{Z})))^{-1} = \left(b_{nn}(\mathcal{Z})\right)^{-1} = \frac{h_1(\mathcal{Z}') - h_2(\mathcal{Z}') + \varepsilon}{h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon}.$$

Similar to (5.3.9), using (5.3.20), (5.3.22), (5.3.23), (5.3.25), and (5.3.20), we obtain that for any $r \in \left[\frac{c_4}{216}\underline{\delta}(\bar{x}), c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}\right]$, $\mathcal{Z} \in B_r(\bar{\mathcal{Z}})$, and $\xi \in \mathbb{R}^n$,

$$|\mathbf{B}(\mathcal{Z}, \xi) - |\xi|^{p-2}\xi| \le \frac{Cr}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}|\xi|^{p-1}, \tag{5.3.27}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2. Now we let $v_2 \in u_2 + W_0^{1,p}(B_r(\bar{\mathcal{Z}}))$ be the unique solution to

$$\begin{cases} -\operatorname{div}_{\mathcal{Z}} \left(|D_{\mathcal{Z}} v_2|^{p-2} D_{\mathcal{Z}} v_2\right) = 0 & \text{in } B_r(\bar{\mathcal{Z}}), \\ \hfill v_2 = u_2 & \text{on } \partial B_r(\bar{\mathcal{Z}}). \end{cases}$$

Using (5.3.27), similar to (5.3.11), we have the following comparison estimate

$$⨍_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}} u_2 - D_{\mathcal{Z}} v_2|^p \le C\Big(\frac{r}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\Big)^{\min\{2,p\}} ⨍_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}} u_2|^p, \tag{5.3.28}$$

where $C > 0$ is a constant depending only on n, p, c_1, and c_2.

For $\bar{x} \in \Omega_{1/2}$ and $r \in (0, \frac{1}{8}\underline{\delta}(\bar{x})^{\frac{1}{2}})$, we define

$$\tilde{\phi}(\bar{x}, r) = \left(⨍_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}} u_2 - (D_{\mathcal{Z}} u_2)_{B_r(\bar{\mathcal{Z}})}|^p\right)^{1/p}. \tag{5.3.29}$$

Then following the same proofs as those of Lemma 5.3.3 and Corollary 5.3.4 with (5.3.28) in place of (5.3.11), we have

Lemma 5.3.5. *Suppose that $\bar{x} \in \Omega_{1/2}$ and u_2 is a solution to (5.3.26). Then there exist constants $C > 0$ depending only on n, p, c_1, and c_2, and $C_\mu > 0$ depending on n, p, c_1, c_2, and μ, such that for any $\mu \in (0,1)$ and $r \in [\frac{c_4}{216}\underline{\delta}(\bar{x}),\, c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}]$, it holds that*

$$\tilde{\phi}(\bar{x}, \mu r) \le C\mu^\alpha \tilde{\phi}(\bar{x}, r) + C_\mu\Big(\frac{r}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\Big)^{\theta_p} \left(⨍_{B_r(\bar{\mathcal{Z}})} |D_{\mathcal{Z}} u_2|^p\right)^{1/p},$$

where $\theta_p = \min\{1, 2/p\}$, α is the same constant as in Lemma 5.3.1, c_4, c_5 are the same constants as in (5.3.4) and (5.3.24), and $\tilde{\phi}$ is defined in (5.3.29). Moreover, for any $\alpha_1 \in (0, \min\{\alpha, \theta_p\})$, there exists a constant $C > 0$ depending only on n, p, c_1, c_2, *and α_1, such that for any $\frac{c_4}{216}\underline{\delta}(\bar{x}) \leq \rho \leq r \leq c_5\underline{\delta}(\bar{x})^{\frac{1}{2}}$, it holds that*

$$\tilde{\phi}(\bar{x},\rho) \leq C\left(\frac{\rho}{r}\right)^{\alpha_1}\tilde{\phi}(\bar{x},r) + C\left(\frac{\rho}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\right)^{\alpha_1}\|D_{\mathcal{Z}}u_2\|_{L^\infty(B_r(\bar{\mathcal{Z}}))}. \tag{5.3.30}$$

5.3.4 Mean oscillation decay estimates

Now we deduce mean oscillation decay estimates by connecting the three different cases of radii r, when $\underline{\delta}(\bar{x})$ is sufficiently small. We let

$$\alpha_1 = \frac{1}{2}\min\{\alpha, \frac{2}{p}\} \tag{5.3.31}$$

and assume

$$\underline{\delta}(\bar{x}) \leq \min\big\{(\frac{c_5}{10c_4})^2, \frac{1}{4+4c_2}\big\}, \tag{5.3.32}$$

where α, c_4, and c_5 are the same constants as in Lemmas 5.3.1, 5.3.3, and 5.3.5, respectively.

Let $\bar{x} \in \Omega_{1/2}$. By (5.1.7), (5.1.9) and (5.3.32), we have

$$\mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \frac{1}{2}(h_1(\bar{x}') - h_2(\bar{x}') + \varepsilon) \leq c_2|\bar{x}'|^2 + \varepsilon \leq 1/4. \tag{5.3.33}$$

and thus

$$\mathrm{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) = \mathrm{dist}(\bar{x}, \partial\Omega_1). \tag{5.3.34}$$

Lemma 5.3.6. *Let $\bar{x} \in \Omega_{1/2}$, u be a solution to (5.1.2), and $r = \frac{1}{8}\underline{\delta}(\bar{x})^{1/2}$. Let ϕ and $\tilde{\phi}$ be defined as in (5.3.1) and (5.3.29) and let α_1 and c_4 be the same constants as in (5.3.31) and (5.3.4). Assume that (5.3.32) holds. Then there exists a constant $C > 0$ depending only on n, p, c_1, and c_2, such that the following holds:*

(i) For any $\rho \in (0, \frac{c_4}{6}\underline{\delta}(\bar{x})]$,

$$\phi(\bar{x}, \rho) \leq C\left(\frac{\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{5.3.35}$$

(ii) For any $\rho \in [\frac{c_4}{216}\underline{\delta}(\bar{x}), r]$,

$$\tilde{\phi}(\bar{x}, \rho) \leq C\left(\frac{\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{5.3.36}$$

Proof. First, we prove assertion (ii). Note that $B_r(\bar{\mathcal{Z}}) \cap Q_1 \subset \tilde{\Lambda}_{\bar{x}}(\Omega_{\bar{x},r})$. By the definition of the extended solution u_2, (5.3.36) clearly holds for $\rho \in [c_5\underline{\delta}(\bar{x})^{1/2}, r]$, where $c_5 \in (0, 1/8)$ is the same constant as in (5.3.24). On the other hand, (5.3.30) directly implies (5.3.36) for $\rho \in [\frac{c_4}{216}\underline{\delta}(\bar{x}), c_5\underline{\delta}(\bar{x})^{1/2})$.

Next, we give the proof of assertion (i). We consider the following three cases:

$$\begin{aligned}
&\operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x}),\\
&\rho < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \frac{c_4}{6}\underline{\delta}(\bar{x}),\\
&\rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x}) < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-).
\end{aligned}$$

Case 1: $\operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$. Since $\bar{x} \in \Omega_{1/2}$, by (5.3.33) and the triangle inequality, we can choose $\hat{x} \in \Gamma_+ \cup \Gamma_-$ with $|\hat{x}| \leq 3/4$, such that $\operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) = |\hat{x} - \bar{x}|$, and thus $\Omega_\rho(\bar{x}) \subset \Omega_{2\rho}(\hat{x}) \subset \Omega_{3\rho}(\bar{x})$.

Since $|\hat{x} - \bar{x}| \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$, by the triangle inequality, we have

$$|\hat{x}'|^2 \leq 2|\bar{x}'|^2 + 2\Big(\frac{c_4}{6}\Big)^2\underline{\delta}(\bar{x})^2, \quad |\bar{x}'|^2 \leq 2|\hat{x}'|^2 + 2\Big(\frac{c_4}{6}\Big)^2\underline{\delta}(\bar{x})^2,$$

which also implies

$$\frac{1}{3}\underline{\delta}(\bar{x}) \leq \underline{\delta}(\hat{x}) \leq 3\underline{\delta}(\bar{x}) \tag{5.3.37}$$

since $c_4 \in (0,1)$. By (5.3.37) and the fact that $|\hat{x}-\bar{x}| \leq \rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$, we also have

$$2|\hat{x}-\bar{x}| \leq 2\rho \leq c_4\underline{\delta}(\hat{x}). \tag{5.3.38}$$

Let $R_1 = c_4\underline{\delta}(\hat{x})$. Then $\Omega_{R_1}(\hat{x}) \subset \Omega_{\frac{3}{2}R_1}(\bar{x})$. Thus we can apply Corollary 5.3.4 at $\hat{x} \in (\Gamma_+ \cup \Gamma_-) \cap \{x \in \mathbb{R}^n : |x'| \leq 3/4\}$ and use (5.3.6) to obtain

$$\begin{aligned}\phi(\bar{x},\rho) &\leq C\psi(\hat{x},2\rho) \leq C\left(\frac{\rho}{R_1}\right)^{\alpha_1}\psi(\hat{x},R_1) + C\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{R_1}(\hat{x}))}\\ &\leq C\left(\frac{\rho}{R_1}\right)^{\alpha_1}\psi(\hat{x},R_1) + C\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}.\end{aligned} \tag{5.3.39}$$

By using (5.3.2) and the change of variables $x \to y$, we have

$$\begin{aligned}\psi(\hat{x},R_1) &\leq \left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p}\\ &\quad + C|\hat{x}'|\,\|Du\|_{L^\infty(\Omega_{R_1}(\hat{x}))}\\ &\leq \left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p}\\ &\quad + CR_1^{1/2}\,\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}.\end{aligned} \tag{5.3.40}$$

By (5.3.37), (5.3.32), and the fact that $c_5 \in (0,1/8)$, it holds that

$$\frac{c_4}{2}\underline{\delta}(\bar{x}) \leq \frac{3}{2}R_1 \leq \frac{9c_4}{2}\underline{\delta}(\bar{x}) \leq \frac{c_5}{2}\underline{\delta}(\bar{x})^{1/2} \leq \frac{1}{2}r. \tag{5.3.41}$$

