

Large-deformation, viscoelastic constitutive modeling of elastomeric foams

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Fingers crossed for my OPT EAD.

TO MY MOM AND WANGWANG

Contents

List of Tables	xi
List of Figures	xiii
1 Introduction	1
1.1 Background on foam materials and constitutive modeling	1
1.2 Thesis contributions	2
1.2.1 Hyperelastic constitutive modeling of elastomeric foams	3
1.2.2 Nonlinear constitutive modeling of viscoelastic foams	3
1.2.3 Applications to closed-cell and open-cell foams	4
1.3 Thesis outline	4
2 Experimental characterization and hyperelastic constitutive modeling of open-cell elastomeric foams	5
2.1 Introduction	5
2.2 Experimental Program	10
2.3 Constitutive theory	18
2.4 Validation experiments and simulations	25
2.4.1 Spherical and conical indentation	25
2.4.2 Simple-shear-like deformation without and with pre-compression .	27
2.4.3 Tension of a specimen with circular holes	31
2.5 Discussion	34
2.5.1 Stability	34
2.5.2 Comparison to other models	36
2.6 Concluding remarks	42
3 Nonlinear constitutive modeling of viscoelastic foams	45
3.1 Introduction	45
3.2 Constitutive theory	55
3.2.1 Equilibrium response	55
3.2.2 Non-equilibrium response	57
3.3 Concluding remarks	61
4 Application of the viscoelastic constitutive model to closed-cell EVA foam	63
4.1 Introduction	63
4.2 Specialization of the viscoelastic constitutive model	64
4.2.1 Specialization of the equilibrium free-energy function	64
4.2.2 Specialization of the non-equilibrium free-energy function	65

4.2.3	Specialization of the flow rule	66
4.2.4	Material parameters for the closed-cell EVA foam	67
4.3	Validation experiments and simulations	68
4.3.1	Spherical indentation	69
4.3.2	Simple-shear-like deformation without and with pre-compression .	72
4.3.3	Tension of a specimen with circular holes	76
4.4	Concluding remarks	78
5	Application of the viscoelastic constitutive model to open-cell Poron XRD foams	81
5.1	Introduction	81
5.2	Specialization of the viscoelastic constitutive model	82
5.2.1	Specialization of the equilibrium free-energy function	82
5.2.2	Specialization of the non-equilibrium free-energy function	83
5.2.3	Specialization of the flow rule	85
5.2.4	Material parameters for the open-cell Poron XRD foams	89
5.3	Validation experiments and simulations	89
5.3.1	Conical indentation	90
5.3.2	Simple-shear-like deformation without and with pre-compression .	94
5.3.3	Tension of a specimen with circular holes	97
5.4	Concluding remarks	99
6	Conclusions and outlook	103
6.1	Concluding remarks	103
6.2	Future directions	105
A	Material parameter estimation procedure – hyperelastic model	109
A.1	Estimation of compression-based parameters	110
A.2	Estimation of tension-based parameters	115
B	Loss of ellipticity	117
C	Experimental program for the closed-cell EVA foam	121
D	Material parameter estimation procedure – viscoelastic model	125
D.1	Equilibrium branch	125
D.2	Non-equilibrium branches	125

List of Tables

2.1	Material parameters for low, moderate, and high density Poron XRD foams	24
4.1	Equilibrium material parameters for the closed-cell EVA foam	67
4.2	Non-equilibrium material parameters for the closed-cell EVA foam	67
5.1	Adjusted equilibrium material parameters for the low, moderate, and high density Poron XRD foams	83
5.2	Material parameters of the non-equilibrium free-energy density functions for the low, moderate, and high density Poron XRD foams	84
5.3	Material parameters of the non-equilibrium flow rules for the low, moderate, and high density Poron XRD foams	85

List of Figures

2.1	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for three different densities of a polyurethane-based, open-cell elastomeric foam	7
2.2	Three images of a representative moderate density Poron XRD foam sample at various scales	12
2.3	Schematic of the top and side views of the experimental setup as deployed on a vibration-isolated optical breadboard.	14
2.4	Example kinematic data collected from simple compression and simple tension experiments on the high density Poron XRD foam	16
2.5	Outline of the experimental data-flow for quality-factor-based digital image correlation (qDIC) data, which is reduced to axial and lateral engineering strain data, and force data that is used to compute the axial engineering stress	17
2.6	Spherical and conical indentation validation tests	26
2.7	Simple-shear-like validation tests both without and with uniaxial pre-compression	29
2.8	Contours of the two-dimensional displacement field on the front face of the specimen from both experiment and simulation for simple-shear-like deformation both without and with pre-compression for the moderate density Poron XRD foam	32
2.9	A tension-dominated validation test for the high density Poron XRD foam consisting of tension of a specimen with circular holes, in which a standard dogbone geometry is modified with six asymmetrically-placed holes	33
2.10	Stability diagrams calculated using a loss-of-ellipticity test as a function of the invariants of the logarithmic strain K_1 and K_2 for several cases of fixed K_3	37
2.11	Comparison of fits obtained using hyperelastic models for porous elastomers in the literature with the proposed approach for the moderate density Poron XRD foam	38
3.1	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam under quasi-static and moderate strain rates	47
3.2	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of low density (144 kg/m^3) under quasi-static and moderate strain rates	48
3.3	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of moderate density (192 kg/m^3) under quasi-static and moderate strain rates	49

3.4	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of high density (240 kg/m^3) under quasi-static and moderate strain rates	50
3.5	Stress-level and amount of hysteresis as a function of the axial engineering strain rate for the closed-cell EVA foam	52
3.6	Stress-level and amount of hysteresis as a function of the axial engineering strain rate for the high density open-cell Poron XRD foam	53
3.7	A rheological schematic of the Maxwell-Wiechert-type nonlinear, finite-deformation, viscoelastic constitutive model.	56
4.1	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam over the strain-rate range 1.0×10^{-3} - $1.0 \times 10^{-1} \text{ s}^{-1}$: Experimental data and constitutive model fits	68
4.2	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam over the strain-rate range 1.0 - $1.0 \times 10^1 \text{ s}^{-1}$: Experimental data and constitutive model fits	69
4.3	Spherical indentation validation tests of the calibrated viscoelastic constitutive model for the closed-cell EVA foam	71
4.4	Simple-shear-like validation tests of the calibrated viscoelastic constitutive model without uniaxial pre-compression for the closed-cell EVA foam . . .	74
4.5	Simple-shear-like validation tests of the calibrated viscoelastic constitutive model with uniaxial pre-compression for the closed-cell EVA foam	75
4.6	A tension-dominated validation test of the calibrated viscoelastic constitutive model for the closed-cell EVA foam	77
5.1	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the low-density Poron XRD foam over the strain-rate range 1.0×10^{-3} - 1.0 s^{-1} : Experimental data and constitutive model fits	86
5.2	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the moderate-density Poron XRD foam over the strain-rate range 1.0×10^{-3} - 1.0 s^{-1} : Experimental data and constitutive model fits	87
5.3	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the high-density Poron XRD foam over the strain-rate range 1.0×10^{-3} - 1.0 s^{-1} : Experimental data and constitutive model fits	88
5.4	Conical indentation validation tests of the calibrated viscoelastic constitutive model for the low-density Poron XRD foam	91
5.5	Conical indentation validation tests of the calibrated viscoelastic constitutive model for the moderate density Poron XRD foam	92
5.6	Conical indentation validation tests of the calibrated viscoelastic constitutive model for the high density Poron XRD foam	93
5.7	Simple-shear-like validation tests of the calibrated viscoelastic constitutive model without uniaxial pre-compression for the moderate density Poron XRD foam	95

5.8	Simple-shear-like validation tests of the calibrated viscoelastic constitutive model with uniaxial pre-compression for the moderate density Poron XRD foam	96
5.9	A tension-dominated validation test of the calibrated viscoelastic constitutive model for the high-density Poron XRD foam	98
5.10	Axial engineering stress versus axial engineering strain curves and lateral engineering strain versus axial engineering strain curves for the low-density Poron XRD foam at a strain-rate of $1.0 \times 10^1 \text{ s}^{-1}$: Experimental data	100
6.1	Preliminary spherical indentation data highlighting the important role of temperature on the mechanical response of polyurethane foams	105
6.2	Representative DIC snapshot of a polyurethane foam specimen undergoing localization during nominally simple compression	106
A.1	Outline of the procedure for estimating the material parameters of the hyperelasticity model using experimental data for simple compression and tension	110
A.2	Demonstration of the procedure for estimating the material parameters of the hyperelasticity model using experimental data for simple compression and tension	113
C.1	Schematic of the isometric view of the high-speed setup as deployed on a vibration-isolated optical breadboard including camera and load frame . . .	124
D.1	Outline of the procedure for estimating the material parameters of the viscoelastic model from experimental data for simple compression and tension at different strain rates.	126
D.2	Demonstration of the procedure for estimating the material parameters of the viscoelasticity model using experimental data for simple compression .	127

Chapter 1

Introduction

1.1 Background on foam materials and constitutive modeling

Elastomeric foams are materials that consist of two phases: an elastomeric matrix and air. The mechanical behavior of the foam is determined by its microstructure and the material properties of its matrix material. Elastomeric foams exhibit mechanical behavior marked by high compressibility and strong coupling between the volumetric and distortional responses. Moreover, due to the viscoelasticity of the matrix material, these materials are usually able to absorb greater amounts of energy than ordinary stiff elastic materials for a certain stress-level and strain-rate. They are commonly used in protective equipment, mattresses, cushions, padding, and packaging materials that benefit from energy absorption capabilities.

The wide applications necessitate an appropriate constitutive model for this class of materials. One major aim of research on elastomeric foams is to develop a hyperelastic model for their equilibrium, elastic response. There are several modeling approaches for the free energy density function: phenomenological curve-fitting approaches that depend on the principal stretches [1–7], Blatz-Ko type hyperelasticity models [8–10] that are based on the principal invariants of the Cauchy-Green deformation tensor, homogenization-based

approaches [11–14] that consider porous elastomers with microstructures consisting of closed-cell pores, fully-micromechanical models based on strut-type unit cells [15–19], or unit cells involving cylindrical or spherical voids [14, 20, 21].

Researchers have also considered modeling the rate-dependent mechanical behavior of elastomeric foams, in which the situation is yet more complex. Most existing finite viscoelasticity models decompose the response into a time-independent hyperelastic part and a time-dependent viscoelastic part. The time-dependent part of finite viscoelasticity models for elastomeric foams are typically formulated using one of two approaches. The first approach is a hereditary integral model, which is motivated by linear viscoelastic models. Hereditary integral models are formulated in terms of relaxation or memory functions [e.g., 5, 22–26]. The second approach is based on the framework of the multiplicative decomposition of the deformation gradient into elastic and inelastic parts [e.g., 27–33]. Few researchers have reported on the modelling of compressible materials [e.g., 34], and most works deal with incompressible or nearly-incompressible materials. Compared to hereditary integral models, the multiplicative decomposition of the deformation gradient leads to models that are thermodynamic consistent and more straightforwardly applied to arbitrary three-dimensional loading histories.

Existing models can capture certain features of the response of viscoelastic foams but are often only applicable to specific loading conditions, e.g., simple compression or only the loading portion of the response. Developing an effective constitutive model that remains predictive for general deformation histories remains challenging despite the ubiquity of these materials.

1.2 Thesis contributions

The overarching motivation of this thesis is to develop a more general and more accessible constitutive modeling framework for elastomeric foam materials. The specific problems

addressed by this thesis are summarized as follows.

1.2.1 Hyperelastic constitutive modeling of elastomeric foams

We propose a methodology for the constitutive modeling of non-localizing, isotropic, open-cell elastomeric foam materials under quasi-static, equilibrium loading. Using data from large-deformation, homogeneous simple compression/tension experiments on three relative densities of a polyurethane-based elastomeric foam, we propose a phenomenological, isotropic, hyperelastic constitutive model. The model is based on the invariants of the logarithmic strain and accounts for high compressibility and strong volumetric-distortional coupling. To validate the predictive capability of the model, we consider three types of validation experiments that involve inhomogeneous deformation: spherical and conical indentation, simple-shear-like deformation both without and with a fixed amount of pre-compression, and tension of a specimen with circular holes. We compare load-displacement responses as well as full displacement fields from the validation experiments against corresponding model predictions obtained using finite-element-based numerical simulations and demonstrate that the model is capable of accurately capturing the experimental response.

1.2.2 Nonlinear constitutive modeling of viscoelastic foams

We propose a methodology for the constitutive modeling of the time-dependent, large-deformation mechanical behavior of isotropic, viscoelastic foam materials under quasi-static and moderate rate loading. The phenomenological modeling framework is motivated by the rheological concept of a “hyperelastic spring” in parallel with several nonlinear Maxwell elements and consists of a time-independent, hyperelastic contribution and several time-dependent, viscoelastic contributions. The hyperelastic modeling is based the contribution described in Section 1.2.1, and for the rate-dependent mechanisms, we employ the framework of the multiplicative decomposition of the deformation gradient into non-equilibrium elastic and viscous parts.

1.2.3 Applications to closed-cell and open-cell foams

Data from large-deformation, homogeneous simple compression/tension experiments under quasi-static and moderate strain-rates for both closed-cell and open-cell elastomeric foams is used to specialize and calibrate the nonlinear, viscoelastic constitutive model. Validation experiments that involve inhomogeneous deformation and a wide range of deformation rates are conducted, which include spherical indentation, simple-shear-like deformation both without and with a fixed amount of pre-compression, and tension of a specimen with circular holes at different deformation rates. The responses from the validation experiments are compared against model predictions obtained from finite-element-based numerical simulations. We show that the experimental response can be accurately captured by the proposed model for both closed-cell and open-cell foam materials.

1.3 Thesis outline

This thesis is outlined around the above three contributions. Chapter 2 presents our methodology for the hyperelastic constitutive modeling of non-localizing, isotropic elastomeric foam materials under quasi-static, equilibrium loading and its application to an open-cell foam material. Chapter 3 presents our methodology for viscoelastic constitutive modeling of non-localizing, isotropic elastomeric foam materials under quasi-static and moderate rate loading. Chapter 4 and Chapter 5 describe the applications of the proposed viscoelastic model to a closed-cell foam and open-cell foams, respectively. We close in Chapter 6 with a summary of the major contributions of the thesis and future research directions.

Chapter 2

Experimental characterization and hyperelastic constitutive modeling of open-cell elastomeric foams

Note: A version of this chapter is published in the Journal of the Mechanics and Physics of Solids. Data and figures have been used with all co-authors' consent.

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2.1 Introduction

Open-cell elastomeric foams are materials consisting of two continuous phases: a polymeric elastomer and a connected gas-filled pore space. Due to the elastomeric matrix and porous structure, these materials are highly compliant and capable of undergoing large, reversible deformations involving substantial volume change. Elastomeric foams are widely used in situations in which compliance and compressibility are necessary, such as in impact

protection and cushioning, and appear in a wide variety of applications.

The mechanical behavior of open-cell elastomeric foams is quite complex, involving large deformations, substantial compressibility, and strong tension/compression asymmetry. Consider, as an illustrative example, the behavior of a commercially-available impact protection foam – a polyurethane-based, open-cell elastomeric foam that will be the focus of our work – whose response to external load is demonstrated in Fig. 2.1 for three different relative densities (Poron XRD, Rogers Corp, Rogers, CT). For each density, Fig. 2.1 shows the experimentally-measured engineering stress versus axial engineering strain response in the left column and the corresponding lateral engineering strain versus axial engineering strain response in the right column for quasi-static (low strain-rate) simple compression/tension (experimental details are discussed in Section 2.2). The response in all three cases is highly nonlinear and displays substantial asymmetry between tension and compression. In particular, in compression, the response is initially linear, but as the axial strain magnitude increases, the response gradually transitions into a “plateau” in which both the axial stress and the lateral strain vary minimally as the axial strain magnitude increases. The plateau regime is generally associated with buckling in the elastomeric-matrix microstructure and concomitant closure of the pore space [35]. Importantly, the plateau regime for Poron XRD is not completely flat, and we observe homogeneous deformation with no evidence of strain localization in the compression experiments. Then, as the material is compressed beyond the plateau regime, rapid stiffening is observed in the axial stress response. However, in tension neither a plateau regime nor a rapidly stiffening response is observed. Upon unloading in both compression and tension, some hysteresis but no permanent set is observed. This is in contrast to the inelastic behavior of crushable foams [e.g., 36–39], which yield and display permanent set upon unloading. In this work, we focus on the mechanics of elastic foams that exhibit neither deformation localization nor permanent set.

Due to the complexity of the mechanical behavior of open-cell elastomeric foams, developing predictive constitutive models remains challenging despite the ubiquity of

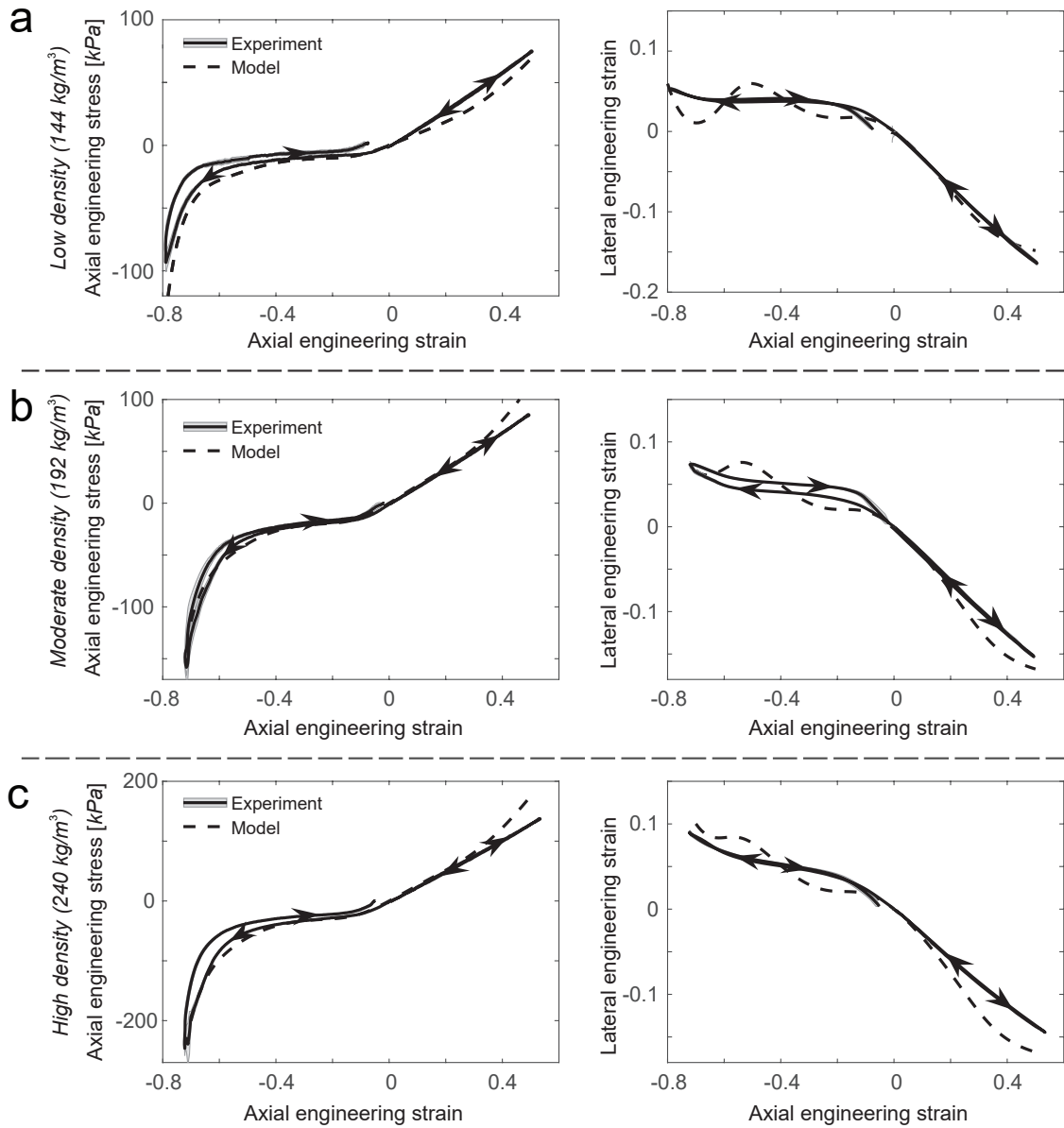


Figure 2.1: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foams of (a) low (144.2 kg/m^3), (b) moderate (192.2 kg/m^3), and (c) high (240.3 kg/m^3) density with the same matrix composition. The solid lines with shaded error areas (one standard deviation, nine experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for (c) the stress axis differs in range from (a) and (b). For all densities, note both the plateau regime in compression in both the stress and lateral strain histories as a function of axial strain, and the small lateral strain magnitudes under large axial strain magnitudes as characteristic features of deformation in open-cell elastomeric foams.

these materials. There is a substantial history of constitutive modeling of porous and foam materials [e.g., 35, 40]. Here, we focus on hyperelastic modeling approaches for capturing the equilibrium, elastic response described in the preceding paragraph. Commonly, Ogden-type hyperelastic models [1, 2], which utilize a phenomenological free-energy density function that depends on the principal stretches, are used to capture data for highly-compressible foams in the large-stretch regime [e.g., 3–7]. While models of this type are capable of describing the features of the equilibrium, hyperelastic response of open-cell elastomeric foams, they can be unwieldy to work with [e.g., 41, 42]. Material parameter fitting is a challenge, and a fitted model typically has minimal extrapolative potential, often providing poor predictions in loading scenarios not used in fitting or becoming unstable beyond the fitted range. Other hyperelasticity models utilize free-energy density functions that depend on the principal invariants of the Cauchy-Green deformation tensors, including the well-known Blatz-Ko hyperelasticity model [8] and its extensions [e.g., 9, 10]. The Blatz-Ko model can capture the response of compressible foams up to moderate stretches but is not equipped to describe large-stretch behavior – specifically, the plateau regime in compression. Other invariant-based approaches include polynomial-type free-energy density functions [22, 24]. Furthermore, homogenization-based approaches [11–14] consider porous elastomers with microstructures consisting of closed-cell pores and result in explicit, invariant-based hyperelastic models that incorporate the effect of the void volume fraction. However, since these models invoke closed-cell microstructures in their derivations and do not account for buckling in elastomeric-matrix ligaments, they cannot be expected to capture the response of highly-compressible, open-cell foams under large compressive deformations – in particular, the response in the plateau regime. Finally, we note that fully-micromechanical models, based on strut-type unit cells [e.g., 15–19] or unit cells involving cylindrical or spherical voids [e.g., 14, 20, 21], have been used to capture the precise behavior of elastomeric foams with different microstructures. While such modeling approaches provide important physical insights, they are computationally expensive to apply

to problems involving length scales much greater than the characteristic size of the porous microstructure, and hence, explicit hyperelastic models are valuable.

The purpose of this chapter is to present a comprehensive methodology for the experimental characterization and constitutive modeling of non-localizing, isotropic, open-cell elastomeric foam materials under quasi-static, equilibrium loading. Initial model development and material parameter estimation are informed by the homogeneous simple compression/tension experiments shown in Fig. 2.1. Accurate full-field kinematic measurements in these experiments are obtained using the quality-factor-based digital image correlation (qDIC) procedure developed in the work of Landauer et al. [43], which is crucial to measure both axial and lateral strains under large deformations. Rather than basing our isotropic hyperelasticity model on the principal invariants of the Cauchy-Green deformation tensors, our hyperelastic model adopts those of Criscione et al. [44], which are based on the invariants of the logarithmic strain [45, 46]. This choice is made on pragmatic grounds, since the logarithmic-strain invariants of Criscione et al. [44] may be associated with specific aspects of deformation, enabling a more straightforward path to choosing and calibrating phenomenological fitting functions that capture experimental data. For perspective, a number of isotropic, logarithmic-strain-based hyperelasticity models have been proposed in the literature. Importantly, the work of Anand [47, 48], showed that a simple quadratic free-energy function employing the logarithmic strain is capable of capturing experimental data for a wide class of materials in the moderate-stretch regime. Since then, various models have been successfully applied to incompressible and nearly-incompressible materials in the large-stretch regime [e.g., 49–52]. The present work extends this literature towards a constitutive modeling approach for highly-compressible materials. We decompose the free-energy function into terms that represent the coupled volumetric/distortional, purely distortional, and purely volumetric contributions to the response and select phenomenological fitting functions for each term that capture the experimental data for open-cell polyurethane foam. We then perform several validation tests that probe the predictive capability of the model

in a variety of deformation modes, namely, compression, shear, and tension-dominated situations. Our intent is to provide an accessible characterization and modeling framework for highly-compressible elastic materials, and we expect that future works will consider other specific, perhaps micromechanically-based, fitting functions and extend the approach to other open-cell elastomeric foam materials.

The remainder of this chapter is organized as follows. In Section 2.2, we describe the experimental program for characterizing the mechanical response of the isotropic, polyurethane-based elastomeric foam, marketed under the “Poron XRD” trademark, using homogeneous simple compression/tension experiments as well as the approach for validation testing. Then, in Section 2.3, we present our hyperelastic constitutive model for the elastic behavior of isotropic, open-cell elastomeric foams and apply the model to Poron XRD. In Section 2.4, the constitutive model is validated in three configurations: first, spherical and conical indentation, second, simple-shear-like deformation both without and with fixed-displacement pre-compression, and third, tension of a specimen with circular holes, where the conditions are specifically chosen to probe the model in compression, shear, and tension-dominated situations. In Section 2.5, we discuss issues of stability and compare the proposed model with other important models from the literature. Finally, we close with a summary and a discussion of the limits of the modeling approach and future research directions in Section 2.6.

2.2 Experimental Program

Note: The experiments described in this section as well as the experiments described in Section 2.4 were performed by Alexander K. Landauer. We include this section here for completeness. Additional experimental detail may be found in Landauer [53].

In this section, we describe the process for experimentally testing the mechanical response of isotropic, elastomeric foams. We perform two classes of experiments: (1)

homogeneous characterization experiments that are utilized to choose constitutive equations and estimate material parameters and (2) inhomogeneous validation experiments that are utilized to test constitutive model predictions. First, regarding characterization testing, we perform simple compression and simple tension experiments. All kinematic data in these experiments are obtained using full-field digital image correlation (DIC), which enables both axial and lateral strains to be obtained. Moreover, we are able to ensure that no unintended inhomogeneous deformation (e.g., specimen barreling or buckling, compaction banding, surface wrinkling, etc.) occurs during characterization experiments. Second, we perform a variety of inhomogeneous validation tests, namely, spherical and conical indentation, simple-shear-like deformation both without and with pre-compression, and tension of a specimen with circular holes, each of which are discussed further in Section 2.4. These validation experiments probe the predictive capability of the constitutive model in compression, shear, and tension-dominated situations.

Foam material The experimental testing procedures and constitutive theory presented in this chapter are intended to be applicable to a large class of isotropic, open-cell elastomeric foams, provided that the material is capable of deforming homogeneously and does not exhibit localized deformation, such as compaction banding. In the present work, we consider three different relative densities of a single type of foam material. Our selection is a polyurethane-based open-cell elastomeric foam developed by Rogers Corporation (Poron XRD, Rogers, CT) for impact mitigation applications, which has been the subject of previous studies in the literature [24, 54]. The three relative densities considered, as specified by the manufacturer, are 144.2 kg/m^3 , 192.2 kg/m^3 , and 240.3 kg/m^3 , which are hereafter referred to as the low, moderate, and high density foams, respectively. To get a sense of the foam microstructure, Fig. 2.2 shows (a) a macro-scale image, (b) a micro-scale image, and (c) a 3D volume render obtained via micro-computed tomography (μCT) for a representative sample of the moderate density foam. The pore structure is highly random, and based on

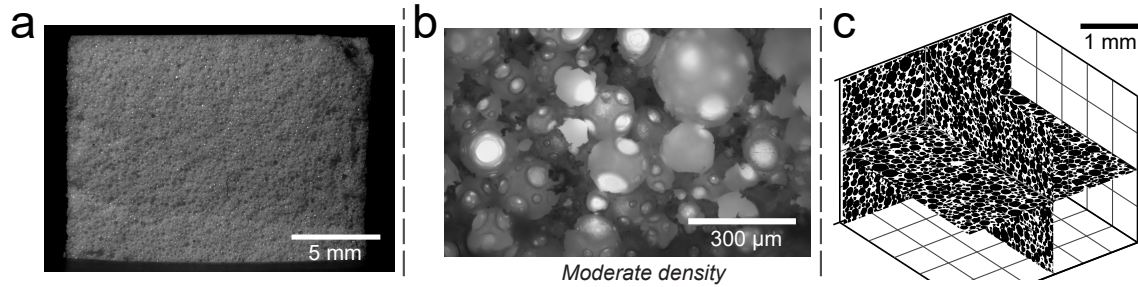


Figure 2.2: Three images of a representative moderate density foam sample at various scales. (a) The cut face of a foam cuboid. (b) Optical microscopy image of the microstructure of the surface of a cut face. (c) Micro-computed tomography 3D volumetric render of the foam microstructure (watershed-based binarization, in which black is void).

optical microscopy measurements, the mean pore diameter is approximately $70\ \mu\text{m}$ with a maximum pore diameter of up to about $500\ \mu\text{m}$ for the moderate density foam.

Specimen preparation The details of specimen preparation depend on the specific type of test, but in general, specimens are collected from as-received approximately 12.9 mm-thick sheets of bulk material, speckled for digital image correlation, and then installed in appropriate fixtures. For compression testing, foam cuboids of approximately $10 \times 10 \times 12.9\ \text{mm}^3$ are excised with a bandsaw. For tension testing, dogbone specimens are cut using a 75W CO₂ laser cutter (Universal Laser Systems Inc., Scottsdale, AZ) to the ASTM D3574 standard geometry for tensile testing of foam polymers. For indentation experiments (Section 2.4.1), $25.4 \times 25.4 \times 12.9\ \text{mm}^3$ samples are laser cut and bonded to an aluminum backer on one of the large faces using a two-part epoxy (ITW Polymers Adhesives, Danvers, MA). For each simple-shear-like experiment (Section 2.4.2), two $30 \times 30 \times 12.9\ \text{mm}^3$ specimens are laser cut, and each large, square face is epoxied to an aluminum backer. Finally, circular holes are cut out of tension specimens to be used for the third type of validation testing (Section 2.4.3). DIC measurements are used to obtain kinematic data in all experiments except for indentation testing, and to prepare specimens for DIC analysis, a face orthogonal to the testing axis is speckle-painted for each specimen. To speckle the specimens, a black base coat of acrylic ink (Liquitex Artist

Materials, Piscataway, NJ) is airbrushed (HP-C Plus, Iwata Medea Inc, Portland, OR) on the surface, followed by a white speckle layer (Golden Artist Colors Inc., New Berlin, NY). For each configuration, the optical train employed is configured such that this technique yields speckles of approximately 3 to 7 pixels in size. Due to the inherent DIC challenges posed by mechanical testing of elastomeric foams – namely, that large speckle distortions and speckle loss due to cell collapse or opening arise when the material undergoes finite deformations – a custom, DIC procedure is utilized [43], which incorporates the concept of cross-correlation quality factors to improve the accuracy of DIC measurements involving distorted speckle patterns undergoing large deformations. Finally, for all experiments, both characterization and validation, three specimens are tested, and each specimen is tested three times (i.e., nine total experiments per condition). For repeated experiments on the same specimen, we ensure that the wait-time between experiments is sufficiently long to allow complete relaxation of any time-dependent material physics, typically 3 hours or more in total test time.

Experimental setup A schematic layout of the optical system and loading apparatus used in the experiments is shown in Fig. 2.3. For all experiments, the loading apparatus and control software are the same as that described in Landauer et al. [43] – namely, a screw-driven linear actuator (Ultra Motion, Cutchogue, NY) and a load cell (LCFD-50, Omega Engineering, Stamford, CT) with a custom LabView (National Instruments, Austin, TX) interface that allows synchronized actuator displacement control, actuator displacement measurement, imaging, and force data collection. The optical system utilized to collect data for two-dimensional DIC is comprised of a custom control computer, a scientific camera (edge5.5, PCO-Tech Inc., Romulus, MI), a long-distance microscopy lens (K2/S, Infinity Photo-Optical Company, Boulder, CO), polarizing filters (American Polarizers Inc., Reading, PA and Edmund Optics, Barrington, NJ), and LED light panels (Lykos Daylight, Manfrotto, South Upper Saddle River, NJ), which is assembled on the same vibration-isolated table as

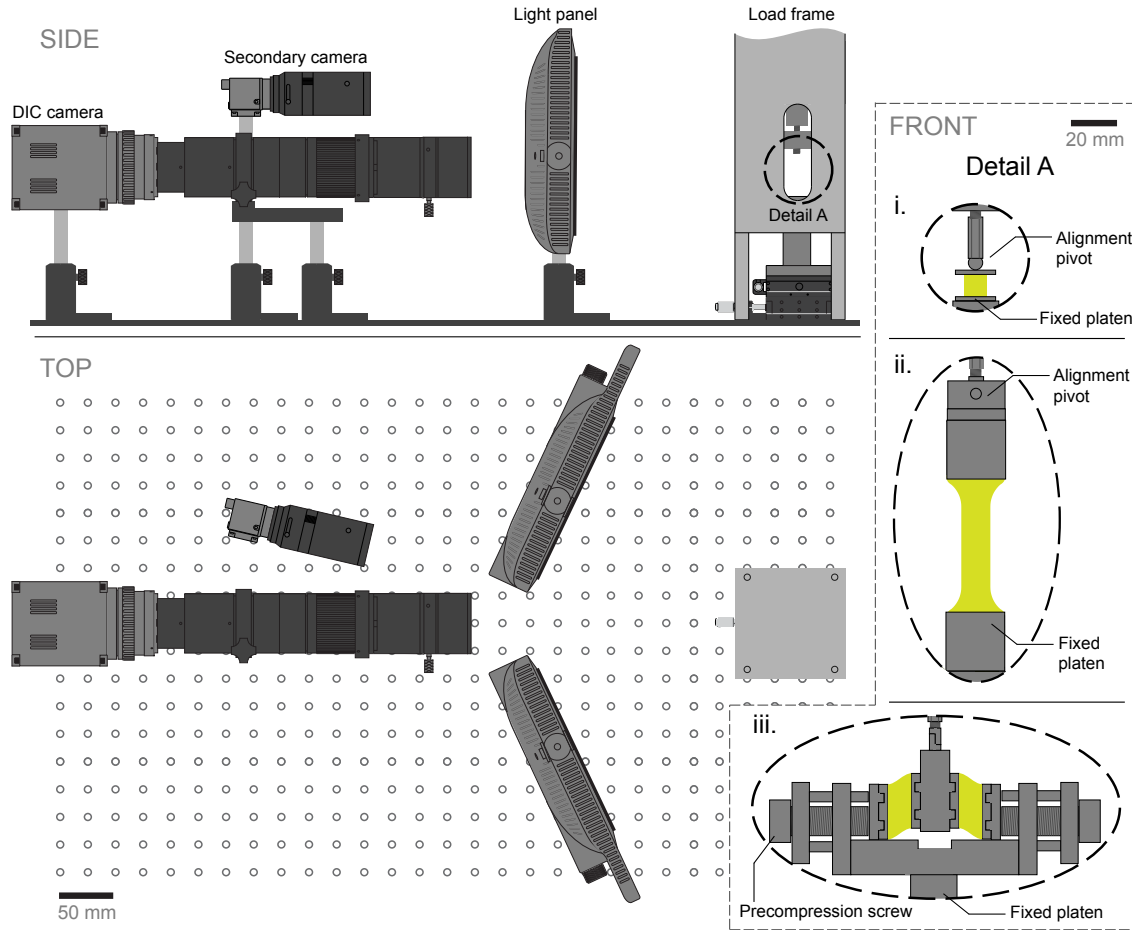


Figure 2.3: Schematic of the top and side views of the experimental setup as deployed on a vibration-isolated optical breadboard. The arrangement of the two cameras with lenses, light panels with linearly polarizing filters, and load frame is shown. Detail A gives front-view diagrams (i.e., as seen by the cameras) of the fixtures used for (i) simple compression, (ii) simple tension, and (iii) simple-shear-like deformation.

the loading apparatus. Additionally, for all experiments, a secondary imaging camera with a field-of-view encompassing the complete specimen and load frame is used to detect any potential anomalous loading conditions (e.g., onset of specimen failure at the grips).

For each experiment type, a custom load-frame insert is used to grip specimens during loading, and the three primary fixtures – simple compression, simple tension, and simple-shear-like deformation – are diagrammed in the insets of Fig. 2.3. By using modular inserts, the control procedures and DIC parameters remain consistent across experiment types. For simple compression, the insert consists of graphitic steel top and bottom platens compressed

via a spherical indenter tip attached to the load cell. The spherical indenter tip acts as a pivot point, allowing any misalignment to be identified and eliminated in the compression experiments. The contact areas between the platens and the specimen are lubricated with graphite powder and spherical glass microbeads (diameter 20-30 μm). For simple tension, screw-actuated wedge-and-pin-type grips – with the gripping-force screw acting as the pin – are attached to the base of the load frame and to the load cell. For the simple-shear-like validation experiments, a symmetric “double-shear” insert, involving two specimens with aluminum backers epoxied to both sides, is utilized, as shown in inset (iii) of Fig. 2.3. The center platen undergoes a vertical displacement prescribed by the linear actuator to shear the specimens. Additionally, horizontal screws on each side of the fixture may be adjusted to subject the specimens to a uniaxial pre-compression, which we set to be equal for both specimens in order to maintain symmetry, and which is held constant during the subsequent shear deformation.

Details of simple compression/tension tests The characterization experiments, i.e., simple compression and tension, are conducted in displacement control to engineering strain limits of -0.75 (axial stretch of $\lambda = 0.25$) in compression and 0.5 (axial stretch of $\lambda = 1.5$) in tension and are loaded and unloaded at a constant, quasi-static engineering strain-rate magnitude of $2.0 \times 10^{-4} \text{ s}^{-1}$.¹ Illustrative example images of reference and deformed specimens for the high density foam are shown in Figs. 2.4(a) and (b) for compression and tension, respectively, where the direction of compression/tension is identified as the x_1 -direction, and the selected regions-of-interest for DIC analysis are outlined with dashed lines. Contours of the resultant two-dimensional displacement fields – i.e., $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ – at sample levels of deformation, namely an axial stretch of $\lambda = 0.35$ for compression and $\lambda = 1.50$ for tension, calculated using the custom qDIC procedure are also shown in Fig. 2.4. In both examples, the contours reveal an axial displacement field u_1 that depends linearly on the

¹ The choice of quasi-static strain-rate was informed by preliminary stress relaxation experiments on the moderate-density foam.

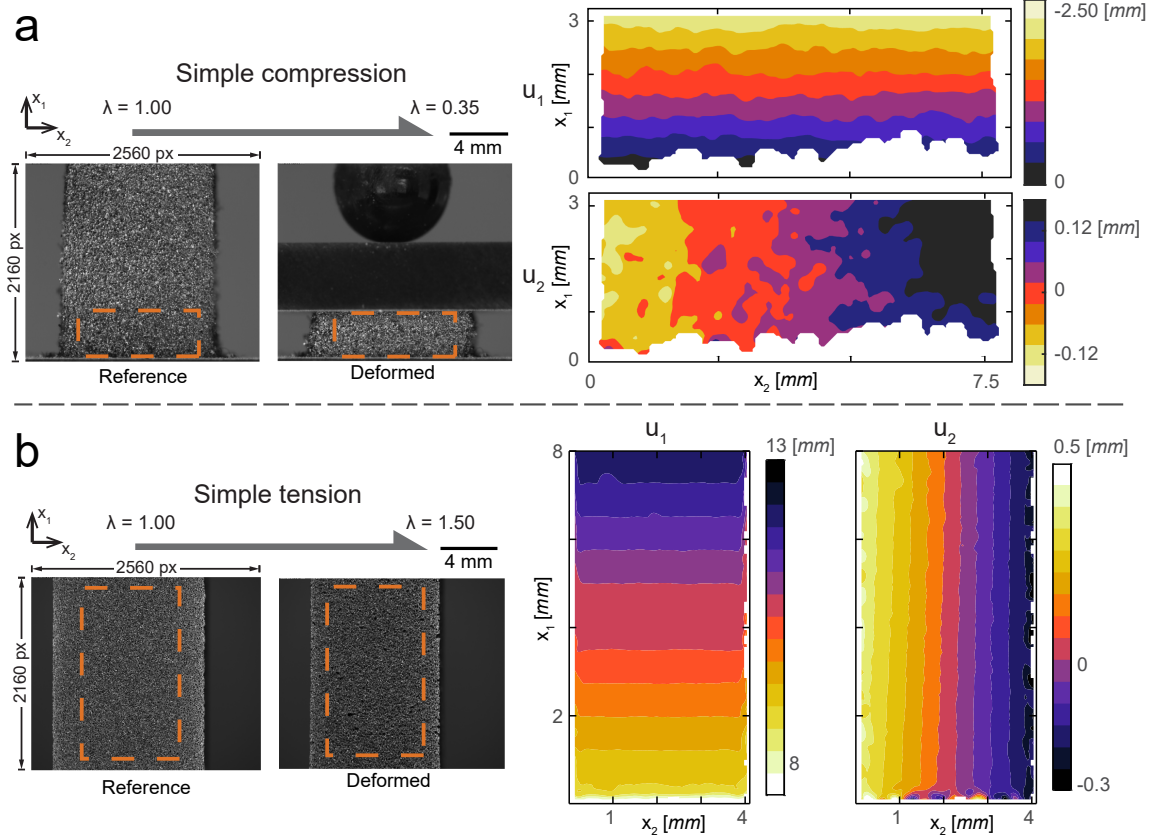


Figure 2.4: Example kinematic data collected from (a) simple compression (adapted from [43]) and (b) simple tension experiments on the high density foam. (Left) Images of the respective specimens in the reference and deformed states with the regions-of-interest for DIC analysis outlined with dashed lines. DIC regions are approximately $1200 \times 350 \text{ px}^2$ for compression and approximately $800 \times 1900 \text{ px}^2$ for tension. (Right) Contours of the DIC-calculated, two-dimensional displacement fields for the axial (u_1) and lateral (u_2) displacement components.

axial coordinate x_1 and a lateral displacement field u_2 that depends linearly on the lateral coordinate x_2 – indicating that the intended homogeneous deformation without shearing and with minimal barreling has been achieved. Importantly, in the compression experiments, we do not observe evidence of strain localization into compaction bands, such as those that often arise during compression of very-low-density, “reticulated” foams [e.g., 55–57]. This is quantified via the histogram-based technique of Wang and Cuitiño [58]. Furthermore, at the spatial resolution of the DIC algorithm using a subset size of $16 \times 16 \text{ px}^2$, inhomogeneities in the displacement field are an order of magnitude smaller than displacement due to the

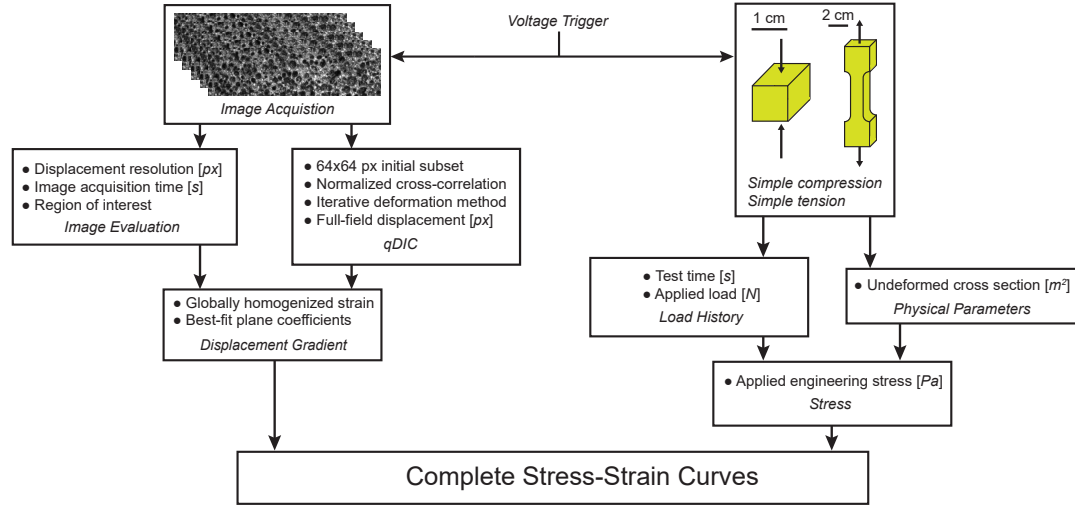


Figure 2.5: Outline of the experimental data-flow for quality-factor-based digital image correlation (qDIC) data, which is reduced to axial and lateral engineering strain data, and force data, which is used to compute the axial engineering stress. Experiments are initiated via a voltage signal provided when the imposed displacement history is begun, ensuring that time-synchronized load and images datasets are collected.

globally applied deformation and of similar scale as the displacement noise floor.

Components of the global displacement gradient tensor are extracted from the displacement fields for each deformation step in each experiment via a global plane-fitting procedure, which is described in detail in Landauer et al. [43], and the components du_1/dx_1 and du_2/dx_2 are identified as the axial engineering strain and lateral engineering strain, respectively. The corresponding axial engineering stress is computed from load-cell data and the previously-measured, referential cross-sectional area. A summary of the experimental data-flow used to reduce experimental data to axial engineering strain, lateral engineering strain, and axial engineering stress data is outlined in Fig. 2.5. The axial engineering stress versus axial engineering strain and lateral engineering strain versus axial engineering strain data are plotted as solid lines in Fig. 2.1 for the (a) low, (b) moderate, and (c) high density foams. The black curves represent the average of the nine instances of each experiment, and the shaded areas surrounding the solid curves represent one standard deviation above and below the average, which indicate good repeatability in the experiments. Arrows indicate

loading and unloading, demonstrating that hysteresis is minimal in the experiments and that the strain-rate is sufficiently low so that the experimentally-measured behavior represents the equilibrium, elastic response of the material. Finally, we note that characterization of the foam material in simple compression along each of the three orthogonal directions of the as-received sheets revealed no discernible difference in the experimentally-measured behavior. This observation, along with the random microstructure of the foam material (Fig. 2.2), supports our choice to model the foam material as isotropic.

2.3 Constitutive theory

In this section, we describe our hyperelastic constitutive model for the elastic behavior of isotropic, elastomeric foams and apply the model to three densities of Poron XRD foam.

Basic kinematics We consider a body B identified with the region of space it occupies in a fixed reference configuration and denote an arbitrary material point within B as $\mathbf{X}$. The referential body B then undergoes a motion $\mathbf{x} = \chi(\mathbf{X}, t)$ to the deformed body $\mathcal{B}_t$ at each time t . The deformation gradient is given by $\mathbf{F} = \nabla \chi$, such that the ratio of referential to deformed volume is strictly greater than zero, i.e., $J = \det \mathbf{F} > 0$.² The displacement field is denoted as $\mathbf{u}(\mathbf{X}, t) = \chi(\mathbf{X}, t) - \mathbf{X}$, and the left and right Cauchy-Green deformation tensors are $\mathbf{B} = \mathbf{F}\mathbf{F}^\top$ and $\mathbf{C} = \mathbf{F}^\top \mathbf{F}$, respectively. The deformation tensors admit the spectral decompositions $\mathbf{B} = \sum_{i=1}^3 \lambda_i^2 \mathbf{l}_i \otimes \mathbf{l}_i$ and $\mathbf{C} = \sum_{i=1}^3 \lambda_i^2 \mathbf{r}_i \otimes \mathbf{r}_i$, where $\{\lambda_i | i = 1, 2, 3\}$ are the principal stretches, $\{\mathbf{l}_i | i = 1, 2, 3\}$ are the principal directions of $\mathbf{B}$, and $\{\mathbf{r}_i | i = 1, 2, 3\}$ are the principal directions of $\mathbf{C}$. Finally, we introduce the logarithmic (Hencky) finite-strain tensor in the deformed body: $\mathbf{E} = \sum_{i=1}^3 (\ln \lambda_i) \mathbf{l}_i \otimes \mathbf{l}_i$.

²*Notation:* The symbols ∇ , Div and Curl denote the gradient, divergence, and curl with respect to the material point $\mathbf{X}$ in the reference body B ; grad , div , and curl denote these operators with respect to the point $\mathbf{x} = \chi(\mathbf{X}, t)$ in the deformed body $\mathcal{B}_t$. We write $\text{Tr} \mathbf{A}$, $\det \mathbf{A}$, and $\text{dev} \mathbf{A}$ to denote the trace, determinant, and deviatoric part of a tensor $\mathbf{A}$, respectively.

An isotropic free-energy function The constitutive response of a hyperelastic material is specified through the free-energy density function ψ . In general, ψ is a function of the right Cauchy-Green tensor $\mathbf{C}$, i.e., $\psi = \hat{\psi}(\mathbf{C})$, but for an isotropic material, the free-energy function is an isotropic function of $\mathbf{C}$ and may be represented as a function of the principal stretches or three independent invariants. Stretch-based, Ogden-type free-energy functions of the form $\psi = \check{\psi}(\lambda_1, \lambda_2, \lambda_3)$ are commonly used to model both incompressible [59] and compressible [1, 2] elastomers; however, as mentioned in Section 2.1, this approach can be unwieldy to work with for highly-compressible materials. Alternatively, invariant-based free-energy functions utilizing the principal invariants of $\mathbf{C}$, i.e., $I_1 = \text{Tr} \mathbf{C}$, $I_2 = (1/2)((\text{Tr} \mathbf{C})^2 - \text{Tr}(\mathbf{C}^2))$, $I_3 = \det \mathbf{C} = J^2$, may be invoked, and indeed, several hyperelastic models for porous elastomers fall in this category [e.g., 8, 11, 14, 22, 24]. However, developing free-energy functions of the form $\psi = \check{\psi}(I_1, I_2, I_3)$ that depend on all three invariants, which is necessary to capture experimental data for compressible foams, is difficult, since it is not straightforward to isolate the effect of each invariant.³ To address this challenge, we develop a free-energy function that depends on invariants of the logarithmic strain tensor $\mathbf{E}$, as proposed by [44]. The logarithmic-strain invariants of Criscione et al. [44], which we denote as $\{K_1, K_2, K_3\}$, represent specific aspects of deformation, enabling a more tractable path to introducing phenomenological fitting functions that capture experimental data. The three invariants are defined as follows:

1. The first invariant is $K_1 = \text{Tr}(\mathbf{E}) = \ln(J) = \ln(\lambda_1 \lambda_2 \lambda_3)$ and represents the volume change of the material with positive values for dilatation and negative values for compaction.
2. The second invariant is $K_2 = |\text{dev}(\mathbf{E})| \geq 0$ and represents the magnitude of constant-volume distortion in the material. In terms of the principal stretches, the second invariant is given by $K_2 = \sqrt{(\ln \lambda_1 - (1/3)K_1)^2 + (\ln \lambda_2 - (1/3)K_1)^2 + (\ln \lambda_3 - (1/3)K_1)^2}$.

³See, for example, Criscione [60] for a discussion of this issue in the context of invariant-based free-energy functions for incompressible materials.

Defining the tensorial direction of deviatoric strain as $\mathbf{N} = \text{dev}(\mathbf{E})/K_2$, such that $\mathbf{N}$ is a deviatoric and unit-magnitude tensor, the logarithmic strain may be expressed as $\mathbf{E} = \frac{1}{3}K_1\mathbf{1} + K_2\mathbf{N}$.

3. The third invariant is $K_3 = 3\sqrt{6}\det(\mathbf{N})$ and represents the mode of distortion. The pre-factor $3\sqrt{6}$ is present so that $-1 \leq K_3 \leq 1$. The third invariant is constant for a given distortion mode (e.g., $K_3 = -1$ in simple compression, $K_3 = 0$ in simple shear, and $K_3 = 1$ in simple tension) and hence, K_3 remains fixed during the simple compression/tension experiments utilized for material characterization. In terms of the principal stretches, the third invariant may be expressed as $K_3 = 3\sqrt{6}(\ln \lambda_1 - (1/3)K_1)(\ln \lambda_2 - (1/3)K_1)(\ln \lambda_3 - (1/3)K_1)/K_2^3$.

As shown by Criscione et al. [44], the Cauchy stress is then given through the derivatives of the free-energy function $\psi = \tilde{\psi}(K_1, K_2, K_3)$ by

$$\mathbf{T} = J^{-1} \left[\frac{\partial \tilde{\psi}}{\partial K_1} \mathbf{1} + \frac{\partial \tilde{\psi}}{\partial K_2} \mathbf{N} + \frac{\partial \tilde{\psi}}{\partial K_3} \frac{1}{K_2} \mathbf{Y} \right] \quad (2.1)$$

where $\mathbf{Y} = 3\sqrt{6}\mathbf{N}^2 - \sqrt{6}\mathbf{1} - 3K_3\mathbf{N}$.⁴

Decomposition of the free-energy function We take the free-energy density function $\tilde{\psi}(K_1, K_2, K_3)$ to be additively decomposed as follows:

$$\tilde{\psi}(K_1, K_2, K_3) = G_0 [X(K_1)K_2^2 + L(K_2, K_3)] + Bf(K_1), \quad (2.2)$$

where G_0 and B are the ground-state shear and bulk moduli, respectively, and $X(K_1)$, $L(K_2, K_3)$, and $f(K_1)$ are phenomenological fitting functions. The interpretation of each term is as follows:

⁴Equation (2.1) may be obtained by recognizing that for an isotropic, hyperelastic material, the Kirchhoff stress $\mathbf{T}_K = J\mathbf{T}$ is given through the derivative of the free-energy function with respect to the spatial logarithmic strain tensor – i.e., $\mathbf{T}_K = \partial \tilde{\psi} / \partial \mathbf{E}$ – and then utilizing the chain rule along with the identities $\partial K_1 / \partial \mathbf{E} = \mathbf{1}$, $\partial K_2 / \partial \mathbf{E} = \mathbf{N}$, and $\partial K_3 / \partial \mathbf{E} = \mathbf{Y} / K_2$.

1. For a material undergoing large volumetric deformation, interaction between the volumetric and distortional contributions to the free energy is expected. The $G_0 X(K_1) K_2^2$ term characterizes this coupling between the volumetric and distortional responses. For simplicity, we assume that this term only involves low-order K_2 -dependence, i.e., that it only depends on K_2 through K_2^2 , and is independent of the third invariant K_3 . The volumetric dependence of this term may be more general through the function $X(K_1)$. More complex forms of volumetric/distortional coupling may certainly be invoked, but we find that this simple form is capable of capturing experimental data while enabling a streamlined material parameter estimation procedure – discussed further in Appendix A. The function $X(K_1)$ may be interpreted as encompassing the volumetric-strain-dependence of the instantaneous shear modulus. We require that $X(K_1 = 0) = 1$, and for the purpose of stability – discussed further in Section 2.5 – we also require that the function $X(K_1)$ remain positive over the intended K_1 -range of application.
2. The $G_0 L(K_2, K_3)$ term represents the higher-order distortional response, i.e., higher-order than the K_2^2 -dependence of the first term, which we take to be uncoupled from the volumetric response for simplicity. This term is intended to capture stiffening behavior as the magnitude of distortion, K_2 , increases, and the inclusion of K_3 -dependence allows for mode-dependence of the higher-order distortional response. We require that $L(K_2 = 0, K_3) = \partial L / \partial K_2|_{K_2=0} = 0$, and that, following Criscione et al. [44], $\partial L / \partial K_3$ goes to zero as order K_2^3 or higher as K_2 goes to zero. Moreover, we require that $\partial L / \partial K_2 > 0$ for all $K_2 > 0$ and $-1 \leq K_3 \leq 1$, so that the function $L(K_2, K_3)$ increases monotonically with K_2 .
3. Finally, $Bf(K_1)$ describes the purely volumetric response. We require that $f(K_1 = 0) = 0$ and $df/dK_1|_{K_1=0} = 0$. Furthermore, for the purpose of stability, we require that $d^2 f / dK_1^2 > 0$ for all K_1 , i.e., that df/dK_1 is a monotonically increasing function

of K_1 .

Using (2.2) in (2.1), we obtain the following expression for the Cauchy stress:

$$\mathbf{T} = J^{-1} \left[\left(G_0 \frac{dX}{dK_1} K_2^2 + B \frac{df}{dK_1} \right) \mathbf{1} + G_0 \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \right) \mathbf{N} + G_0 \frac{\partial L}{\partial K_3} \frac{1}{K_2} \mathbf{Y} \right]. \quad (2.3)$$

Note that the first term in (2.3), namely, the term involving $G_0(dX/dK_1)K_2^2\mathbf{1}$, represents a shear-induced mean stress that arises due to the coupled volumetric/distortional response.

Specialization of the free-energy function To fit to the experimental data collected in Section 2.2 for low, moderate, and high density Poron XRD, we choose specific forms for the phenomenological functions appearing in (2.2). We emphasize that, while these forms are intended to capture data for Poron XRD, future modification may be necessary to apply the modeling approach to other foam materials.

1. Based on experimental data, we find that a form for the function $X(K_1)$ that linearly depends on K_1 but with different slopes before and after the onset of the plateau regime in compression is sufficient to capture the response. Accordingly, we adopt the following simple phenomenological form for dX/dK_1 :

$$\frac{dX}{dK_1} = \frac{1}{2} (X'_1 + X'_2) + \frac{1}{2} (X'_1 - X'_2) \tanh \left[\frac{1}{\Delta_K} (K_1 - K_1^0) \right], \quad (2.4)$$

where $K_1^0 < 0$ is the value of K_1 denoting the transition to the plateau regime, $\Delta_K > 0$ is a parameter denoting the K_1 -range across which the assumed hyperbolic-tangent-type transition in the value of dX/dK_1 transitions from X'_1 prior to the plateau regime in compression (i.e., $K_1 \gtrsim K_1^0$) to X'_2 after the onset of the plateau regime in compression (i.e., $K_1 \lesssim K_1^0$). To capture experimental data, the value of X'_2 is small compared to X'_1 . Integrating (2.4) with respect to K_1 and applying the condition $X(K_1 = 0) = 1$, we

obtain the following phenomenological form for $X(K_1)$:

$$X(K_1) = \frac{1}{2} (X'_1 + X'_2) K_1 + \frac{\Delta_K}{2} (X'_1 - X'_2) \ln \left[\frac{\cosh((K_1 - K_1^0)/\Delta_K)}{\cosh(K_1^0/\Delta_K)} \right] + 1. \quad (2.5)$$

2. The higher-order distortional response is taken to be given through

$$L(K_2, K_3) = C_0 K_2^p + C_1 (1 + K_3) K_2^q, \quad (2.6)$$

where $C_0 > 0$ and $C_1 > 0$ are constant parameters and $p > 2$ and $q > 2$ are higher-order exponents. We note that since $K_3 = -1$ in compression, the second term does not affect the response in compression. This term is present to allow for different higher-order responses in compression and tension to be specified.

3. Finally, the purely volumetric response function is fitted to

$$\frac{df}{dK_1} = \frac{J^{C_2} - 1}{C_2} + C_3 J \left[J^{-r} - \left(\frac{1 - J_{\min}}{J - J_{\min}} \right)^r \right], \quad (2.7)$$

where $J = \exp(K_1)$ is the volume ratio and $J_{\min}$, r , C_2 , and C_3 are constant parameters. We note that the parameter $J_{\min}$ represents the minimum allowable value of J and that df/dK_1 diverges as J approaches $J_{\min}$ from above. The parameter $J_{\min}$ is limited to be within the range $0 < J_{\min} < 1$ and may be interpreted as $J_{\min} = 1 - \phi_0$, where ϕ_0 is the void volume fraction of the undeformed foam. The first term in (2.7) is intended to capture the response in tension and prior to the plateau regime in compression, while the second term is intended to capture the rapidly stiffening behavior in compression. We note that $d^2 f / dK_1^2|_{K_1=0}$ does not precisely equal one. However, since the first term is intended to capture the response prior to the onset of the plateau regime, its contribution will dominate the second derivative at $K_1 = 0$, so that $d^2 f / dK_1^2|_{K_1=0} \approx 1$, and B approximately represents the ground-state bulk modulus.⁵ Upon integrating

⁵ Using (2.7), the exact expression for the ground-state bulk modulus is $B_0 = B(d^2 f / dK_1^2|_{K_1=0}) = B[1 +$

Density	Density-dependent				Density-independent									
	G_0 [kPa]	B [kPa]	$J_{\min}$	C_1	K_1^0	Δ_K	X'_1	X'_2	C_0	p	q	r	C_2	C_3
Low	34.5	58.7	0.12	2.5	-0.21	0.2	3.7	0.22	0.1	4	5	2	9	0.026
Moderate	65.2	117.4	0.16	1.9										
High	102.0	193.8	0.19	1.9										

Table 2.1: Material parameters for low, moderate, and high density Poron XRD foams. In total, fourteen parameters are utilized, with four density-dependent parameters and ten density-independent parameters.

(2.7) with respect to K_1 (for $r \neq 1$) and applying the condition $f(K_1 = 0) = 0$, we obtain the following functional form for f :

$$f = \frac{J^{C_2} - C_2 \ln J - 1}{C_2^2} + \frac{C_3}{r-1} \left[\frac{-1}{J^{r-1}} + \frac{(1 - J_{\min})^r}{(J - J_{\min})^{r-1}} + J_{\min} \right]. \quad (2.8)$$

Material parameters for Poron XRD We estimate the material parameters appearing in our constitutive model – a total of fourteen parameters – for Poron XRD from the simple compression/tension data in Fig. 2.1. For ease of presentation, the discussion of our heuristic procedure for material parameter estimation is relegated to Appendix A. The material parameters for all three densities of Poron XRD determined using this procedure are listed in Table 2.1. Four of the material parameters are density-dependent – G_0 , B , $J_{\min}$, and C_1 – and the remaining ten parameters are density-independent and are not refit for each of the three foam densities. Figure 2.1 compares simple compression/tension experimental data (solid lines) with corresponding calculations using the fitted model (dashed lines). The quality of the comparison between the experimental data and the fitted model for both the axial engineering stress and the lateral engineering strain data and for all three densities is quite reasonable.

$C_3 r J_{\min} / (1 - J_{\min})$. To fit the rapidly stiffening behavior in compression, we take $C_3 \ll 1$, and hence, $B_0 \approx B$.

2.4 Validation experiments and simulations

In this section, we validate the predictive capability of our hyperelastic constitutive theory by comparing model predictions against experimental measurements in settings involving inhomogeneous deformation. To probe the model in a variety of situations, i.e., compression, shear, and tension-dominated situations, we consider three specific types of validation tests: (1) spherical and conical indentation, (2) simple-shear-like deformation both without and with a fixed amount of pre-compression, and (3) tension of a specimen with circular holes. The constitutive model of Section 2.3 has been numerically implemented in Abaqus/Standard [61] using a user-material (UMAT) subroutine, and throughout this section, we obtain model predictions in inhomogeneous deformation scenarios using finite-element calculations in Abaqus/Standard.

2.4.1 Spherical and conical indentation

We begin by considering indentation – a validation case that involves compression-dominated deformation directly beneath the indenter – for both spherical and conical indenter geometries. As discussed in Section 2.2, indentation experiments are conducted on foam specimens with a square $25.4 \times 25.4 \text{ mm}^2$ cross section and a height of $H = 12.9 \text{ mm}$. One square face of each specimen is constrained by a bonded aluminum backer, and the specimen is indented on the opposite unconstrained face. Spherical indentation is performed using a steel indenter tip with a radius of $r = 3.97 \text{ mm}$, and conical indentation is performed using an aluminum indenter tip with a base angle of $\theta = 30^\circ$. Photographs of the experimental setup prior to deformation for spherical and conical indentation are shown in Figs. 2.6(a) and (c), respectively. Experiments are performed at an indenter displacement-rate of $\dot{\delta} = 2.0 \times 10^{-3} \text{ mm/s}$ for spherical indentation and $\dot{\delta} = 2.5 \times 10^{-3} \text{ mm/s}$ for conical indentation. The maximum indenter displacements achieved during spherical and conical indentation are around 4 mm in all cases. The experimentally-measured indenter force P versus displacement δ relations

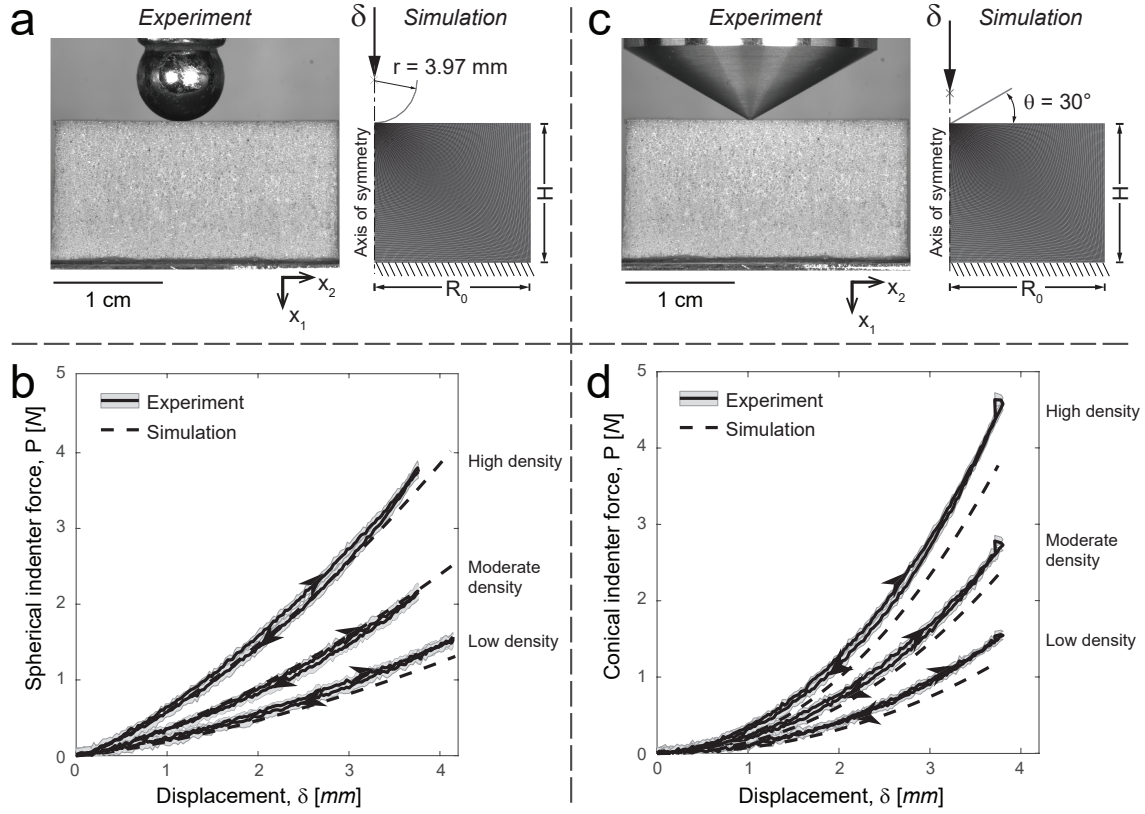


Figure 2.6: Spherical and conical indentation validation tests. Snapshots of the experimental setup prior to deformation and the corresponding finite-element configurations are given for (a) spherical and (c) conical indentation. Indenter force P versus indenter displacement δ for (b) spherical and (d) conical indentation. Solid lines with shaded error regions represent experimental data, and dashed lines are the predictions from the constitutive model.

for spherical and conical indentation are shown as solid lines in Figs. 2.6(b) and (d), respectively. In both cases, data for all three foam densities are shown. As in Fig. 2.1, the shaded regions surrounding the solid curves represent one standard deviation above and below the average across all nine instances of the experiment, and the arrows indicate loading and unloading. Hysteresis is minimal, demonstrating that the chosen indenter displacement-rates are sufficiently slow.

For the corresponding finite-element simulations, we idealize the geometry as axisymmetric for computational efficiency. This involves idealizing the cuboid foam specimen as a disk of radius R_0 and height $H = 12.9$ mm. The outer radius is set to be $R_0 = 14.4$ mm,

which is the average value of the outer radius of the square specimen over the angular coordinate.⁶ The axisymmetric finite-element configurations for spherical and conical indentation are shown in Figs. 2.6(a) and (c), respectively. The foam substrate is meshed using 17,236 Abaqus-CAX4H elements (four-node, axisymmetric, quadrilateral, constant-pressure elements) and the indenters are idealized as rigid surfaces. Contact between the foam substrate and the indenter is approximated as rough – i.e., no slip – in both cases.⁷ Regarding boundary conditions, to model the fixed-base boundary condition, all displacement components are prescribed to be zero on the bottom face of the mesh. The left face of the mesh is coincident with the axis of symmetry, and hence, the horizontal displacement is set to zero along this face. The right face of the mesh is left unconstrained. The calculated indenter force P versus displacement δ relations – obtained without any parameter adjustment – are included as dashed lines for all three foam densities in Figs. 2.6(b) and (d). Generally, there is good agreement between the experimentally-measured and simulated responses, especially for the case of spherical indentation.⁸

2.4.2 Simple-shear-like deformation without and with pre-compression

Next, we validate aspects of the model related to shear-dominated deformation. We perform two types of experiments, carried out on foam cuboids with a square cross-section of side-length $W = 30$ mm and a height of $H = 12.9$ mm. First, we subject foam cuboids to simple-shear-like deformation, in which one square face is translated parallel to the other by a shear displacement δ . In these experiments, the square faces are constrained by

⁶We have performed selected three-dimensional simulations, in which the experimental geometry is exactly represented, and the corresponding simulated indenter force versus displacement result is indistinguishable from the axisymmetric result, verifying that the axisymmetric idealization is valid.

⁷We have performed corresponding simulations with frictionless interaction between the foam substrate and the indenter, and the difference between the rough and frictionless cases is negligible, indicating that the role of surface friction is small in indentation of compressible foam materials.

⁸We note that for the conical indentation case, the indenter forces predicted by the model are slightly lower than the corresponding experimental data, which may be rationalized as follows. To optimize the stability of the model, the fitted volumetric response function, df/dK_1 , slightly underestimates experimental data. The under-prediction in the conical indenter force is associated with the large volumetric deformations that occur during conical indentation, as compared to spherical indentation, which is slightly less volumetrically-dominated.

bonded aluminum backers, and the shear displacement is applied by moving one backer at a rate of $\dot{\delta}/H = 2.0 \times 10^{-4} \text{ s}^{-1}$ to a maximum shear displacement of $\delta/H = 0.5$ and then unloading, while the other backer remains fixed, and the distance between the backers is held at H . Notably, all rectangular side-faces remain unconstrained, so that edge-effects are present, and hence, deformation is simple-shear-like but inhomogeneous. In the second type of experiment, we consider simple-shear-like deformation with a fixed uniaxial pre-compression as an example of coupled distortional/volumetric deformation. The simple-shear-like experiment with pre-compression is carried out in two steps. First, the specimen is compressed, such that the deformed distance between the constrained faces is $0.75H$. Specimens are then allowed to rest for at least 40 minutes, while the pre-compression is held fixed in order to allow complete relaxation of viscoelastic material behavior. Second, the specimen is sheared by a shear displacement δ at the same rate of $\dot{\delta}/H = 2.0 \times 10^{-4} \text{ s}^{-1}$ to a maximum shear displacement of $\delta/H = 0.375$ and then unloaded, while the distance between the constrained faces is held fixed at $0.75H$. In both types of experiments, we utilize the double-shear fixture pictured in Fig. 2.7(a), which is akin to a double-lap or three-bar (e.g., ASTM D4255 Procedure B) shear design. In this configuration, two specimens are tested simultaneously in order to maintain symmetry, and therefore, the resultant shear force P is identified as half the force applied to the moving center platen. The measured normalized shear force P/W^2 versus shear displacement δ relations for all three foams densities and both without and with pre-compression are shown as solid lines in Fig. 2.7(c). As before, the shaded regions denote the range of one standard deviation above and below the average experimental measurement, and the arrows denote loading and unloading, demonstrating minimal hysteresis. Two features are notable in the force-displacement relations. First, in shear-dominated deformation, no plateau region is observed, and the slope of the force-displacement relation increases slightly as deformation progresses. Second, when pre-compression is applied, the force-displacement relation becomes more compliant, illustrating the interaction between volumetric and distortional aspects of the

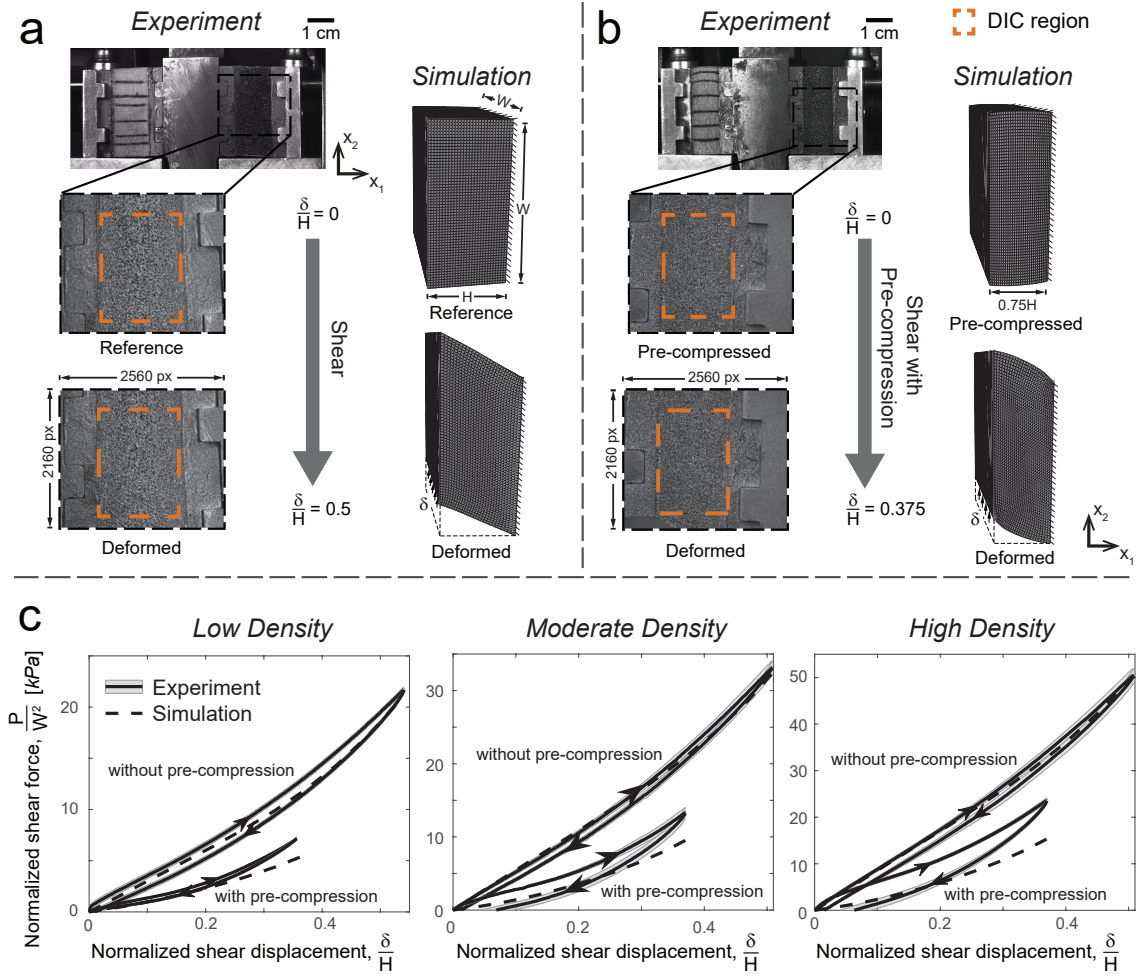


Figure 2.7: Simple-shear-like validation tests both without and with uniaxial pre-compression. (a) Snapshot of the double-shear fixture with two specimens prior to deformation and the experimental region-of-interest used for DIC imaging (approximately $1300 \times 1200 \text{ px}^2$) in both the reference and the deformed states for the case without pre-compression. The corresponding finite-element configuration is shown on the right, also in both the reference and the deformed states. (b) Experimental images and finite-element configuration for the case with pre-compression (DIC region approximately $950 \times 1200 \text{ px}^2$), displaying the pre-compressed and the deformed, sheared states. (c) Normalized shear force P/W^2 versus normalized shear displacement δ/H both without and with pre-compression and for all three densities. Solid lines with shaded error regions represent experimental data, and dashed lines are simulated predictions of the constitutive model.

material response.