Since $\Omega_{R_1}(\hat{x}) \subset \Omega_{\frac{3}{2}R_1}(\bar{x})$, by using (5.3.6), (5.3.22)–(5.3.25), (5.3.41), the change of variables $x \to \mathcal{Z}$, and the triangle inequality, we also have

$$\begin{aligned}
&\left(\fint_{\Omega_{R_1}(\hat{x})} |D_{x'}u|^p + |D_{x_n}u - (D_{x_n}u)_{\Omega_{R_1}(\hat{x})}|^p\right)^{1/p} \\
&\leq C\frac{|\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))|^{1/p}}{|\Omega_{R_1}(\hat{x})|^{1/p}}\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p} \\
&\quad + C\,\frac{R_1}{\underline{\delta}(\bar{x})^{\frac{1}{2}}}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))} \\
&\leq C\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p} \\
&\quad + CR_1^{1/2}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}.
\end{aligned} \tag{5.3.42}$$

Without loss of generality, we assume $\hat{x} \in \Gamma_-$ and thus $\hat{\mathcal{Z}} := \tilde{\Lambda}_{\bar{x}}(\hat{x}) \in \tilde{\Gamma}_-$. We denote $B_R^+(\hat{\mathcal{Z}}) := B_R(\hat{\mathcal{Z}}) \cap \{\mathcal{Z} \in \mathbb{R}^n : \mathcal{Z}_n > \hat{\mathcal{Z}}_n\}$ for any $R > 0$. By (5.3.25) and (5.3.38), we have $B_{2R_1}(\hat{\mathcal{Z}}) \subset B_{3R_1}(\bar{\mathcal{Z}}) \subset B_r(\bar{\mathcal{Z}}) \subset \mathcal{C}_1 = \{(\mathcal{Z}', \mathcal{Z}_n) \in \mathbb{R}^n : |\mathcal{Z}'| < 1\}$. Since $c_4 \leq \min\{c_1/8, 1/8\}$, by (5.1.8) and (5.3.37), we know that $B_{2R_1}(\hat{\mathcal{Z}}) \cap Q_1 \equiv B_{2R_1}^+(\hat{\mathcal{Z}})$. Again by (5.3.25), we also have $B_{R_1/2}^+(\hat{\mathcal{Z}}) \subset \tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x})) \subset B_{2R_1}^+(\hat{\mathcal{Z}})$. Therefore, by the triangle inequality and the definition of u_2,

$$\begin{aligned}
&\left(\fint_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{\tilde{\Lambda}_{\bar{x}}(\Omega_{R_1}(\hat{x}))}|^p\right)^{1/p} \\
&\leq C\left(\fint_{B_{2R_1}^+(\hat{\mathcal{Z}})} |D_{\mathcal{Z}'}u_2|^p + |D_{\mathcal{Z}_n}u_2 - (D_{\mathcal{Z}_n}u_2)_{B_{2R_1}^+(\hat{\mathcal{Z}})}|^p\right)^{1/p} \\
&= C\left(\fint_{B_{2R_1}(\hat{\mathcal{Z}})} |D_{\mathcal{Z}}u_2 - (D_{\mathcal{Z}}u_2)_{B_{2R_1}(\hat{\mathcal{Z}}_0)}|^p\right)^{1/p} \leq C\tilde{\phi}(\bar{x}, 3R_1).
\end{aligned} \tag{5.3.43}$$

Combining (5.3.39), (5.3.40), (5.3.42), and (5.3.43), we have

$$\begin{aligned}
&\phi(\bar{x}, \rho) \\
&\leq C\left(\frac{\rho}{R_1}\right)^{\alpha_1}\left(\tilde{\phi}(\bar{x}, 3R_1) + R_1^{1/2}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}\right) + C\,\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{\frac{3}{2}R_1}(\bar{x}))}.
\end{aligned} \tag{5.3.44}$$

Note that by (5.3.37), $R_1^{-\alpha_1} R_1^{1/2} \leq R_1^{-\alpha_1/2} \leq C r^{-\alpha_1}$. Thus (5.3.36) with $3R_1$ in place of ρ and (5.3.44) directly imply (5.3.35).

Case 2: $\rho < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-) \leq \frac{c_4}{6}\underline{\delta}(\bar{x})$. Let $R_2 = \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-)$. Then we can apply the estimate (5.3.35) in Case 1 with R_2 in place of ρ to obtain

$$\phi(\bar{x}, R_2) \leq C \left(\frac{R_2}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{5.3.45}$$

By (5.3.34), $B_{R_2}(\bar{x}) \subset \Omega_1$, and we can apply Lemma 5.3.1 to get

$$\phi(\bar{x}, \rho) \leq C \left(\frac{\rho}{R_2}\right)^{\alpha_1} \phi(\bar{x}, R_2). \tag{5.3.46}$$

Combining (5.3.45) and (5.3.46) yields (5.3.35).

Case 3: $\rho \leq \frac{c_4}{6}\underline{\delta}(\bar{x}) < \operatorname{dist}(\bar{x}, \Gamma_+ \cup \Gamma_-)$. Let $R_3 = \frac{c_4}{6}\underline{\delta}(\bar{x})$. Then by (5.3.34), $B_{R_3}(\bar{x}) \subset \Omega_1$. By Lemma 5.3.1, we have

$$\phi(\bar{x}, \rho) \leq C \left(\frac{\rho}{R_3}\right)^{\alpha_1} \phi(\bar{x}, R_3). \tag{5.3.47}$$

Similar to Case 1, by using (5.3.22)–(5.3.25), change of variables, and the triangle inequality, we obtain

$$\begin{aligned}\phi(\bar{x}, R_3) &\leq C \frac{|\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))|^{1/p}}{|B_{R_3}(\bar{x})|^{1/p}} \left(\fint_{\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))} |D_{\mathcal{Z}} u_2 - (D_{\mathcal{Z}} u_2)_{\tilde{\Lambda}_{\bar{x}}(B_{R_3}(\bar{x}))}|^p \right)^{1/p} \\ &\qquad + C\, \frac{R_3}{\underline{\delta}(\bar{x})^{\frac{1}{2}}} \|Du\|_{L^\infty(B_{R_3}(\bar{x}))} \\ &\leq C\tilde{\phi}(\bar{x}, 2R_3) + \frac{CR_3}{r} \|Du\|_{L^\infty(B_{R_3}(\bar{x}))}.\end{aligned} \tag{5.3.48}$$

By (5.3.36) with $2R_3$ in place of ρ, we also have

$$\tilde{\phi}(\bar{x}, 2R_3) \leq C \left(\frac{R_3}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{\bar{x},r})}. \tag{5.3.49}$$

Combining (5.3.47), (5.3.48), and (5.3.49) yields (5.3.35). The proof is completed. $\square$

It is straightforward to see the following lower bound of $|\Omega_\rho(\bar{x})|$ from the proof of Lemma 5.3.6 (assertion (i), case 1) and (5.3.6), which would be useful in the proof of Proposition 5.1.5.

Lemma 5.3.7. *Let $\bar{x} \in \Omega_{1/2}$ and $c_4 = c_4(n, c_1, c_2) \in (0,1)$ be the same constant as in (5.3.4). Assume that (5.3.32) holds. Then there exists a constant $c > 0$ depending only on n, such that for any $\rho \in (0, \frac{c_4}{6}\underline{\delta}(\bar{x}))$, it holds that*

$$|\Omega_\rho(\bar{x})| \geq c\rho^n.$$

5.3.5 Proof of Proposition 5.1.5

Now we are ready to prove Proposition 5.1.5.

Proof of Proposition 5.1.5. Let $x_0 \in \Omega_{1/4}$ and we prove the proposition around $x = x_0$. We recall $\underline{\delta}(x_0) = \varepsilon + |x_0'|^2$. Let $x \in \Omega_{x_0, \sqrt{\underline{\delta}(x_0)}/4}$. By the triangle inequality,

$$|x'|^2 \leq 2|x_0'|^2 + \underline{\delta}(x_0)/8, \quad |x_0'|^2 \leq 2|x'|^2 + \underline{\delta}(x_0)/8.$$

Therefore, for any $x \in \Omega_{x_0, \sqrt{\underline{\delta}(x_0)}/4}$, it holds that

$$\frac{\underline{\delta}(x_0)}{3} \leq \underline{\delta}(x) \leq 3\underline{\delta}(x_0). \tag{5.3.50}$$

We denote

$$c_0 := \frac{1}{3} \min\Big\{\big(\frac{c_5}{10c_4}\big)^2, \frac{1}{4 + 4c_2}\Big\},$$

where c_4 and c_5 are the same constants as in (5.3.4) and (5.3.24). Thus $c_0 > 0$ depends only on n, p, c_1, and c_2. When $\underline{\delta}(x_0) > c_0$, by (5.3.50), we can apply classical results of

Hölder regularity of the gradient for the p-Laplace equation to get (5.1.21) with $x = x_0$, and some $\beta = \beta(n,p) \in (0,1)$ and $C = C(n,p,c_1,c_2) > 0$. Let $x_1, x_2 \in \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4}$ and we denote

$$\rho := |x_1 - x_2|.$$

It suffices to show that when $\underline{\delta}(x_0) \le c_0$,

$$|Du(x_1) - Du(x_2)| \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})} \tag{5.3.51}$$

holds for some constant $C > 0$ depending only on n, p, c_1, and c_2, where $\alpha_1 \in (0,1)$ is the same constant as in (5.3.31). From now on, we assume

$$\underline{\delta}(x_0) \le c_0 = \frac{1}{3}\min\big\{(\frac{c_5}{10c_4})^2, \frac{1}{4+4c_2}\big\}. \tag{5.3.52}$$

By (5.3.50), we have

$$\frac{\underline{\delta}(x_0)}{3} \le \underline{\delta}(x_1) \le 3\underline{\delta}(x_0) \quad \text{and} \quad \frac{\underline{\delta}(x_0)}{3} \le \underline{\delta}(x_2) \le 3\underline{\delta}(x_0) \tag{5.3.53}$$

and therefore (5.3.52) also implies that (5.3.32) holds for $\bar{x} = x_1, x_2 \in \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4} \subset \Omega_{1/2}$. Thus, we can apply both Lemmas 5.3.6 and 5.3.7 with $\bar{x} = x_1$ and with $\bar{x} = x_2$.

We consider two different cases: $\rho \le \frac{c_4}{36}\underline{\delta}(x_0)$ and $\rho > \frac{c_4}{36}\underline{\delta}(x_0)$.

Case 1: $\rho \le \frac{c_4}{36}\underline{\delta}(x_0)$. By (5.3.53), we also have

$$\rho \le \frac{c_4}{12}\underline{\delta}(x_1) \quad \text{and} \quad \rho \le \frac{c_4}{12}\underline{\delta}(x_2). \tag{5.3.54}$$

For any $x \in \Omega_\rho(x_2)$, by the triangle inequality

$$
\begin{aligned}
|Du(x_1) - Du(x_2)| \leq & |Du(x_1) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x_2) - (Du)_{\Omega_\rho(x_2)}| \\
& + |Du(x) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x) - (Du)_{\Omega_\rho(x_2)}|.
\end{aligned}
$$

We then take the L^p average over $x \in \Omega_\rho(x_2) \subset \Omega_{2\rho}(x_1)$ and use Lemma 5.3.7, the Lebesgue differentiation theorem, and the triangle inequality to get

$$
\begin{aligned}
& |Du(x_1) - Du(x_2)| \\
& \leq |Du(x_1) - (Du)_{\Omega_{2\rho}(x_1)}| + |Du(x_2) - (Du)_{\Omega_\rho(x_2)}| + C\phi(x_1, 2\rho) + C\phi(x_2, \rho) \\
& \leq C \sum_{j=0}^{\infty} \phi(x_1, 2^{1-j}\rho) + C \sum_{j=0}^{\infty} \phi(x_2, 2^{-j}\rho),
\end{aligned}
\tag{5.3.55}
$$

where ϕ is the mean oscillation of Du defined in (5.3.1).

By (5.3.53) and the triangle inequality, we know that $\Omega_{x_1,r} \subset \Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2} \subset \Omega_1$, where $r = \frac{1}{8}\underline{\delta}(x_1)^{1/2}$. Since (5.3.54) holds, we can apply (5.3.35) with x_1 in place of $\bar{x}$ and $2^{1-j}\rho$ in place of ρ to obtain

$$
\begin{aligned}
& \sum_{j=0}^{\infty} \phi(x_1, 2^{1-j}\rho) \leq C \sum_{j=0}^{\infty} \left(\frac{2^{1-j}\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{x_1,r})} \\
& \leq C \left(\frac{\rho}{r}\right)^{\alpha_1} \|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})} \leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.
\end{aligned}
\tag{5.3.56}
$$

Similarly, we also have

$$
\sum_{j=0}^{\infty} \phi(x_2, 2^{-j}\rho) \leq C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{5.3.57}
$$

Combining (5.3.55), (5.3.56), and (5.3.57) yields (5.3.51).

Case 2: $\rho > \frac{c_4}{36}\underline{\delta}(x_0)$. By (5.3.53), we also have

$$\rho > \frac{c_4}{108}\underline{\delta}(x_1) \quad \text{and} \quad \rho > \frac{c_4}{108}\underline{\delta}(x_2). \tag{5.3.58}$$

With x_1 in place of $\bar{x}$ in Section 5.3.3, we denote the new coordinate by $\xi = \tilde{\Lambda}_{x_1}(x)$ and set $\xi_1 := (\xi_1', \xi_1^n) := \tilde{\Lambda}_{x_1}(x_1)$, where $\tilde{\Lambda}_{x_1}$ is defined as in (5.3.18). Similarly, with x_2 in place of $\bar{x}$, we denote another coordinate by $\eta := (\eta_2', \eta_2^n) := \tilde{\Lambda}_{x_2}(x)$ and set $\eta_2 = \tilde{\Lambda}_{x_2}(x_2)$. Let $u_2^{(1)}$ and $u_2^{(2)}$ be the extended solutions in the coordinates ξ and η defined as in Section 5.3.3, respectively. As in Section 5.3.3, we also define the mean oscillation of extended solutions in the two coordinates ξ and η by

$$\tilde{\phi}(x_1, r) = \left(\fint_{B_r(\xi_1)} |D_\xi u_2^{(1)} - (D_\xi u_2^{(1)})_{B_r(\xi_1)}|^p \right)^{1/p}$$

and

$$\tilde{\phi}(x_2, r) = \left(\fint_{B_r(\eta_2)} |D_\eta u_2^{(2)} - (D_\eta u_2^{(2)})_{B_r(\eta_2)}|^p \right)^{1/p}.$$

Let us first briefly describe our ideas to prove (5.3.51) in this case. By the triangle inequality,

$$\begin{aligned} |Du(x_1) - Du(x_2)| \le |Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| + |Du(x_2) - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}| \\ + |(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|, \end{aligned} \tag{5.3.59}$$

where $c_6 > 0$ is a constant which will be specified later. We will estimate the third term on the right-hand side of (5.3.59) using careful change of variables $\xi \to \eta$ and estimate the first two terms using iteration of Lemma 5.3.6. A delicate transition from the original coordinates x to the new coordinates ξ (or η) is also needed when the radius (inside the iteration procedure) is at the scale of $\underline{\delta}(x_0)$.

Note that by (5.3.18), $\Phi := \tilde{\Lambda}_{x_2}\tilde{\Lambda}_{x_1}^{-1}$ is indeed a dilation of the coordinate ξ in the

ξ^n direction, namely,

$$\eta = (\eta', \eta^n) = \Phi(\xi) = \big(\xi',\ \frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon}\,\xi^n\big).$$

By (5.1.7), (5.1.8), and (5.3.53), we have

$$\frac{c_1}{9c_2} \le \frac{c_1|x_2'|^2 + \varepsilon}{c_2|x_1'|^2 + \varepsilon} \le \frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} \le \frac{c_2|x_2'|^2 + \varepsilon}{c_1|x_1'|^2 + \varepsilon} \le \frac{9c_2}{c_1}. \tag{5.3.60}$$

This implies that Φ and Φ^{-1} are bounded independent of ε. Moreover, similar to (5.3.21), using (5.3.53) one can also show that

$$\begin{aligned}
&\Big|\frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} - 1\Big| \le 2c_2\frac{(|x_1'| + |x_2'|)|x_1' - x_2'|}{c_1|x_1'|^2 + \varepsilon} \le C\underline{\delta}(x_0)^{-1/2}\rho,\\
&\Big|\frac{h_1(x_1') - h_2(x_1') + \varepsilon}{h_1(x_2') - h_2(x_2') + \varepsilon} - 1\Big| \le 2c_2\frac{(|x_1'| + |x_2'|)|x_1' - x_2'|}{c_1|x_2'|^2 + \varepsilon} \le C\underline{\delta}(x_0)^{-1/2}\rho,
\end{aligned} \tag{5.3.61}$$

where $C > 0$ is a constant depending only on c_1 and c_2. Note that (5.3.61) directly implies

$$|D\Phi - I_n| \le C\underline{\delta}(x_0)^{-1/2}\rho \quad \text{and} \quad |D\Phi^{-1} - I_n| \le C\underline{\delta}(x_0)^{-1/2}\rho. \tag{5.3.62}$$

By the definitions of the extended solutions $u_2^{(1)}$ and $u_2^{(2)}$, we also know that for any $\xi, \eta \in \mathbb{R}^n$ with $|\xi'| < 1$ and $|\eta'| < 1$, we have

$$u_2^{(1)}(\xi) = u_2^{(2)}(\Phi(\xi)), \quad u_2^{(2)}(\eta) = u_2^{(1)}(\Phi^{-1}(\eta)).$$

Without loss of generality, we assume

$$\frac{h_1(x_2') - h_2(x_2') + \varepsilon}{h_1(x_1') - h_2(x_1') + \varepsilon} \ge 1,$$

and thus we have $\Phi^{-1}(B_\rho(\eta_2)) \subset B_\rho(\xi_2)$, where we denote $\xi_2 := (\xi_2', \xi_2^n) := \Phi^{-1}(\eta_2)$.

Clearly $\xi_1' = x_1'$ and $\xi_2' = x_2'$. Therefore, by using (5.1.7), (5.1.9), (5.3.58), and the triangle inequality, for any $\xi \in \Phi^{-1}(B_\rho(\eta_2))$, it holds that

$$\begin{aligned}|\xi - \xi_1| &\le |\xi - \xi_2| + |\xi_2 - \xi_1| \le \rho + |\xi_2' - \xi_1'| + |\xi_2^n - \xi_1^n| \\ &\le 2\rho + h_1(x_1') - h_2(x_1') + \varepsilon \\ &\le 2\rho + c_2|x_1'|^2 + \varepsilon \le c_6\rho,\end{aligned}$$

where $c_6 > 2$ is a constant depending only on n, c_1, and c_2. Thus $\Phi^{-1}(B_\rho(\eta_2)) \subset B_\rho(\xi_2) \subset B_{c_6\rho}(\xi_1)$.

Let $c_5 \in (0, 1/8)$ be the same constant as in Lemma 5.3.5. From now on, we assume

$$\rho \le \min\{c_5/4, 1/(64c_6)\}\underline{\delta}(x_0)^{1/2}, \tag{5.3.63}$$

since otherwise (5.3.51) clearly holds. Note that (5.3.58), (5.3.63), and (5.3.53) directly imply

$$\frac{c_4}{108}\underline{\delta}(x_2) < \rho \le c_5\underline{\delta}(x_2)^{1/2} < \frac{1}{8}\underline{\delta}(x_2)^{1/2} \tag{5.3.64}$$

and

$$\frac{c_4}{108}\underline{\delta}(x_1) < c_6\rho \le \frac{1}{16}\underline{\delta}(x_1)^{1/2}. \tag{5.3.65}$$

Now we estimate the three terms on the right-hand side of (5.3.59) separately.