For the corresponding finite-element simulations, a fully three-dimensional model is utilized. The reference finite-element mesh, consisting of 97,200 Abaqus-C3D8H elements (eight-node, three-dimensional, hexahedral, constant-pressure elements) is shown

in Fig. 2.7(a). Regarding boundary conditions, for simple-shear-like deformation *without* pre-compression, the right face of the mesh in Fig. 2.7(a) represents the fixed backer, on which all displacement components are prescribed to be zero. The left face of the mesh represents the moved backer, on which the vertical displacement u_2 is prescribed, while the other displacement components are constrained – i.e., $u_1 = 0$, $u_2 = \delta$, and $u_3 = 0$. The remaining faces are traction-free. The deformed mesh after a shear displacement of $\delta/H = 0.5$ is shown in Fig. 2.7(a), which upon close inspection, reveals the inhomogeneous nature of deformation near the boundaries in the simulation. For simple-shear-like deformation *with* pre-compression, pre-compression is applied by prescribing $u_1 = -0.25H$ and $u_2 = u_3 = 0$ on the right face of the mesh, while holding all displacement components at zero on the left face of the mesh. The deformed mesh after pre-compression is shown in 2.7(b), displaying some bulging in the vicinity of the traction-free faces. Then, a shear displacement is applied by fixing $u_1 = -0.25H$ and $u_2 = u_3 = 0$ on the right face of the mesh and prescribing $u_2 = \delta$ and $u_1 = u_3 = 0$ on the left face of the mesh. The deformed mesh after a shear displacement of $\delta/H = 0.375$ is shown in Fig. 2.7(b), displaying significant inhomogeneous deformation. The calculated shear force P versus shear displacement δ relations are included as dashed lines in Fig. 2.7(c) for each density both without and with pre-compression. For all densities, the model captures the features of the response both without and with pre-compression. In particular, it predicts both the lack of a plateau region in the response during shear-dominated deformation and the increased compliance with pre-compression. The ability of the model to capture the effect of pre-compression on the response in simple-shear-like deformation provides a partial validation of the coupled volumetric/distortional contribution to the free-energy function (2.2).

In addition to P versus δ relations, we also compare experimentally-measured and numerically-simulated displacement fields on the front face of the specimen. The DIC regions-of-interest are outlined with dashed lines in the reference, undeformed state and the

deformed state at the maximum level of shear displacement for simple-shear-like deformation without and with pre-compression in Figs. 2.7(a) and (b), respectively, and contours of the two-dimensional, DIC-calculated displacement field – i.e., $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ – on the front face of the specimen in a representative experiment on the moderate density foam are shown in the left columns of Figs. 2.8(a) and (b). While in Fig. 2.8(a), the fields represent the displacement from the reference state to the deformed state, the fields in Fig. 2.8(b) for simple-shear-like deformation with pre-compression represent the displacement from the pre-compressed state to the deformed, sheared state.⁹ The corresponding simulated displacement fields are shown in the right columns of Figs. 2.8(a) and (b), demonstrating that the salient features of the experimental displacement fields are captured by the model. In particular, inhomogeneous deformation occurring due to edge-effects are clearly apparent in the u_1 -fields in both experiments and simulations. Overall, the magnitudes and distributions of both the u_1 - and u_2 -components of the displacement field are consistent between experiment and simulation.

2.4.3 Tension of a specimen with circular holes

Finally, we conduct a validation test to evaluate the predictive capability of the constitutive model in a tension-dominated setting. Given the favorable comparisons of model predictions and experiments across all three densities in the cases of indentation and simple-shear-like deformation, we focus only on the high density foam in this final test. To obtain inhomogeneous deformation fields, we take inspiration from the work of Wang and Chester [62] and cut circular holes in our tension specimens. Specifically, we begin with the dogbone geometry used in the simple tension characterization experiments described in Section 2.2 and laser-cut six, non-overlapping circular holes, each with a radius of 2.5 mm, at arbitrarily-chosen locations in the gage-section. This creates a purposefully asymmetric geometry,

⁹This exception in the definition of the displacement field is only made in Fig. 2.8(b), and for notational economy, we continue to denote the displacement components as u_1 and u_2 .

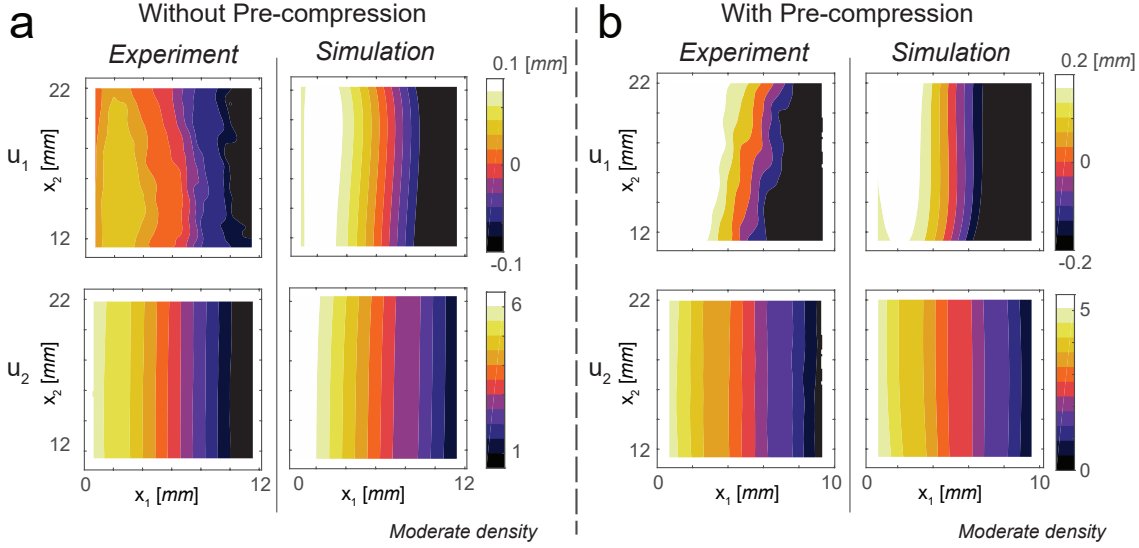


Figure 2.8: Contours of the two-dimensional displacement field – i.e., $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ – on the front face of the specimen in the DIC region-of-interest from both experiment and simulation for simple-shear-like deformation (a) without pre-compression and at a shear displacement of $\delta/H = 0.5$ and (b) with pre-compression and at a shear displacement of $\delta/H = 0.375$ for the moderate density foam. For simple-shear-like deformation with pre-compression, the fields represent the displacement from the pre-compressed state to the deformed, sheared state.

which is shown in Fig. 2.9(a). These modified specimens are then subjected to a tension-testing procedure analogous to that described in Section 2.2 for homogeneous simple tension experiments. The bottom grip is fixed, and the top grip moves upward at a displacement rate of $\dot{\delta} = 2.0 \times 10^{-3}$ mm/s. The measured tension force P as a function of the top grip displacement δ is shown in Fig. 2.9(b) as a solid line. As before, the shaded area denotes the range of one standard deviation above and below the average experimental measurement, and the arrows denote loading and unloading, demonstrating minimal hysteresis.

In the corresponding finite-element simulations, we model only the region between the grips, consisting of the gage-section and the section of the specimen between the grips and the gage-section. The specimen is modeled in three dimensions, matching the experimental geometry, and meshed with 185,216 Abaqus-C3D8H three-dimensional elements, as shown in Fig. 2.9(a). The front and back faces of the mesh as well as the side faces and the interior faces of the holes are traction-free. Regarding displacement boundary conditions, the bottom

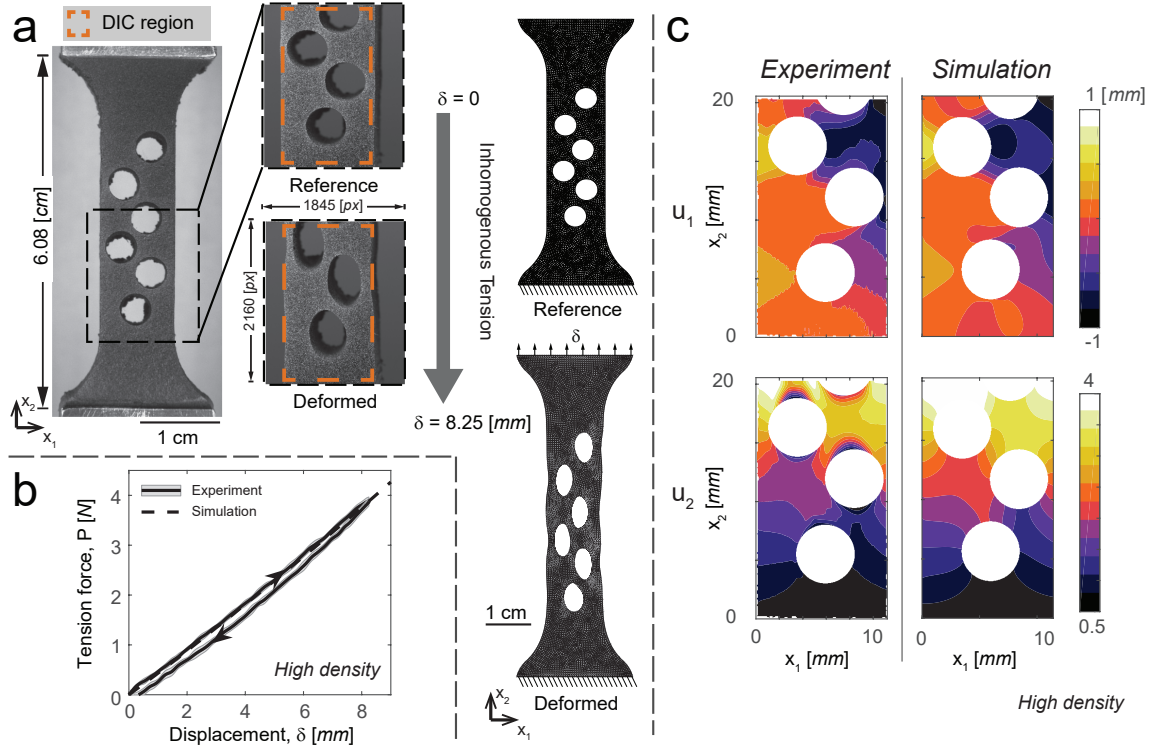


Figure 2.9: A tension-dominated validation test for the high density foam consisting of tension of a specimen with circular holes, in which a standard dogbone geometry is modified with six asymmetrically-placed 2.5 mm radius holes. (a) Snapshot of the experimental specimen prior to deformation and the experimental region-of-interest used for DIC imaging in both the reference and the deformed states (DIC region approximately 1100×2000 px²) as well as the corresponding finite-element configurations. (b) The top-grip force P versus displacement δ , in which solid lines with shaded error regions represent experimental data, and dashed lines are simulated predictions of the constitutive model. (c) Contours of the two-dimensional displacement field – i.e., $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ – on the front face of the specimen in the DIC region-of-interest from both experiment and simulation.

of the mesh is fully constrained – i.e., $u_1 = u_2 = u_3 = 0$ – and on the top surface of the mesh the vertical displacement u_2 is prescribed, while the other displacement components are constrained – i.e., $u_1 = 0$, $u_2 = \delta$, and $u_3 = 0$. Fig. 2.6(b) shows the numerically-predicted force-displacement response as a dashed line, showing excellent agreement with the experimentally-measured relation.

As in the simple-shear-like validation case, we also compare experimentally-measured and numerically-simulated displacement fields on the front face of the specimen. The DIC region-of-interest is shown both in the reference state as well as the deformed state after

a top-grip displacement of $\delta = 8.25$ mm in Fig. 2.9(a), outlined with dashed lines, and contours of the two-dimensional, DIC-calculated displacement field – i.e., $u_1(x_1, x_2)$ and $u_2(x_1, x_2)$ – on the front face of the specimen in a representative experiment are plotted in Fig. 2.9(c). The corresponding simulated front-face displacement fields are plotted alongside the experimental results in Fig. 2.9(c). There is strong quantitative agreement between the experimentally-measured and simulated displacement fields throughout the region-of-interest. The notable differences, specifically, the displacement gradients near the tops and bottoms of the holes in the experimental results, arise as an artifact of incorporating internal material-void boundaries within an Eulerian framework for the hybrid, cumulative-iterative displacement summation scheme used in the qDIC algorithm (see Landauer et al. [43]) and subsequent plotting in the reference configuration.

2.5 Discussion

2.5.1 Stability

A common drawback of stretch-based, compressible hyperelastic models [1, 2] is unstable behavior that limits the deformation range over which the model may be robustly applied. In this section, we examine the stability of the proposed hyperelastic model and illustrate the limits of stable behavior. Here, we assess stability by testing the ellipticity of the quasi-static governing equations – i.e., testing that the eigenvalues of the acoustic tensor in the deformed configuration remain positive for given homogeneous base states of deformation [e.g., 63–66]. For a homogeneous base state of deformation, the spatial acoustic tensor for a hyperelastic material is

$$Q_{ik}(\mathbf{n}) = \mathbb{C}_{ijkl}n_jn_l \quad \text{with} \quad \mathbb{C}_{ijkl} = J^{-1}F_{jn}F_{lq}\frac{\partial^2\psi}{\partial F_{in}\partial F_{kq}}, \quad (2.9)$$

where $\mathbb{C}_{ijkl}$ are the spatial moduli, evaluated at the base state, and $\mathbf{n}$ is a unit vector. Details of the steps leading to (2.9) as well as an expression for the spatial moduli $\mathbb{C}_{ijkl}$ that is appropriate for use with a logarithmic-strain-based free-energy function are given in Appendix B. If the eigenvalues of $\mathbf{Q}(\mathbf{n})$ for a given base state of deformation remain positive for all unit vectors $\mathbf{n}$, the ellipticity of the governing equations is maintained, and homogeneous deformation is stable. However, if an eigenvalue of $\mathbf{Q}(\mathbf{n})$ is negative for any unit vector $\mathbf{n}$, the localization of deformation into bands becomes possible. To test for stability, we consider base states of deformation described by a spatial logarithmic strain tensor, $\mathbf{E}$, with invariants, $\{K_1, K_2, K_3\}$, and principal directions, $\{\mathbf{l}_1, \mathbf{l}_2, \mathbf{l}_3\}$, and all unit vectors $\mathbf{n}$ in three dimensions – i.e., $\mathbf{n} = \cos \theta \sin \gamma \mathbf{l}_1 + \sin \theta \sin \gamma \mathbf{l}_2 + \cos \gamma \mathbf{l}_3$ over the ranges $0 \leq \theta \leq 2\pi$ and $0 \leq \gamma \leq \pi$. In this way, we may identify whether unstable behavior is possible for a given combination of invariants, $\{K_1, K_2, K_3\}$.¹⁰

Representative stability maps, generated using the free-energy density function introduced in Section 2.3 and the material parameters for the moderate density foam from Table 2.1, are shown in Fig. 2.10. Three specific modes of loading are considered in Fig. 2.10: (a) $K_3 = 1$ (simple tension), (b) $K_3 = -1$ (simple compression), and (c) $K_3 = 0$ (simple shear). Stable combinations of $\{K_1, K_2, K_3\}$ are denoted as white regions, while combinations of $\{K_1, K_2, K_3\}$ in which at least one eigenvalue of $\mathbf{Q}(\mathbf{n})$ is negative for at least one unit vector $\mathbf{n}$ are denoted as gray regions. In Figs. 2.10(a) and (b), the deformation paths for simple tension and simple compression are shown as solid lines, and in Fig. 2.10(c), an idealized homogeneous loading path for simple shear, representative of the inhomogeneous simple-shear-like deformation of Section 2.4.2, is shown. In all cases, the deformation paths traverse stable states.

While large regions of stable behavior are evident in Fig. 2.10, unstable behavior is also observed. The sources of unstable behavior may be linked to features of the model. First, the unstable regions in the bottom left corners of Figs. 2.10(a), (b), and (c) arise because

¹⁰ We also utilize the analytical necessary and sufficient conditions for strong ellipticity of Dacorogna [67] as an independent verification of the results presented in this section.

$X(K_1)$ is negative in this range when the material parameters for the moderate density foam are utilized. The phenomenological form for $X(K_1)$ (2.5) is an increasing function of K_1 , which is enforced to be positive over the fitted range; however, for sufficiently negative values of K_1 , $X(K_1) < 0$, leading to unstable behavior. If necessary, the functional form (2.5) may be extended to be positive for all K_1 to increase stable behavior. Second, since $dX/dK_1 > 0$ for all K_1 , the shear-induced mean stress, $G_0(dX/dK_1)K_2^2$, is positive and may be of arbitrarily large magnitude as K_2 increases, leading to unstable behavior. In particular, the unstable region in the upper right domain of Fig. 2.10(b) is due to this effect. This feature is due to the low-order K_2 -dependence of the coupled volumetric/distortional response and cannot be remedied by modifying the forms of the fitting functions. Instead, it would require an alternate structure for the decomposition of the free-energy function (2.2). Finally, unstable behavior for high values of K_2 can arise due to the higher-order distortional response function, $L(K_2, K_3)$, and its dependence on K_3 . In particular, the unstable behavior in the top region of Fig. 2.10(c) is due to this effect. The functional form (2.6) may be extended with higher-order K_2 -terms – perhaps motivated by locking behavior in the distortional response – to suppress unstable behavior for large values of K_2 . In sum, the model exhibits robust, largely-stable behavior as evidenced by Fig. 2.10 as well as the numerical solutions obtained across a diverse set of inhomogeneous deformation scenarios in Section 2.4.

2.5.2 Comparison to other models

In this section, we compare the ability of the proposed model to capture experimental data with that of other important hyperelastic models for porous elastomers in the literature. We begin with the classic model of Blatz and Ko [8] – a phenomenological, invariant-based free-energy function for porous elastomers – which depends on the second and third principal invariants of $\mathbf{C}$ as follows:

$$\psi = \frac{G_0}{2} \left(\frac{I_2}{I_3} + 2\sqrt{I_3} - 5 \right), \quad (2.10)$$

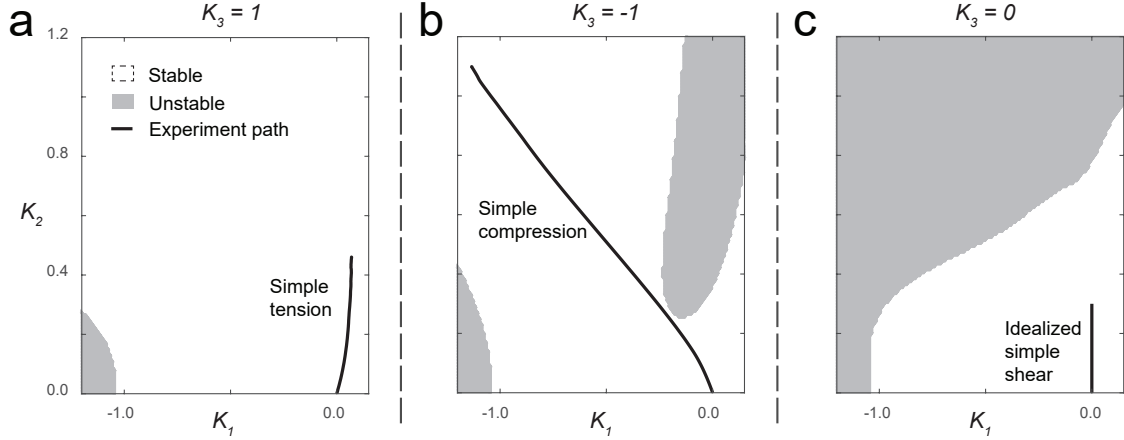


Figure 2.10: Stability diagrams calculated using a loss-of-ellipticity test as a function of the invariants of the logarithmic strain K_1 and K_2 for several cases of fixed K_3 : (a) $K_3 = 1$ (simple tension), (b) $K_3 = -1$ (simple compression), and (c) $K_3 = 0$ (simple shear). The white regions indicate stable behavior, while gray regions indicate the potential for unstable behavior. Solid lines correspond to representative deformation paths from simple tension, simple compression, and idealized simple shear, respectively.

where G_0 is the ground-state shear modulus of the foam. We note that (2.10) represents a simplified version of a broader three-parameter model and implies a ground-state Poisson's ratio of 0.25. Since the ground-state Poisson's ratio of the moderate density Poron XRD foam is about 0.27, we utilize the simplified form (2.10) along with the previously-fitted value of $G_0 = 65.2$ kPa (Table 2.1). The resulting engineering stress versus axial strain and lateral strain versus axial strain response in simple compression/tension for the Blatz-Ko model is shown in Figs. 2.11(a) and (b) using square ($\square$) markers along with the experimental data for the moderate density foam. As expected for the highly-porous foam under consideration, the Blatz-Ko model does a reasonable job capturing data in the low to moderate strain regime, i.e., axial engineering strains between approximately -0.1 and 0.2, but is not applicable beyond this regime.

Next, we consider two homogenization-based models for porous elastomers with spherical voids: the model of Danielsson et al. [11] and the more recent model of Shrimali et al. [14]. Both models consider microstructures consisting of closed-cell voids and do not account for buckling in the elastomeric-matrix microstructure during compression. Therefore,

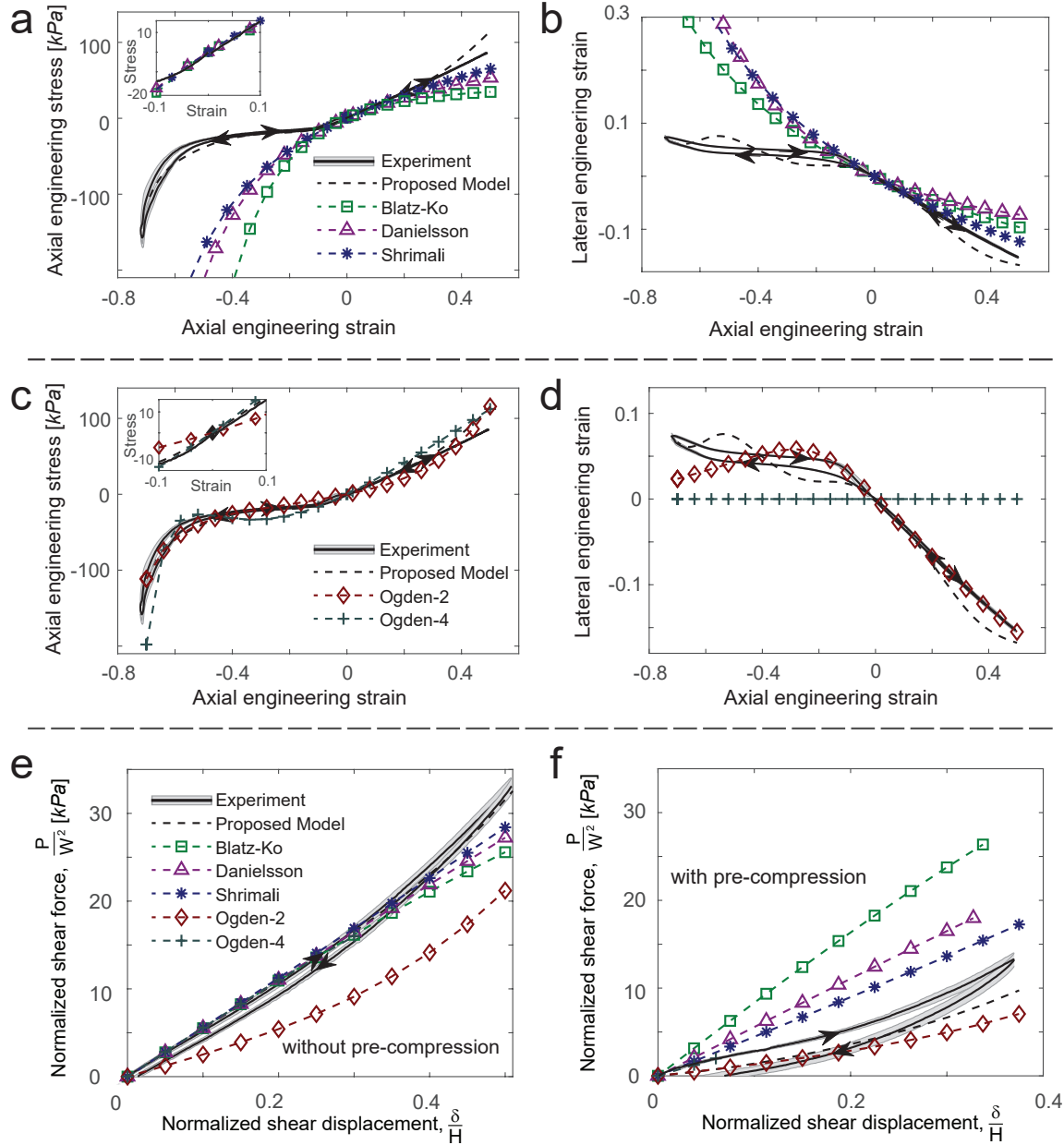


Figure 2.11: Comparison of fits obtained using hyperelastic models for porous elastomers in the literature with the proposed approach for the moderate density Poron XRD foam. (a) Axial engineering stress and (b) lateral engineering strain versus axial engineering strain fits for the proposed model, the [8] model (Blatz-Ko), the Danielsson et al. [11] model (Danielsson), and the Shrimali et al. [14] model (Shrimali). (c) Axial engineering stress and (d) lateral engineering strain versus axial engineering strain fits for a two-term Ogden-Storakers fit (Ogden-2) and a four-term Ogden-Storakers fit (Ogden-4) [1, 2]. (*Insets (a) and (c)*) Close-up of the moderate-strain regime. Normalized shear force P/W^2 versus normalized shear displacement δ/H for simple-shear-like deformation both (e) without and (f) with pre-compression for all models considered as well as experimental data for the moderate density Poron XRD foam.

they are limited in scope to low to moderate porosity foams, and we do not expect these models to capture experimental data for the highly-porous, open-cell foam under consideration. However, since these models are two of only a handful of explicit, compressible hyperelasticity models, we include them in our discussion for the sake of comparison. First, the approach of Danielsson et al. [11], when specialized to the case of a neo-Hookean matrix, yields the following free-energy function that depends on the invariants $I_1 = \text{Tr} \mathbf{C}$ and $J = \det \mathbf{F}$:

$$\psi = \frac{G_m}{2} \left[I_1 \left(2 - \frac{1}{J} - \frac{\phi_0 + 2(J-1)}{J^{2/3}(1 + (J-1)/\phi_0)^{1/3}} \right) - 3(1 - \phi_0) \right], \quad (2.11)$$

where G_m is the ground-state shear modulus of the matrix material and ϕ_0 is the void volume fraction of the undeformed porous material. The ground-state shear modulus of the foam material is then $G_0 = G_m(1 - \phi_0)$. From Table 2.1, we take $G_0 = 65.2$ kPa and estimate $\phi_0 = 1 - J_{\min} = 0.84$, so that $G_m = 407.5$ kPa. The corresponding engineering stress versus axial strain and lateral strain versus axial strain response in simple compression/tension is shown in Figs. 2.11(a) and (b) using triangle ($\triangle$) markers. We note that $\phi_0 = 1 - J_{\min}$ is a rough estimate of the initial void volume fraction. For this reason, we have considered values of ϕ_0 over the range $0.75 \leq \phi_0 \leq 0.95$ with $G_0 = 65.2$ kPa fixed and confirmed that within this range, the precise value of ϕ_0 has a minimal impact on the response shown in Figs. 2.11(a) and (b). Second, the homogenization-based free-energy function of Shrivalli et al. [14] – also specialized to the case of a neo-Hookean matrix – is

$$\begin{aligned} \psi = & \frac{3G_m(1 - \phi_0)}{2(3 + 2\phi_0)}(I_1 - 3) \\ & + \frac{3G_m}{2J^{1/3}} \left[2J - 1 - \frac{(1 - \phi_0)J^{1/3}(3J^{2/3} + 2\phi_0)}{3 + 2\phi_0} - \frac{\phi_0^{1/3}J^{1/3}(2J + \phi_0 - 2)}{(J - 1 + \phi_0)^{1/3}} \right], \end{aligned} \quad (2.12)$$

where G_m and ϕ_0 continue to denote the ground-state shear modulus of the matrix material and the undeformed void volume fraction. The ground-state shear modulus of the foam

material for this model is $G_0 = 3G_m(1 - \phi_0)/(3 + 2\phi_0)$, and again, we take $G_0 = 65.2$ kPa and $\phi_0 = 0.84$, so that $G_m = 635.7$ kPa for this case. The corresponding response is included in Figs. 2.11(a) and (b) using star (*) markers. As for the model of Danielsson et al. [11], we have confirmed that the response shown in Figs. 2.11(a) and (b) for the model of Shrimali et al. [14] is minimally affected as the value of ϕ_0 is varied over the range $0.75 \leq \phi_0 \leq 0.95$, while $G_0 = 65.2$ kPa remains fixed. As expected, both models yield similar predictions and are capable of capturing data in the moderate strain regime but do not capture the response beyond this regime – especially the response in the plateau regime in compression.

Finally, we consider the stretch-based Ogden-Storakers model [1, 2]. This is the most commonly utilized phenomenological hyperelasticity model for highly-compressible foams and is built in to Abaqus [61] as the “hyperfoam” material. The free-energy function for the Ogden-Storakers model is given through the principal stretches $\{\lambda_1, \lambda_2, \lambda_3\}$ and $J = \lambda_1 \lambda_2 \lambda_3$ as

$$\psi = \sum_{n=1}^N \frac{2G_n}{\alpha_n^2} \left[\lambda_1^{\alpha_n} + \lambda_2^{\alpha_n} + \lambda_3^{\alpha_n} - 3 + \frac{1}{\beta_n} \left(J^{-\alpha_n \beta_n} - 1 \right) \right], \quad (2.13)$$

where N is the number terms and $\{G_n, \alpha_n, \beta_n\}$ are material properties for each term. We consider two-term and four-term fits, which we refer to as Ogden-2 and Ogden-4, respectively. For both cases, we utilize the material parameter fitting routines built in to Abaqus. The six parameters of the Ogden-2 fit were determined to be $\{G_1 = 24.4$ kPa, $\alpha_1 = 8.31$, $\beta_1 = -0.154$, $G_2 = 4.18$ kPa, $\alpha_2 = -1.39$, $\beta_2 = 7.71\}$, and the corresponding fitted response is shown in Figs. 2.11(c) and (d) using diamond ($\diamond$) markers. Although the Ogden-2 fit captures the overall shape of the axial stress and lateral strain versus axial strain responses, the accuracy of the fit in the moderate-strain regime is sacrificed, as shown in the inset of Fig. 2.11(c). Regarding the Ogden-4 fit, we found that to enable the nonlinear fitting routine in Abaqus to converge, the lateral strain response must be fixed to zero, so that all $\beta_n = 0$ (see Fig. 2.11(d)). Hence, there are eight fitted parameters in total for the Ogden-4 fit, which were determined to be $\{G_1 = 568$ kPa, $\alpha_1 = 0.867$, $G_2 = 610$ kPa, $\alpha_2 = 0.859$, $G_3 =$

$-1.71 \text{ MPa}, \alpha_3 = 0.165, G_4 = 627 \text{ kPa}, \alpha_4 = -0.477\}$. Although the axial stress versus axial strain response of the Ogden-4 fit – shown in Fig. 2.11(c) using plus (+) markers – more closely matches experimental data in the moderate-strain regime compared to the Ogden-2 fit, the response is non-monotonic in compression, which suggests unstable behavior. Indeed, the loss-of-ellipticity test reveals unstable behavior for a large set of deformation states for the Ogden-4 fit. For this class of highly-compressible, open-cell elastomeric foams, it is typically quite difficult to obtain a fit using stretch-based models that captures both the axial stress and lateral strain versus axial strain responses and behaves stably over a wide range of deformation states, and the inclusion of additional terms does not change this observation. For example, we have considered five-term and six-term Ogden-type fits, while taking all $\beta_n = 0$ so that the fits involve ten and twelve parameters, respectively, and we observe unstable behavior similar to that of the Ogden-4 fit.

We also compare predictions of all models considered in this section against the experimental data from inhomogeneous simple-shear-like deformation both without and with pre-compression described in Section 2.4.2. Model predictions are obtained using finite-element simulations, and the calculated shear force P versus shear displacement δ relations are shown in Figs. 2.11(e) and (f) along with experimental data for the moderate density foam. The Blatz-Ko model ($\square$) does not capture the increased compliance in shear with pre-compression, and while both homogenization-based models ($\triangle$ and $*$) do predict some increase in compliance with pre-compression, they do not quantitatively capture the normalized force versus displacement response in simple-shear-like deformation with pre-compression. The Ogden-2 fit ($\diamond$) does not capture the experimental data for simple-shear-like deformation without pre-compression but is consistent with the data for simple-shear-like deformation with pre-compression. Finally, due to instabilities in the fitted model, numerical solutions using the Ogden-4 fit (+) cannot be obtained beyond moderate shear displacements – namely, $\delta/H = 0.3$ and $\delta/H = 0.08$ for simple-shear-like deformation without and with pre-compression, respectively. These results illustrate the difficulties

in robustly capturing experimental data for highly-compressible, open-cell elastomeric foams and the improved predictions enabled by the proposed approach.

2.6 Concluding remarks

Summary In this work, we utilized an extensive set of experiments on three densities of the polyurethane-based, open-cell elastomeric foam “Poron XRD”. Experiments consisted of homogeneous simple compression/tension as well as three types of inhomogeneous experiments: spherical and conical indentation, simple-shear-like deformation without and with pre-compression, and tension of a specimen with circular holes. A phenomenological, isotropic, finite-deformation, hyperelastic constitutive model based on invariants of the logarithmic, Hencky strain was proposed. A key simplifying assumption of our modeling approach is that volumetric/distortional coupling only involves low-order dependence on the magnitude of distortional deformation – which leads to a more straightforward interpretation of the fitting functions and a systematic path for material parameter estimation from simple compression/tension data (see Appendix A). In compression and tension, the fitted model for each of the three densities of Poron XRD faithfully captures the nonlinear stress versus axial strain and lateral strain versus axial strain responses, in particular, the tension/compression asymmetry featuring a nearly-flat plateau regime in compression. The constitutive model was implemented in Abaqus [61] using a user-material subroutine, which was used to obtain model predictions in inhomogeneous deformation settings for the purpose of validation. The model predictions were shown to be consistent with experimental data in the inhomogeneous validation cases across compression, shear, and tension-dominated settings. The user-material subroutine implementation of the constitutive model and input files for each validation case have been made available (<https://github.com/HenannResearchGroup>).

Limitations Our hyperelastic model was developed to capture the equilibrium response of open-cell, isotropic elastomeric foams with relatively high void volume fraction. As such,

the model has several operational limitations arising from assumptions made in model development. First, many elastomeric foam materials exhibit strain-rate-dependent, viscoelastic material behavior, whereas the proposed model is rate-independent. Second, the polymeric matrix material of the foam may be temperature sensitive, in some cases dramatically so, and this effect may be coupled with the rate-sensitivity of the matrix material. All experiments in the present work were conducted under approximately isothermal conditions at room temperature. Accordingly, the model also assumes isothermal conditions, and the effect of variable temperature is not captured. To use the model at elevated or reduced temperature, a new set of experiments at the temperature of interest would be required. Third, depending upon the microstructure, elastomeric foam materials may exhibit strain localization into compaction bands under uniform external loading. The foam material considered in the present work did not exhibit localization, and localization behavior is not accounted for in the model.

Future work There are numerous avenues for future work. This study focused on a single polyurethane-based, open-cell elastomeric foam material over a moderate range of densities. We expect that the mechanical behavior of other open-cell foam materials may be well captured by the proposed model, so long as the material does not exhibit discernible strain localization. Future work will apply the proposed modeling approach to other elastomeric foam materials. We expect that improvements will be made to the phenomenological fitting functions utilized in the present work, either driven by additional experimental data for different elastomeric foam materials or motivated by micromechanical arguments. Furthermore, while the model has been tested in compression, shear, and tension-dominated situations, model validation has not exhaustively spanned all combinations of the invariants $\{K_1, K_2, K_3\}$. For example, it remains to test the proposed model in equibiaxial tension, in which $K_1 > 0$ and $K_3 = -1$ (cf., e.g., the simple compression, $K_1 < 0$ and $K_3 = -1$, or simple tension, $K_1 > 0$ and $K_3 = 1$, modes used in model calibration). Beyond hyperelastic modeling

of elastomeric foams, we envision that the proposed modeling framework may be extended to incorporate the effects of strain-rate and temperature sensitivity by adapting existing finite viscoelasticity frameworks for monolithic elastomers [e.g., 27, 68]. Incorporating the effects of strain-rate sensitivity will be the focus of the subsequent chapters. Finally, localization arising from microstructural defects is a key aspect of deformation in very-low-density foams, and developing a model that captures this behavior will be the subject of future work.

Chapter 3

Nonlinear constitutive modeling of viscoelastic foams

3.1 Introduction

Due to the viscoelastic behavior of the matrix material, many elastomeric foams display a highly dissipative, time-dependent mechanical response, manifesting in strain-rate-dependence and hysteresis under reversed loading. These materials are widely used in situations in which both compliance and energy dissipation are necessary, e.g., impact protection, cushioning, and vibration damping, and appear in a wide variety of specific applications, such as equipment for impact protection. In order to best harness the dissipative potential of this class of materials, a predictive constitutive model for their rate-dependent, large-deformation mechanical behavior is needed for use in design.

The complex mechanical behavior of viscoelastic elastomeric foams involves rate-dependence, energy dissipation, large deformations, volumetric compressibility, and substantial tension/compression asymmetry. The purpose of this chapter is to outline the framework of a nonlinear viscoelastic constitutive model that is applicable to a wide variety of elastomeric foams. We focus on the rate-dependent behavior of two types of viscoelastic foams:

(1) the closed-cell EC-2048 Ethylene-Vinyl Acetate (EVA) foam developed by Reebok Corporation (Reebok, Adidas, Boston, MA) and (2) the open-cell polyurethane-based foam Poron XRD foam, described in Chapter 2. The experimentally-measured mechanical response of EVA under quasi-static and moderate strain-rates (1.0×10^{-3} - $1.0 \times 10^1 \text{ s}^{-1}$) is demonstrated in Fig. 3.1, and similar data for three different densities of Poron XRD over the strain-rate range (1.0×10^{-3} - 1.0 s^{-1}) is shown in Figs. 3.2, 3.3 and 3.4. (The experimental data sets shown in these figures were obtained by Jialiang Tao and Alexander K. Landauer, using the experimental procedures described in Chapter 2.2 and Appendix C.) For each foam and each strain rate, these figures show the experimentally-measured engineering stress versus axial engineering strain response in the left column and the corresponding lateral engineering strain versus axial engineering strain response in the right column for simple compression/tension. As noted previously, the responses for all materials and strain rates are highly nonlinear and display substantial asymmetry between tension and compression. Moreover, the stress/strain response is rate-dependent and displays hysteresis upon reversed loading with the amount of hysteresis and the average stress-level increasing as the strain rate increases.