We first estimate the term $|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|$ in (5.3.59). For any $\xi \in \Phi^{-1}(B_\rho(\eta_2))$, by the triangle inequality,

$$\begin{aligned}&|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}| \\ &\le |(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - D_\xi u_2^{(1)}(\xi)| + |D_\xi u_2^{(1)}(\xi) - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|.\end{aligned}$$

We then take the L^p average over $\xi \in \Phi^{-1}(B_\rho(\eta_2)) \subset B_{c_6\rho}(\xi_1)$ and use (5.3.60), (5.3.62), and the triangle inequality to get

$$
\begin{aligned}
&|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}| \\
&\le C\tilde{\phi}(x_1, c_6\rho) + \Big(\fint_{\Phi^{-1}(B_\rho(\eta_2))} |D_\xi u_2^{(1)}(\xi) - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|^p d\xi\Big)^{1/p} \\
&\le C\tilde{\phi}(x_1, c_6\rho) + \Big(\fint_{B_\rho(\eta_2)} |D_\eta u_2^{(2)}(\eta)\, D\Phi - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}|^p d\eta\Big)^{1/p} \\
&\le C\tilde{\phi}(x_1, c_6\rho) + C\tilde{\phi}(x_2, \rho) + C\underline{\delta}(x_0)^{-1/2}\rho\, \|D_\eta u_2^{(2)}\|_{L^\infty(B_\rho(\eta_2))}.
\end{aligned}
\tag{5.3.66}
$$

By (5.3.64) and (5.3.25) we know that $|D\tilde{\Lambda}_{x_2}| \le C$ in $\Omega_{x_2,\rho}$ and thus

$$
\|D_\eta u_2^{(2)}\|_{L^\infty(B_\rho(\eta_2))} \le C\|Du\|_{L^\infty(\Omega_{x_2,\rho})} \le C\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{5.3.67}
$$

By (5.3.65), we can apply (5.3.36) with x_1 in place of $\bar{x}$ and $c_6\rho$ in place of ρ, and use (5.3.53) to get

$$
\begin{aligned}
\tilde{\phi}(x_1, c_6\rho) &\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/4})} \\
&\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.
\end{aligned}
\tag{5.3.68}
$$

Similarly, we have

$$
\begin{aligned}
\tilde{\phi}(x_2, \rho) &\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_2,\sqrt{\underline{\delta}(x_0)}/4})} \\
&\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.
\end{aligned}
\tag{5.3.69}
$$

Combining (5.3.66), (5.3.67), (5.3.68) and (5.3.69), we obtain

$$
|(D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)} - (D_\eta u_2^{(2)})_{B_\rho(\eta_2)}| \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{5.3.70}
$$

Next, we estimate the term $|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}|$ in (5.3.59). We define $\rho_j := c_6 2^{-j}\rho$ for $j \in \mathbb{N}$. Because of (5.3.58), we let $j_1 \ge 1$ be the integer such that

$$
\rho_{j_1} \ge \frac{c_4}{216}\underline{\delta}(x_1), \quad \rho_{j_1+1} < \frac{c_4}{216}\underline{\delta}(x_1). \tag{5.3.71}
$$

Then by using the triangle inequality and Lemma 5.3.7, we have

$$\begin{aligned}
&|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| \\
&\le |Du(x_1) - (Du)_{\Omega_{\rho_{j_1+1}}(x_1)}| + |(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}| \\
&\quad + |(D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)} - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| \\
&\le C\sum_{j=j_1+1}^{\infty} \phi(x_1,\rho_j) + C\sum_{j=0}^{j_1} \tilde\phi(x_1,\rho_j) + |(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|.
\end{aligned} \tag{5.3.72}$$

Using Lemma 5.3.7 and Hölder's inequality, we obtain

$$\begin{aligned}
&|(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}| \\
&\le C\Big(\fint_{\Omega_{\rho_{j_1}}(x_1)} |Du(x) - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|^p dx\Big)^{1/p} \\
&\le C\Big(\fint_{\tilde\Lambda_{x_1}(\Omega_{\rho_{j_1}}(x_1))} |D_\xi u_2^{(1)}(\xi)\, B(\xi) - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}|^p d\xi\Big)^{1/p},
\end{aligned} \tag{5.3.73}$$

where as in Section 5.3.3, we denote

$$B(\xi) = D\tilde\Lambda_{x_1}(\tilde\Lambda_{x_1}^{-1}(\xi)),$$

and use (5.3.25) in the last line. By (5.3.23) with x_1 in place of $\bar x$, from (5.3.73) we deduce

$$\begin{aligned}
&|(Du)_{\Omega_{\rho_{j_1+1}}(x_1)} - (D_\xi u_2^{(1)})_{B_{\rho_{j_1}}(\xi_1)}| \le C\tilde\phi(x_1, 2\rho_{j_1}) + \frac{C\rho_{j_1}}{\underline{\delta}(x_1)^{1/2}}\|Du\|_{L^\infty(\Omega_{x_1,\rho_{j_1}})} \\
&\le C\tilde\phi(x_1,\rho_{j_1-1}) + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,c_6\rho})}.
\end{aligned} \tag{5.3.74}$$

Here in the last inequality we also used (5.3.71) and (5.3.58). Combining (5.3.72), (5.3.74) and using (5.3.63), we get

$$
\begin{aligned}
&|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_1)}| \\
&\le C\sum_{j=j_1+1}^{\infty} \phi(x_1,\rho_j) + C\sum_{j=0}^{j_1} \tilde{\phi}(x_1,\rho_j) + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})}.
\end{aligned} \tag{5.3.75}
$$

We recall (5.3.58) and (5.3.65). Therefore, by applying Lemma 5.3.6 with x_1 in place of $\bar{x}$ and ρ_j in place of ρ, and using (5.3.75) and (5.3.53), we obtain

$$
\begin{aligned}
&|Du(x_1) - (D_\xi u_2^{(1)})_{B_{c_6\rho}(\xi_2)}| \\
&\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})} + C\rho^{1/2}\|Du\|_{L^\infty(\Omega_{x_1,\sqrt{\underline{\delta}(x_0)}/4})} \\
&\le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}.
\end{aligned} \tag{5.3.76}
$$

Similarly, it also holds that

$$
|Du(x_2) - (D_\xi u_2^{(2)})_{B_\rho(\xi_2)}| \le C\underline{\delta}(x_0)^{-\alpha_1/2}\rho^{\alpha_1}\|Du\|_{L^\infty(\Omega_{x_0,\sqrt{\underline{\delta}(x_0)}/2})}. \tag{5.3.77}
$$

Combining (5.3.59), (5.3.70), (5.3.76), and (5.3.77) yields (5.3.51). The proof is completed. □

5.4 Estimates of $U_1^\varepsilon - U_2^\varepsilon$

In the following, we use the $C^{1,\beta}$ estimate we derive in Proposition 5.1.5 to obtain an asymptotic expansion of Du_ε in terms of $U_1^\varepsilon - U_2^\varepsilon$, for arbitrary constants U_1^ε and U_2^ε (Proposition 5.4.1). When $p \ge (n+1)/2$, for the specific U_1^ε and U_2^ε in (5.1.2), $U_1^\varepsilon - U_2^\varepsilon$ will converges to 0 as $\varepsilon \to 0$, and we compute in Theorem 5.4.4 the rate of convergence using information of the flux $\mathcal{F}$ defined in (5.1.6). When $p < (n+1)/2$, $U_1^\varepsilon - U_2^\varepsilon$ will converges to $U_1 - U_2$ as shown in Theorem 5.2.5.

Proposition 5.4.1. *Let h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $p > 1$, $n \ge 2$,*

$\varepsilon \in [0, 1/4)$, U_1^ε, U_2^ε be arbitrary constants with $|U_i^\varepsilon| \le \|\varphi\|_{L^\infty(\partial\Omega)}$, and $u_\varepsilon \in W^{1,p}(\widetilde{\Omega}^\varepsilon)$ be a solution of

$$\begin{cases} -\operatorname{div}(|Du_\varepsilon|^{p-2}Du_\varepsilon) = 0 & \text{in } \widetilde{\Omega}^\varepsilon, \\ \qquad\qquad\qquad u_\varepsilon = U_i^\varepsilon & \text{on } \partial\mathcal{D}_i^\varepsilon,\ i = 1, 2, \\ \qquad\qquad\qquad u_\varepsilon = \varphi & \text{on } \partial\Omega. \end{cases}$$

There exist positive constants $\beta \in (0, 1)$, C_1 and C_2 depending only on n, p, c_1, and c_2, such that

$$Du_\varepsilon(x) = \left(0', \frac{U_1^\varepsilon - U_2^\varepsilon}{\delta(x)}\right) + \mathbf{f}_1(x, \varepsilon) \quad \text{for} \ \ x \in \overline{\Omega}_{1/4}^\varepsilon, \tag{5.4.1}$$

where δ is defined in (5.1.11),

$$|\mathbf{f}_1(x, \varepsilon)| \le C_1 \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\right). \tag{5.4.2}$$

Proof. By mean value theorem, we know that for any $x = (x', x_n) \in \overline{\Omega}_{1/4}^\varepsilon$, there exist $\zeta(x') \in (-\frac{\varepsilon}{2} + h_2(x'), \frac{\varepsilon}{2} + h_1(x'))$ and $y(x) = (x', \zeta(x')) \in \Omega_{1/4}^\varepsilon$, such that

$$D_n u_\varepsilon(y(x)) = (U_1^\varepsilon - U_2^\varepsilon)/\delta(x).$$

Let

$$\mathbf{f}_1(x, \varepsilon) := (D_{x'}u_\varepsilon(x), D_n u_\varepsilon(x) - D_n u_\varepsilon(y(x))). \tag{5.4.3}$$

By Propositions 5.1.5 and 5.2.2, we have

$$\begin{aligned} |D_n u_\varepsilon(x) - D_n u_\varepsilon(y(x))| &\le C|x_n - \zeta(x')|^\beta \delta(x)^{-\beta/2} \|Du_\varepsilon\|_{L^\infty(\Omega_{x,\sqrt{\delta(x)}/2})} \\ &\le C_1 \left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\right). \end{aligned} \tag{5.4.4}$$

Let $z(x) = (x', \varepsilon/2 + h_1(x'))$. Since $u_\varepsilon \equiv U_1^\varepsilon$ on $\overline{\mathcal{D}}_1^\varepsilon$, by Proposition 5.2.2, we have

$$\begin{aligned} |D_{x'}u(z(x))| &\le C|x'||Du(z(x))| \\ &\le C_1|x'|\left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\right). \end{aligned} \tag{5.4.5}$$

Similar to (5.4.4), we also have

$$|D_{x'}u(x) - D_{x'}u(z(x))| \le C_1\left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\right). \tag{5.4.6}$$

Therefore, by (5.4.5), (5.4.6), and the triangle inequality, we have

$$|D_{x'}u(x)| \le C_1\left(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(x)} + \|\varphi\|_{L^\infty(\partial\Omega)} e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\right).$$

This completes the proof of Proposition 5.4.1. □

Following the proof of [43, Proposition 2.1] with slight modification, we have the following proposition, which will be proved in the Appendix.