Importantly, the quantitative nature of rate-dependence is different in the closed-cell EVA foam than in the open-cell Poron XRD foams. As shown in Fig. 3.1, for the closed-cell EVA foam, rate-dependence is less obvious when comparing the stress/strain curves for the lowest three strain rates (1.0×10^{-3} - $1.0 \times 10^{-1} \text{ s}^{-1}$) than when comparing the stress/strain curves for the highest two strain rates (1.0 s^{-1} and $1.0 \times 10^1 \text{ s}^{-1}$). The stress/strain curves for the lowest three strain rates are quite similar, indicating a low degree of rate-dependence. In contrast, the effect of rate-dependence is significant in the stress/strain curves across all four rates for all three densities of the open-cell Poron XRD foam. This difference in the nature of rate-dependence for the two foam materials creates challenges for developing a model that is applicable to both materials. Another interesting feature of the stress/strain curves that presents a modeling challenge is that for the moderate and high density open-cell

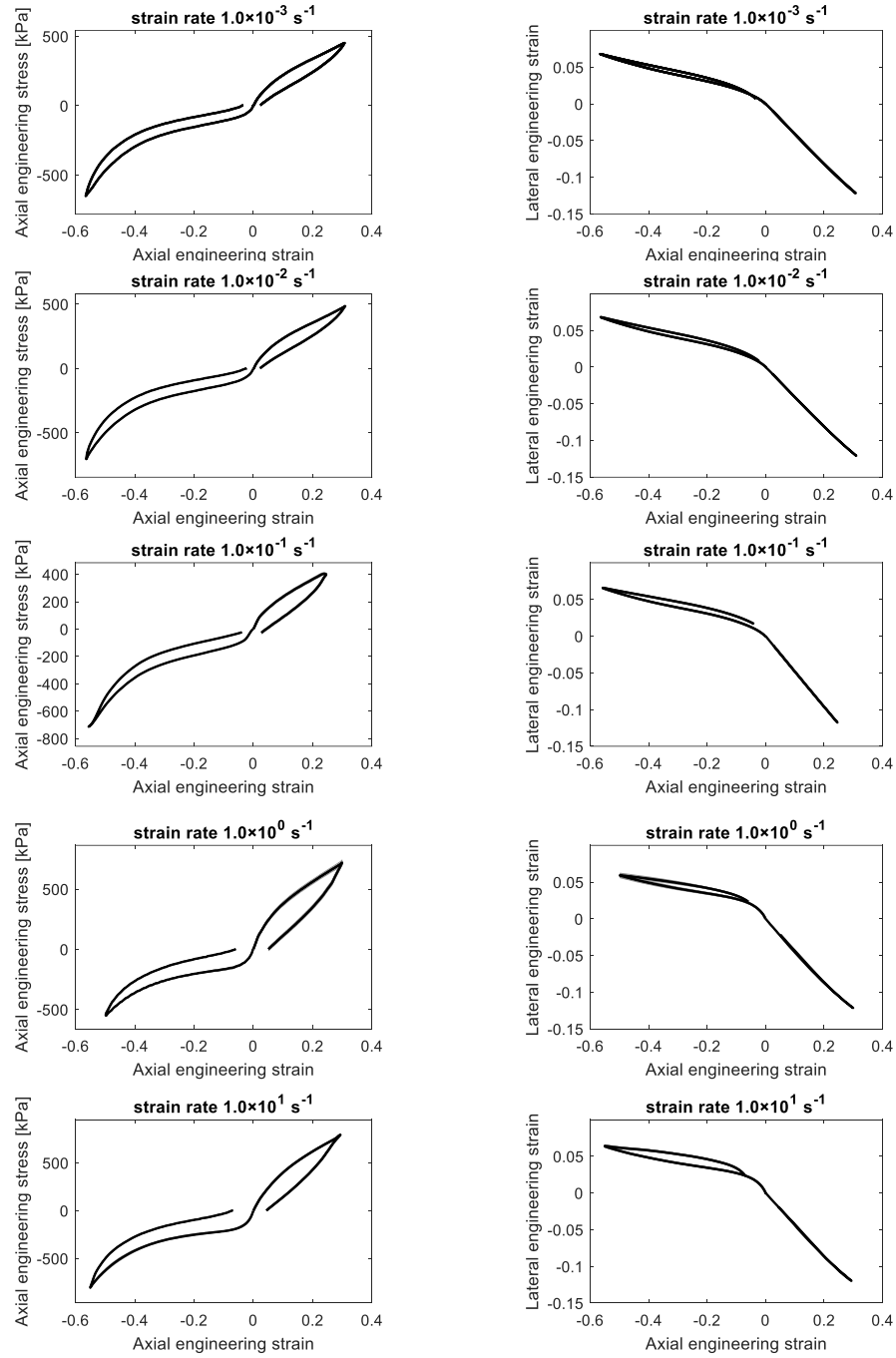


Figure 3.1: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam under quasi-static and moderate strain rates. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data. Note that for different rates, the stress axes differ in range. Experimental data was obtained by Jialiang Tao.

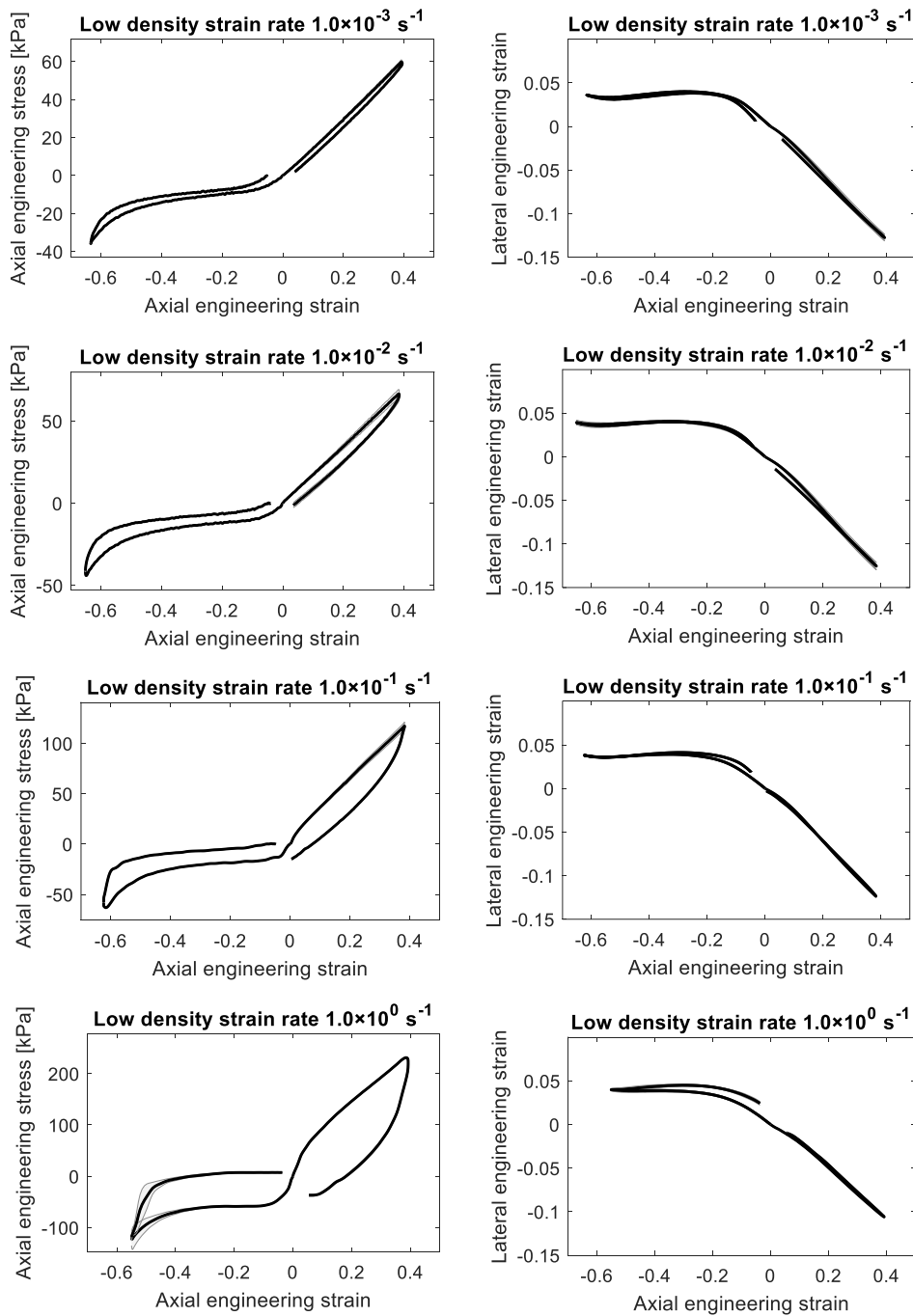


Figure 3.2: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of low density (144 kg/m^3) under quasi-static and moderate strain rates. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data. Note that for different rates, the stress axes differ in range. Experimental data was obtained by Alexander K. Landauer and Jialiang Tao.

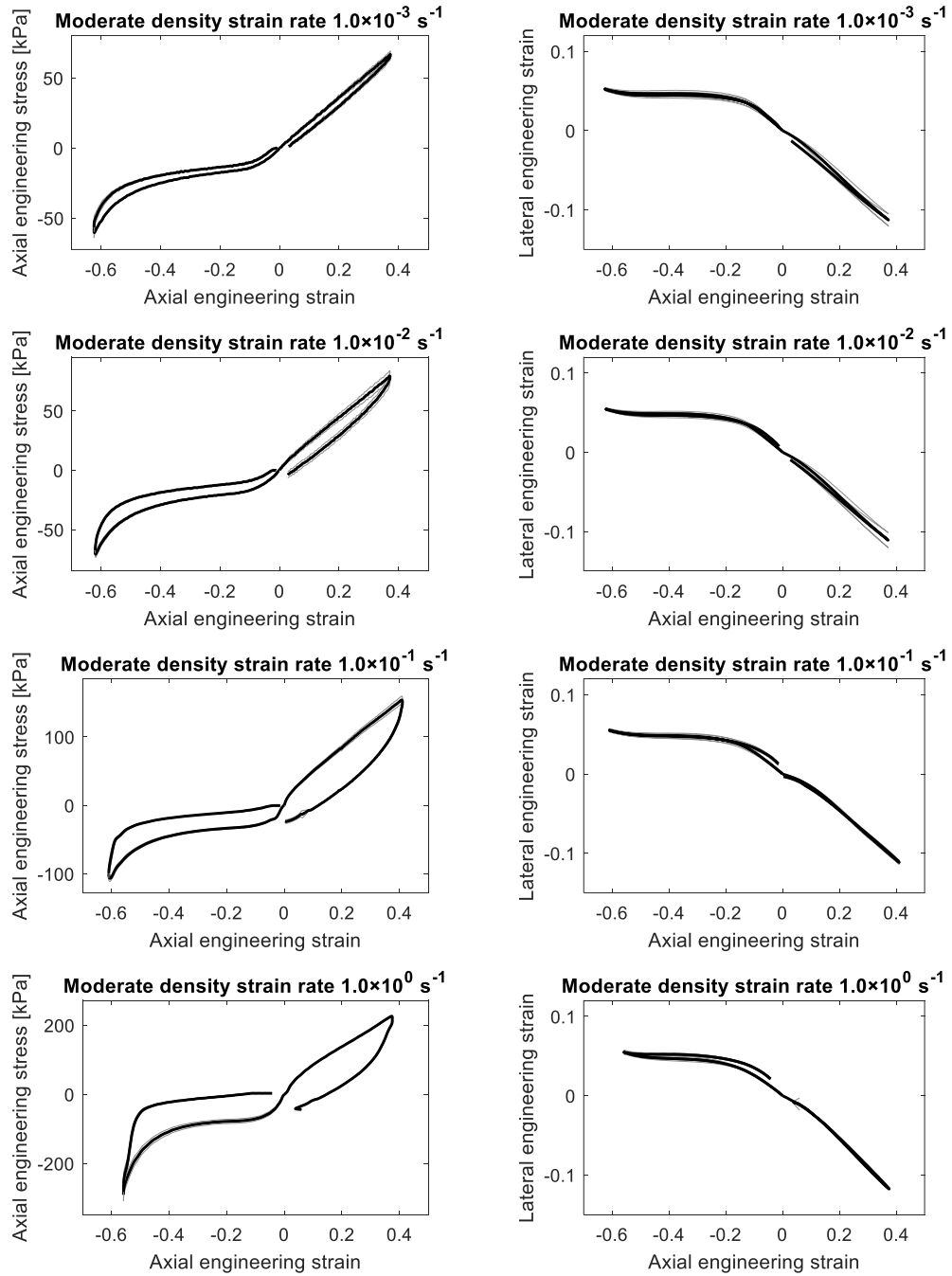


Figure 3.3: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of moderate density (192 kg/m³) under quasi-static and moderate strain rates. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data. Note that for different rates, the stress axes differ in range. Experimental data was obtained by Alexander K. Landauer and Jialiang Tao.

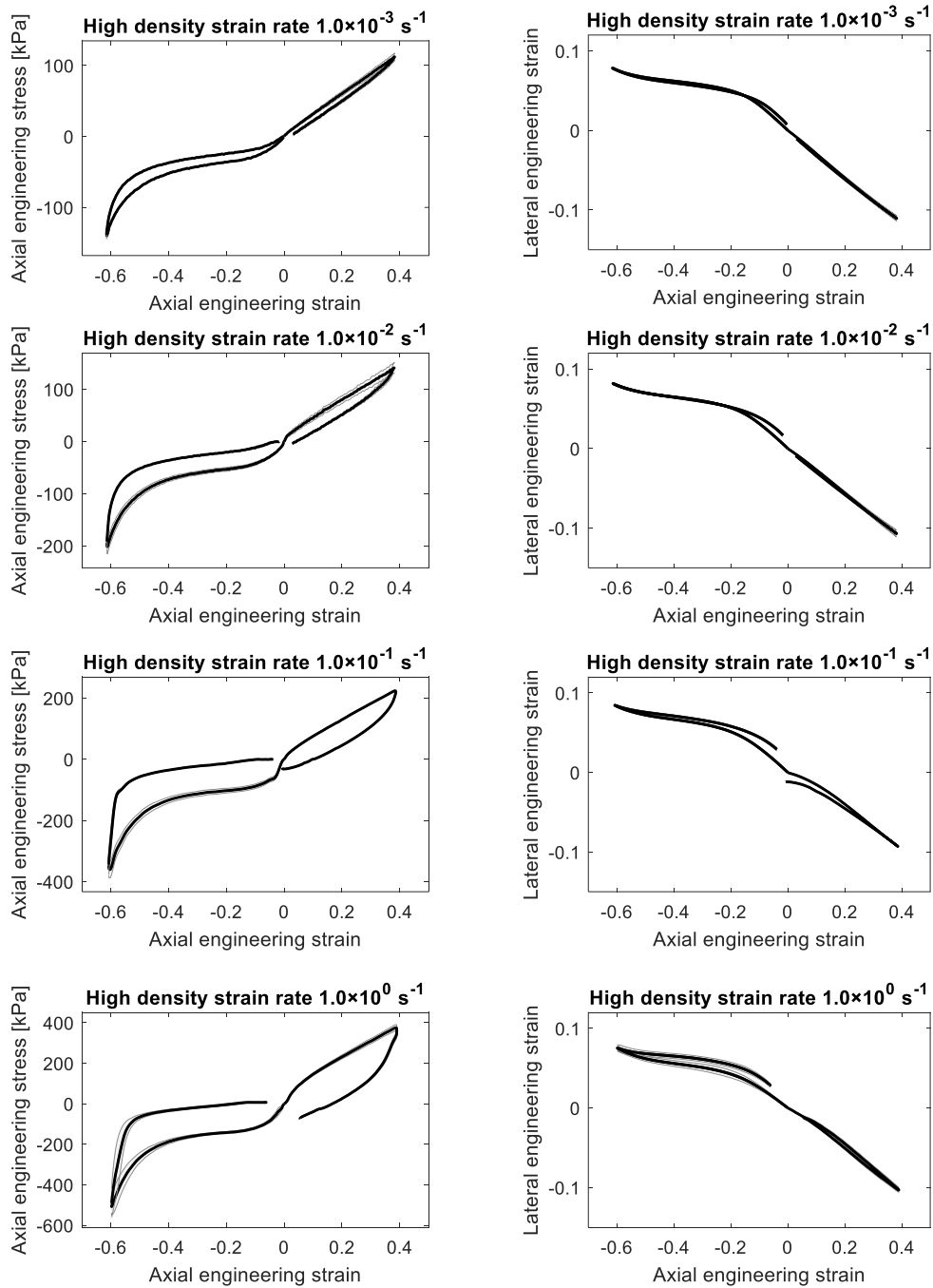


Figure 3.4: Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for polyurethane-based, open-cell elastomeric foam of high density (240 kg/m³) under quasi-static and moderate strain rates. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data. Note that for different rates, the stress axes differ in range. Experimental data was obtained by Alexander K. Landauer and Jialiang Tao.

Poron XRD foams, shown in Figs. 3.3 and 3.4, the transitions from the plateau regime to the stiffening regime happen at a lower magnitude of axial strain for the highest strain rate of 1.0 s^{-1} .

Next, we quantify the dependence of the average stress-level and the amount of hysteresis in compression on the engineering strain rate. Denoting the axial engineering strain as ϵ , we define the average stress-level $\bar{\sigma}$ as the average axial stress magnitude over the axial engineering strain range -0.4 to 0 from the loading portion of the curve in compression:

$$\bar{\sigma} = \frac{1}{0.4} \int_{-0.4}^0 |\sigma(\epsilon)| d\epsilon. \quad (3.1)$$

This axial engineering strain range is chosen so that $\bar{\sigma}$ is indicative of the stress-level in the plateau regime. We note that our definition of the average stress-level (3.1) is similar to the definition of specific strain energy in Koumlis and Lamberson [69]. The amount of hysteresis $\mathcal{D}$ is then calculated as the area between the loading and unloading portions of the axial stress/strain curve in compression:

$$\mathcal{D} = \int_{\min(\epsilon)}^0 |\sigma_{\text{loading}}(\epsilon) - \min(\sigma_{\text{unloading}}(\epsilon), 0)| d\epsilon. \quad (3.2)$$

Fig. 3.5(a) demonstrates the dependence of the average stress-level $\bar{\sigma}$ on the axial engineering strain rate in semi-logarithmic scale for the closed-cell EVA foam. We note that there is a qualitative change in the dependence for the highest strain rate, which shows a significantly higher average stress-level. This is in contrast to the foam material of Koumlis and Lamberson [69], which displays a more linear dependence of average stress-level on the logarithm of the engineering strain rate (see Fig. 12(b) of [69]). Regarding the amount of hysteresis, Fig. 3.5(b) shows that $\mathcal{D}$ remains nearly uniform for the strain-rate range 1.0×10^{-3} - $1.0 \times 10^{-1} \text{ s}^{-1}$, and an increase in $\mathcal{D}$ is only observed for the highest strain rates. Figure 3.6 demonstrates the dependence of the average stress-level and the amount of hysteresis for the high density open-cell Poron XRD foam. We only show the trends for

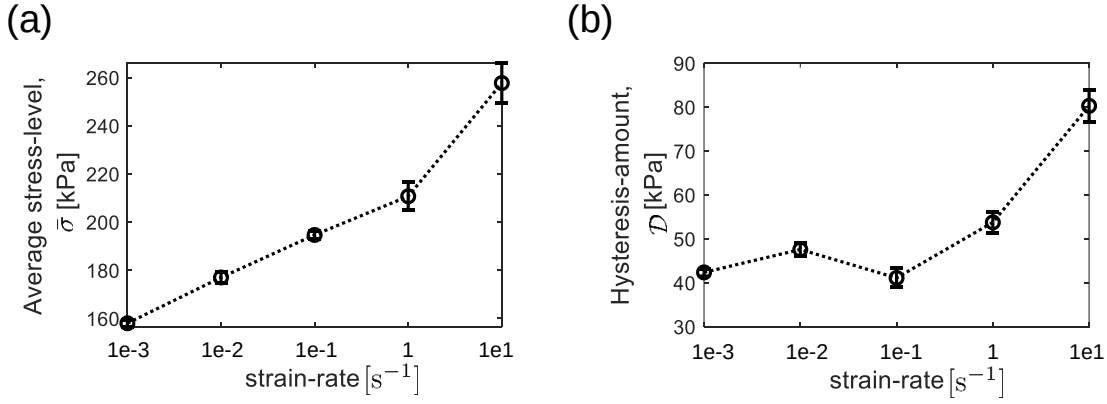


Figure 3.5: Stress-level and amount of hysteresis as a function of the axial engineering strain rate for closed-cell EVA foam. (a) Average stress-level versus logarithm of the axial engineering strain rate. (b) Amount of hysteresis versus logarithm of the axial engineering strain rate. Error bars indicate one standard deviation in three experiments at the same strain rate.

the high density foam since all three densities display similar behaviors. We note that both the average stress-level and the amount of hysteresis exhibit a more linear dependence in the semi-logarithmic scale in contrast to the closed-cell EVA foam but similar to the foam material of Koumlis and Lamberson [69]. Therefore, different elastomeric foam materials can display different features in their rate-dependence, and these differences pose challenges for developing a general model framework that can be applied to a broad class of different foam materials.

Most existing large-deformation nonlinear viscoelasticity models employ a decomposition of the material response into two parts: a time-independent hyperelastic part and a time-dependent viscoelastic part. For the hyperelastic part, rubber elasticity models are commonly employed using familiar material models, such as Ogden-type hyperelastic models [e.g., 5, 27], invariant-based models [e.g., 25, 29], and other modified hyperelastic models [e.g., 23, 30].

Then, there are two prevalent approaches in the literature for formulating the time-dependent part of a large-deformation nonlinear viscoelasticity model. The first is the hereditary integral modeling approach, which is motivated by linear viscoelastic modeling.

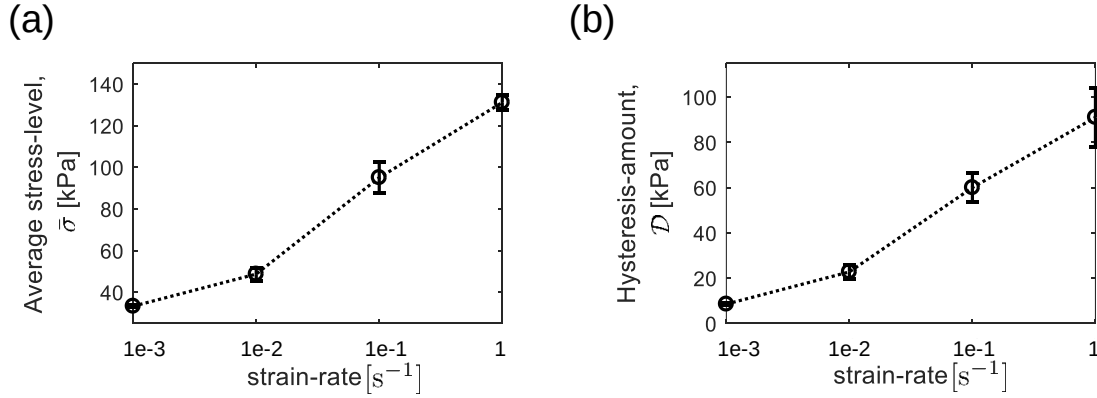


Figure 3.6: Stress-level and amount of hysteresis as a function of the axial strain rate for high density open-cell Poron XRD foam. (a) Average stress-level versus logarithm of the axial engineering strain rate. (b) Amount of hysteresis versus logarithm of the axial engineering strain rate. Error bars indicate one standard deviation in three experiments at the same strain rate.

Hereditary integral models are formulated in terms of relaxation or memory functions [e.g., 5, 22–26]. Most nonlinear viscoelastic constitutive models of this type have been experimentally calibrated and validated using only loading curves, and only a few works deal with the unloading curves. For example, Ju et al. [25] modeled the viscoelastic contribution to the stress using a hereditary integral model with different loading and unloading parameters. Most hereditary-integral-based models utilize a Zener model form, which is composed of a spring (rate-independent part) in parallel with a Maxwell element (rate-dependent part). Briody et al. [5] and Yang and Shim [24] modelled the strain-rate sensitivity of foam materials using a number of Maxwell elements with different relaxation times. Several works consider a wide strain-rate range. For example, Yang and Shim [24] characterized Poron XRD foams over the strain-rate range 1.0×10^{-2} – 1.0×10^3 s⁻¹, and Esposito et al. [26] modeled a closed-cell foam for strain rates spanning over six decades. However, most hereditary-integral-based models are limited to a narrow strain-rate range. In general, while the hereditary-integral-based modeling approach can capture nonlinear viscoelastic behavior in compression, it is difficult to establish thermodynamic consistency,

and it is difficult to generalize the predictive capability of such models to arbitrary three-dimensional loading histories.

The second approach is based on the framework of the multiplicative decomposition of the deformation gradient into elastic and inelastic parts. Nearly all nonlinear viscoelastic models that follow this approach have been applied to incompressible or nearly-incompressible materials [e.g., 27–33], and few researchers have reported on the modelling of highly-compressible foam materials [e.g., 34]. Within this model structure, Zener-type models [e.g., 27] and models involving an arbitrary number of parallel Maxwell elements [e.g., 70] are widely used. Like existing hereditary-integral-based models, only several works [e.g., 32] have described the rate-dependent viscoelastic behavior of elastomeric materials over a wide strain-rate range (e.g., 10×10^{-3} – $10 \times 10^1 \text{ s}^{-1}$ for polyurea in Shim and Mohr [32]), and most multiplicative-decomposition-based models have only been experimentally validated for narrow ranges of strain-rates. Compared to hereditary integral models, the multiplicative-decomposition-based approach leads to models that are thermodynamically consistent and may be straightforwardly applied to arbitrary three-dimensional loading histories.

The present work focuses on the modeling of the loading and unloading response of highly compressible elastomeric foams over a wide range of strain-rates from $1.0 \times 10^{-3} \text{ s}^{-1}$ to $1.0 \times 10^1 \text{ s}^{-1}$ and at large compressive strain magnitudes (up to 0.9 true strain). This chapter lays out a general modeling framework that is intended for a broad class of foam materials. The model framework consists of a time-independent, hyperelastic contribution and several time-dependent, viscoelastic contributions. The equilibrium hyperelastic part is based on the model of Chapter 2. For the rate-dependent mechanisms, we employ the rheological concept of parallel Maxwell elements and the framework of the multiplicative decomposition of the deformation gradient into non-equilibrium elastic and viscous parts. In this chapter, the modeling framework is left in a general form that may then be applied to

different specific foam materials to capture the different behaviors demonstrated in Figs. 3.1-3.4. Then, in Chapters 4 and 5, the framework will be specialized to the closed-cell EVA foam and the open-cell Poron XRD foams, respectively.

3.2 Constitutive theory

In this section, we describe our isotropic, viscoelastic model for nonlinear, finite-deformation, rate-dependent behavior of elastomeric foams. Our constitutive modeling approach is based upon a decomposition of the response into a hyperelastic, equilibrium response and a series of dissipative, non-equilibrium contributions. For conceptual clarity, this approach may be thought of as a nonlinear and finite-deformation generalization of the Maxwell-Wiechert model of linear viscoelasticity, shown schematically in Fig. 3.7.¹ With reference to the schematic, the left branch represents the equilibrium, hyperelastic response of the material, which is described by the model introduced in Chapter 2 and recapped in Section 3.2.1. Then, the non-equilibrium, time-dependent material response is described by N nonlinear, Maxwell-like, non-equilibrium mechanisms in parallel with the equilibrium branch. Each mechanism – denoted by α – is described by the model discussed in Section 3.2.2.

3.2.1 Equilibrium response

The equilibrium response is based on the model described in Chapter 2.

Basic kinematics We consider a body B identified with the region of space it occupies in a fixed reference configuration and denote an arbitrary material point within B as $\mathbf{X}$. The referential body B then undergoes a motion $\mathbf{x} = \chi(\mathbf{X}, t)$ to the deformed body $\mathcal{B}_t$ at each time t . The deformation gradient is given by $\mathbf{F} = \nabla \chi$, such that the ratio of referential to

¹To be clear, this illustration is only meant to guide ideas – our approach is not that of linear viscoelasticity. Our model is nonlinear and accounts for coupled distortional and volumetric deformation. Motivating nonlinear finite-deformation constitutive equations through rheological analogs is common in the literatures for polymer viscoplasticity [e.g., 71–76] and viscoelasticity of incompressible elastomers [e.g., 27, 68, 77, 78].

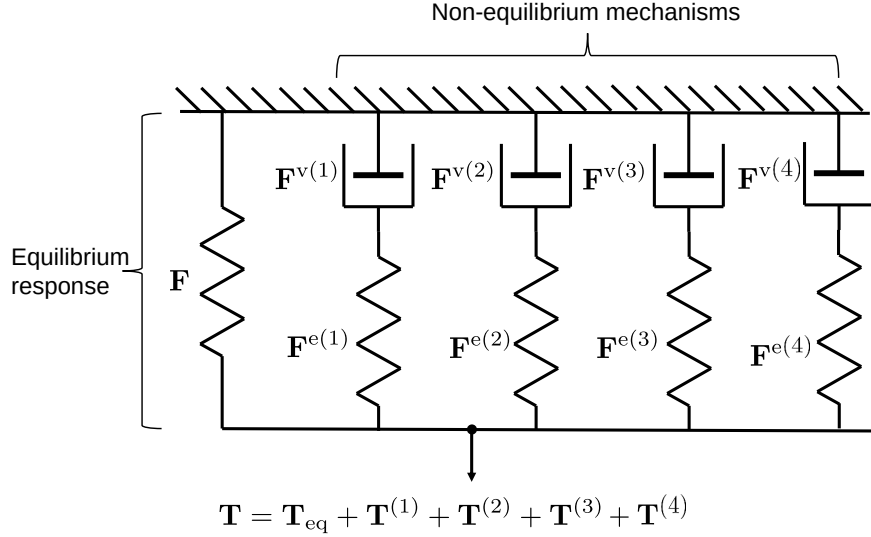


Figure 3.7: A rheological schematic of the Maxwell-Wiechert-type nonlinear, finite-deformation, viscoelastic constitutive model.

deformed volume is strictly greater than zero, i.e., $J = \det \mathbf{F} > 0$.² The displacement field is denoted as $\mathbf{u}(\mathbf{X}, t) = \chi(\mathbf{X}, t) - \mathbf{X}$. The left polar decomposition of $\mathbf{F}$ is $\mathbf{F} = \mathbf{V}\mathbf{R}$, where $\mathbf{V}$ is the left stretch tensor and $\mathbf{R}$ is the rotation tensor. The spectral decomposition of $\mathbf{V}$ is $\mathbf{V} = \sum_{i=1}^3 (\lambda_i) \mathbf{l}_i \otimes \mathbf{l}_i$, where $\{\lambda_i | i = 1, 2, 3\}$ are the principal stretches and $\{\mathbf{l}_i | i = 1, 2, 3\}$ are the principal directions of $\mathbf{V}$. Finally, the logarithmic (Hencky) finite-strain tensor in the deformed body is given as $\mathbf{E} = \sum_{i=1}^3 (\ln \lambda_i) \mathbf{l}_i \otimes \mathbf{l}_i$.

Equilibrium free-energy function The equilibrium, hyperelastic response is specified through a free-energy density function ψ_{eq} that depends on the invariants of the logarithmic strain tensor $\mathbf{E}$ as proposed by Criscione et al. [44], which we denote as $\{K_1, K_2, K_3\}$. The first invariant $K_1 = \text{Tr}(\mathbf{E}) = \ln(J)$ represents the amount of dilatation, and the second invariant $K_2 = |\text{dev}(\mathbf{E})| \geq 0$ is the magnitude of constant-volume distortion. The logarithmic strain tensor may then be expressed in terms of the invariants K_1 and K_2 as $\mathbf{E} = \frac{1}{3} K_1 \mathbf{1} + K_2 \mathbf{N}$, where $\mathbf{N} = \text{dev}(\mathbf{E})/K_2$ is the tensorial direction of deviatoric strain. The third invariant

²*Notation:* The symbols ∇ , Div and Curl denote the gradient, divergence, and curl with respect to the material point $\mathbf{X}$ in the reference body $\mathcal{B}$; grad , div , and curl denote these operators with respect to the point $\mathbf{x} = \chi(\mathbf{X}, t)$ in the deformed body $\mathcal{B}_t$. We write $\text{Tr} \mathbf{A}$, $\det \mathbf{A}$, and $\text{dev} \mathbf{A}$ to denote the trace, determinant, and deviatoric part of a tensor $\mathbf{A}$, respectively.

$K_3 = 3\sqrt{6}\det(\mathbf{N}) \in [-1, 1]$ represents the distortion mode and is constant for a given distortion mode, e.g., simple compression ($K_3 = -1$), simple shear ($K_3 = 0$), or simple tension ($K_3 = 1$). As shown by Criscione et al. [44], the Cauchy stress is then obtained through the derivatives of the free-energy function $\psi_{\text{eq}} = \tilde{\psi}_{\text{eq}}(K_1, K_2, K_3)$ utilizing the chain rule and yielding

$$\mathbf{T}_{\text{eq}} = J^{-1} \left[\frac{\partial \tilde{\psi}_{\text{eq}}}{\partial K_1} \mathbf{1} + \frac{\partial \tilde{\psi}_{\text{eq}}}{\partial K_2} \mathbf{N} + \frac{\partial \tilde{\psi}_{\text{eq}}}{\partial K_3} \frac{1}{K_2} \mathbf{Y} \right] \quad (3.3)$$

where $\mathbf{Y} = 3\sqrt{6}\mathbf{N}^2 - \sqrt{6}\mathbf{1} - 3K_3\mathbf{N}$.

Decomposition of the free-energy function We take the equilibrium free-energy density function $\tilde{\psi}_{\text{eq}}(K_1, K_2, K_3)$ to be additively decomposed as follows:

$$\tilde{\psi}_{\text{eq}}(K_1, K_2, K_3) = G_0 [X_{\text{eq}}(K_1)K_2^2 + L_{\text{eq}}(K_2, K_3)] + Bf_{\text{eq}}(K_1), \quad (3.4)$$

where G_0 and B are the ground-state, equilibrium shear and bulk moduli respectively, and $X_{\text{eq}}(K_1)$, $L_{\text{eq}}(K_2, K_3)$ and $f_{\text{eq}}(K_1)$ are phenomenological fitting functions. The interpretation of each term is discussed in Chapter 2.3. Using (3.4) in (3.3), we obtain the following expression for the equilibrium Cauchy stress:

$$\mathbf{T}_{\text{eq}} = J^{-1} \left[\left(G_0 \frac{dX_{\text{eq}}}{dK_1} K_2^2 + B \frac{df_{\text{eq}}}{dK_1} \right) \mathbf{1} + G_0 \left(2X_{\text{eq}}(K_1)K_2 + \frac{\partial L_{\text{eq}}}{\partial K_2} \right) \mathbf{N} + G_0 \frac{\partial L_{\text{eq}}}{\partial K_3} \frac{1}{K_2} \mathbf{Y} \right]. \quad (3.5)$$

Specific forms for the phenomenological fitting functions $X_{\text{eq}}(K_1)$, $L_{\text{eq}}(K_2, K_3)$ and $f_{\text{eq}}(K_1)$ may then be adopted for different foam materials. In what follows in Chapters 4 and 5, we will utilize the specific forms introduced in Chapter 2.3.

3.2.2 Non-equilibrium response

The time-dependent, viscous material response is described by a series of N nonlinear, Maxwell-like, non-equilibrium mechanisms in parallel with the equilibrium response

(Fig. 3.7). For each non-equilibrium branch – denoted by α – the deformation gradient, $\mathbf{F}$, is multiplicatively decomposed into elastic and viscous parts: $\mathbf{F} = \mathbf{F}^{e(\alpha)} \mathbf{F}^{v(\alpha)}$, where $\mathbf{F}^{e(\alpha)}$ is the non-equilibrium elastic distortion, representing the deformation of spring α , with $J^{e(\alpha)} = \det \mathbf{F}^{e(\alpha)} > 0$, and $\mathbf{F}^{v(\alpha)}$ is the viscous distortion, representing the deformation of dashpot α .

Non-equilibrium free energy function Analogous to the equilibrium free energy, the free energy due to each non-equilibrium spring is based on the elastic logarithmic strain tensor $\mathbf{E}^{e(\alpha)}$, utilizing the following definitions: $\mathbf{F}^{e(\alpha)} = \mathbf{V}^{e(\alpha)} \mathbf{R}^{e(\alpha)}$, left polar decomposition of $\mathbf{F}^{e(\alpha)}$; $\mathbf{V}^{e(\alpha)} = \sum_{i=1}^3 (\lambda_i^{e(\alpha)}) \mathbf{I}_i^{e(\alpha)} \otimes \mathbf{I}_i^{e(\alpha)}$, spectral decomposition of $\mathbf{V}^{e(\alpha)}$; and $\mathbf{E}^{e(\alpha)} = \sum_{i=1}^3 (\ln \lambda_i^{e(\alpha)}) \mathbf{I}_i^{e(\alpha)} \otimes \mathbf{I}_i^{e(\alpha)}$, elastic logarithmic strain tensor. Analogous to the three invariants K_1 , K_2 , and K_3 defined for the logarithmic finite-strain tensor $\mathbf{E}$, we define three invariants for the elastic logarithmic finite-strain tensor $\mathbf{E}^{e(\alpha)}$, denoted as $K_1^{e(\alpha)}$, $K_2^{e(\alpha)}$, and $K_3^{e(\alpha)}$. The non-equilibrium contribution to the Cauchy stress for each mechanism α is then given through the derivatives of the non-equilibrium free energy function $\tilde{\psi}_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}, K_2^{e(\alpha)}, K_3^{e(\alpha)})$ by

$$\mathbf{T}^{(\alpha)} = J^{e(\alpha)-1} \left[\frac{\partial \tilde{\psi}_{\text{neq}}^{(\alpha)}}{\partial K_1^{e(\alpha)}} \mathbf{1} + \frac{\partial \tilde{\psi}_{\text{neq}}^{(\alpha)}}{\partial K_2^{e(\alpha)}} \mathbf{N}^{e(\alpha)} + \frac{\partial \tilde{\psi}_{\text{neq}}^{(\alpha)}}{\partial K_3^{e(\alpha)}} \frac{1}{K_2^{e(\alpha)}} \mathbf{Y}^{e(\alpha)} \right], \quad (3.6)$$

where $\mathbf{N}^{e(\alpha)} = \text{dev}(\mathbf{E}^{e(\alpha)})/K_2^{e(\alpha)}$ and $\mathbf{Y}^{e(\alpha)} = 3\sqrt{6}\mathbf{N}^{e(\alpha)^2} - \sqrt{6}\mathbf{1} - 3K_3^{e(\alpha)}\mathbf{N}^{e(\alpha)}$.