Proposition 5.4.2. *Let $n \ge 2$, $p \ge (n+1)/2$, h_1, h_2 be C^2 functions satisfying (5.1.7)-(5.1.9), $\varepsilon \in (0,1)$, and $u_\varepsilon \in W^{1,p}(\Omega)$ be the solution of (5.1.2). Then there exist positive constants C_1, C_2 depending only on n, p, c_1, c_2, and $\|\varphi\|_{L^\infty}$, such that for any $r \in (0,1)$,*

$$\left|\mathcal{F} - \lim_{\varepsilon\to 0_+}\int_{\Gamma^\varepsilon_{-,r}} |Du_\varepsilon|^{p-2}Du_\varepsilon \cdot \nu\right| \le C_1 e^{-\frac{C_2}{r}}, \tag{5.4.7}$$

where $\mathcal{F}$ is given in (5.1.6).

Remark 5.4.3. The proof of Proposition 5.4.2 relies on the facts that $|Du_0| \le C_1 e^{-\frac{C_2}{r}}$ and $\|u_\varepsilon - u_0\|_{C^{1,\alpha}(K)} \to 0$ as $\varepsilon \to 0$ for any $K \subset\subset \Omega \setminus \left(\bigcup_{0<\varepsilon\le\varepsilon_0} (\mathcal{D}_1^\varepsilon \cup \mathcal{D}_2^\varepsilon) \cup \{0\}\right)$ with $\varepsilon_0 > 0$, where u_0 is the minimizer of (5.1.3) with $\varepsilon = 0$. See Proposition 5.2.2 and

Theorem 5.2.5. However, Du_0 may be unbounded when $p < (n+1)/2$. This fact was overlooked in [20, 21].

With the help of Propositions 5.2.2 and 5.4.2, we are able to derive the rate of convergence for $U_1^\varepsilon - U_2^\varepsilon$ when $\varepsilon \to 0$.

Theorem 5.4.4. *Let $p \geq (n+1)/2$, U_1^ε and U_2^ε be the constants in (5.1.2). Then it holds that*

$$\lim_{\varepsilon \to 0_+} (U_1^\varepsilon - U_2^\varepsilon)\Theta(\varepsilon)^{-1} = \operatorname{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)}, \tag{5.4.8}$$

where $\Theta(\varepsilon)$ is given in (5.1.12), K is given in (5.1.13), and $\mathcal{F}$ is given in (5.1.6).

Proof. By Proposition 5.4.1, we have

$$\left| Du_\varepsilon \cdot \nu(y) - \frac{U_1^\varepsilon - U_2^\varepsilon}{\delta(y)} \right| \leq C_1 \Big(\frac{|U_1^\varepsilon - U_2^\varepsilon|}{\delta^{1-\beta/2}(y)} + e^{-\frac{C_2}{\sqrt{\varepsilon + |y'|}}} \Big) \quad \text{for} \quad y \in \Gamma^\varepsilon_{-,1/4}, \tag{5.4.9}$$

where δ is defined in (5.1.11).

We will show that every sequence converges to the same limit.

First, Let $\{\varepsilon_j\}_{j \in \mathbb{N}}$ be a decreasing sequence such that $\varepsilon_j \to 0$ as $j \to \infty$ and $U_1^{\varepsilon_j} \geq U_2^{\varepsilon_j}$ for every $j \in \mathbb{N}$. By (5.4.9), for any $\tau \in (0, 1/2)$ and $r \in (0, 1/4)$, we have

$$\begin{aligned}
(1-\tau)\Big(\frac{U_1^{\varepsilon_j} - U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1 - C\delta_j^{\beta/2}(y)) - C(\tau)e^{-\frac{C_2}{\sqrt{\varepsilon_j + |y'|}}} &\leq |Du_{\varepsilon_j}|^{p-2} Du_{\varepsilon_j} \cdot \nu(y) \\
\leq (1+\tau)\Big(\frac{U_1^{\varepsilon_j} - U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1 + C\delta_j^{\beta/2}(y)) + C(\tau)e^{-\frac{C_2}{\sqrt{\varepsilon_j + |y'|}}} &\quad \text{for } y \in \Gamma^{\varepsilon_j}_{-,1/4},
\end{aligned}$$

where $\delta_j(y) := \varepsilon_j + h_1(y') - h_2(y')$, $C, C_2 > 0$ depends only on n, p, c_1, c_2, β, $\|\varphi\|_{L^\infty}$, and $\operatorname{dist}(\mathcal{D}_1^{\varepsilon_j} \cup \mathcal{D}_2^{\varepsilon_j}, \partial\Omega)$, and $C(\tau)$ additionally depends on τ. By the change of variables $dS = \sqrt{1+|D_{x'}h_2(y')|^2}dy'$, (5.1.7), and (5.1.9), we have for any $r < 1/4$,

$$\begin{aligned}
&(1-\tau)\int_{|y'|<r}\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1-C\delta_j^{\beta/2}(y))\sqrt{1+|D_{x'}h_2(y')|^2}\,dy' \\
&\quad - C(\tau)\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy' \\
&\le \int_{\Gamma_{-,r}^{\varepsilon_j}} |Du_{\varepsilon_j}|^{p-2}Du_{\varepsilon_j}\cdot\nu\,dS \\
&\le (1+\tau)\int_{|y'|<r}\Big(\frac{U_1^{\varepsilon_j}-U_2^{\varepsilon_j}}{\delta_j(y)}\Big)^{p-1}(1+C\delta_j^{\beta/2}(y))\sqrt{1+|D_{x'}h_2(y')|^2}\,dy' \\
&\quad + C(\tau)\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy'. \qquad (5.4.10)
\end{aligned}$$

Note that

$$\int_{|y'|<r} e^{-\frac{C_2}{\sqrt{\varepsilon_j}+|y'|}}\,dy' \le Cr^{(n-1)}. \qquad (5.4.11)$$

Moreover, for $p \ge (n+1)/2$,

$$\lim_{r\to 0_+}\lim_{\varepsilon\to 0_+}\int_{|y'|<r}\Big(\frac{\Theta(\varepsilon)}{\delta(y)}\Big)^{p-1}dy' = \frac{1}{K}. \qquad (5.4.12)$$

The verification of (5.4.12) follows from direct computations, which will be given in the Appendix (Lemma 6.2.1). By taking the limit as $j\to\infty$ in (5.4.10) first, then taking $r\to 0$ and $\tau \to 0$, from (5.4.7), (5.4.10), (5.4.11), and (5.4.12) we get

$$\lim_{j\to\infty}\big((U_1^{\varepsilon_j}-U_2^{\varepsilon_j})\Theta(\varepsilon_j)^{-1}\big)^{p-1} = K\mathcal{F}. \qquad (5.4.13)$$

Similarly, if $\{\varepsilon_j\}_{j\in\mathbb{N}}$ is a decreasing sequence such that $\varepsilon_j \to 0$ as $j\to\infty$ and $U_1^{\varepsilon_j} \le U_2^{\varepsilon_j}$ for every $j\in\mathbb{N}$, then we have

$$\lim_{j\to\infty}\big((U_2^{\varepsilon_j}-U_1^{\varepsilon_j})\Theta(\varepsilon_j)^{-1}\big)^{p-1} = -K\mathcal{F}. \qquad (5.4.14)$$

Therefore, we conclude that if $\mathcal{F} > 0$, then for any decreasing sequence $\varepsilon_j \to 0_+$, there exists $j_0 \in \mathbb{N}$ such that $U_1^{\varepsilon_j} \geq U_2^{\varepsilon_j}$ for every $j \geq j_0$, since otherwise there exists a decreasing subsequence $\{\varepsilon_{j_k}\}$ such that $U_1^{\varepsilon_{j_k}} < U_2^{\varepsilon_{j_k}}$, then from (5.4.14) we should have $\mathcal{F} \leq 0$. Thus if $\mathcal{F} > 0$, (5.4.13) implies (5.4.8). Similarly, (5.4.8) also holds when $\mathcal{F} < 0$. For the remaining case when $\mathcal{F} = 0$, let $\{\varepsilon_j\}$ be any decreasing sequence such that $\varepsilon_j \to 0$ as $j \to \infty$. Then there exists a decreasing subsequence $\{\varepsilon_{j_k}\}$ such that either $U_1^{\varepsilon_{j_k}} \leq U_2^{\varepsilon_{j_k}}$ holds for every $k \in \mathbb{N}$ or $U_1^{\varepsilon_{j_k}} \geq U_2^{\varepsilon_{j_k}}$ holds for every $k \in \mathbb{N}$. Thus (5.4.13) (or (5.4.14)) implies that

$$\lim_{k\to\infty} (U_1^{\varepsilon_{j_k}} - U_2^{\varepsilon_{j_k}})\Theta(\varepsilon_{j_k})^{-1} = 0.$$

Therefore, (5.4.8) is also true when $\mathcal{F} = 0$. The proof of the theorem is completed. □

5.5 Proofs of Theorem 5.1.1 and Proposition 5.1.4

In this section, we give the proofs of Theorem 5.1.1 and Proposition 5.1.4.

Proof of Theorem 5.1.1. First, we consider the case when $p \geq (n+1)/2$. Let

$$f_0(\varepsilon) := (U_1^\varepsilon - U_2^\varepsilon)/\Theta(\varepsilon) - \operatorname{sgn}(\mathcal{F})(K|\mathcal{F}|)^{1/(p-1)}. \tag{5.5.1}$$

By Theorem 5.4.4 and (5.4.2), we know that

$$\lim_{\varepsilon\to 0} f_0(\varepsilon) = 0,$$

and

$$|\mathbf{f}_1(x,\varepsilon)| \leq C_1\Big(\delta(x)^{\beta/2-1}\Theta(\varepsilon)(|K\mathcal{F}|^{1/(p-1)} + |f_0(\varepsilon)|) + \|\varphi\|_{L^\infty(\partial\Omega)}e^{-\frac{C_2}{\sqrt{\varepsilon}+|x'|}}\Big),$$

where $\mathbf{f}_1$ is defined as in (5.4.3). Then (5.1.15) follows from (5.4.1), (5.5.1), and the above.