Decomposition of the non-equilibrium free-energy function We take the constitutive equation for the non-equilibrium free-energy density function $\tilde{\psi}_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}, K_2^{e(\alpha)}, K_3^{e(\alpha)})$ for each mechanism α to be additively decomposed as follows:

$$\begin{aligned} \tilde{\psi}_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}, K_2^{e(\alpha)}, K_3^{e(\alpha)}) = \\ G_{\text{neq}}^{(\alpha)} \left[X_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)})(K_2^{e(\alpha)})^2 + L_{\text{neq}}^{(\alpha)}(K_2^{e(\alpha)}, K_3^{e(\alpha)}) \right] + B_{\text{neq}}^{(\alpha)} f_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}) \end{aligned} \quad (3.7)$$

where $G_{\text{neq}}^{(\alpha)}$ and $B_{\text{neq}}^{(\alpha)}$ are the ground-state shear and bulk moduli, respectively, for each mechanism α , and $X_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)})$, $L_{\text{neq}}^{(\alpha)}(K_2^{e(\alpha)}, K_3^{e(\alpha)})$, and $f_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)})$ are phenomenological fitting functions for each mechanism α . In general, these fitting functions may have different specific forms from their equilibrium counterparts $X_{\text{eq}}(K_1)$, $L_{\text{eq}}(K_2, K_3)$, and $f_{\text{eq}}(K_1)$ and from one another, but in the applications of Chapters 4 and 5, in order to reduce complexity, we will utilize the same phenomenological fitting functions introduced in Chapter 2.3. Then, using (3.7) in (3.6), we obtain the following expression for non-equilibrium contribution to the Cauchy stress for each mechanism α :

$$\begin{aligned} \mathbf{T}^{(\alpha)} = J^{e(\alpha)-1} & \left[\left(G_{\text{neq}}^{(\alpha)} \frac{dX_{\text{neq}}^{(\alpha)}}{dK_1^{e(\alpha)}} (K_2^{e(\alpha)})^2 + B_{\text{neq}}^{(\alpha)} \frac{df_{\text{neq}}^{(\alpha)}}{dK_1^{e(\alpha)}} \right) \mathbf{1} \right. \\ & + G_{\text{neq}}^{(\alpha)} \left(2X_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}) K_2^{e(\alpha)} + \frac{\partial L_{\text{neq}}^{(\alpha)}}{\partial K_2^{e(\alpha)}} \right) \mathbf{N}^{e(\alpha)} \\ & \left. + G_{\text{neq}}^{(\alpha)} \frac{\partial L_{\text{neq}}^{(\alpha)}}{\partial K_3^{e(\alpha)}} \frac{1}{K_2^{e(\alpha)}} \mathbf{Y}^{e(\alpha)} \right]. \end{aligned} \quad (3.8)$$

The total Cauchy stress is then given by $\mathbf{T} = \mathbf{T}_{\text{eq}} + \sum_{\alpha} \mathbf{T}^{(\alpha)}$ [73].

Flow rule The evolution of $\mathbf{F}^{v(\alpha)}$ for each non-equilibrium mechanism α (which describes the response of each dashpot in Fig. 3.7) is given by

$$\dot{\mathbf{F}}^{v(\alpha)} = \mathbf{D}^{v(\alpha)} \mathbf{F}^{v(\alpha)}, \quad \mathbf{F}^{v(\alpha)}(\mathbf{X}, t=0) = \mathbf{1}, \quad (3.9)$$

where $\mathbf{D}^{v(\alpha)}$ is the viscous stretching. The constitutive equation for the viscous stretching for each mechanism α is given through the Mandel stress for that mechanism $\mathbf{M}^{e(\alpha)}$, which is related to the non-equilibrium Cauchy stress for each mechanism $\mathbf{T}^{(\alpha)}$ through

$$\mathbf{M}^{e(\alpha)} = J^{e(\alpha)} \mathbf{R}^{e(\alpha)\top} \mathbf{T}^{(\alpha)} \mathbf{R}^{e(\alpha)}. \quad (3.10)$$

We define the equivalent shear stress for each mechanism α as

$$\bar{\tau}^{(\alpha)} = \sqrt{\text{dev}(\mathbf{M}^e(\alpha)) : \text{dev}(\mathbf{M}^e(\alpha)) / 2} \quad (3.11)$$

and adopt the following constitutive equation for $\mathbf{D}^{v(\alpha)}$:

$$\begin{aligned} \mathbf{D}^{v(\alpha)} &= \dot{\gamma}^{v(\alpha)} \frac{\text{dev}(\mathbf{M}^e(\alpha))}{2\bar{\tau}^{(\alpha)}} + \frac{1}{3} \dot{\epsilon}^{v(\alpha)} \text{sign}\left(\text{Tr}(\mathbf{M}^e(\alpha))\right) \mathbf{1}, \\ \dot{\gamma}^{v(\alpha)} &= \dot{\gamma}_0^{(\alpha)} \left(\frac{\bar{\tau}^{(\alpha)}}{\tau_0^{(\alpha)}} \right)^{\frac{1}{m^{(\alpha)}}}, \\ \dot{\epsilon}^{v(\alpha)} &= \dot{\epsilon}_0^{(\alpha)} \left(\frac{|\text{Tr}(\mathbf{M}^e(\alpha))|}{\sigma_0^{(\alpha)}} \right)^{\frac{1}{m^{(\alpha)}}}, \end{aligned} \quad (3.12)$$

where $\dot{\gamma}^{v(\alpha)}$ and $\dot{\epsilon}^{v(\alpha)}$ are functions of Mandel stress invariants and denote the equivalent shear viscous strain rate and the volumetric viscous strain rate, respectively. The material parameter $0 < m^{(\alpha)} \leq 1$ represents a strain-rate sensitivity exponent for each mechanism α . The parameter $\dot{\gamma}_0^{(\alpha)} > 0$ is a reference shear strain rate; $\dot{\epsilon}_0^{(\alpha)} > 0$ is a reference volumetric strain rate; $\tau_0^{(\alpha)} > 0$ is a reference shear stress; and $\sigma_0^{(\alpha)} > 0$ is a reference mean stress. We note that only one of $\{\dot{\gamma}_0^{(\alpha)}, \tau_0^{(\alpha)}\}$ is an independent fitting parameter for the deviatoric response, and similarly, only one of $\{\dot{\epsilon}_0^{(\alpha)}, \sigma_0^{(\alpha)}\}$ is an independent fitting parameter for the volumetric response, so that the viscous response of each non-equilibrium mechanism α involves three fitting parameters. Let $\eta^{(\alpha)} = \bar{\tau}^{(\alpha)} / \dot{\gamma}^{v(\alpha)}$ and $\kappa^{(\alpha)} = |\text{Tr}(\mathbf{M}^e(\alpha))| / 3\dot{\epsilon}^{v(\alpha)}$ denote the shear and bulk viscosities, respectively, which are in general, non-Newtonian viscosities that depend on the respective viscous strain rates $\dot{\gamma}^{v(\alpha)}$ and $\dot{\epsilon}^{v(\alpha)}$. If we take $m^{(\alpha)} = 1$, the non-equilibrium mechanism reduces to a Newtonian branch, and if we take $m^{(\alpha)} < 1$, the mechanism is a shear thinning branch. In the limit of $m^{(\alpha)} \rightarrow 0$, the branch becomes rate-independent. In principle, all non-equilibrium mechanisms may be non-Newtonian branches. However, to reduce model complexity and the number of adjustable

material parameters, we favor utilizing Newtonian branches over non-Newtonian branches in practice. Therefore, for a given material, as a rule of thumb, we constrain the number of shear thinning branches to be one or zero. From our practical experience of applying this framework (Chapters 4 and 5), one shear thinning branch is sufficient to capture experimental data, and we recommend this approach in order to reduce model complexity.

3.3 Concluding remarks

In this chapter, the framework for a phenomenological, isotropic, finite-deformation, viscoelastic constitutive model was proposed based on a multi-mechanism, multiplicative-decomposition-based strategy. We expect that this modeling approach may be applied to a variety of foam materials so long as the material does not exhibit discernible strain localization. In Chapter 4 and Chapter 5, we apply the model to closed-cell EVA foam and open-cell Poron XRD foam, respectively.

Chapter 4

Application of the viscoelastic constitutive model to closed-cell EVA foam

4.1 Introduction

The model framework outlined in Chapter 3 is quite general and may be applied to different specific foam materials by adopting specialized forms for the constitutive functions. We aim to demonstrate that this framework is sufficiently general so that it may be applied to a wide variety of foam materials with different microstructures (closed-cell and open-cell) and different densities. In this chapter, we focus on the closed-cell EC-2048 Ethylene-Vinyl Acetate (EVA) foam developed by Reebok Corporation (Reebok, Adidas, Boston, MA) and used in athletic shoe midsoles. The mechanical response of EVA in simple compression/tension under quasi-static and moderate strain-rates (1.0×10^{-3} - $1.0 \times 10^1 \text{ s}^{-1}$) is shown in Fig. 3.1. All experimental data for the closed-cell EVA foam was collected by Jialiang Tao, and details of the simple compression/tension experiments are given in Appendix C. In Section 4.2, the viscoelastic constitutive framework of Chapter 3 is

specialized to the EVA foam, and the fit of the calibrated viscoelastic constitutive model to the data of Fig. 3.1 is presented. In Section 4.3, we then perform several validation tests that involve inhomogeneous deformation to probe the predictive capability of the model in a variety of deformation modes and deformation rates. Our intent is to demonstrate an accessible modeling framework for compressible and viscoelastic foam materials.

4.2 Specialization of the viscoelastic constitutive model

To capture the experimental data of Fig. 3.1 for the closed-cell EVA foam, we utilize one equilibrium branch representing the hyperelastic contribution and four non-equilibrium branches representing the rate-dependent viscoelastic contributions.

4.2.1 Specialization of the equilibrium free-energy function

For the equilibrium free energy function ψ_{eq} , we adopt the same specific functional forms as in Chapter 2 [79] for the phenomenological functions appearing in (3.4), which are summarized below.

1. The distortional/volumetric-coupling response function X_{eq} :

$$X_{\text{eq}}(K_1) = \frac{1}{2} (X'_1 + X'_2) K_1 + \frac{\Delta_K}{2} (X'_1 - X'_2) \ln \left[\frac{\cosh((K_1 - K_1^0)/\Delta_K)}{\cosh(K_1^0/\Delta_K)} \right] + 1. \quad (4.1)$$

The function $X_{\text{eq}}(K_1)$ depends linearly on K_1 but with different slopes before and after the onset of the plateau regime in compression. The parameter $K_1^0 < 0$ is the value of K_1 at which the transition to the plateau regime occurs, and $\Delta_K > 0$ is a parameter denoting the K_1 -range of the transition to the plateau regime. The parameters X'_1 and X'_2 are the slopes of $X_{\text{eq}}(K_1)$ before and after the plateau regime in compression, respectively.

2. The higher-order distortional response function L_{eq} :

$$L_{\text{eq}}(K_2, K_3) = C_0 K_2^p + C_1 (1 + K_3) K_2^q, \quad (4.2)$$

where $C_0 > 0$ and $C_1 > 0$ are constant parameters and $p > 2$ and $q > 2$ are higher-order exponents. We note that the second term does not affect the response in compression ($K_3 = -1$) and allows for different higher-order responses in compression and tension.

3. The purely volumetric response function f_{eq} :

$$f_{\text{eq}} = \frac{J^{C_2} - C_2 \ln J - 1}{C_2^2} + \frac{C_3}{r-1} \left[\frac{-1}{J^{r-1}} + \frac{(1 - J_{\min})^r}{(J - J_{\min})^{r-1}} + J_{\min} \right]. \quad (4.3)$$

where $J = \exp(K_1)$ is the volume ratio and $J_{\min}$, r , C_2 , and C_3 are constant parameters. The parameter $J_{\min}$ represents the minimum allowable value of J , and df_{eq}/dK_1 diverges as J approaches $J_{\min}$ from above. The parameter $J_{\min}$ is in the range $0 < J_{\min} < 1$.

4.2.2 Specialization of the non-equilibrium free-energy function

For the non-equilibrium branches, we use the same functional forms for $X_{\text{neq}}^{(\alpha)}$, $L_{\text{neq}}^{(\alpha)}$, and $f_{\text{neq}}^{(\alpha)}$ in (3.7) that we do for X_{eq} , L_{eq} , and f_{eq} of the equilibrium branch, specified in Section 4.2.1. For simplicity, we constrain the parameters in $X_{\text{neq}}^{(\alpha)}$ and $f_{\text{neq}}^{(\alpha)}$, $\{K_1^0, \Delta_K, X_1', X_2', C_2, C_3, r, J_{\min}\}$, to be the same as the parameters appearing in X_{eq} and f_{eq} . To further reduce the model complexity, we constrain the ratio of the non-equilibrium bulk and shear moduli for each branch $B_{\text{neq}}^{(\alpha)}/G_{\text{neq}}^{(\alpha)} = B/G_0$ to be equal across all non-equilibrium branches and equal to the ratio for the equilibrium response. To accommodate experimental data, we allow for different fitting parameters in the higher-order distortional response $L_{\text{neq}}^{(\alpha)}$:

$$L_{\text{neq}}^{(\alpha)}(K_2^{e(\alpha)}, K_3^{e(\alpha)}) = C_0 (K_2^{e(\alpha)})^p + C_{1\text{neq}}^{(\alpha)} (1 + K_3^{e(\alpha)}) (K_2^{e(\alpha)})^{q_{\text{neq}}^{(\alpha)}}. \quad (4.4)$$

The parameters C_0 and p are the same as those appearing in (4.2) for all branches, but we let $L_{\text{neq}}^{(\alpha)}$ have branch-dependent parameters $C_{\text{ineq}}^{(\alpha)}$ and $q_{\text{neq}}^{(\alpha)}$, where $C_{\text{ineq}}^{(\alpha)} \geq 0$ and $q_{\text{neq}}^{(\alpha)} > 2$ can be non-integer. This adds two additional fitting parameters for each non-equilibrium mechanism.

4.2.3 Specialization of the flow rule

To fit to the experiment data of Fig. 3.1, we utilize $N = 4$ non-equilibrium branches. One non-equilibrium branch is a shear thinning branch, and three are Newtonian branches. Conceptually, each non-equilibrium branch roughly corresponds to one decade of strain rate beyond the lowest strain rate considered. The shear-thinning, non-equilibrium branch corresponds a strain rate of $1.0 \times 10^{-2} \text{ s}^{-1}$ and is invoked to account for the low amount of rate-sensitivity observed at low strain rates for the EVA foam. The three Newtonian branches then correspond to the strain rates 1.0×10^{-1} - $1.0 \times 10^1 \text{ s}^{-1}$ and capture the transition to increased rate sensitivity at elevated strain rates (see Fig. 3.5).

For all three Newtonian branches, since $m^{(\alpha)} = 1$, we have that the shear viscosity $\eta^{(\alpha)} = \bar{\tau}^\alpha / \dot{\gamma}^{v(\alpha)}$ and the bulk viscosity $\kappa^{(\alpha)} = |\text{Tr}(\mathbf{M}^e(\alpha))| / 3\dot{\epsilon}^{v(\alpha)}$ are constants. We reduce the number of parameters by taking the relaxation time of the distortional and volumetric responses to be equal, i.e., $t_0^{(\alpha)} = \eta^{(\alpha)} / G_{\text{neq}}^{(\alpha)} = \kappa^{(\alpha)} / B_{\text{neq}}^{(\alpha)}$ for each branch. So for each Newtonian branch, we have two additional material parameters: $\{G_{\text{neq}}^{(\alpha)}, t_0^{(\alpha)}\}$, with $\{B_{\text{neq}}^{(\alpha)}, \eta^{(\alpha)}, \kappa^{(\alpha)}\}$ then given through $B_{\text{neq}}^{(\alpha)} = (B_0/G_0)G_{\text{neq}}^{(\alpha)}$, $\eta^{(\alpha)} = G_{\text{neq}}^{(\alpha)}t_0^{(\alpha)}$, and $\kappa^{(\alpha)} = B_{\text{neq}}^{(\alpha)}t_0^{(\alpha)}$, respectively.

For the shear thinning branch, we reduce the number of parameters by taking $\tau_0^{(\alpha)} / G_{\text{neq}}^{(\alpha)} = \sigma_0^{(\alpha)} / 3B_{\text{neq}}^{(\alpha)}$. This is inspired by taking the relaxation time of the distortional and volumetric responses to be equal in the Newtonian branches. Because $\dot{\gamma}_0^{(\alpha)}$ is not an independent fitting parameter distinct from $\tau_0^{(\alpha)}$, we fix $\tau_0^{(\alpha)} = 100 \text{ kPa}$. To further reduce model complexity, we constrain $\dot{\epsilon}_0^{(\alpha)} = \dot{\gamma}_0^{(\alpha)}$. Therefore, for the shear thinning branch we have three additional fitting parameters: $\{G_{\text{neq}}^{(\alpha)}, m^{(\alpha)}, \dot{\gamma}_0^{(\alpha)}\}$.

Equilibrium													
G_0 [kPa]	B [kPa]	$J_{\min}$	C_1	K_1^0	Δ_K	X'_1	X'_2	C_0	p	q	r	C_2	C_3
500	1000	0.16	1.2	-0.1	0.12	7	0.64	0.35	4	3	20	0.21	2

Table 4.1: Equilibrium material parameters for the closed-cell EVA foam. In total, fourteen parameters are utilized in addition to the non-equilibrium material parameters given in Table 4.2.

Non-equilibrium								
$G_{\text{neq}}^{(1)}$ [kPa]	$m^{(1)}$	$\dot{\gamma}_0^{(1)}$	$C_{\text{1neq}}^{(1)}$	$q_{\text{neq}}^{(1)}$	$G_{\text{neq}}^{(2)}$ [kPa]	$t_0^{(2)}$ [s]	$C_{\text{1neq}}^{(2)}$	$q_{\text{neq}}^{(2)}$
500	0.2	500	0	/	90	50	0	/
$G_{\text{neq}}^{(3)}$ [kPa]	$t_0^{(3)}$ [s]	$C_{\text{1neq}}^{(3)}$	$q_{\text{neq}}^{(3)}$		$G_{\text{neq}}^{(4)}$ [kPa]	$t_0^{(4)}$ [s]	$C_{\text{1neq}}^{(4)}$	$q_{\text{neq}}^{(4)}$
80	2	2.5	2.1		50	0.1	0	/

Table 4.2: Non-equilibrium material parameters for the closed-cell EVA foam. In general, seventeen parameters are introduced in addition to the equilibrium material parameters given in Table 4.1, but for the EVA foam, only eleven of these are utilized.

4.2.4 Material parameters for the closed-cell EVA foam

We utilize a heuristic procedure, which is described in detail in Appendix D, to determine the material parameters of the viscoelastic model using the simple compression/tension data of Fig. 3.1. The parameters determined using this procedure are listed in Table 4.1 for the equilibrium parameters and Table 4.2 for the non-equilibrium parameters. Figures 4.1 and 4.2 compare the simple compression/tension experimental data (solid lines) with corresponding model fits (dashed lines). The comparison quality between the experimental data and the fitted model for both the axial engineering stress and the lateral engineering strain data and for all five strain-rates is quite reasonable. Seventeen additional parameters have been introduced across all four non-equilibrium branches in addition to the fourteen equilibrium material parameters. However, for the EVA foam, only one non-equilibrium branch utilizes non-zero values for $C_{\text{1neq}}^{(\alpha)}$ and $q_{\text{neq}}^{(\alpha)}$, so practically, only eleven non-equilibrium parameters are invoked.

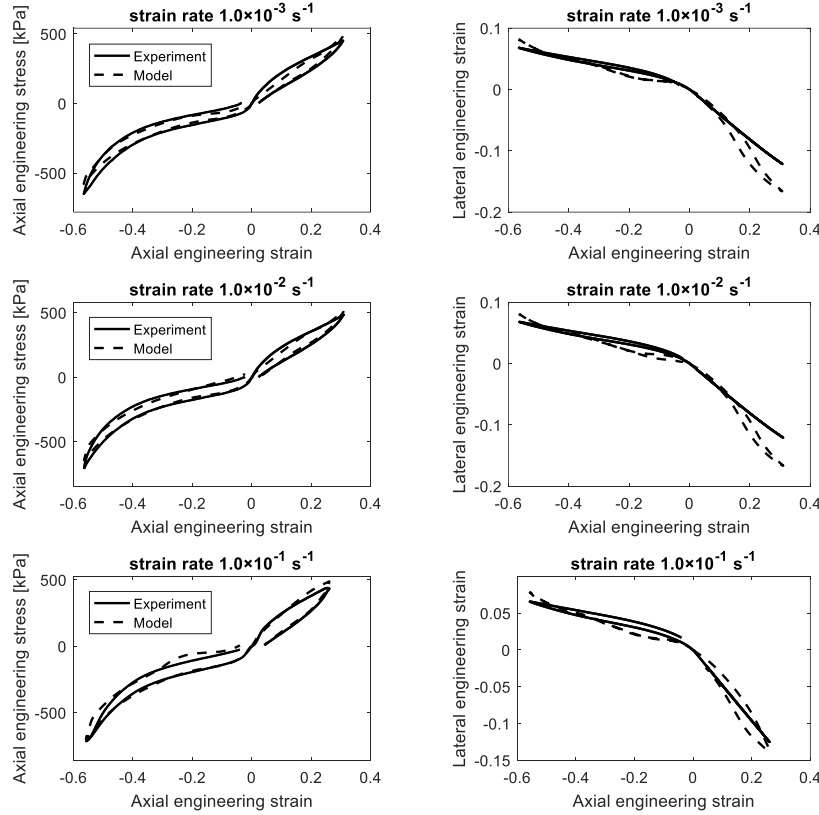


Figure 4.1: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam over the strain-rate range 1.0×10^{-3} - $1.0 \times 10^{-1} \text{ s}^{-1}$. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for different rates, the stress axes differ in range.

4.3 Validation experiments and simulations

The constitutive model of Section 3.2 has been numerically implemented in Abaqus/Explicit [61] using a user-material (VUMAT) subroutine. To probe the predictive capability of our viscoelastic model in various situations involving compression, shear, and tension-dominated deformation, a series of experiments involving inhomogeneous deformation have been carried out (Appendix C). The model predictions obtained using finite-element calculations in Abaqus/Explicit are compared against the responses measured from experiments. Three specific validation tests are done: (1) spherical indentation, (2) simple-shear-like deformation

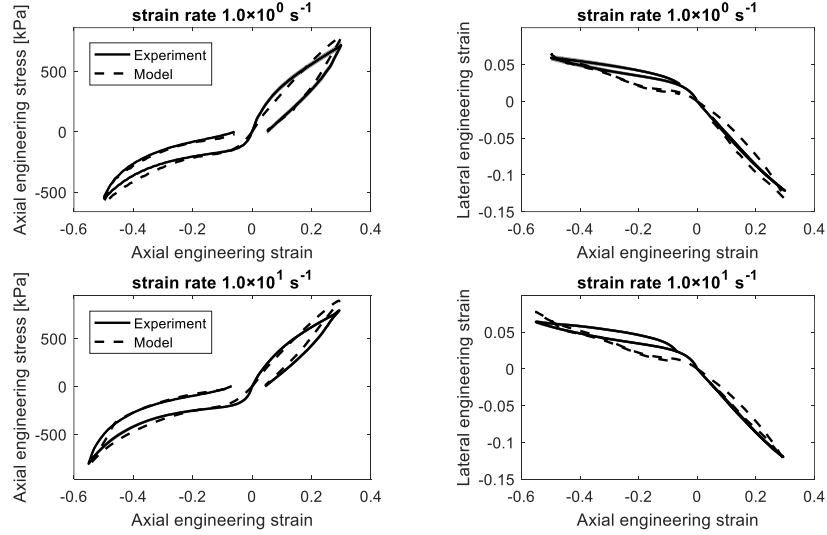


Figure 4.2: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the closed-cell EVA elastomeric foam over the strain-rate range $1.0\text{-}1.0 \times 10^1 \text{ s}^{-1}$. The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for different rates, the stress axes differ in range.

both without and with a fixed amount of pre-compression, and (3) tension of a specimen with circular holes. For all validation tests, we use displacement control, and the displacement loading profile is chosen based on two criterion: (1) The local strain-rate fields achieved in the validation tests should remain within the fitted strain-rate range everywhere.¹ (2) We seek to probe a wide strain-rate range.

4.3.1 Spherical indentation

Spherical indentation is a validation case that involves compression-dominated deformation directly beneath the spherical indenter. Indentation experiments are conducted on sixteen foam specimens with an average rectangular $81.2 \times 53.5 \text{ mm}^2$ cross section and a height of $H = 14.0 \text{ mm}$. We use a bonded aluminum backer to constrain the bottom rectangular face

¹This is determined by examining the strain-rate fields in simulations under the assumption that there is no strain localization.

of each specimen. The specimen is indented on the top unconstrained face. Spherical indentation is performed using a steel indenter tip with a radius of $r = 3.97$ mm. A photograph of the experimental setup prior to deformation for spherical indentation is shown in Fig. 4.3(a). Experiments are performed using a sinusoidal displacement history shown in Fig. 4.3(b) and at different cyclic frequencies $f = 0.3$ Hz, 0.03 Hz and 0.003 Hz. The input history is achieved using the load frame which is used for the high-strain-rate tension/compression experiments described in Appendix C. The maximum indenter displacements achieved during spherical indentation are around 4 mm in all cases. The experimentally-measured indenter force P versus time t and indenter force P versus displacement δ relations are shown as solid lines in Figs. 4.3(c). Data for all three cyclic frequencies are shown. The shaded regions surrounding the solid curves represent one standard deviation above and below the average across all sixteen instances of the experiment.

For the corresponding finite-element simulations, the geometry is idealized as axisymmetric for computational efficiency. We idealize the cuboid foam specimen as a disk of radius R_0 and height $H = 14.0$ mm. The outer radius is set to be $R_0 = 37.2$ mm, which is chosen by matching the area of the cross section.² The axisymmetric finite-element configurations for spherical indentation are shown in Fig. 4.3(a). The foam substrate is meshed using 13,020 Abaqus-CAX4R elements (four-node, axisymmetric, quadrilateral, reduced-integration elements) and the indenter is idealized as a rigid surface. We set the contact between the foam substrate and the indenter as rough (i.e., no slip) in our simulations.³ Regarding boundary conditions, all displacement components are prescribed to be zero on the bottom face of the mesh to match the fixed-bottom boundary condition. The left face of the mesh is also the axis of symmetry. Hence the horizontal displacement is set to zero along this face. We leave the right face of the mesh unconstrained. The calculated indenter

²It has been verified that simulated indenter force versus displacement results acquired from three-dimensional simulations with the exact experimental geometry are indistinguishable from the axisymmetric result, which suggests that the axisymmetric idealization is valid.

³We have performed corresponding simulations with frictionless interaction between the foam substrate and the indenter. We verified that the difference between the rough and frictionless cases is negligible, which indicates that the role of surface friction is small in indentation of compressible foam materials.

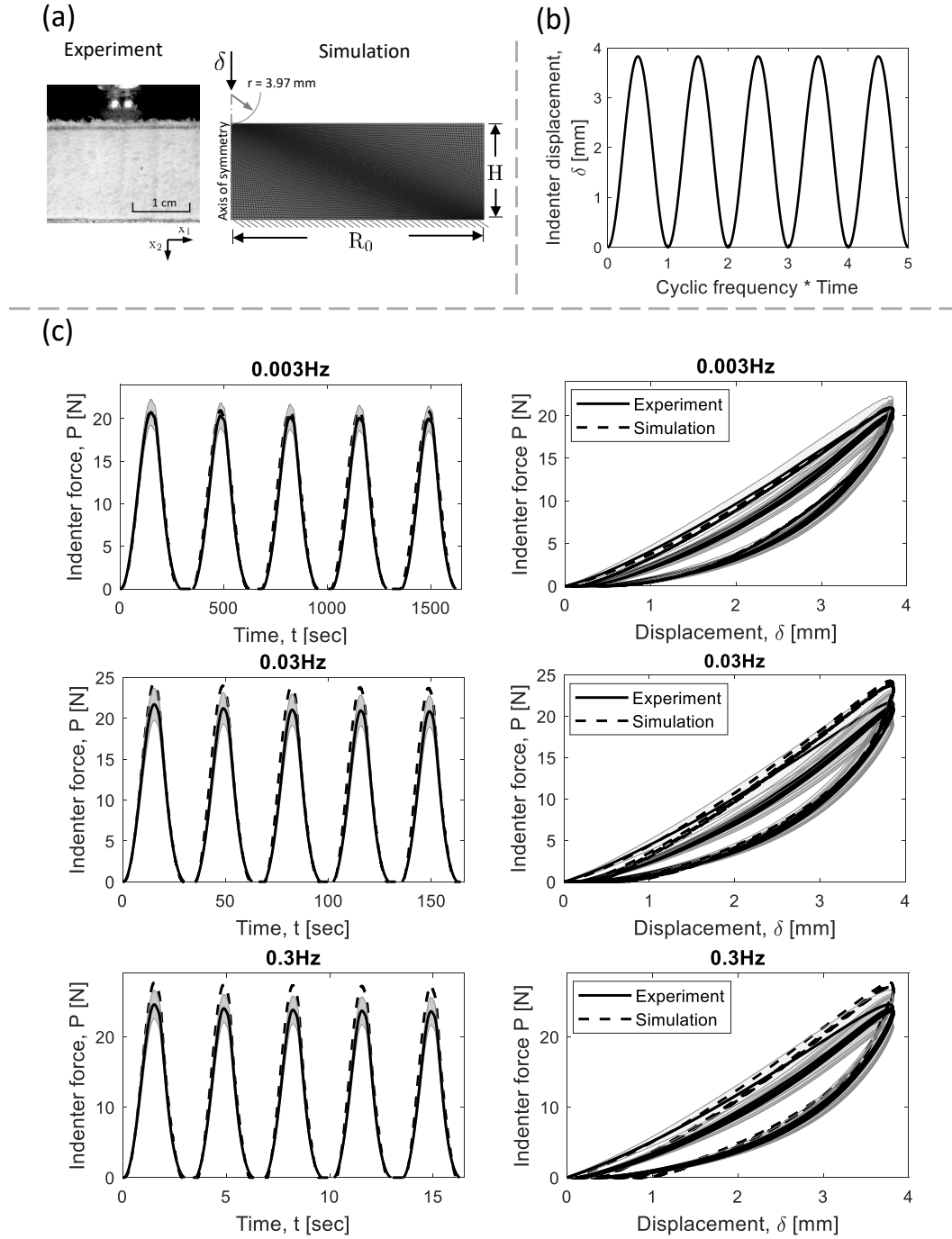


Figure 4.3: Spherical indentation validation tests. (a) Snapshots of the experimental setup prior to deformation and the corresponding finite-element configuration for spherical indentation. (b) Indenter displacement input. (c) Indenter force P versus time t and indenter force P versus indenter displacement δ for spherical indentation. Solid lines with shaded error regions represent experimental data, and dashed lines are the predictions from the constitutive model.

force P versus displacement δ relations – obtained without any parameter adjustment – are included as dashed lines for all three frequencies in Fig. 4.3(c). Generally, there is good agreement between the experimentally-measured and simulated responses.

4.3.2 Simple-shear-like deformation without and with pre-compression

The viscoelastic model is also validated in shear-dominated deformation. Two types of experiments are performed on foam cuboids with a rectangular cross-section of side-lengths $W_1 = 30$ mm and $W_2 = 20$ mm and a height of $H = 10$ mm. In the first type of experiment, foam cuboids are subjected to simple-shear-like deformation, in which one rectangular face is moved parallel to the other by a shear displacement δ , as shown in Fig. 4.4(a). The rectangular faces are constrained by bonded aluminum backers, and the shear displacement is applied by moving one backer with a sinusoidal displacement history. Three different cyclic frequencies are considered $f = 1$ Hz, 0.1 Hz and 0.01 Hz to probe different rates of deformation. The input history is shown in Fig. 4.4(b). The maximum shear displacement achieved is $\delta/H = 0.3$. The other backer remains fixed during the process, and the distance between the backers is held at H . Though this deformation is simple-shear-like, since all rectangular side-faces remain unconstrained, this deformation is inhomogeneous because of the presence of edge effects. Secondly, to test an example of coupled distortional/volumetric deformation, we consider simple-shear-like deformation with a fixed uniaxial pre-compression. The simple-shear-like experiment with pre-compression is performed in two steps. First, the specimen is compressed, such that the deformed distance between the constrained rectangular faces is $0.75H$. To allow complete relaxation from the previous compression step, specimens are then rested for at least six hours while the pre-compression is held fixed. Second, the specimen is sheared by a shear displacement δ with a sinusoidal displacement history while the distance between the constrained faces is held fixed at $0.75H$, and three different cyclic frequencies $f = 1$ Hz, 0.1 Hz and 0.01 Hz are considered. The maximum shear displacements reached are $\delta/H = 0.225$ for all the 3

frequencies. In both types of experiments, the double-shear fixture pictured in Fig. 4.4(a) is utilized, and the load frame used in the low-strain-rate tension and compression experiments is used to apply the designated shear deformation. In this setup, to maintain symmetry, two specimens are tested simultaneously. The resultant shear force P is identified as half the force applied to the moving center platen. The measured shear force P versus shear displacement δ relations for all three frequencies without and with pre-compression are shown as solid lines in Fig. 4.4(c) and Fig. 4.5(c), respectively. Several features are notable in the force-displacement relations. First, the force-displacement relation becomes more compliant when pre-compression is applied across all loading frequencies. Second, there is no plateau in force-displacement relation. Third, the amount of hysteresis increases with strain rate.

We utilize a fully three-dimensional model for the corresponding finite-element simulations. Figure 4.4(a) shows the reference finite-element mesh, consisting of 52,521 Abaqus-C3D8R elements (eight-node, three-dimensional, hexahedral, reduced-integration elements). For simple-shear-like deformation *without* pre-compression, the right face of the mesh is fixed to represent the fixed backer. The left face represents the moved backer. The vertical displacement u_2 of the left face is prescribed, while the other displacement components are constrained – i.e., $u_1 = 0$, $u_2 = \delta(t)$, and $u_3 = 0$. The remaining faces of the mesh are traction-free. Fig. 4.4(a) shows the deformed mesh after a shear displacement of $\delta/H = 0.3$, which – upon close inspection – reveals the inhomogeneous nature of deformation near the boundaries in the simulation. For simple-shear-like deformation *with* pre-compression, we apply pre-compression by prescribing $u_1 = -0.25H$ and $u_2 = u_3 = 0$ on the right face of the mesh, while fixing all displacement components at zero on the left face of the mesh. Figure 4.5(a) shows the deformed mesh after pre-compression, displaying some bulging in the vicinity of the traction-free faces. Then, we apply a shear displacement by keeping $u_1 = -0.25H$ and $u_2 = u_3 = 0$ on the right face of the mesh and prescribing $u_2 = \delta(t)$ and $u_1 = u_3 = 0$ on the left face of the mesh. Figure 4.5(a) shows the deformed

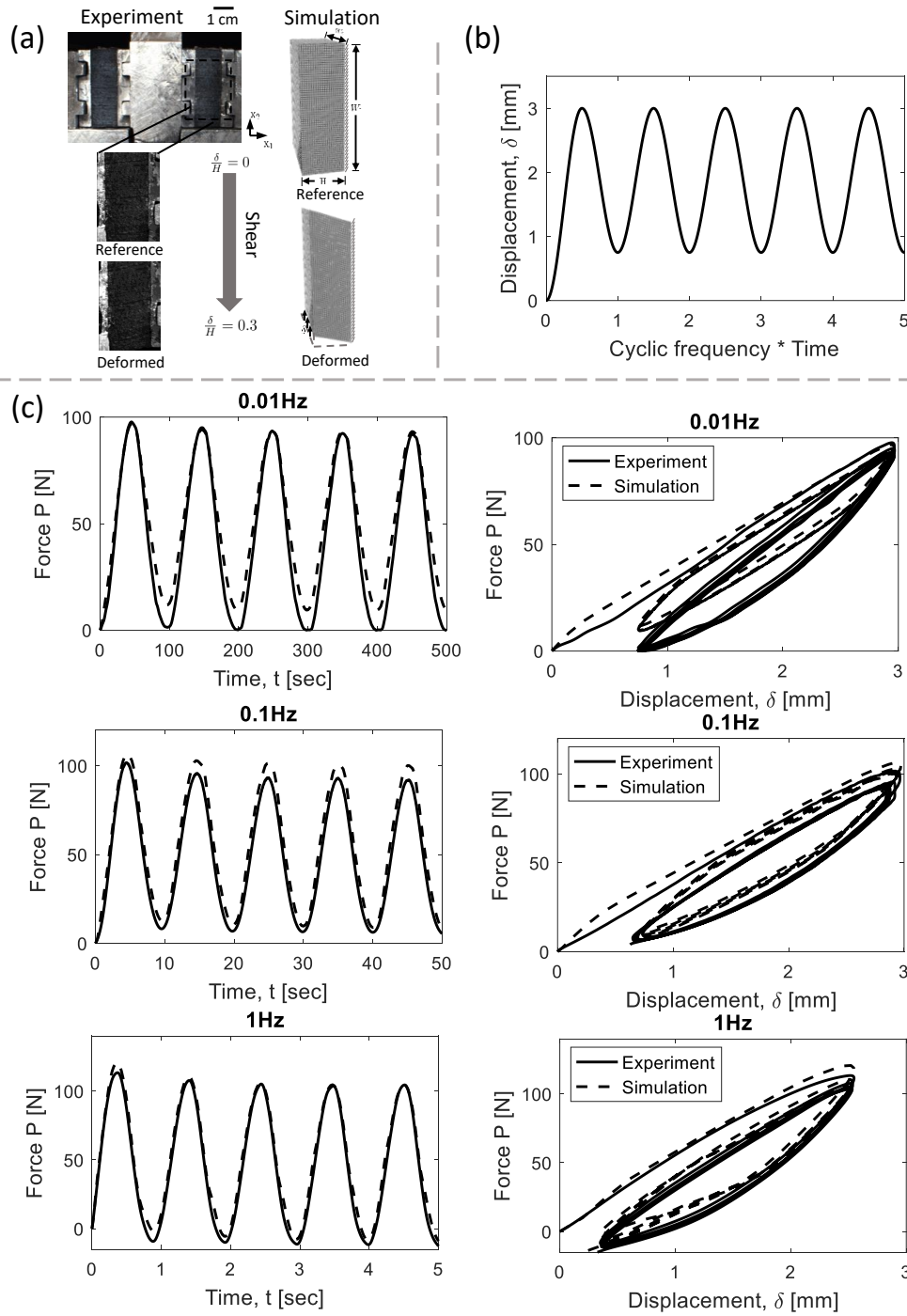


Figure 4.4: Simple-shear-like validation tests without uniaxial pre-compression. (a) Snapshot of the double-shear fixture with two specimens prior to deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) Shear force P versus shear displacement δ , in which solid lines represent experimental data, and dashed lines are simulated predictions of the constitutive model.

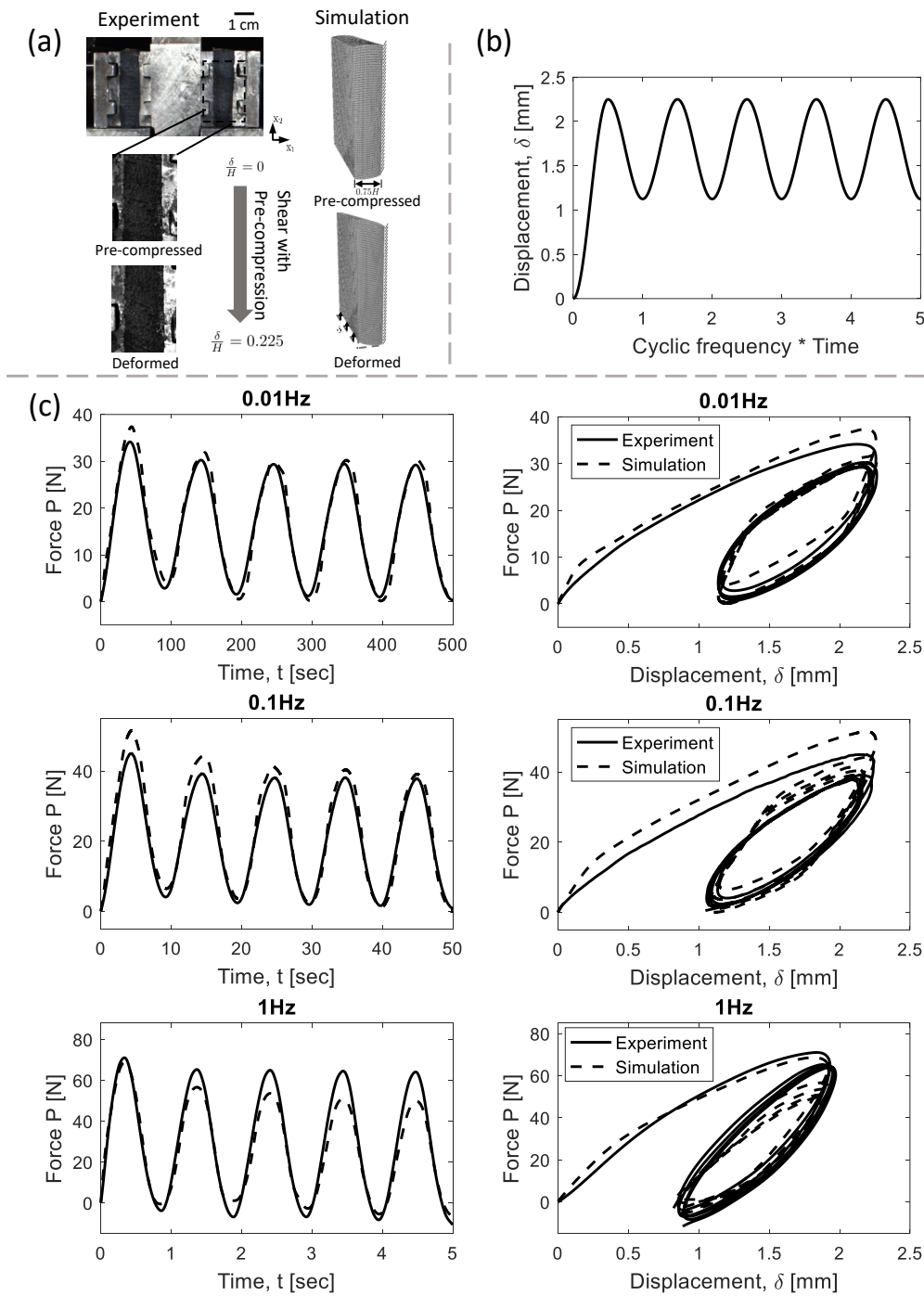


Figure 4.5: Simple-shear-like validation tests with uniaxial pre-compression. (a) Snapshot of the double-shear fixture with two specimens prior to shear deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) Shear force P versus shear displacement δ , in which solid lines represent experimental data, and dashed lines are simulated predictions of the constitutive model.

mesh after a shear displacement of $\delta/H = 0.215$, which displays significant inhomogeneous deformation. The calculated shear force P versus time t relations and the calculated shear force P versus shear displacement δ relations for each frequency are included as dashed lines in Fig. 4.4(c) for shear without pre-compression and in Fig. 4.5(c) for shear with pre-compression. The model captures the features of the response both without and with pre-compression at all frequencies. In particular, it predicts the correct amount of hysteresis and cyclic stress relaxation. The model also captures the more compliant force-displacement relation when pre-compression is applied.

4.3.3 Tension of a specimen with circular holes

Finally, a validation test is conducted to validate aspects of model related to tension-dominated deformation. We consider a validation test on a specimen with circular holes to obtain inhomogeneous deformation fields. Specifically, we begin with the specimens which follow the ASTM D3574 standard used in the simple tension characterization experiments described in Appendix C. Six non-overlapping circular holes with a radius of 2 mm are die-cut at locations that are arbitrarily chosen within the gauge section of the tension specimen. This procedure creates an asymmetric geometry, which is shown in Fig. 4.6(a). The modified specimen is then subjected to a sinusoidal cyclic loading for five cycles. The displacement loading history is shown in Fig. 4.6(b). We note that the amplitude of the sinusoidal loading is decreased after the first cycle. The displacement profile is chosen to avoid possible specimen buckling caused by compressive forces. The bottom grip is fixed, and the top grip moves at three different frequencies: 0.01 Hz, 0.1 Hz and 1 Hz. The 0.01 Hz and 0.1 Hz experiments are carried out via the load frame used for the low-strain-rate compression/tension experiments, and the 1 Hz experiment is carried out using the load frame used for the high-strain-rate experiments. Like the shear experiments, the tension validation experiments are only done on a single specimen. The measured tension force P as a function of top grip displacement δ relations are shown in Fig. 4.6(b) as solid lines.

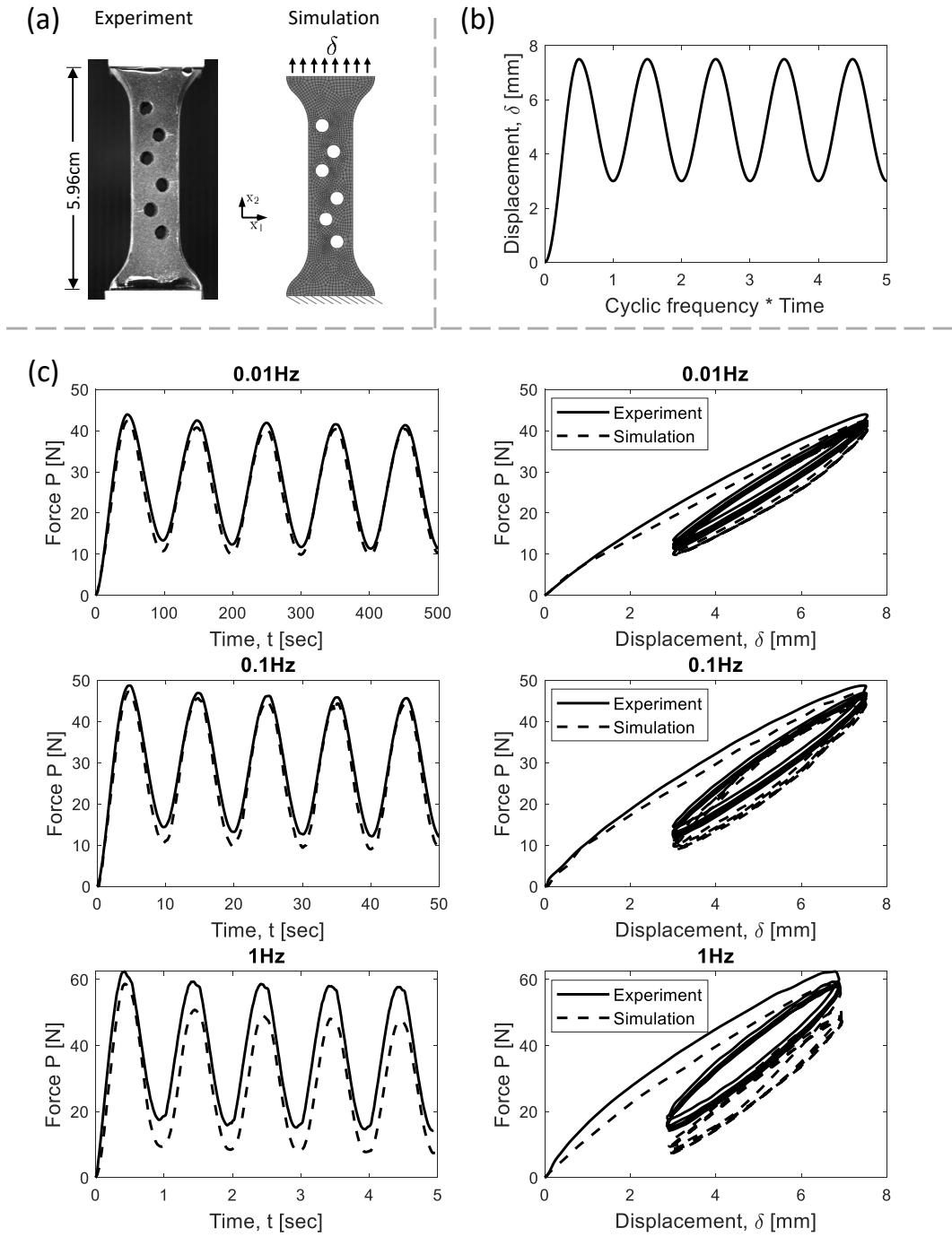


Figure 4.6: A tension-dominated validation test. (a) Snapshot of the experimental specimen prior to deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) The top-grip force P versus displacement δ , in which solid lines represent experimental data, and dashed lines are simulated predictions of the constitutive model.

In the corresponding finite-element simulations, we model only the region between the grips, consisting of the gage-section and the section of the specimen between the grips and the gage-section. As shown in Fig. 4.6(a), The specimen is modeled in three dimensions, matching the experimental geometry, and is meshed with 35,055 Abaqus-C3D8R three-dimensional elements. The front and back faces of the mesh as well as the side faces and the interior faces of the holes are traction-free. Regarding boundary conditions, the bottom of the mesh is fully constrained – i.e., $u_1 = u_2 = u_3 = 0$ – and on the top surface of the mesh the vertical displacement u_2 is prescribed, while the other displacement components are constrained – i.e., $u_1 = 0$, $u_2 = \delta(t)$, and $u_3 = 0$. Fig. 4.6(c) shows the numerically-predicted responses as dashed lines. The left column shows the force-time response and the right column shows the force-displacement response. We can see the model predictions show excellent agreement with the experimentally-measured relations at all three frequencies. The model predicts the correct force level and amount of cyclic softening. We note that the force predicted for the loading frequency of 1 Hz is slightly lower than the corresponding experimental data, which may be rationalized as follows. As shown in Figs. 4.1 and 4.2, the model fits for the higher strain rates (1.0 s^{-1} and $1.0 \times 10^1 \text{ s}^{-1}$) in simple tension are slightly under the experimental data, which is consistent with what we see in the tension validation.

4.4 Concluding remarks

In this chapter, we applied the model proposed in Chapter 3 to the closed-cell EVA foam over strain-rate range 10^{-3} - 10^1 s^{-1} . The fitted model captures the observed rate-dependence of the engineering stress/strain response, including hysteretic behavior upon unloading, asymmetry in compression/tension, and the lateral strain versus axial strain response. The proposed constitutive theory is able to accommodate the changes in rate-dependence with increasing strain rate, i.e., the foam material we focus on in this chapter is less rate-dependent

for strain rates below 10^1 s^{-1} than for strain rates above. We demonstrated that the model is able to capture the response in inhomogenous validation cases across compression, shear, and tension-dominated settings and at various deformation rates.

Chapter 5

Application of the viscoelastic constitutive model to open-cell Poron XRD foams

5.1 Introduction

In this chapter, we focus on the rate-dependent behavior of the open-cell polyurethane-based elastomeric foams described in Chapter 2, whose response to external load under moderate strain-rates (1.0×10^{-3} - 1.0 s^{-1}) is demonstrated in Figs. 3.2, 3.3, and 3.4 for each of the three densities. This experimental data was collected by Alexander K. Landauer and Jialiang Tao using the procedures discussed in Chapter 2.2 and Appendix C. For each rate, Figs. 3.2, 3.3, and 3.4 show the experimentally-measured engineering stress versus axial engineering strain response in the left column and the corresponding lateral engineering strain versus axial engineering strain response in the right column for simple compression/tension. As noted previously, the responses for all four rates are highly nonlinear and asymmetric between tension and compression. Moreover, the stress/strain response is rate-dependent

and displays hysteresis upon reversed loading with the amount of hysteresis and the stress-level increasing as the strain-rate increases. The rate-dependence is very clear over all four strain-rates. Importantly, as the strain-rate increases, the transition from the linear regime to the plateau regime becomes sharper, especially in the low density and moderate density materials. Another interesting feature is that for the moderate-density and high-density open-cell Poron XRD foams, the transitions from the plateau regime to the stiffening regime happen at a lower axial strain magnitude for a strain-rate of 1.0 s^{-1} . We also note that in the experiments, the initial slopes of the lateral engineering strain versus axial engineering strain curves in simple tension and compression are not consistent for some of the strain-rates. We expect that this is due to experimental error since the magnitude of the lateral engineering strain is small, and the initial slope is sensitive to experimental errors – especially in compression. We take our estimates for the ground-state Poisson's ratio to be the rate-independent values obtained in Chapter 2. In Section 5.2, the viscoelastic framework is specialized to the open-cell Poron XRD foams, and the fits of the calibrated constitutive model to all three densities of Poron XRD are presented. In Section 5.3, we apply our established validation procedure to test the predictive capability of the fitted model over different deformation modes and deformation rates.

5.2 Specialization of the viscoelastic constitutive model

To fit to the experimental data collected for the open-cell Poron XRD foams, we utilize one equilibrium branch that represents the hyperelastic contribution and four non-equilibrium branches that represent the rate-dependent viscoelastic contributions.

5.2.1 Specialization of the equilibrium free-energy function

For the equilibrium free-energy function ψ_{eq} , we continue to use the functional forms introduced in Chapter 2. To capture the quasi-static experimental data shown in Section 3.1 for

Density	Density-dependent				Density-independent									
	G_0 [kPa]	B [kPa]	$J_{\min}$	C_1	K_1^0	Δ_K	X'_1	X'_2	C_0	p	q	r	C_2	C_3
Low	33.0	56.1	0.12	3.5	-0.21	0.2	3.7	0.22	0.1	4	5	2	9	0.026
Moderate	57.0	102.6	0.14	1.3										
High	100.0	190.0	0.19	1.0										

Table 5.1: Adjusted equilibrium material parameters for the low, moderate, and high density Poron XRD foams.

the open-cell Poron XRD foams (estimated from the data for a strain rate of $1.0 \times 10^{-3} \text{ s}^{-1}$), we aim to maintain the material parameters appearing in the equilibrium free-energy function (3.4) the same as in our fit in Chapter 2. However, due to the change of some environmental variables such as temperature between the experiments of Chapter 2 and Chapter 3, it is necessary to make several adjustments to the material parameters in the equilibrium free-energy density function. Adjusted material parameters are obtained using the calibration procedure of Appendix A and are given in Table 5.1. Comparing the parameters in Table 5.1 with those in Table 2.1, we note that only the density-dependent parameters have been adjusted, while the density-independent parameters remain unchanged.

5.2.2 Specialization of the non-equilibrium free-energy function

For the non-equilibrium branches, we use the same functional forms for $X_{\text{neq}}^{(\alpha)}$, $L_{\text{neq}}^{(\alpha)}$, and $f_{\text{neq}}^{(\alpha)}$ in the non-equilibrium free-energy density function (3.7) as we do for X_{eq} , L_{eq} , and f_{eq} in the equilibrium free-energy density function (3.4), which are specified in Section 4.2.1. However, the parameters appearing in the non-equilibrium branches are allowed to be different from their equilibrium counterparts, and we use the subscript $_{\text{neq}}$ to denote the material parameters in (3.7) and to differentiate them from the material parameters in the equilibrium free-energy density function (3.4). For the most part, the parameters are the same in all non-equilibrium branches, but in a few cases, branch-dependent material parameters are utilized. A superscript (α) is used to denote any branch-dependent material parameters. Branch-dependent material parameters are necessitated by the observation in

Density	Branch-dependent				Branch-independent									
	$C_{3\text{neq}}^{(1)}$	$C_{3\text{neq}}^{(2)}$	$C_{3\text{neq}}^{(3)}$	$C_{3\text{neq}}^{(4)}$	$C_{1\text{neq}}$	$K_{1\text{neq}}^0$	ΔK_{neq}	$X'_{1\text{neq}}$	$X'_{2\text{neq}}$	$C_{0\text{neq}}$	p_{neq}	q_{neq}	r_{neq}	$C_{2\text{neq}}$
Low	0.007	0.007	0.007	0.007	5.0	-0.073	0.1	10.2	0.54	0.22	3	3	5	24
Moderate	0.007	0.018	0.018	0.018	1.3	-0.073	0.1	10.2	0.54	0.22	3	3	5	24
High	0.026	0.065	0.065	0.065	1.0	-0.21	0.2	3.7	0.22	0.1	4	5	2	9

Table 5.2: Material parameters of the non-equilibrium free-energy density functions for the low, moderate, and high density Poron XRD foams. We use the subscript $_{\text{neq}}$ to denote the non-equilibrium material parameters and to differentiate them from the material parameters in the equilibrium free-energy density function. In total, fourteen new density-dependent parameters are utilized with four branch-dependent parameters for each density and ten branch-independent parameters for the low density and moderate density foam materials.

the experimental data shown in Figs. 5.1, 5.2, and 5.3 that the onset of the stiffening regime happens at a lower axial strain magnitude as the strain-rate increases to 1.0 s^{-1} for the high density and moderate density foams. To accommodate this behavior, we take the parameter $C_{3\text{neq}}^{(\alpha)}$ in the function $f_{\text{neq}}^{(\alpha)}$ to be branch-dependent:

$$f_{\text{neq}}^{(\alpha)} = \frac{J^{\text{e}(\alpha)} C_{2\text{neq}} - C_{2\text{neq}} \ln J^{\text{e}(\alpha)} - 1}{C_{2\text{neq}}^2} + \frac{C_{3\text{neq}}^{(\alpha)}}{r_{\text{neq}} - 1} \left[\frac{-1}{J^{\text{e}(\alpha)} r_{\text{neq}} - 1} + \frac{(1 - J_{\min})^{r_{\text{neq}}}}{(J^{\text{e}(\alpha)} - J_{\min})^{r_{\text{neq}} - 1}} + J_{\min} \right] \quad (5.1)$$

where $C_{3\text{neq}}^{(\alpha)}$ is a branch-dependent scaling factor for the term which is intended to capture the rapidly stiffening behavior in compression. This adds one additional parameter for each non-equilibrium mechanism. We note that the parameter $J_{\min}$ is taken to be the same as the parameter $J_{\min}$ in f_{eq} for all non-equilibrium mechanisms.

For simplicity, we constrain the rest of the non-equilibrium material parameters appearing in the functions $f_{\text{neq}}^{(\alpha)}$, $X_{\text{neq}}^{(\alpha)}$, and $L_{\text{neq}}^{(\alpha)}$ to be branch-independent. To further reduce the model complexity, we constrain the ratio of the non-equilibrium bulk and shear moduli for each branch $B_{\text{neq}}^{(\alpha)} / G_{\text{neq}}^{(\alpha)} = B / G_0$ to be equal across all non-equilibrium branches and equal to the ratio for the equilibrium response for all three densities. As noted previously in Section 5.1, the transition from the linear regime to the plateau regime becomes sharper as the strain

Non-equilibrium flow rule								
Density	$G_{\text{neq}}^{(1)}$ [kPa]	$t_0^{(1)}$ [s]	$G_{\text{neq}}^{(2)}$ [kPa]	$t_0^{(2)}$ [s]	$G_{\text{neq}}^{(3)}$ [kPa]	$t_0^{(3)}$ [s]	$G_{\text{neq}}^{(4)}$ [kPa]	$t_0^{(4)}$ [s]
Low	10.0	20	55.0	2	260.0	0.07	270.0	0.02
Moderate	35.0	30	45.0	3	320.0	0.08	350.0	0.03
High	40.0	60	120.0	6	120.0	0.6	400.0	0.04

Table 5.3: Material parameters of the non-equilibrium flow rules for the low, moderate, and high density Poron XRD foams. In total, eight density-dependent parameters are utilized.

rate increases for the low density and moderate density foam materials. This indicates that in order to capture the experimental data, it is necessary to have a different set of branch-independent material parameters in the non-equilibrium functions $X_{\text{neq}}^{(\alpha)}$, $f_{\text{neq}}^{(\alpha)}$ and $L_{\text{neq}}^{(\alpha)}$ than in the equilibrium functions X_{eq} , L_{eq} , and f_{eq} for the low density and moderate density foams. Accordingly, we utilize the loading stress-strain curve in simple compression at a strain-rate of 1.0 s^{-1} to determine the non-equilibrium parameters appearing in $X_{\text{neq}}^{(\alpha)}$, $L_{\text{neq}}^{(\alpha)}$, and $f_{\text{neq}}^{(\alpha)}$ for the low density and moderate density foams. For the high density foam, we do not observe substantial rate-dependence in the shape of this transition to the plateau regime, and therefore, we constrain all the branch-independent material parameters in $X_{\text{neq}}^{(\alpha)}$, $f_{\text{neq}}^{(\alpha)}$, and $L_{\text{neq}}^{(\alpha)}$ to be the same as the equilibrium parameters in X_{eq} , L_{eq} , and f_{eq} .

5.2.3 Specialization of the flow rule

To fit to the experimental data, we utilize $N = 4$ non-equilibrium branches, and all branches are taken to be Newtonian branches. For Newtonian branches, since $m^{(\alpha)} = 1$, the two material parameters are the shear viscosity $\eta^{(\alpha)} = \bar{\tau}^\alpha / \dot{\gamma}^{\nu(\alpha)}$ and the bulk viscosity $\kappa^{(\alpha)} = |\text{Tr}(\mathbf{M}^{\text{e}(\alpha)})| / 3\dot{\epsilon}^{\nu(\alpha)}$. We reduce the number of parameters by taking the relaxation time of the distortional and volumetric responses to be equal, i.e., $t_0^{(\alpha)} = \eta^{(\alpha)} / G_{\text{neq}}^{(\alpha)} = \kappa^{(\alpha)} / B_{\text{neq}}^{(\alpha)}$ for each branch. Therefore, for each Newtonian branch, we have two additional material parameters: $\{G_{\text{neq}}^{(\alpha)}, t_0^{(\alpha)}\}$.

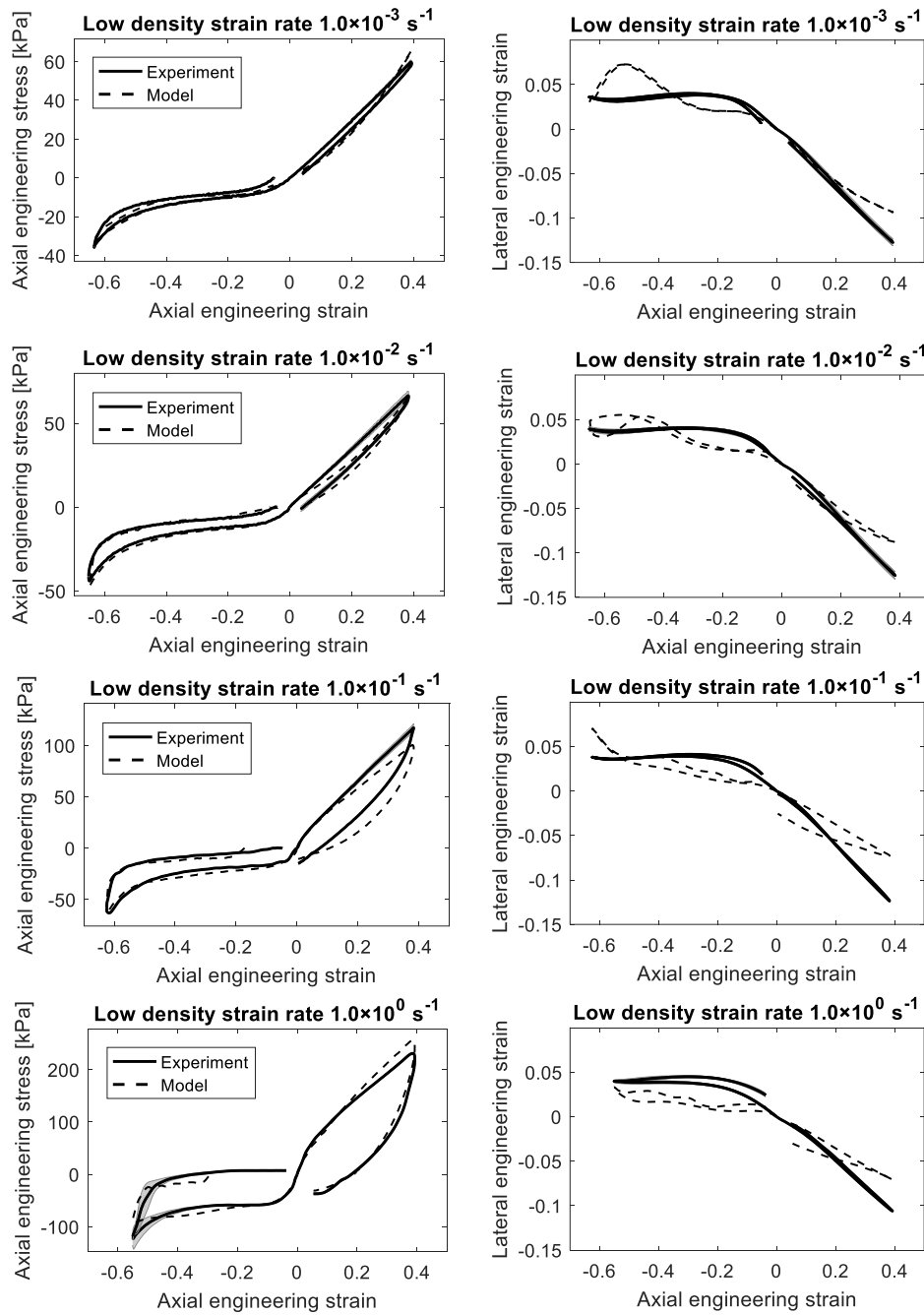


Figure 5.1: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the polyurethane-based, open-cell elastomeric foam of low density (144 kg/m^3) over the strain-rate range 1.0×10^{-3} – 1.0 s^{-1} . The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for different rates, the stress axes differ in range.

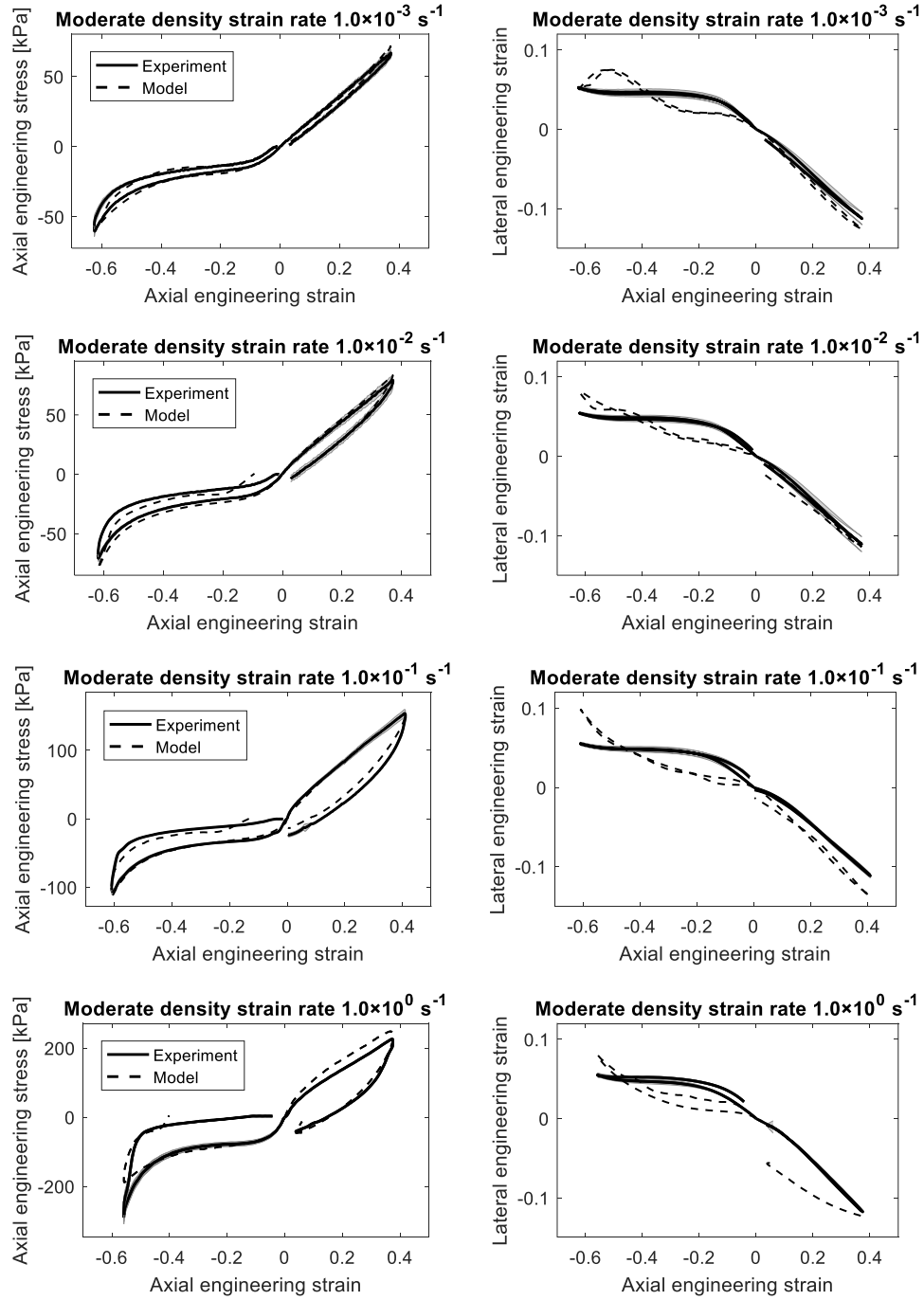


Figure 5.2: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the polyurethane-based, open-cell elastomeric foam of moderate density (192 kg/m³) over the strain-rate range 1.0×10^{-3} - 1.0 s^{-1} . The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for different rates, the stress axes differ in range.

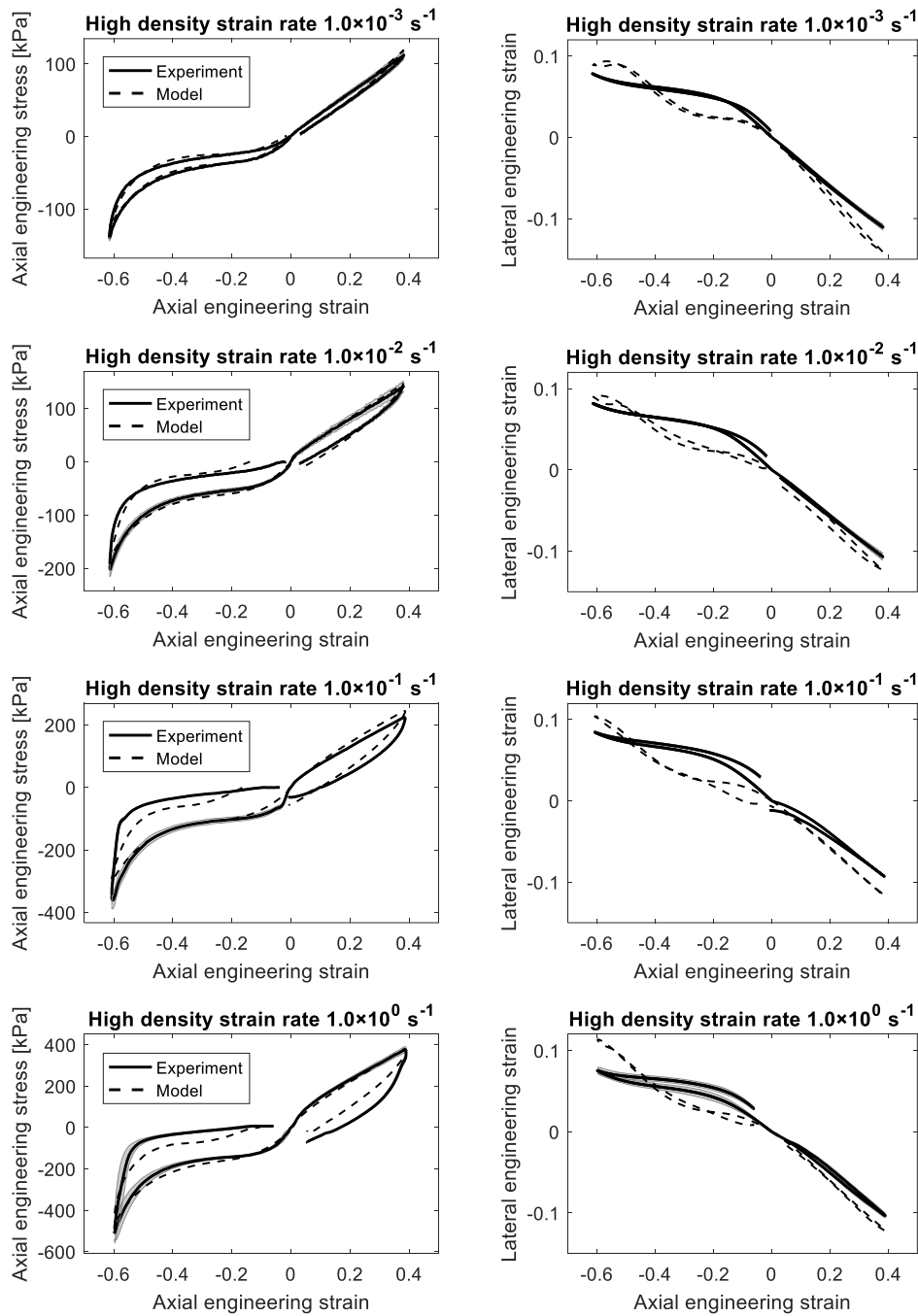


Figure 5.3: (Left) Axial engineering stress versus axial engineering strain curves and (right) lateral engineering strain versus axial engineering strain curves for the polyurethane-based, open-cell elastomeric foam of high density (240 kg/m^3) over the strain-rate range 1.0×10^{-3} - 1.0 s^{-1} . The solid lines with shaded error areas (one standard deviation, three experiments) indicate experimental data, and the dashed lines are constitutive model fits. Note that for different rates, the stress axes differ in range.

5.2.4 Material parameters for the open-cell Poron XRD foams

We utilize a heuristic procedure, which is described in Appendix D, to determine the material parameters of the viscoelastic model using simple compression/tension data. The material parameters appearing in the non-equilibrium free-energy density functions are listed in Table 5.2, consisting of fourteen additional density-dependent parameters. The material parameters appearing in the non-equilibrium flow rules are listed in Table 5.3, consisting of eight new density-dependent parameters. In total, 22 new parameters are utilized to fit four decades of strain-rate. Figures 5.1, 5.2, and 5.3 compare the simple compression/tension experimental data (solid lines) with corresponding model fits (dashed lines) for all three densities. The comparison quality between the experimental data and the fitted model for both the axial engineering stress and the lateral engineering strain data for all four strain-rates and all three densities is quite reasonable.

5.3 Validation experiments and simulations

The constitutive model of Section 5.2 has been numerically implemented in Abaqus/Explicit [61] using a user-material (VUMAT) subroutine. To probe the predictive capability of our viscoelastic model in various situations involving compression, shear, and tension-dominated deformation, a series of experiments that involve inhomogeneous deformation have been carried out. The model predictions obtained using finite-element calculations in Abaqus/Explicit are compared against the responses measured from experiments. Three specific validation tests are done: (1) conical indentation, (2) simple-shear-like deformation both without and with a fixed amount of pre-compression, and (3) tension of a specimen with circular holes. For all validation tests, we use displacement control, and the displacement loading profile is chosen based on two criteria: (1) The local strain-rate fields achieved in the validation tests should remain within the fitted strain-rate range everywhere. (2) We aim to probe a wide strain-rate range.

5.3.1 Conical indentation

The experimental setup and specimen used for conical indentation are the same as described in Section 2.4.1. Indentation experiments are conducted on foam specimens for all three densities with a square $25.4 \times 25.4 \text{ mm}^2$ cross section and a height of $H = 12.9 \text{ mm}$. Conical indentation is performed using an aluminum indenter tip with a base angle of $\theta = 30^\circ$. A photograph of the experimental setup prior to deformation for conical indentation is shown in Fig. 5.4(a). Experiments are performed using a sinusoidal displacement history as illustrated in Fig. 5.4(b) and at different cyclic frequencies $f = 0.3 \text{ Hz}$, 0.03 Hz , and 0.003 Hz . The input displacement history is achieved using the same load frame that was used for the low strain-rate tension/compression experiments. The maximum indenter displacement achieved during conical indentation is around 4 mm in all cases. The experimentally-measured indenter force P versus time t and indenter force P versus displacement δ relations for the low density, moderate density and high density foams are shown as solid lines in Figs. 5.4(c), 5.5, and 5.6, respectively. Data for all three displacement frequencies are shown. The shaded regions surrounding the solid curves represent one standard deviation above and below the average across three repetitions on the same specimen.