Next, we consider the case when $1 < p < (n+1)/2$. We define

$$g_0(\varepsilon) := U_1^\varepsilon - U_2^\varepsilon - (U_1 - U_2),$$

and $\mathbf{g}_1(x,\varepsilon) = \mathbf{f}_1(x,\varepsilon)$ as in (5.4.3). By Theorem 5.2.5,

$$\lim_{\varepsilon\to 0} g_0(\varepsilon) = 0,$$

Similar as above, (5.1.16) follows. □

Proof of Proposition 5.1.4. For the case when $p \geq (n+1)/2$, by the symmetry of the domain and uniqueness of the solution, we know that $u_0(x', x_n) = -u_0(x', -x_n)$. Therefore, $U_0 = u(x', 0) = 0$. By the strong maximum principle, $u_0 > 0$ in $\widetilde{\Omega}^0 \cap \{x_n > 0\}$. Then by the Hopf lemma, $Du_0 \cdot \nu > 0$ on $\partial\mathcal{D}_1^0$, and hence $\mathcal{F} > 0$.

For the case when $1 < p < (n+1)/2$, we prove by contradiction. Assume $U_1 \leq U_2$. Since $u_0(x', x_n) = -u_0(x', -x_n)$, we know that $u_0(x', 0) = 0$ and $U_1 = -U_2$. Therefore, $U_1 \leq 0$ and u_0 achieves minimum in B_5^+ on $\partial\mathcal{D}_1^0$. Hence $Du_0 \cdot \nu > 0$ on $\partial\mathcal{D}_1^0$ by the Hopf lemma. This implies

$$\int_{\partial\mathcal{D}_1^0} |Du_0|^{p-2} Du_0 \cdot \nu > 0,$$

which contradicts to $(5.1.5)_3$. □

CHAPTER 6

Appendix

6.1 Proof of Proposition 5.4.2

In the first part of the appendix, we prove Proposition 5.4.2. The proof essentially follows those of [43, Proposition 2.1] and [21, Lemma 5.1]. Our estimate is sharper due to a better estimate on $|Du_0|$ (Proposition 5.2.2).

Proof of Proposition 5.4.2. Similar to the proof of Theorem 5.2.5, for small $r \in (0, 1/2)$, we take a smooth surface η so that $\mathcal{D}_1^\varepsilon$ is surrounded by $\Gamma_{-,s}^0 \cup \eta$. See Figure 5.1. We denote the surface

$$\Sigma_r^\varepsilon := \left\{ x \in \mathbb{R}^n : |x'| = r, -\frac{\varepsilon}{2} + h_2(x') < x_n < h_2(x') \right\}.$$

Since $\int_{\partial \mathcal{D}_1^0} |Du_0|^{p-2} Du_0 \cdot \nu = \mathcal{F}$ and $\int_{\partial \mathcal{D}_1^\varepsilon} |Du_\varepsilon|^{p-2} Du_\varepsilon \cdot \nu = 0$, by integration by parts, we have

$$-\int_{\Gamma_{-,r}^0} |Du_0|^{p-2} Du_0 \cdot \nu + \int_\eta |Du_0|^{p-2} Du_0 \cdot \nu = \mathcal{F}, \tag{6.1.1}$$

and

$$-\int_{\Gamma^{\varepsilon}_{-,r}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu+\int_{\Sigma^{\varepsilon}_{r}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu+\int_{\eta}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu=0. \tag{6.1.2}$$

Note that the minus signs appear because ν on $\Gamma^0_{-,r}$ and $\Gamma^\varepsilon_{-,r}$ are defined to be pointing upwards, while ν on η and Σ^ε_r are pointing away from $\mathcal{D}^\varepsilon_1$. By (5.2.5), we have $|Du_0(x)|\le C_1e^{-\frac{C_2}{r}}$ in $\overline{\Omega^0_r}$, and hence

$$\left|\int_{\Gamma^{0}_{-,r}}|Du_0|^{p-2}Du_0\cdot\nu\right|\le C_1e^{-\frac{C_2}{r}} \tag{6.1.3}$$

for some positive ε-independent constants C_1 and C_2. By Theorem 5.2.5, we have

$$\lim_{\varepsilon\to 0_+}\int_{\eta}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu=\int_{\eta}|Du_0|^{p-2}Du_0\cdot\nu. \tag{6.1.4}$$

By (5.2.7), we have $|Du_\varepsilon(x)|\le C(\varepsilon+|x'|^2)^{-1}$ in $\Omega^0_{1/2}$. Therefore,

$$\begin{aligned}\left|\int_{\Sigma^{\varepsilon}_{r}}|Du_\varepsilon|^{p-2}Du_\varepsilon\cdot\nu\right|&\le C\int_{|x'|=r}\int_{-\varepsilon/2+h_2(x')}^{h_2(x')}\Big(\frac{1}{\varepsilon+|x'|^2}\Big)^{p-1}dx_ndS\\&\le C\frac{\varepsilon r^{n-2}}{(\varepsilon+r^2)^{p-1}}.\end{aligned} \tag{6.1.5}$$

Finally, (5.4.7) follows directly from (6.1.1)-(6.1.5). Proposition 5.4.2 is proved. □

6.2 Proof of Eq. (5.4.12)

In the following, we verify (5.4.12).

Lemma 6.2.1. (5.4.12) *holds when* $p\ge(n+1)/2$.

Proof. We only give the proof for the case when $n\ge 3$. The case $n=2$ follows similarly and is simpler. After a rotation of coordinates if necessary, we may assume that

$$D^2_{x'}(h_1 - h_2)(0') = \text{diag }(\lambda_1, \dots, \lambda_{n-1}).$$

First, we replace $\delta(y)$ in the denominator with the quadratic polynomial $\varepsilon + \sum_{i=1}^{n-1} \lambda_i y_i^2/2$. By (5.1.8), (5.1.9), and the fact that h_1, h_2 are C^2, we estimate

$$\begin{aligned}
&\Big|\int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\delta(y)}\Big)^{p-1} - \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy'\Big| \\
&= \Theta(\varepsilon)^{p-1} \left|\int_{|y'|<r} \frac{\Big(\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2\Big)^{p-1} - \delta(y)^{p-1}}{\Big[\delta(y)\Big(\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2\Big)\Big]^{p-1}} dy'\right| \\
&\leq C\Theta(\varepsilon)^{p-1} \int_{|y'|<r} \frac{h(r)|y'|^2(\varepsilon + |y'|^2)^{p-2}}{(\varepsilon + |y'|)^{2p-2}} dy' \\
&\leq Ch(r) \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy',
\end{aligned}$$

where $h(r)$ is the modulus of continuity of $D^2_{x'}(h_1 - h_2)$, and hence $h(r) \to 0$ as $r \to 0$, and C is some positive constant independent of ε and r. Therefore, it suffices to show that for any $r > 0$,

$$\lim_{\varepsilon \to 0_+} \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy' = \frac{1}{K}. \tag{6.2.1}$$

In the spherical coordinates, for $y' \in \mathbb{R}^{n-1}$, we write

$$y_1 = \sqrt{\frac{2}{\lambda_1}} s \cos\theta_1, \quad y_2 = \sqrt{\frac{2}{\lambda_2}} s \sin\theta_1 \cos\theta_2, \quad y_3 = \sqrt{\frac{2}{\lambda_3}} s \sin\theta_1 \sin\theta_2 \cos\theta_3, \dots,$$

$$y_{n-2} = \sqrt{\frac{2}{\lambda_{n-2}}} s \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \cos\theta_{n-2},$$

$$y_{n-1} = \sqrt{\frac{2}{\lambda_{n-1}}} s \sin\theta_1 \sin\theta_2 \cdots \sin\theta_{n-3} \sin\theta_{n-2},$$

where $s \in [0, \infty)$, $\theta_1, \theta_2, \dots, \theta_{n-3} \in [0, \pi]$ and $\theta_{n-2} \in [0, 2\pi)$. For convenience of notation, we denote $\Sigma := [0,\pi]^{n-3} \times [0, 2\pi)$. By this change of variables,

$$\begin{aligned}
&\int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy' \\
&= \frac{2^{\frac{n-1}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r}{\varphi(\theta)}} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + s^2}\Big)^{p-1} s^{n-2} J(\theta)\, ds d\theta, \qquad (6.2.2)
\end{aligned}$$

where

$$\varphi(\theta) = \Big(\frac{2}{\lambda_1}\cos^2\theta_1 + \frac{2}{\lambda_2}\sin^2\theta_1\cos^2\theta_2 + \cdots + \frac{2}{\lambda_{n-1}}\sin^2\theta_1 \cdots \sin^2\theta_{n-2}\Big)^{\frac{1}{2}},$$

and

$$J(\theta) = \sin^{n-3}\theta_1 \sin^{n-4}\theta_2 \cdots \sin\theta_{n-3}.$$

Note that

$$\sqrt{\frac{2}{\max \lambda_i}} \le \varphi(\theta) \le \sqrt{\frac{2}{\min \lambda_i}}.$$

When $p > (n+1)/2$, $\Theta(\varepsilon) = \varepsilon^{\frac{2p-n-1}{2(p-1)}}$. By the change of variables $t = \varepsilon^{-1}s^2$, the right-hand side of (6.2.2) becomes

$$\frac{2^{\frac{n-3}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^{\frac{r^2}{\varphi^2(\theta)\varepsilon}} \frac{t^{\frac{n-3}{2}}}{(1+t)^{p-1}}\, dt J(\theta) d\theta.$$

Since $(n-3)/2 - (p-1) = (n-2p-1)/2 < -1$, the integral converges as $\varepsilon \to 0$. Therefore,

$$\begin{aligned}
\lim_{\varepsilon \to 0_+} \int_{|y'|<r} \Big(\frac{\Theta(\varepsilon)}{\varepsilon + \sum_{i=1}^{n-1} \frac{\lambda_i}{2} y_i^2}\Big)^{p-1} dy' &= \frac{2^{\frac{n-3}{2}}}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} \int_\Sigma \int_0^\infty \frac{t^{\frac{n-3}{2}}}{(1+t)^{p-1}}\, dt J(\theta) d\theta \\
&= \frac{2^{\frac{n-3}{2}} |\mathbb{S}^{n-2}|}{(\lambda_1 \cdots \lambda_{n-1})^{\frac{1}{2}}} B\Big(\frac{n-1}{2}, \frac{2p-n-1}{2}\Big), \quad (6.2.3)
\end{aligned}$$

where B is the beta function. Recalling the identities

$$|\mathbb{S}^{n-2}| = \frac{2\pi^{\frac{n-1}{2}}}{\Gamma\big(\frac{n-1}{2}\big)}, \quad B\Big(\frac{n-1}{2}, \frac{2p-n-1}{2}\Big) = \frac{\Gamma\big(\frac{n-1}{2}\big)\Gamma\big(p-\frac{n-1}{2}\big)}{\Gamma(p-1)}, \tag{6.2.4}$$

and plugging them into (6.2.3), we have proved (6.2.1) for the case when $p > (n+1)/2$.