For the corresponding finite-element simulations, we utilize the same dimensions, boundary conditions, and finite-element mesh described in Section 2.4.1. The axisymmetric finite-element configuration for conical indentation is shown in Fig. 5.4(a). The calculated indenter force P versus time t and indenter force P versus displacement δ relations are included as dashed lines for all three frequencies in Figs. 5.4(c), 5.5, and 5.6 for each density. Generally, there is good agreement between the experimentally-measured and simulated responses. Notably, the quality of our model predictions depends on the density with the maximum discrepancy occurring for the low density foam. We hypothesize that this is due to a film on the top surface of the specimen. The film is a thin layer of dense material, which leads to extra stiffening when deformed by the indenter. This extra stiffening is more significant when the foam density is low and the loading frequency is low. Since our model

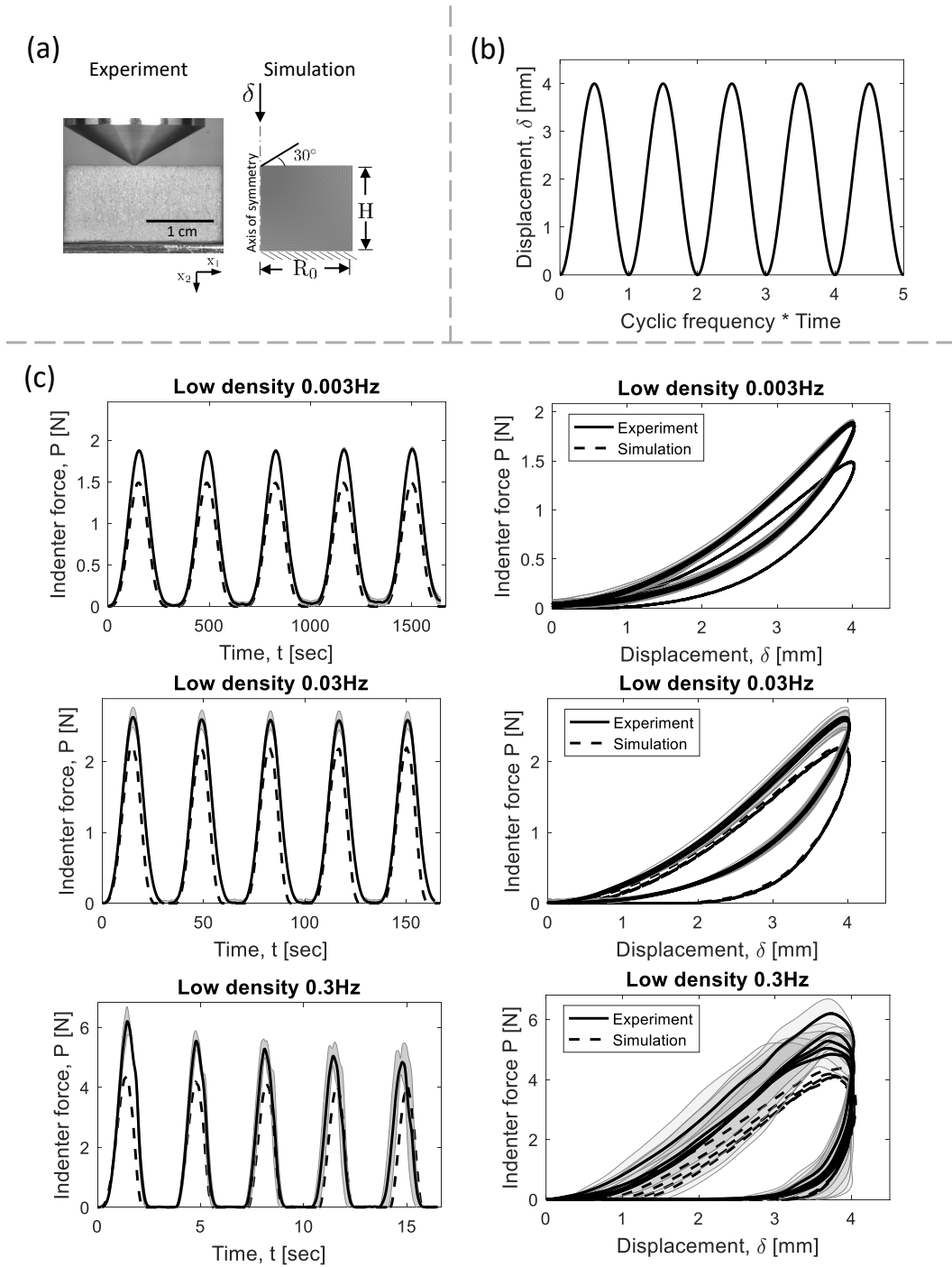


Figure 5.4: Conical indentation validation tests for the low density Poron XRD foam. (a) Snapshots of the experimental setup prior to deformation and the corresponding finite-element configuration for conical indentation. (b) Indenter displacement input. (c) Indenter force P versus time t and indenter force P versus indenter displacement δ for conical indentation. Solid lines with shaded error regions represent experimental data, and dashed lines are the predictions from the constitutive model.

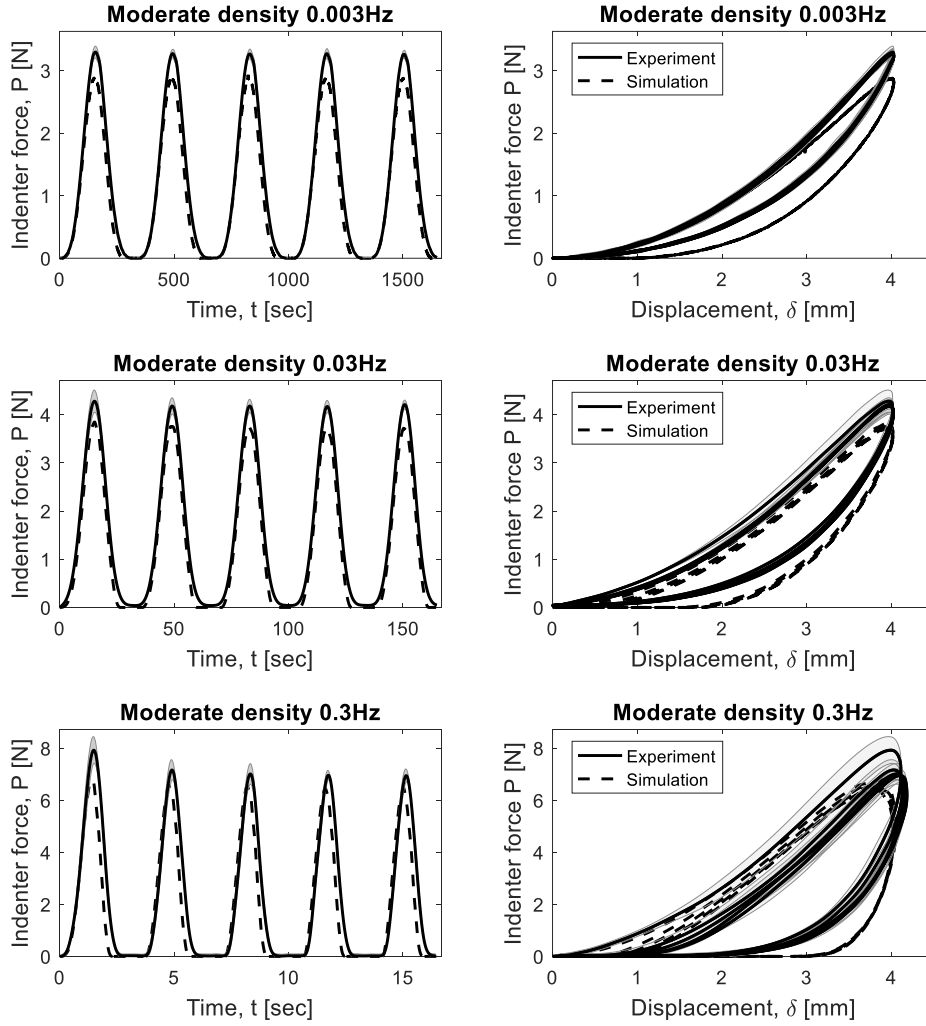


Figure 5.5: Conical indentation validation tests for the moderate density Poron XRD foam. Indenter force P versus time t and indenter force P versus indenter displacement δ for conical indentation. Solid lines with shaded error regions represent experimental data, and dashed lines are the predictions from the constitutive model.

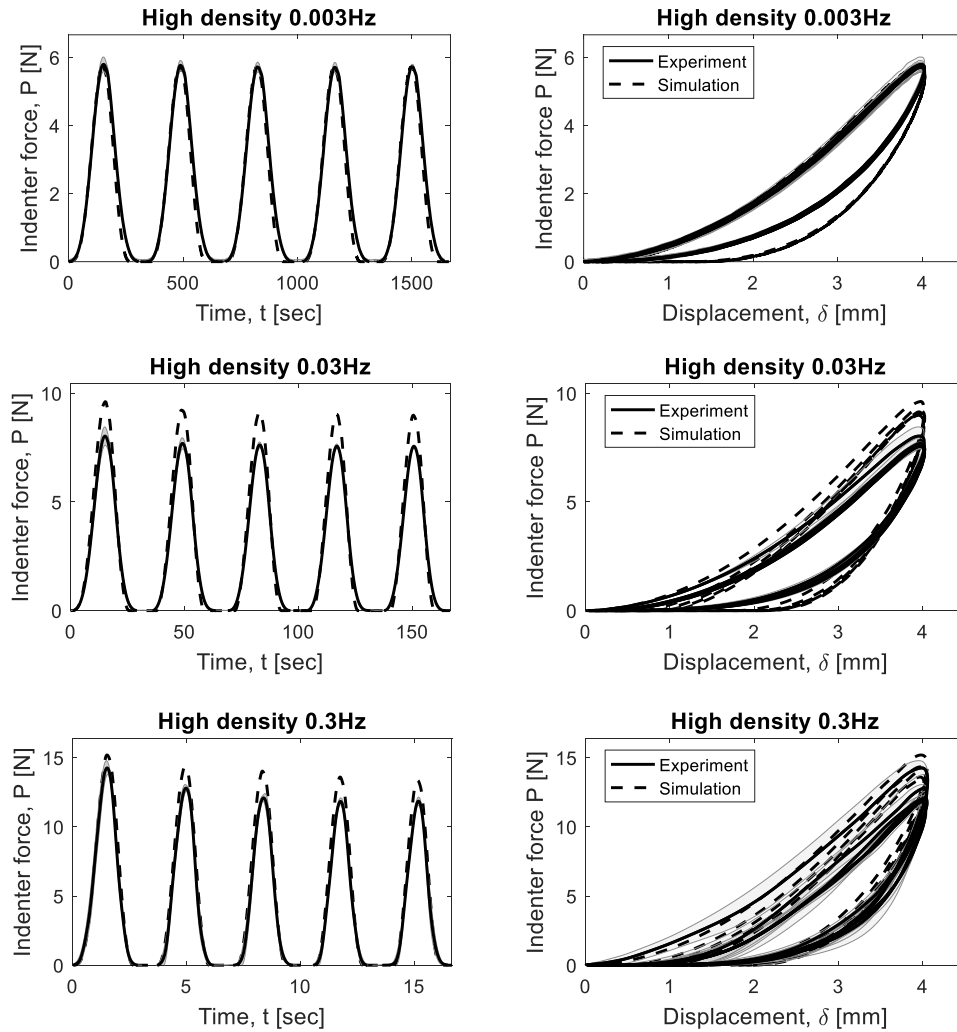


Figure 5.6: Conical indentation validation tests for the high density Poron XRD foam. Indenter force P versus time t and indenter force P versus indenter displacement δ for conical indentation. Solid lines with shaded error regions represent experimental data, and dashed lines are the predictions from the constitutive model.

and simulations do not take this film into consideration, model predictions underestimate experimental data by the greatest amount in the low density and low frequency case.

5.3.2 Simple-shear-like deformation without and with pre-compression

The viscoelastic model is also validated in shear-dominated deformation. For this validation test, we focus on the moderate-density Poron XRD foam. As described in Section 2.4.2, simple-shear-like experiments without and with pre-compression are performed on foam cuboids with a square cross-section of side-length $W = 30$ mm and a height of $H = 12.9$ mm. Photographs of the experimental setups without and with pre-compression are shown in Figs. 5.7(a) and 5.8(a), respectively. The shear displacement δ is applied with a sinusoidal displacement history with three different cyclic frequencies $f = 1$ Hz, 0.1 Hz and 0.01 Hz. The input histories are shown in Figs. 5.7(b) and 5.8(b). The maximum shear displacement achieved is $\delta/H = 0.4$ for the case without pre-compression and $\delta/H = 0.3$ for the case with pre-compression. In both types of experiments, the double-shear fixture pictured in Fig. 5.7(a) and the load frame used in the low strain-rate characterization experiments are used to apply the designated shear deformation history. The measured shear force P versus shear displacement δ relations for all three frequencies without and with pre-compression are shown as solid lines in Figs. 5.7(c) and 5.8(c), respectively. Two features are notable in the force-displacement relations. First, as discussed in Chapter 2, at quasi-static loading rates, the Poron XRD foams show a more compliant force-displacement relation when pre-compression is applied. However, this feature becomes less pronounced as the loading frequency increases. Second, as for the quasi-static behavior, there is no plateau in force-displacement relation.

We utilize the same finite-element simulation setup as described in Section 2.4.2. Figure 5.7(a) shows the reference finite-element mesh and the deformed mesh after a shear displacement of $\delta/H = 0.4$. Figure 5.8(a) shows the deformed mesh after pre-compression and the deformed mesh after a shear displacement of $\delta/H = 0.3$. The calculated shear

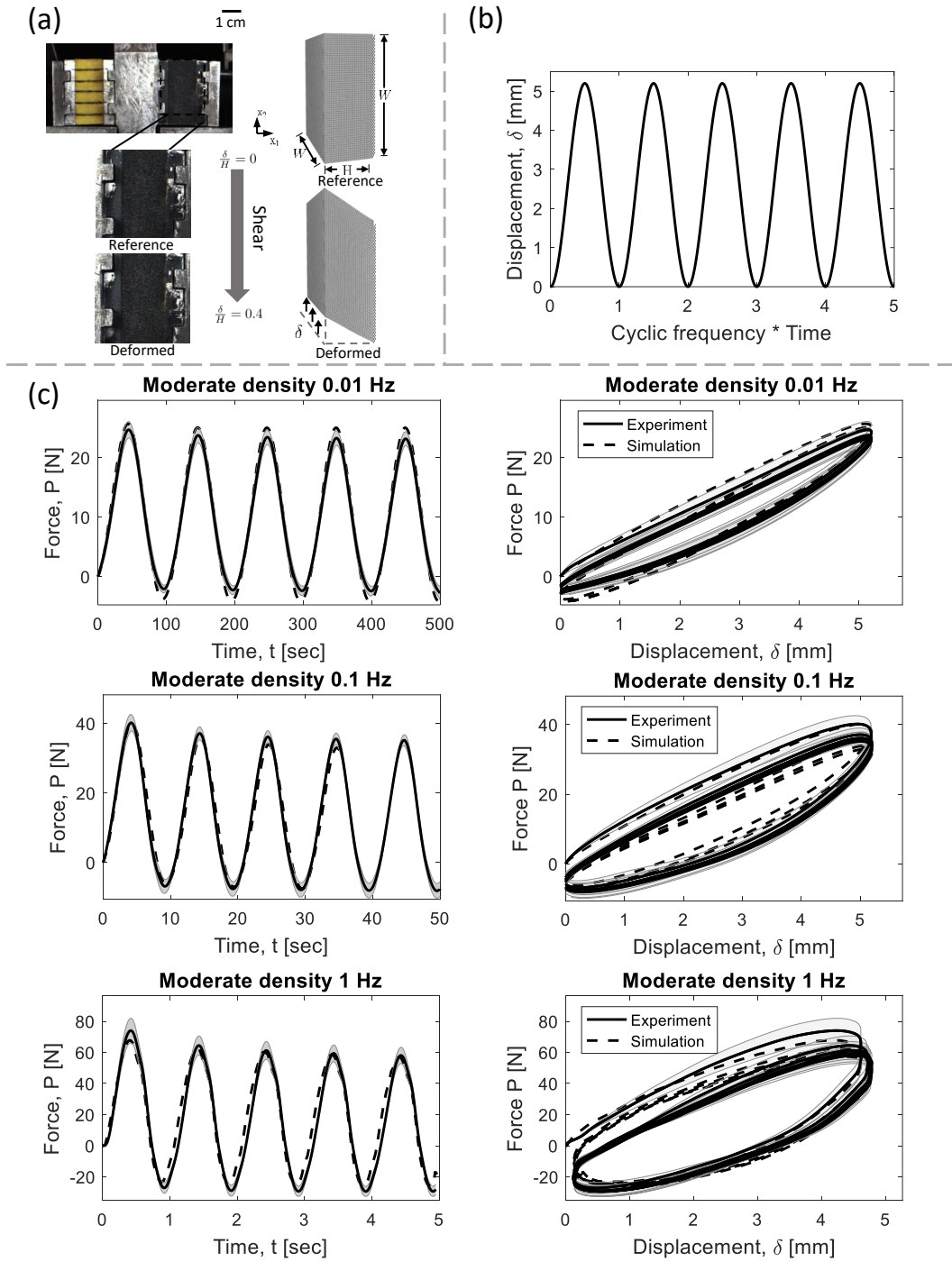


Figure 5.7: Simple-shear-like validation tests without uniaxial pre-compression for the moderate density Poron XRD foam. (a) Snapshot of the double-shear fixture with two specimens prior to deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) Shear force P versus shear displacement δ , in which solid lines with shaded error regions represent experimental data, and dashed lines are simulated predictions of the constitutive model.

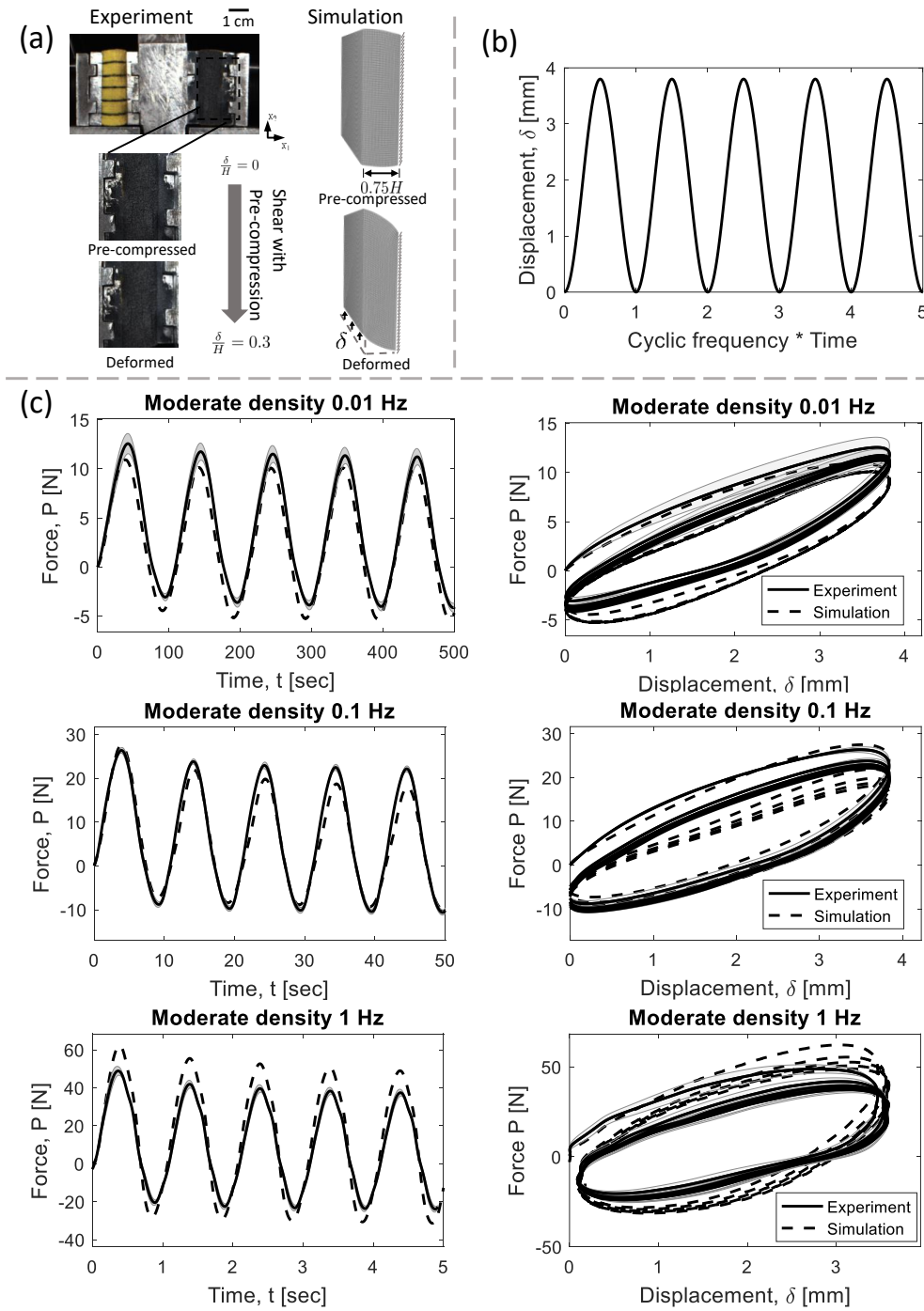


Figure 5.8: Simple-shear-like validation tests with uniaxial pre-compression for the moderate density Poron XRD foam. (a) Snapshot of the double-shear fixture with two specimens prior to shear deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) Shear force P versus shear displacement δ , in which solid lines with shaded error regions represent experimental data, and dashed lines are simulated predictions of the constitutive model.

force P versus time t and shear force P versus shear displacement δ relations for each frequency are included as dashed lines in Fig. 5.7(c) for shear without pre-compression and in Fig. 5.8(c) for shear with pre-compression. The model captures the features of the response both without and with pre-compression at all loading frequencies. In particular, the model predicts a stiffer force-displacement relation as the loading frequency increases, which is consistent with experimental data. This is because the non-equilibrium free-energy density function $\tilde{\psi}_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)}, K_2^{e(\alpha)}, K_3^{e(\alpha)})$ is not a function of the total K_1 . For shear with pre-compression, we allow the specimen relax for several hours before shear deformation, which means that $K_1^{e(\alpha)} = 0$ for all branches at the beginning of shear deformation and $K_1^{e(\alpha)}$ remains near zero in all branches during shear deformation. Therefore, none of the non-equilibrium branches experience increased compliance with pre-compression. As the loading frequency increases, the viscous behavior becomes more dominant and the non-equilibrium branches contribute more to the total resultant shear force. Therefore, the more compliant force-displacement relation for the case with pre-compression caused by the equilibrium branch becomes less significant with increasing loading frequency.

5.3.3 Tension of a specimen with circular holes

Finally, aspects of model related to tension-dominated deformation are validated. As for the shear validation tests, the tension validation test focuses on a single density of Poron XRD – for this case, the high-density foam material. The experimental setup and specimen geometry used for tension-dominated validation are the same as described in Section 2.4.3. The specimen is shown in Fig. 5.9(a) and is subjected to a cyclic loading for five cycles. The displacement loading history is shown in Fig. 5.9(b). We note that the amplitude of the cyclic loading is decreased after the first cycle. Since no signs of specimen buckling are observed in the experiments, the displacement returns to zero in the last cycle, causing a compressive force to develop at the end of the five loading cycles. The bottom grip is fixed, and the top grip moves at three different cyclic frequencies: 0.01 Hz, 0.1 Hz and

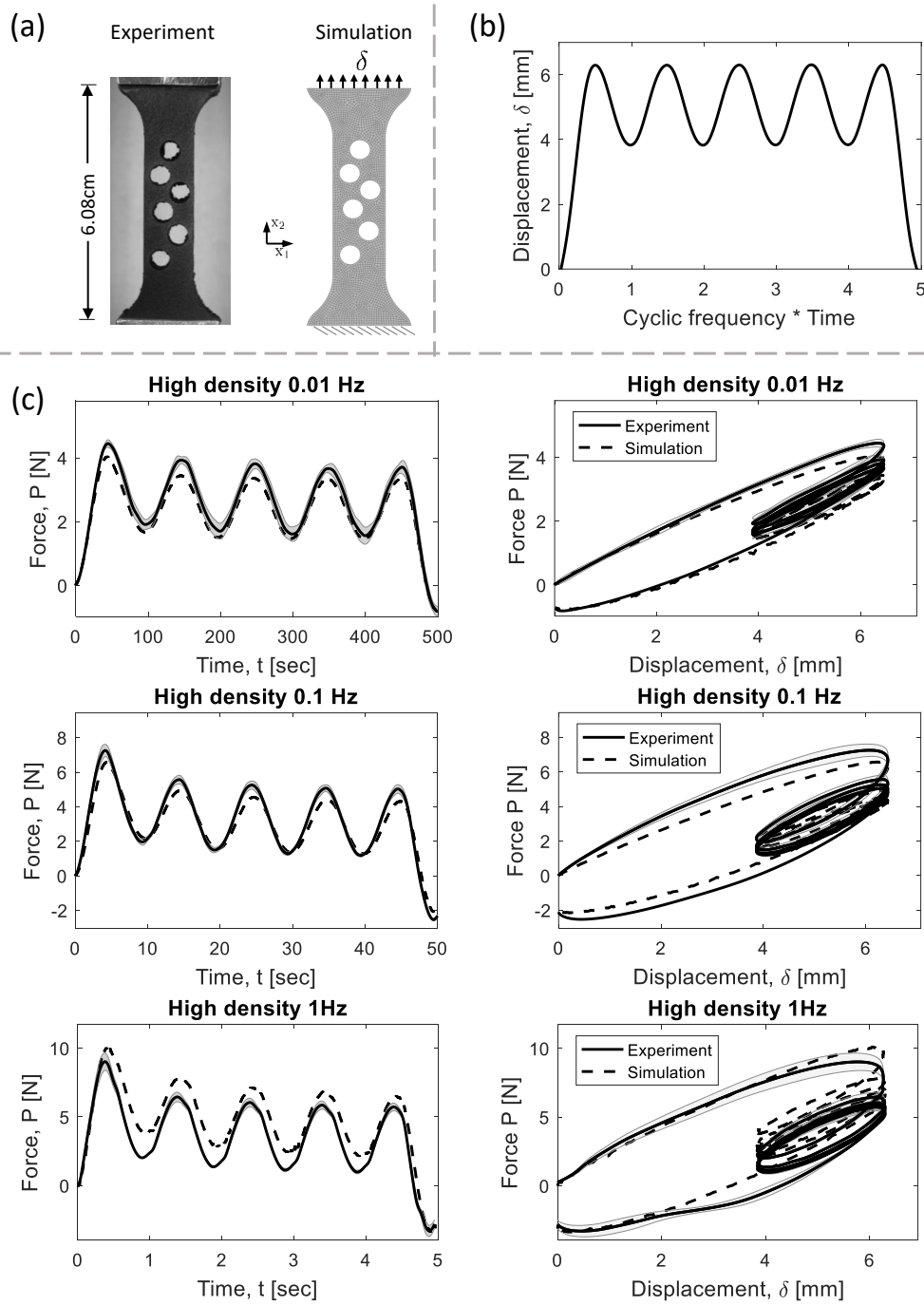


Figure 5.9: A tension-dominated validation test for the high-density Poron XRD foam. (a) Snapshot of the experimental specimen prior to deformation and the corresponding finite-element configuration. (b) Displacement loading history. (c) The top-grip force P versus displacement δ , in which solid lines with shaded error regions represent experimental data, and dashed lines are simulated predictions of the constitutive model.

1 Hz. The 0.01 Hz and 0.1 Hz experiments are carried out using the load frame used for the low-strain-rate compression/tension experiments, and the 1 Hz experiment is carried out using the load frame used for the high-strain-rate experiments. The measured tension force P as a function of top grip displacement δ relations are shown in Fig. 5.9(c) as solid lines with shaded error regions. As before, the shaded area denotes the range of one standard deviation above and below the average experimental measurement.

The corresponding finite-element configuration (geometry, mesh, and boundary conditions) is the same as described in Section 2.4.3 and is shown in Fig. 5.9(a). Figure 5.9(c) shows the numerically-predicted responses as a dashed lines. The left column shows the force-time response, and the right column shows the force-displacement response. We can see that the model predictions show excellent agreement with the experimentally-measured relations at all three cyclic frequencies. The model predicts the correct force level, the correct amount of cyclic relaxation, and a compressive force arising at the end of the last cycle.

5.4 Concluding remarks

Summary In this chapter, we applied the viscoelastic model proposed in Chapter 3 to open-cell Poron XRD foams of three densities over the strain-rate range 10^{-3} - 10^0 s^{-1} . The fitted model captures the observed rate-dependence of the engineering stress/strain response, including hysteretic behavior upon unloading, asymmetry in compression/tension, as well as the lateral strain versus axial strain response. Particularly, the calibrated constitutive theory is able to accommodate the sharpening of the transition from the linear regime to the plateau regime and the earlier transition from the plateau regime to the stiffening regime as the strain-rate increases for the Poron XRD foams.

The fitted model has been validated against a comprehensive set of inhomogeneous deformation experiments. The experiments consist of conical indentation, simple-shear-like

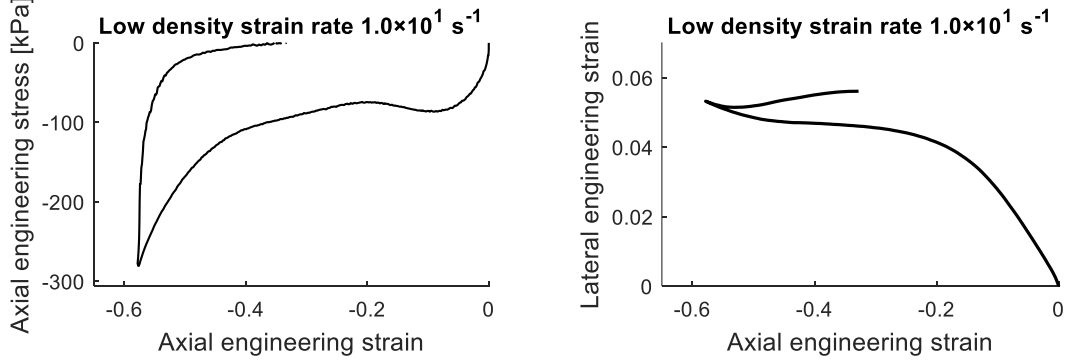


Figure 5.10: Experimental data for (Left) axial engineering stress versus axial engineering strain and (right) lateral engineering strain versus axial engineering strain for the polyurethane-based, open-cell elastomeric foam of low density (144 kg/m^3) at a strain-rate of $1.0 \times 10^1 \text{ s}^{-1}$.

deformation without and with pre-compression, and tension of a specimen with circular holes at various deformation rates. The quality of comparison between model predictions and experimental data in all validation experiments is quite reasonable.

Limitations and future work In this chapter, we have focused on a single polyurethane-based, open-cell elastomeric foam material over a moderate range of densities and a moderate range of strain-rates. We expect that the mechanical behavior of other open-cell foam materials may be well captured by the proposed model, so long as the material does not exhibit discernible strain localization. In the current work, the viscoelastic model was fit to experimental data over the strain-rate range from $1.0 \times 10^{-3} \text{ s}^{-1}$ to 1.0 s^{-1} , but at a strain-rate of $1.0 \times 10^1 \text{ s}^{-1}$, we observe possible strain localization in simple compression experiments on Poron XRD. As shown in Fig. 5.10 for the low-density Poron XRD foam, there is a peak in the plateau regime of the engineering stress-strain curve, and non-monotonic behavior is typically associated with localization of deformation. To be able to capture the material behavior at and above a strain-rate of $1.0 \times 10^1 \text{ s}^{-1}$, it will be necessary to account for strain localization in the model.

Also, as noted in Section 5.2.1, we adjusted the equilibrium fit parameters because the

experiments presented in this chapter and the quasi-static experiments presented in Chapter 2 were performed at different locations, and we believe that slight differences in ambient temperature led to slight differences in the stress-strain response. Environmental variables such as temperature can have a significant effect on the material behavior, and future work will extend the current model framework to incorporate temperature sensitivity.

Chapter 6

Conclusions and outlook

6.1 Concluding remarks

Elastomeric foams are an important class of engineering materials with wide applications that pose significant challenges in constitutive modeling. The current predictive capability for these materials – especially in the large-deformation regime – lags behind other classes of engineering materials, such as metals, plastics, and homogeneous elastomers. The work in this thesis addressed this limitation and proposed a large-deformation, viscoelastic constitutive modeling framework for elastomeric foams. The main conclusions and contributions of this thesis are summarized as follows:

- A phenomenological, isotropic, hyperelastic constitutive model based on invariants of the logarithmic, Hencky strain was proposed for the quasi-static, large-deformation mechanical behavior of elastomeric foams. We applied the model to three densities of the polyurethane-based, open-cell elastomeric foam “Poron XRD.” The model was fitted to homogeneous simple compression/tension experimental data and validated against three types of inhomogeneous experiments: spherical and conical indentation, simple-shear-like deformation without and with pre-compression, and tension of a specimen with circular holes. The model predictions were shown to be consistent

with experimental data in the inhomogeneous validation cases. A key simplifying assumption of our modeling approach is that volumetric/distortional coupling only involves low-order dependence on the magnitude of distortional deformation – which leads to a more straightforward interpretation of the fitting functions and a systematic path for material parameter estimation from simple compression/tension data. Since most of the material parameters are density-independent, we may straightforwardly apply the model to a foam made of the same matrix material but of different densities.

- A phenomenological, isotropic, viscoelastic constitutive model based on a multi-mechanism, multiplicative-decomposition based strategy was proposed for the time-dependent, large-deformation mechanical behavior of elastomeric foams. A key simplifying assumption of our modeling approach is that the non-equilibrium free-energy density function has the same functional form as the equilibrium free-energy density function, i.e., the hyperelastic model discussed above. Another key simplifying assumption is that for the flow rule, we constrain the number of non-Newtonian, shear thinning branches to be one or zero, and the remaining branches are Newtonian, which allows a more straightforward interpretation of the fitting parameters for the flow rule and a reduced number of adjustable material parameters. By invoking only one non-Newtonian, shear thinning branch and allowing the equilibrium free-energy function and the non-equilibrium free-energy function to have different material parameters, we have enabled the viscoelastic constitutive model to be flexible enough to accommodate different rate-dependent behaviors.
- We applied the proposed viscoelastic constitutive model to two different types of foam material - a closed-cell foam and several densities of an open-cell foam – which exhibit notably different rate-dependent behaviors. For both materials, the viscoelastic model was fitted to homogeneous simple compression/tension experimental data over quasi-static and moderate strain-rate ranges. As for the hyperelastic model,

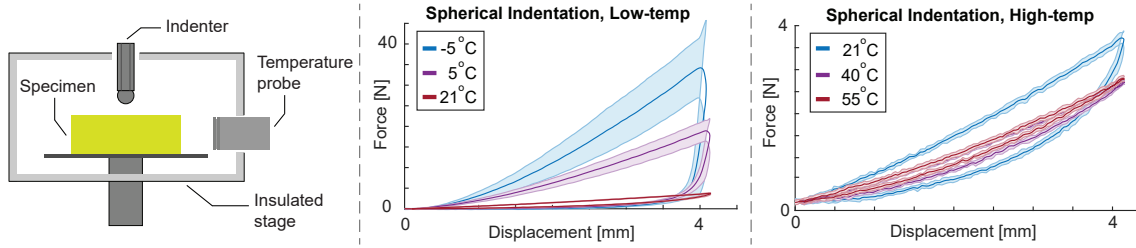


Figure 6.1: Preliminary spherical indentation data highlighting the important role of temperature on the mechanical response of polyurethane foams. (Left) A schematic of the temperature-controlled indentation chamber. (Middle and Right) The measured indentation force versus displacement over a range of temperatures, illustrating a significant increase in the apparent stiffness and hysteresis of the response with decreasing temperature. All data is for the polyurethane-based foam Poron XRD of moderate (192 kg/m^3) density and was obtained by Alexander K. Landauer [53].

we validated the fitted viscoelastic model against three types of inhomogeneous experiments: spherical/conical indentation, simple-shear-like deformation without and with pre-compression, and tension of a specimen with circular holes. The validation experiments were performed over a range of loading frequencies to probe the predictive capability of the model over the wide strain-rate range to which the model was fit. The comparisons of model predictions and experimental data were shown to be quite reasonable for both the closed-cell foam and the open-cell foams. In sum, we demonstrated that the proposed constitutive model may be applied to both open-cell and closed-cell foams and is able to accommodate different rate-dependent behaviors.

6.2 Future directions

While significant progress has been made, there are many directions for future work. We highlight three research directions below.

- Temperature plays an important role in the response of elastomeric foams. As demonstrated by Landauer [53] and shown in Fig. 6.1, in spherical indentation experiments at an equivalent nominal strain-rate (indentation speed divided by specimen thickness) of $5 \times 10^{-2} \text{ s}^{-1}$, the temperature has a significant effect on both the force level and

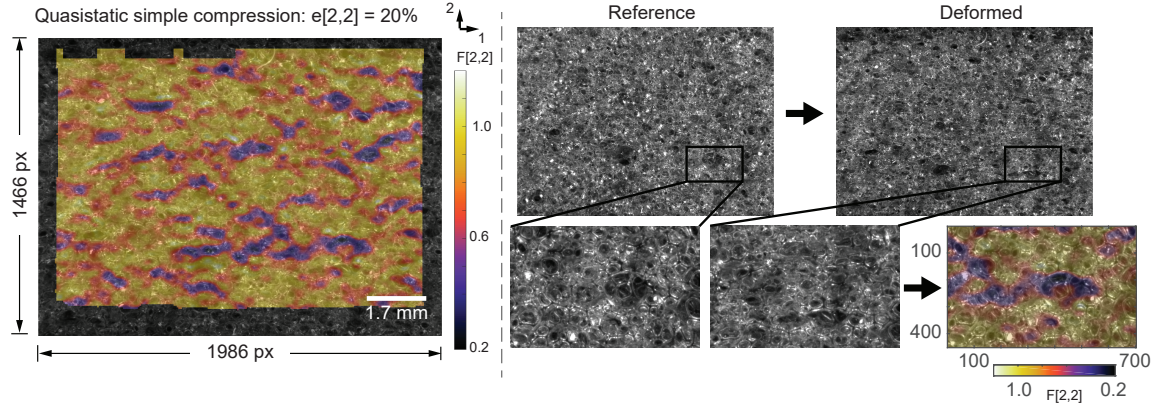


Figure 6.2: Representative DIC snapshot of a polyurethane foam specimen undergoing localization during nominally simple compression. Local pore collapse is clearly visible in the magenta colored regions indicative of significant levels of local compressive strain (shown as the vertical component of the deformation gradient F_{22} in the absence of significant rotations). The higher resolution close-ups on the right show a typical microstructural overlay with the DIC strain data showing proper co-localization of pore closure and large compressive strain. Images obtained by Alexander K. Landauer [53].

the amount of hysteresis. However, the role of temperature is absent from the model proposed in this thesis as well as constitutive models in the literature. To incorporate the effect of temperature, it is necessary to adopt temperature-dependent functional forms for the equilibrium free-energy function (3.4), the non-equilibrium free-energy function (3.7), and the flow rule (3.9). One potential approach is to follow works in the literature on the temperature-dependence of viscoelasticity and viscoplasticity in homogeneous polymeric materials [cf., e.g., 72, 75, 80–82].