When $p = (n+1)/2$, $\Theta(\varepsilon) = |\ln\varepsilon|^{-\frac{1}{p-1}}$. By the change of variables $w = \varepsilon^{-1/2}s$, the right-hand side of (6.2.2) becomes

$$\begin{aligned}
&\frac{2^{\frac{n-1}{2}}|\ln\varepsilon|^{-1}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}}\int_\Sigma\int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\frac{w^{n-2}}{(1+w^2)^{\frac{n-1}{2}}}\,dwJ(\theta)\,d\theta\\
&=\frac{2^{\frac{n-1}{2}}|\ln\varepsilon|^{-1}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}}\int_\Sigma\int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\frac{w}{1+w^2}\,dwJ(\theta)\,d\theta\\
&+\frac{2^{\frac{n-1}{2}}|\ln\varepsilon|^{-1}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}}\int_\Sigma\int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\left(\frac{w^{n-2}}{(1+w^2)^{\frac{n-1}{2}}}-\frac{w}{1+w^2}\right)dwJ(\theta)\,d\theta\\
&=: \mathrm{I}+\mathrm{II}.
\end{aligned}$$

By direct computations,

$$\begin{aligned}
\mathrm{I} &= \frac{2^{\frac{n-3}{2}}|\ln\varepsilon|^{-1}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}}\int_\Sigma\int_0^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\,d\ln(1+w^2)J(\theta)\,d\theta\\
&=\frac{2^{\frac{n-3}{2}}|\ln\varepsilon|^{-1}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}}\int_\Sigma\Big[\ln\big(\varepsilon+\frac{r^2}{\varphi^2(\theta)}\big)-\ln\varepsilon\Big]J(\theta)\,d\theta.
\end{aligned}$$

Therefore, by (6.2.4),

$$\lim_{\varepsilon\to 0_+}\mathrm{I} = \frac{2^{\frac{n-3}{2}}|\mathbb{S}^{n-2}|}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}} = \frac{(2\pi)^{\frac{n-1}{2}}}{(\lambda_1\cdots\lambda_{n-1})^{\frac{1}{2}}\Gamma\big(\frac{n-1}{2}\big)}. \tag{6.2.5}$$

To estimate II, we split the integral over $(0, \frac{r}{\varphi(\theta)\sqrt{\varepsilon}})$ into $(0,1)$ and $(1, \frac{r}{\varphi(\theta)\sqrt{\varepsilon}})$, and denote them by II_1 and II_2, respectively. It is easily seen that $|\mathrm{II}_1| \le C|\ln\varepsilon|^{-1}$. To estimate II_2, we have

$$|\mathrm{II}_2| \leq C|\ln\varepsilon|^{-1}\int_\Sigma\int_1^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\frac{w\Big[w^{n-3}-(1+w^2)^{\frac{n-3}{2}}\Big]}{(1+w^2)^{\frac{n-1}{2}}}\,dw J(\theta)\,d\theta.$$

By the mean value theorem, there exists a $\xi \in (w^2, 1+w^2)$, such that

$$w^{n-3}-(1+w^2)^{\frac{n-3}{2}} = -\frac{n-3}{2}\xi^{\frac{n-5}{2}}.$$

Note that $(w^2, 1+w^2) \subset (w^2, 2w^2)$ when $w \geq 1$. Therefore,

$$\left|\int_1^{\frac{r}{\varphi(\theta)\sqrt{\varepsilon}}}\frac{w\Big[w^{n-3}-(1+w^2)^{\frac{n-3}{2}}\Big]}{(1+w^2)^{\frac{n-1}{2}}}\,dw\right| \leq C\int_1^\infty\frac{w^{n-4}}{(1+w^2)^{\frac{n-1}{2}}} \leq C,$$

which implies $|\mathrm{II}_2| \leq C|\ln\varepsilon|^{-1}$. Hence, by (6.2.5) and the estimate $|\mathrm{II}_1|+|\mathrm{II}_2| \leq C|\ln\varepsilon|^{-1}$, we have proved (6.2.1) for the case when $p=(n+1)/2$. $\square$

Bibliography

[1] H. Ammari, G. Ciraolo, H. Kang, H. Lee, and K. Yun, *Spectral analysis of the Neumann-Poincaré operator and characterization of the stress concentration in anti-plane elasticity*, Arch. Ration. Mech. Anal. **208** (2013), no. 1, 275–304.

[2] H. Ammari, B. Davies, and S. Yu, *Close-to-touching acoustic subwavelength resonators: eigenfrequency separation and gradient blow-up*, Multiscale Model. Simul. **18** (2020), no. 3, 1299–1317.

[3] H. Ammari, H. Kang, H. Lee, J. Lee, and M. Lim, *Optimal estimates for the electric field in two dimensions*, J. Math. Pures Appl. (9) **88** (2007), no. 4, 307–324.

[4] H. Ammari, H. Kang, and M. Lim, *Gradient estimates for solutions to the conductivity problem*, Math. Ann. **332** (2005), no. 2, 277–286.

[5] S. N. Antontsev and J. F. Rodrigues, *On stationary thermo-rheological viscous flows*, Ann. Univ. Ferrara Sez. VII Sci. Mat. **52** (2006), no. 1, 19–36.

[6] I. Babuška, B. Andersson, P.J. Smith, and K. Levin, *Damage analysis of fiber composites. I. Statistical analysis on fiber scale*, Comput. Methods Appl. Mech. Engrg. **172** (1999), no. 1-4, 27–77.

[7] E. Bao, H.G. Li, Y.Y. Li, and B. Yin, *Derivative estimates of solutions of elliptic systems in narrow regions*, Quart. Appl. Math. **72** (2014), no. 3, 589–596.

[8] E. Bao, Y.Y. Li, and B. Yin, *Gradient estimates for the perfect conductivity problem*, Arch. Ration. Mech. Anal. **193** (2009), no. 1, 195–226.

[9] ———, *Gradient estimates for the perfect and insulated conductivity problems with multiple inclusions*, Comm. Partial Differential Equations **35** (2010), no. 11, 1982–2006.

[10] G. I. Barenblatt, *On self-similar motions of a compressible fluid in a porous medium*, Akad. Nauk SSSR. Prikl. Mat. Meh. **16** (1952), 679–698.

[11] V. Bögelein, *Higher integrability for weak solutions of higher order degenerate parabolic systems*, Ann. Acad. Sci. Fenn. Math. **33** (2008), no. 2, 387–412.

[12] E. Bonnetier and F. Triki, *Pointwise bounds on the gradient and the spectrum of the Neumann-Poincaré operator: the case of 2 discs*, Multi-scale and high-contrast PDE: from modelling, to mathematical analysis, to inversion, 2012, pp. 81–91.

[13] ______, *On the spectrum of the Poincaré variational problem for two close-to-touching inclusions in 2D*, Arch. Ration. Mech. Anal. **209** (2013), no. 2, 541–567.

[14] T. Brander, J. Ilmavirta, and M. Kar, *Superconductive and insulating inclusions for linear and non-linear conductivity equations*, Inverse Probl. Imaging **12** (2018), no. 1, 91–123.

[15] D. Breit, A. Cianchi, L. Diening, and S. Schwarzacher, *Global Schauder estimates for the p-Laplace system*, Arch. Ration. Mech. Anal. **243** (2022), no. 1, 201–255.

[16] B. Budiansky and G. F. Carrier, *High Shear Stresses in Stiff-Fiber Composites*, Journal of Applied Mechanics **51** (1984), no. 4, 733–735.

[17] L. A. Caffarelli and Q. Huang, *Estimates in the generalized Campanato-John-Nirenberg spaces for fully nonlinear elliptic equations*, Duke Math. J. **118** (2003), no. 1, 1–17.

[18] Y. Capdeboscq and S.C. Yang Ong, *Quantitative Jacobian determinant bounds for the conductivity equation in high contrast composite media*, Discrete Contin. Dyn. Syst. Ser. B **25** (2020), no. 10, 3857–3887.

[19] J. Choi and H. Dong, *Gradient estimates for Stokes systems in domains*, Dyn. Partial Differ. Equ. **16** (2019), no. 1, 1–24.

[20] G. Ciraolo and A. Sciammetta, *Gradient estimates for the perfect conductivity problem in anisotropic media*, J. Math. Pures Appl. (9) **127** (2019), 268–298.

[21] ______, *Stress concentration for closely located inclusions in nonlinear perfect conductivity problems*, J. Differential Equations **266** (2019), no. 9, 6149–6178.

[22] E. DiBenedetto, *$C^{1+\alpha}$ local regularity of weak solutions of degenerate elliptic equations*, Nonlinear Anal. **7** (1983), no. 8, 827–850.

[23] ______, *Degenerate parabolic equations*, Universitext, Springer-Verlag, New York, 1993.

[24] E. DiBenedetto and J. Manfredi, *On the higher integrability of the gradient of weak solutions of certain degenerate elliptic systems*, Amer. J. Math. **115** (1993), no. 5, 1107–1134.

[25] H. Dong and S. Kim, *On C^1, C^2, and weak type-$(1,1)$ estimates for linear elliptic operators*, Comm. Partial Differential Equations **42** (2017), no. 3, 417–435.

[26] H. Dong, J. Lee, and S. Kim, *On conormal and oblique derivative problem for elliptic equations with Dini mean oscillation coefficients*, Indiana Univ. Math. J. **69** (2020), no. 6, 1815–1853.

[27] H. Dong and H.G. Li, *Optimal estimates for the conductivity problem by Green's function method*, Arch. Ration. Mech. Anal. **231** (2019), no. 3, 1427–1453.

[28] H. Dong, Y.Y. Li, and Z. Yang, *Gradient estimates for the insulated conductivity problem: The non-umbilical case*, 2022. arXiv:2203.10081.

[29] ———, *Optimal gradient estimates of solutions to the insulated conductivity problem in dimension greater than two*, J. Eur. Math. Soc., published online first (2024).

[30] H. Dong and Z. Li, *Classical solutions of oblique derivative problem in nonsmooth domains with mean Dini coefficients*, Trans. Amer. Math. Soc. **373** (2020), no. 7, 4975–4997.

[31] H. Dong and J. Xiong, *Boundary gradient estimates for parabolic and elliptic systems from linear laminates*, Int. Math. Res. Not. IMRN **17** (2015), 7734–7756.

[32] H. Dong, Z. Yang, and H. Zhu, *Asymptotics of the solution to the perfect conductivity problem with p-laplacian*, 2023. arXiv:2311.11541, to appear in Math. Ann.

[33] ———, *The insulated conductivity problem with p-Laplacian*, Arch. Ration. Mech. Anal. **247** (2023), no. 5, 95.

[34] H. Dong and H. Zhang, *On an elliptic equation arising from composite materials*, Arch. Ration. Mech. Anal. **222** (2016), no. 1, 47–89.

[35] H. Dong and H. Zhu, *Gradient estimates for singular parabolic p-Laplace type equations with measure data*, Calc. Var. Partial Differential Equations **61** (2022), no. 3, Paper No. 86, 41.

[36] ———, *Gradient estimates for singular p-laplace type equations with measure data*, J. Eur. Math. Soc., published online first (2023).

[37] F. Duzaar and G. Mingione, *Gradient continuity estimates*, Calc. Var. Partial Differential Equations **39** (2010), no. 3-4, 379–418.

[38] ———, *Gradient estimates via linear and nonlinear potentials*, J. Funct. Anal. **259** (2010), no. 11, 2961–2998.

[39] ———, *Gradient estimates via non-linear potentials*, Amer. J. Math. **133** (2011), no. 4, 1093–1149.

[40] A. Garroni and R. V. Kohn, *Some three-dimensional problems related to dielectric breakdown and polycrystal plasticity*, R. Soc. Lond. Proc. Ser. A Math. Phys. Eng. Sci. **459** (2003), no. 2038, 2613–2625.

[41] E. Giusti, *Direct methods in the calculus of variations*, World Scientific Publishing Co., Inc., River Edge, NJ, 2003.

[42] Y. Gorb, *Singular behavior of electric field of high-contrast concentrated composites*, Multiscale Model. Simul. **13** (2015), no. 4, 1312–1326.

[43] Y. Gorb and A. Novikov, *Blow-up of solutions to a p-Laplace equation*, Multiscale Model. Simul. **10** (2012), no. 3, 727–743.

[44] C. Hamburger, *Regularity of differential forms minimizing degenerate elliptic functionals*, J. Reine Angew. Math. **431** (1992), 7–64.

[45] L. I. Hedberg and Th. H. Wolff, *Thin sets in nonlinear potential theory*, Ann. Inst. Fourier (Grenoble) **33** (1983), no. 4, 161–187.

[46] M. I. Idiart, *The macroscopic behavior of power-law and ideally plastic materials with elliptical distribution of porosity*, Mech. Res. Commun. **35** (2008), no. 8, 583–588.

[47] H. Kang, M. Lim, and K. Yun, *Asymptotics and computation of the solution to the conductivity equation in the presence of adjacent inclusions with extreme conductivities*, J. Math. Pures Appl. (9) **99** (2013), no. 2, 234–249.

[48] ———, *Characterization of the electric field concentration between two adjacent spherical perfect conductors*, SIAM J. Appl. Math. **74** (2014), no. 1, 125–146.

[49] J. B. Keller, *Stresses in Narrow Regions*, Journal of Applied Mechanics **60** (1993), no. 4, 1054–1056.

[50] J. Kim and M. Lim, *Electric field concentration in the presence of an inclusion with eccentric core-shell geometry*, Math. Ann. **373** (2019), no. 1-2, 517–551.

[51] J. Kinnunen and J. L. Lewis, *Higher integrability for parabolic systems of p-Laplacian type*, Duke Math. J. **102** (2000), no. 2, 253–271.

[52] N. V. Krylov, *Nonlinear elliptic and parabolic equations of the second order*, Mathematics and its Applications (Soviet Series), vol. 7, D. Reidel Publishing Co., Dordrecht, 1987. Translated from the Russian by P. L. Buzytsky [P. L. Buzytskiĭ].

[53] ______, *On Bellman's equations with VMO coefficients*, Methods Appl. Anal. **17** (2010), no. 1, 105–121.

[54] T. Kuusi and G. Mingione, *New perturbation methods for nonlinear parabolic problems*, J. Math. Pures Appl. (9) **98** (2012), no. 4, 390–427.

[55] ______, *Gradient regularity for nonlinear parabolic equations*, Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) **12** (2013), no. 4, 755–822.

[56] ______, *Linear potentials in nonlinear potential theory*, Arch. Ration. Mech. Anal. **207** (2013), no. 1, 215–246.

[57] ______, *Guide to nonlinear potential estimates*, Bull. Math. Sci. **4** (2014), no. 1, 1–82.

[58] ______, *A nonlinear Stein theorem*, Calc. Var. Partial Differential Equations **51** (2014), no. 1-2, 45–86.

[59] ______, *Riesz potentials and nonlinear parabolic equations*, Arch. Ration. Mech. Anal. **212** (2014), no. 3, 727–780.

[60] ______, *The Wolff gradient bound for degenerate parabolic equations*, J. Eur. Math. Soc. (JEMS) **16** (2014), no. 4, 835–892.

[61] O. Levy and R. V. Kohn, *Duality relations for non-Ohmic composites, with applications to behavior near percolation*, J. Statist. Phys. **90** (1998), no. 1-2, 159–189.

[62] H.G. Li, *Asymptotics for the Electric Field Concentration in the Perfect Conductivity Problem*, SIAM J. Math. Anal. **52** (2020), no. 4, 3350–3375.

[63] H.G. Li, Y.Y. Li, and Z. Yang, *Asymptotics of the gradient of solutions to the perfect conductivity problem*, Multiscale Model. Simul. **17** (2019), no. 3, 899–925.

[64] H.G. Li, F. Wang, and L. Xu, *Characterization of electric fields between two spherical perfect conductors with general radii in 3D*, J. Differential Equations **267** (2019), no. 11, 6644–6690.

[65] Y.Y. Li and Z. Yang, *Gradient estimates of solutions to the conductivity problem with flatter insulators*, Anal. Theory Appl. **37** (2021), no. 1, 114–128.

[66] ———, *Gradient estimates of solutions to the insulated conductivity problem in dimension greater than two*, Math. Ann. **385** (2023), no. 3-4, 1775–1796.

[67] G. M. Lieberman, *Regularized distance and its applications*, Pacific J. Math. **117** (1985), no. 2, 329–352.

[68] ———, *Boundary regularity for solutions of degenerate elliptic equations*, Nonlinear Anal. **12** (1988), no. 11, 1203–1219.

[69] ———, *The natural generalization of the natural conditions of Ladyzhenskaya and Ural′tseva for elliptic equations*, Comm. Partial Differential Equations **16** (1991), no. 2-3, 311–361.

[70] ———, *Sharp forms of estimates for subsolutions and supersolutions of quasilinear elliptic equations involving measures*, Comm. Partial Differential Equations **18** (1993), no. 7-8, 1191–1212.

[71] ———, *Second order parabolic differential equations*, World Scientific Publishing Co., Inc., River Edge, NJ, 1996.

[72] M. Lim and K. Yun, *Blow-up of electric fields between closely spaced spherical perfect conductors*, Comm. Partial Differential Equations **34** (2009), no. 10-12, 1287–1315.

[73] X. Markenscoff, *Stress amplification in vanishingly small geometries*, Computational Mechanics **19** (1996), no. 1, 77–83.

[74] G. Mingione, *Gradient potential estimates*, J. Eur. Math. Soc. (JEMS) **13** (2011), no. 2, 459–486.

[75] ———, *Nonlinear measure data problems*, Milan J. Math. **79** (2011), no. 2, 429–496.

[76] Q.-H. Nguyen and N. C. Phuc, *Good-λ and Muckenhoupt-Wheeden type bounds in quasilinear measure datum problems, with applications*, Math. Ann. **374** (2019), no. 1-2, 67–98.

[77] ———, *Pointwise gradient estimates for a class of singular quasilinear equations with measure data*, J. Funct. Anal. **278** (2020), no. 5, 108391, 35.

[78] ———, *Existence and regularity estimates for quasilinear equations with measure data: the case $1 < p \leq (3n-2)/(2n-1)$*, Anal. PDE **15** (2022), no. 8, 1879–1895.

[79] ———, *A comparison estimate for singular p-Laplace equations and its consequences*, Arch. Ration. Mech. Anal. **247** (2023), no. 3, Paper No. 49, 24.

[80] J.-T. Park and P. Shin, *Regularity estimates for singular parabolic measure data problems with sharp growth*, J. Differential Equations **316** (2022), 726–761.

[81] M. Růžička, *Electrorheological fluids: modeling and mathematical theory*, Lecture Notes in Mathematics, vol. 1748, Springer-Verlag, Berlin, 2000.

[82] P.-M. Suquet, *Overall potentials and extremal surfaces of power law or ideally plastic composites*, J. Mech. Phys. Solids **41** (1993), no. 6, 981–1002.

[83] P. Tolksdorf, *Regularity for a more general class of quasilinear elliptic equations*, J. Differential Equations **51** (1984), no. 1, 126–150.

[84] N. S. Trudinger, *On Harnack type inequalities and their application to quasilinear elliptic equations*, Comm. Pure Appl. Math. **20** (1967), 721–747.

[85] K. Uhlenbeck, *Regularity for a class of non-linear elliptic systems*, Acta Math. **138** (1977), no. 3-4, 219–240.

[86] J. L. Vázquez, *A strong maximum principle for some quasilinear elliptic equations*, Appl. Math. Optim. **12** (1984), no. 3, 191–202.

[87] ———, *Smoothing and decay estimates for nonlinear diffusion equations: Equations of porous medium type*, Oxford Lecture Series in Mathematics and its Applications, vol. 33, Oxford University Press, Oxford, 2006.

[88] B. Weinkove, *The insulated conductivity problem, effective gradient estimates and the maximum principle*, Math. Ann. **385** (2023), no. 1-2, 1–16.

[89] K. Yun, *Estimates for electric fields blown up between closely adjacent conductors with arbitrary shape*, SIAM J. Appl. Math. **67** (2007), no. 3, 714–730.

[90] ———, *Optimal bound on high stresses occurring between stiff fibers with arbitrary shaped cross-sections*, J. Math. Anal. Appl. **350** (2009), no. 1, 306–312.

[91] ———, *An optimal estimate for electric fields on the shortest line segment between two spherical insulators in three dimensions*, J. Differential Equations **261** (2016), no. 1, 148–188.