- In this thesis, we focused on foam materials over strain-rate ranges in which we do not observe strain localization into compaction bands. However, strain localization can occur in very-low-density foams, as shown by Landauer [53] and illustrated in Fig 6.2 for a different very-low-density polyurethane foam (not Poron XRD) under quasi-static compression of a nominal engineering strain magnitude of 20%. The left image presents a DIC-microstructural overlay, where local strain data (visualized using the vertical component of the deformation gradient F_{22}) is displayed on top of the foam's microstructural data. Significant local compression exists in local

areas within the foam, which, upon further examination, co-localizes with local pore closures (see progression of insets to the right). Incorporating a mechanism for localization in the constitutive model would expand our current model to become predictive for the widest array of polyurethane foams. In the current model, we carefully chose the functional forms in the equilibrium free-energy density function and the non-equilibrium free-energy density functions so that the model was stable over the entire intended range-of-use. One approach to account for instability in localizing foams would be to purposefully design a strain energy function that exhibits a loss of ellipticity during compressive loading. As a specific starting point, one could modify the fitting functions $f_{\text{eq}}(K_1)$ and $f_{\text{neq}}^{(\alpha)}(K_1^{e(\alpha)})$, so that df_{eq}/dK_1 and $df_{\text{neq}}^{(\alpha)}/dK_1^{e(\alpha)}$ are non-monotonic – in contrast to the monotonic versions utilized in this thesis. This potential approach is similar to the hyperelastic energy limiter approach of Volokh [83–87] in that softening behavior is introduced to the hyperelastic response but different in that it does not introduce a limit on the strain energy.

- Data-driven, model-free approaches for constitutive modeling of materials have drawn significant attention in recent years [e.g., 88–95]. Model-free approaches promise a systematic and end-to-end process for constitutive modeling and are a potential alternative to data-driven phenomenological models, based on an explicit constitutive model, such as the one proposed in this thesis. Some existing model-free works rely directly on experimental data for strain-stress pairs [e.g., 89–91]. When the availability of experimental data is limited, these models are predictive only in limited deformation modes and are therefore not necessarily an attractive alternative to explicit models. As an alternative, techniques such as digital image correlation make the full displacement fields available in experiments involving inhomogeneous deformation. To utilize the indirect deformation field and force data to develop a constitutive model, one may formulate the problem as a global force optimization problem [e.g., 93, 95]. One approach to enforce physical constraints such as equilibrium is to use a Lagrangian

multiplier in the objective functions [89, 94, 95]. To account for time-dependent behavior, one may give the previous state variables as input variables [90, 92] or pass the deformation history to the model utilizing a Recurrent Neural Network [91]. While most model-free approaches result in a uninterpretable black-box material model, the work of Flaschel et al. [95] provides a penalty solution to automatically select explicit features in the strain energy density function. Using data from inhomogeneous experiments in conjunction with a penalty-type solution using the model framework proposed in this thesis may provide an expeditious route to applying the model to new foam materials based on a minimal number of experiments.

Appendix A

Material parameter estimation procedure – hyperelastic model

The constitutive model described in Section 3.2 involves 14 material parameters that are used to represent the hyperelastic behavior of non-localizing, open-cell elastomeric foams. This Appendix provides a heuristic procedure for estimating the values of these parameters for a generic elastomeric foam material. The procedure is illustrated by estimating the material parameters for the high density Poron XRD foam, using the experimental data from simple compression/tension experiments shown in Fig. 2.1(c).

The first step of the procedure is to determine the ground-state shear and bulk moduli, G_0 and B . The ground-state Young's modulus and Poisson's ratio are extracted from the experimental data via a linear least squares regression to the compression and tension experimental data in the axial engineering strain range of approximately -0.02 to 0.02 and used to calculate G_0 and B . For the high density foam, $G_0 = 102.0$ kPa and $B = 193.8$ kPa. Second, the phenomenological functions $X(K_1)$, $L(K_2, K_3)$, and $f(K_1)$ are iteratively fit to data extracted from the axial stress and lateral strain versus axial strain data in simple compression, in which $K_3 = -1$ is fixed. The parameters used to fit the compression response consist of $\{X'_1, K_1^0, \Delta_K, X'_2\}$ for $X(K_1)$, $\{C_0, p\}$ for $L(K_2, K_3)$, and $\{J_{\min}, C_2, C_3, r\}$ for $f(K_1)$.

Finally, the remaining parameters – namely, $\{C_1, q\}$ for $L(K_2, K_3)$ – are then estimated from the simple tension experimental data, in which $K_3 = 1$ is fixed. The processes for estimating the compression-based and tension-based parameters are discussed in the following sections, and Figure A.1 outlines the steps in this workflow.

A.1 Estimation of compression-based parameters

In simple compression, the principal stretches are $\lambda_1 = \lambda_2 = \lambda_{\text{lat}} > 1$ and $\lambda_3 = \lambda_{\text{ax}} < 1$, and the logarithmic strain invariants are $K_1 = \ln(\lambda_{\text{ax}} \lambda_{\text{lat}}^2)$, $K_2 = (\sqrt{6}/3) \ln(\lambda_{\text{lat}}/\lambda_{\text{ax}})$, and $K_3 = -1$, so that the histories of K_1 and K_2 during compression may be determined from the kinematic measurements. The principal engineering stresses are $S_1 = S_2 = 0$ and

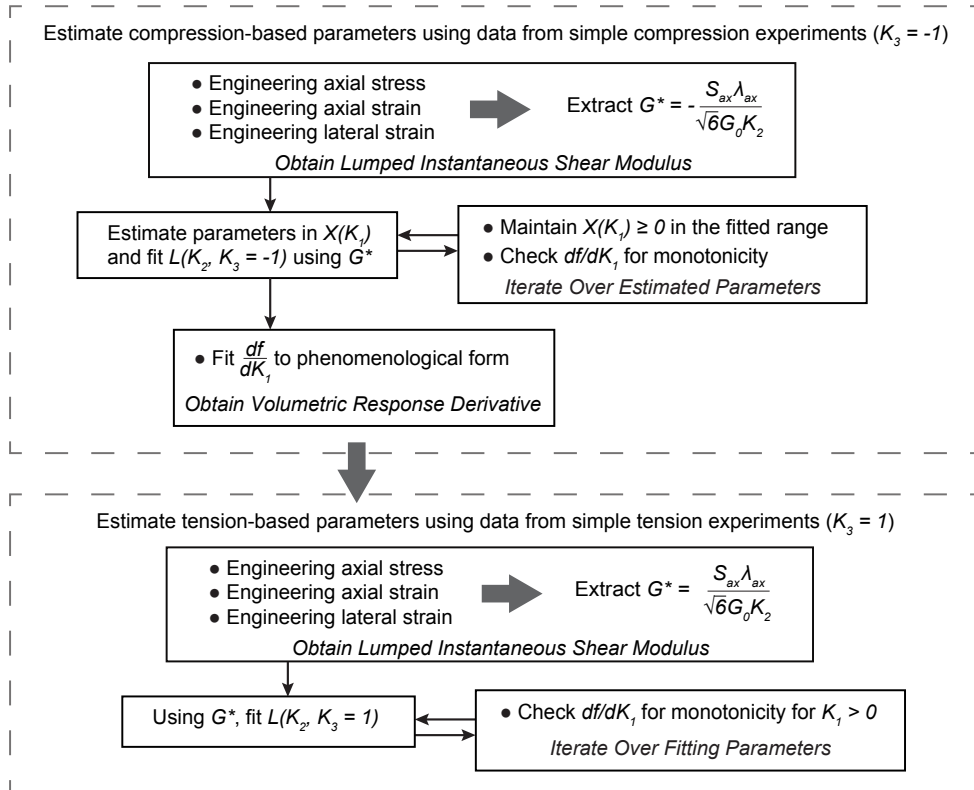


Figure A.1: Outline of the procedure for estimating material parameters from experimental data for simple compression and tension. Compression-based parameters are estimated first, followed by the estimation of tension-based parameters.

$S_3 = S_{\text{ax}} < 0$. Using (3.5) and the definition of the first Piola stress, $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-\top}$, the components of the constitutive equation for the stress are

$$\begin{aligned} S_{\text{ax}} &= \frac{1}{\lambda_{\text{ax}}} \left[\left(G_0 \frac{dX}{dK_1} K_2^2 + B \frac{df}{dK_1} \right) - \frac{2}{\sqrt{6}} G_0 \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \Big|_{K_3=-1} \right) \right], \\ 0 &= \frac{1}{\lambda_{\text{lat}}} \left[\left(G_0 \frac{dX}{dK_1} K_2^2 + B \frac{df}{dK_1} \right) + \frac{1}{\sqrt{6}} G_0 \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \Big|_{K_3=-1} \right) \right]. \end{aligned} \quad (\text{A.1})$$

Next, combining (A.1)₁ and (A.1)₂ in order to eliminate df/dK_1 and rearranging, we obtain

$$G^* = X(K_1) + \frac{1}{2K_2} \frac{\partial L}{\partial K_2} \Big|_{K_3=-1} = -\frac{S_{\text{ax}} \lambda_{\text{ax}}}{\sqrt{6} G_0 K_2}. \quad (\text{A.2})$$

where G^* is a normalized lumped instantaneous shear modulus. All quantities on the right-hand-side of (A.2) – namely, the histories of S_{ax} , λ_{ax} , and K_2 during simple compression as well as the ground-state shear modulus G_0 – are known, so that (A.2) may be used to determine the history of G^* during simple compression, which may, in turn, be used to estimate the parameters appearing in $X(K_1)$ and $L(K_2, K_3)$. The measured G^* history during simple compression, projected onto the G^* - K_1 -plane and the G^* - K_2 -plane, is shown as solid lines in Figs. A.2(a) and (b), respectively. An iterative process is necessary to estimate the material parameters appearing in the $X(K_1)$ and $L(K_2, K_3 = -1)$ functions, while ensuring that – for the purpose of stability – a monotonically increasing fitting function for df/dK_1 may be used to capture data. A method for choosing an initial guess for the parameters appearing in $X(K_1)$ is as follows:

1. The parameter X'_1 is the slope of $X(K_1)$ prior to the plateau regime in compression. The value of X'_1 may be estimated from the slope of the projection of G^* versus K_1 at $K_1 = 0$, as illustrated in Fig. A.2(a).
2. The parameter K_1^0 is the value of K_1 denoting the transition to the plateau regime in $X(K_1)$. The value of K_1^0 may be estimated as the center of the K_1 -range, in which the

slope of G^* versus K_1 in the projection of Fig. A.2(a) changes rapidly during simple compression, as illustrated in Fig. A.2(a).

3. The parameter Δ_K is the width of the K_1 -range across which the value of dX/dK_1 transitions from X'_1 to X'_2 . The value of Δ_K may be estimated from the width of the transition range in K_1 for G^* versus K_1 in the projection of Fig. A.2(a) during simple compression, as illustrated in Fig. A.2(a).
4. The parameter X'_2 is the slope of $X(K_1)$ after the onset of the plateau regime in compression. The value of X'_2 is estimated to be small compared to X'_1 – i.e., $X'_2 \sim O(X'_1/10)$. For the purpose of stability, it is important that the choice of this parameter results in $X(K_1) > 0$ over the entire fitted range of K_1 .

Guided by Fig. A.2(a), we estimate the material parameters appearing in the $X(K_1)$ function to be $X'_1 = 3.7$, $K_1^0 = -0.21$, $\Delta_K = 0.2$, and $X'_2 = 0.22$ for the Poron XRD foam material.

Once the parameters of the $X(K_1)$ function are estimated, the next step is to estimate the parameters of the $L(K_2, K_3 = -1)$ function, namely, C_0 and p . Rearranging (A.2), we obtain

$$\left. \frac{\partial L}{\partial K_2} \right|_{K_3=-1} = 2K_2 \left(-\frac{S_{ax}\lambda_{ax}}{\sqrt{6}G_0K_2} - X(K_1) \right). \quad (\text{A.3})$$

Using the measured data and the previously-estimated parameters, (A.3) may be used to determine C_0 and p , which we have estimated to be $C_0 = 0.1$ and $p = 4$ for the Poron XRD foam material. Projections of the fitted G^* history during simple compression, i.e., $G^* = X(K_1) + (1/(2K_2))\partial L/\partial K_2|_{K_3=-1}$, using the estimated material parameters are shown as dashed lines in Figs. A.2(a) and (b).

With initial estimates for X'_1 , K_1^0 , Δ_K , X'_2 , C_0 , and p in hand, we check that a monotonically increasing fitting function for df/dK_1 may be used to capture data. Rearranging

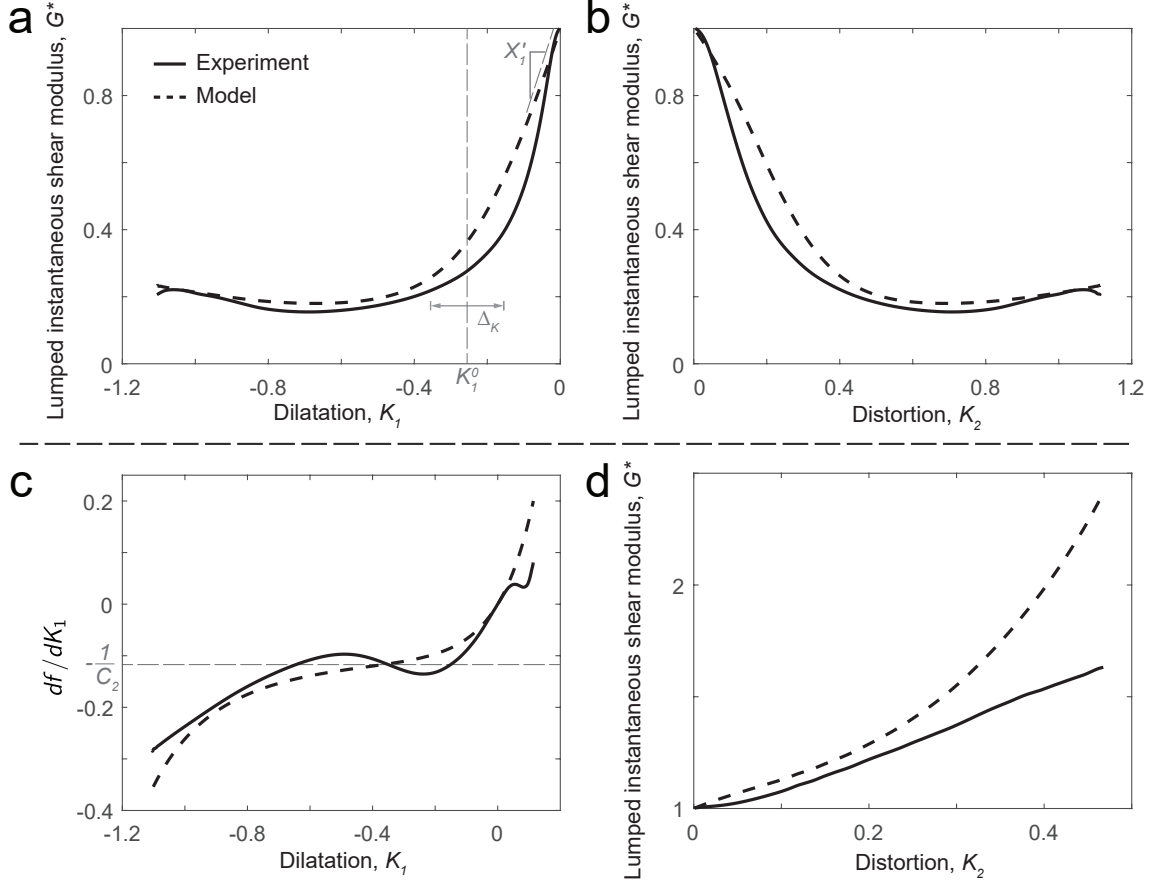


Figure A.2: Projections of the normalized lumped instantaneous shear modulus, G^* , plotted as a function of (a) the dilatation, K_1 , and (b) the distortion, K_2 , for a simple compression loading history ($K_3 = -1$). (c) The first derivative of the volumetric response function $f(K_1)$ with respect to K_1 , df/dK_1 , plotted as a function of K_1 . (d) Projection of the normalized lumped instantaneous shear modulus, G^* , plotted as a function of the distortion, K_2 , for a simple tension loading history ($K_3 = 1$). Solid lines denote experimental data for the high density Poron XRD foam material; dashed lines denote the model fit; and interpretations of the material parameters X_1' , K_1^0 , Δ_K , and C_2 are illustrated.

(A.1)₂, we obtain the following expression for df/dK_1 :

$$\frac{df}{dK_1} = -\frac{G_0}{B} \left[\frac{1}{\sqrt{6}} \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \Big|_{K_3=-1} \right) + \frac{dX}{dK_1} K_2^2 \right] \quad (\text{A.4})$$

Again, all quantities on the right-hand-side of (A.4) are either measured data or previously-fitted parameters and are therefore known. The df/dK_1 history during simple compression (A.4) is shown as a solid line in Fig. A.2(c), using the previously-estimated parameters. This

history is non-monotonic¹ but may be reasonably captured by a monotonically-increasing fitting function. If the df/dK_1 history is too non-monotonic, the material parameters of the $X(K_1)$ function should be adjusted. Typically, the parameter X'_1 should be decreased from its initial estimated value, and the parameter X'_2 should be increased – all while ensuring that $X(K_1) \geq 0$ over the fitted K_1 -range. Then, the $L(K_2, K_3 = -1)$ function is refit, and the resulting df/dK_1 history may be checked again for near-monotonicity. In this way, the material parameters of the $X(K_1)$ function are iteratively adjusted until the df/dK_1 history may be reasonably captured by a monotonically-increasing fitting function. Once this is achieved, the parameters $J_{\min}$, C_2 , C_3 , and r may be estimated. The parameter $J_{\min}$ represents the minimum allowable value of J and may be interpreted as $J_{\min} = 1 - \phi_0$, where ϕ_0 is the void volume fraction. Therefore, $\ln J_{\min}$ is the minimum allowable value of K_1 , and df/dK_1 diverges as $K_1 \rightarrow \ln J_{\min}$. The parameter C_2 sets the value of df/dK_1 in the plateau region. More specifically, the value of df/dK_1 in the plateau region is slightly more negative than $(-1/C_2)$, as illustrated in Fig. A.2(c). Finally, the parameters C_3 and r are adjustable parameters that may be chosen so that the model response matches the stiffening behavior after the plateau regime. The values of the parameters for the high density Poron XRD foam material, determined using this procedure, are $J_{\min} = 0.19$, $C_2 = 9$, $C_3 = 0.026$, and $r = 2$, and the fitted df/dK_1 function is plotted as a dashed line in Fig. A.2(c).²

¹We note that the non-monotonic behavior in the df/dK_1 history of Fig. A.2(c) arises due to the presence of the shear-induced mean stress term, $(dX/dK_1)K_2^2$, in (A.4).

² We note that the material parameter estimation procedure described in this section ensures that the fitted engineering stress versus axial engineering strain response remains monotonic in simple compression in order to maximize stable behavior. However, the model fit to the lateral engineering strain versus axial engineering strain response can be slightly non-monotonic (see Fig. 2.1). Since non-monotonicity in the lateral strain response does not affect stability, and since the oscillations in the lateral strain response are on the order of 0.01 whereas the axial strain varies over an order of 0.1, we do not prioritize the elimination of the relatively low amplitude oscillations in the fitted lateral strain response in the present work.

A.2 Estimation of tension-based parameters

Once the response in simple compression has been fit, experimental data from simple tension is used to estimate the remaining parameters in the $L(K_2, K_3)$ function, namely, C_1 and q . As in the material parameter estimation procedure described above for simple compression, estimating these parameters is an iterative process, since it must be ensured that df/dK_1 may be reasonably captured by a monotonically-increasing fitting function over the positive K_1 -range of application. In simple tension, the principal stretches are $\lambda_1 = \lambda_{\text{ax}} > 1$ and $\lambda_2 = \lambda_3 = \lambda_{\text{lat}} < 1$; the logarithmic strain invariants are $K_1 = \ln(\lambda_{\text{ax}}\lambda_{\text{lat}}^2)$, $K_2 = (\sqrt{6}/3)\ln(\lambda_{\text{ax}}/\lambda_{\text{lat}})$, and $K_3 = 1$; and the principal engineering stresses are $S_1 = S_{\text{ax}} > 0$ and $S_2 = S_3 = 0$. The components of the constitutive equation for the stress are then

$$\begin{aligned} S_{\text{ax}} &= \frac{1}{\lambda_{\text{ax}}} \left[\left(G_0 \frac{dX}{dK_1} K_2^2 + B \frac{df}{dK_1} \right) + \frac{2}{\sqrt{6}} G_0 \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \Big|_{K_3=1} \right) \right], \\ 0 &= \frac{1}{\lambda_{\text{lat}}} \left[\left(G_0 \frac{dX}{dK_1} K_2^2 + B \frac{df}{dK_1} \right) - \frac{1}{\sqrt{6}} G_0 \left(2X(K_1)K_2 + \frac{\partial L}{\partial K_2} \Big|_{K_3=1} \right) \right], \end{aligned} \quad (\text{A.5})$$

and combining (A.5)₁ and (A.5)₂ in order to eliminate df/dK_1 and rearranging, we obtain the normalized lumped instantaneous shear modulus, G^* , in tension:

$$G^* = X(K_1) + \frac{1}{2K_2} \frac{\partial L}{\partial K_2} \Big|_{K_3=1} = \frac{S_{\text{ax}}\lambda_{\text{ax}}}{\sqrt{6}G_0K_2}, \quad (\text{A.6})$$

which may be rearranged as

$$\frac{\partial L}{\partial K_2} \Big|_{K_3=1} = 2K_2 \left(\frac{S_{\text{ax}}\lambda_{\text{ax}}}{\sqrt{6}G_0K_2} - X(K_1) \right). \quad (\text{A.7})$$

Using the measured data and previously-estimated parameters, (A.7) may be used to estimate the remaining two parameters appearing in the $L(K_2, K_3 = 1)$ function, C_1 and q . Then,

rearranging (A.5)₂, we obtain the following expression for df/dK_1 during simple tension:

$$\frac{df}{dK_1} = \frac{G_0}{B} \left[\frac{1}{\sqrt{6}} \left(2X(K_1)K_2 + \left. \frac{\partial L}{\partial K_2} \right|_{K_3=1} \right) - \frac{dX}{dK_1} K_2^2 \right]. \quad (\text{A.8})$$

Using the estimated values for the parameters C_1 and q , (A.8) is used to check that the df/dK_1 history is nearly-monotonic for $K_1 > 0$. If this is not the case, the value of C_1 should be increased from its initial estimated value. If it is necessary to substantially increase C_1 to obtain a monotonic df/dK_1 history for $K_1 > 0$, so that the G^* history from simple tension is significantly overestimated, the estimation of the compression-based parameters should be repeated using a reduced value for X'_1 , and in this way, the parameters are iteratively adjusted until the df/dK_1 history is nearly monotonic. Using this process, we estimate the parameters for the high density Poron XRD foam material to be $C_1 = 1.9$ and $q = 5$, and the resulting fitted G^* history during simple tension, projected in the G^* - K_2 -plane, is shown in Fig. A.2(d) as a dashed line, along with the measured G^* history as a solid line. It is evident that a compromise is made in which G^* is overestimated in simple tension in order to obtain a suitable df/dK_1 history.

Appendix B

Loss of ellipticity

In this appendix, we provide background for the loss of ellipticity analysis utilized in Section 2.5.1. We consider a time-dependent, incremental process superimposed upon a time-independent base state of deformation [96]. The base state is described by a motion field $\chi(\mathbf{X})$, which satisfies the governing equations. Then, denoting small increments with a superposed dot, the incremental displacement field is expressed referentially as $\dot{\mathbf{u}}(\mathbf{X}, t)$ or, alternatively, expressed spatially as $\dot{\mathbf{u}}(\chi^{-1}(\mathbf{x}), t)$. The associated incremental deformation gradient and incremental spatial displacement gradient fields are $\dot{\mathbf{F}} = \nabla \dot{\mathbf{u}}$ and $\dot{\mathbf{H}} = \nabla \dot{\mathbf{u}} = \dot{\mathbf{F}}\mathbf{F}^{-1}$, respectively. The first Piola stress in the base state is given by $\mathbf{S} = J\mathbf{T}\mathbf{F}^{-\top} = \partial\psi/\partial\mathbf{F}$, and the increments in the first Piola stress are obtained by linearizing this constitutive equation: $\dot{\mathbf{S}} = \mathbb{C}_{\mathbf{r}}\dot{\mathbf{F}}$, where $\mathbb{C}_{\mathbf{r}} = \partial^2\tilde{\psi}/\partial\mathbf{F}\partial\mathbf{F}$ are the material moduli, evaluated in the base state. The stress increments, pushed forward to the deformed body, are given by $\dot{\mathbf{T}} = J^{-1}\dot{\mathbf{S}}\mathbf{F}^{\top} = \mathbb{C}\dot{\mathbf{H}}$, where

$$\mathbb{C}_{ijkl}(\mathbf{x}) = J^{-1}F_{jn}F_{lq}\frac{\partial^2\psi}{\partial F_{in}\partial F_{kq}}, \quad (\text{B.1})$$

are the spatial moduli, evaluated at the base state. The incremental form of the equations of motion are

$$\text{div } \dot{\mathbf{T}} = \text{div } (\mathbb{C}\dot{\mathbf{H}}) = \rho \frac{\partial^2 \dot{\mathbf{u}}}{\partial t^2}, \quad (\text{B.2})$$

where $\rho = J^{-1}\rho_r$ is the spatial mass density, ρ_r is the referential mass density, and $\partial(\bullet)/\partial t$ is the spatial time derivative. Straightforward application of the chain rule leads to the following expression for the spatial tangents:

$$\mathbb{C}_{ijkl} = \frac{1}{2J} \mathbb{D}_{ijmn} \mathbb{L}_{mnpq} \mathbb{B}_{pqkl} - T_{il} \delta_{jk} \quad (\text{B.3})$$

with

$$\mathbb{D} = \frac{\partial \mathbf{T}_K}{\partial \mathbf{E}}, \quad \mathbb{L} = \frac{\partial \ln(\mathbf{B})}{\partial \mathbf{B}}, \quad \text{and} \quad \mathbb{B}_{ijkl} = \delta_{ik} B_{jl} + \delta_{jk} B_{il}, \quad (\text{B.4})$$

where $\mathbf{T}_K = J\mathbf{T}$ is the Kirchhoff stress. For a free-energy function based on logarithmic strain invariants, $\tilde{\psi}(K_1, K_2, K_3)$, straightforward calculations involving the chain rule lead to the following expression for the constitutive contribution to the spatial tangents:

$$\begin{aligned} \mathbb{D}_{ijkl} = \frac{\partial T_{K,ij}}{\partial E_{kl}} &= \frac{\partial^2 \tilde{\psi}}{\partial K_1^2} \delta_{ij} \delta_{kl} + \left(\frac{\partial^2 \tilde{\psi}}{\partial K_2^2} - \frac{\partial \tilde{\psi}}{\partial K_2} \frac{1}{K_2} - \frac{\partial \tilde{\psi}}{\partial K_3} \frac{3K_3}{K_2^2} \right) N_{ij} N_{kl} + \frac{\partial^2 \tilde{\psi}}{\partial K_3^2} \frac{1}{K_2^2} Y_{ij} Y_{kl} \\ &+ \left(\frac{\partial^2 \tilde{\psi}}{\partial K_1 \partial K_2} - 2\sqrt{6} \frac{\partial \tilde{\psi}}{\partial K_3} \frac{1}{K_2^2} \right) (\delta_{ij} N_{kl} + N_{ij} \delta_{kl}) \\ &+ \frac{\partial^2 \tilde{\psi}}{\partial K_1 \partial K_3} \frac{1}{K_2} (\delta_{ij} Y_{kl} + Y_{ij} \delta_{kl}) \\ &+ \left(\frac{\partial^2 \tilde{\psi}}{\partial K_2 \partial K_3} \frac{1}{K_2} - 3 \frac{\partial \tilde{\psi}}{\partial K_3} \frac{1}{K_2^2} \right) (N_{ij} Y_{kl} + Y_{ij} N_{kl}) \\ &+ \left(\frac{\partial \tilde{\psi}}{\partial K_2} \frac{1}{K_2} - \frac{\partial \tilde{\psi}}{\partial K_3} \frac{3K_3}{K_2^2} \right) \left(\mathbb{I}_{ijkl} - \frac{1}{3} \delta_{ij} \delta_{kl} \right) \\ &+ \frac{3\sqrt{6}}{2} \frac{\partial \tilde{\psi}}{\partial K_3} \frac{1}{K_2^2} (\delta_{ik} N_{jl} + \delta_{il} N_{jk} + N_{ik} \delta_{jl} + N_{il} \delta_{jk}). \end{aligned} \quad (\text{B.5})$$

Then, we consider a homogeneous base state of deformation in an infinite domain – so that the base-state motion field χ varies linearly in $\mathbf{X}$, and the base-state deformation gradient $\mathbf{F}$, spatial logarithmic strain tensor $\mathbf{E}$, and spatial moduli $\mathbb{C}$ are all spatially constant – and

consider solutions for the incremental displacement field in the following form:

$$\dot{\mathbf{u}} = \hat{\mathbf{u}} g(k(\mathbf{n} \cdot \mathbf{x} - ct)), \quad (\text{B.6})$$

where $k\mathbf{n}$ is a wave vector, k is the wave-vector magnitude, $\mathbf{n}$ is the unit vector along the direction of the wave vector, c is the wave speed, $\hat{\mathbf{u}}$ is the amplitude of the incremental displacement, and g is a continuous and sufficiently differentiable (but otherwise arbitrary) function. Upon substituting (B.6) into (B.2), we obtain the following eigenvalue problem: $\mathbf{Q}(\mathbf{n})\hat{\mathbf{u}} = \rho c^2 \hat{\mathbf{u}}$ with eigenvalues ρc^2 and eigenvectors $\hat{\mathbf{u}}$, where $Q_{ik}(\mathbf{n}) = \mathbb{C}_{ijkl}n_j n_l$ is the acoustic tensor in the deformed configuration for a unit vector $\mathbf{n}$, which is utilized to assess stability in Section 2.5.1.

Appendix C

Experimental program for the closed-cell EVA foam

Note: The experiments described in this appendix were performed by Jialiang Tao. We include this appendix for completeness.

Specimen preparation All specimens are cut from the same batch of 17.5mm-thickness foam sheets, which are made in 165% BPC, 170°C mold temperature and in the same mold. For different tests, geometries and dimensions of specimens vary as different attachments and setups are applied. For compression testing, foam cuboids of approximately $11 \times 11 \times 12.7 \text{ mm}^3$ are cut with a bandsaw. For tension testing, dogbone-shaped specimens are cut using a die (Pioneer-Dietecs Corporation, Weymouth, MA) following the ASTM D3574 standard geometry for tensile testing of foam polymers but with shorter side lengths in order to fit grip attachments. For indentation experiments, $50 \times 50 \times 15 \text{ mm}^3$ samples are cut with a bandsaw and bonded to aluminum base plates using a two-part epoxy (Loctite, Henkel Corp., Hartford, CT). For shear experiments with or without compression, two $30 \times 20 \times 10 \text{ mm}^3$ specimens are cut with a bandsaw, and the two large faces are attached to two side plates on the shear device via a two-part epoxy. For inhomogeneous tension tests, circular holes are cut from tension specimens which follow the ASTM D3574 standard. For

DIC purposes, speckle patterns are applied with a black base coat of acrylic ink (Liquitex Artist Materials, Piscataway, NJ) using an airbrush (HP-C Plus, Iwata Medea Inc, Portland, OR) on the surface, followed by a white speckle layer (Golden Artist Colors Inc., New Berlin, NY). A customized DIC algorithm is utilized [43], which incorporates the concept of cross-correlation quality factors to improve the accuracy of DIC measurements. Finally, for repeated experiments on the same specimen, we ensure that the wait-time between experiments is sufficiently long (more than eight hours) to allow complete relaxation of the specimen.

Experimental setup In order to fulfill the experimental requirements for different strain rates, two different load frames are utilized in our experiments. For low strain-rate experiments within the range 1.0×10^{-3} - $1.0 \times 10^{-1} \text{ s}^{-1}$, we utilize the identical experiment setup employed in our previous work [79]. The load frame has a slow and low-capacity linear actuator (Ultra Motion, Cutchogue, NY) and a load cell (LCFD-50, Omega Engineering, Stamford, CT). A customized LabView (National Instruments, Austin, TX) interface is developed to synchronize actuator movement control, actuator displacement measurement, camera, and force data collection. The optical system utilized to collect data for DIC is comprised of a scientific camera (edge5.5, PCO-Tech Inc., Romulus, MI), a long-distance microscopy lens (K2/S, Infinity Photo-Optical Company, Boulder, CO), polarizing filters (American Polarizers Inc., Reading, PA and Edmund Optics, Barrington, NJ), and LED light panels (Lykos Daylight, Manfrotto, South Upper Saddle River, NJ), which is assembled on the same vibration-isolated table as the loading apparatus. Additionally, for all experiments, a secondary imaging camera with a field-of-view encompassing the complete specimen and load frame is used to detect any potential anomalous loading conditions (e.g., onset of specimen failure at the grips).

For the experiments in the moderate strain-rate range 1.0 - $1.0 \times 10^1 \text{ s}^{-1}$, we utilize a different load frame which has a faster, high-capacity linear actuator (Linmot, Lake Geneva,

WI) and a quartz-based dynamic load cell (1050C, Dytran Instruments, Chatsworth, CA). A schematic diagram of the load frame used in the moderate strain-rate experiments is shown in Fig. C.1. The synchronization of motor movement control, actuator displacement, camera, and force data collection is controlled by the driver and the specialized control software Linmot Talk. The optical system is composed of a lens (ZEISS Milvus 2/100m, Carl Zeiss A, White Plains, NY), a teleconverter (Bower SX7DGN, BowerUSA, Long Island, NY) and a close-up lens (The Tiffen Company, LLC, Hauppauge, NY). For lighting purposes, two lighting panels (Lykos Daylight, Manfrotto, South Upper Saddle River, NJ) are used. All the devices for this setup are installed on a passive vibration-isolated optics table (Thorlabs Inc., Newton, NJ).

For simple compression or tension experiments, to obtain uniform and non-localized displacement fields, two different fixtures are used for specimens. For compression experiments, the specimen is sandwiched between two polished steel platens. In order to eliminate moments applied by the platens, which may cause shear deformation during compression experiments, we use a spherical indenter tip to contact the top platen moment-free. The contact areas between the platens and the specimen are lubricated with a mixture of graphite and glass beads (diameter 20-30 μm) to mitigate barreling effects. For simple tension, two grips are attached to the base of the load frame, and load cells are installed on both ends of the specimen.

Details of simple compression/tension experiments The simple compression and tension experiments are controlled to reach an engineering strain limit of -0.55 (axial stretch of $\lambda = 0.45$) in compression and 0.25 (axial stretch of $\lambda = 1.25$) in tension. In both experiments, specimens are loaded and unloaded at a constant engineering strain-rate. Since no significant inhomogeneous deformation is observed, components of the global displacement gradient tensor, du_1/dx_1 and du_2/dx_2 are fitted by 2D linear plane fitting according to the displacement fields for each deformation step during each experiment. The force data is

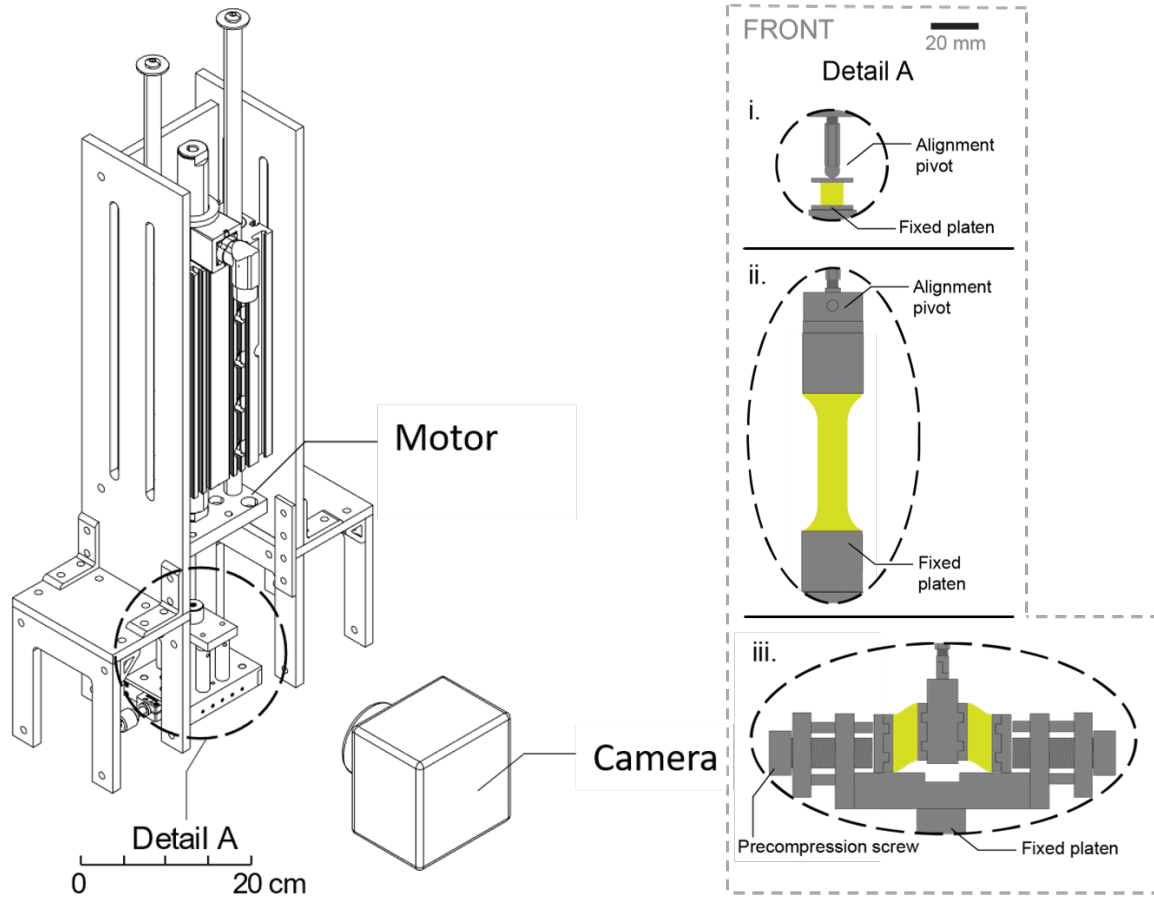


Figure C.1: Schematic of the isometric view of the high-speed setup as deployed on a vibration-isolated optical breadboard including camera and load frame. Detail A gives front-view diagrams (i.e., as seen by the cameras) of the fixtures used for (i) simple compression, (ii) simple tension, and (iii) shear.

read directly from the load cell. Then, the engineering stress data is computed by dividing force data by the measured undeformed cross-sectional area of the specimens.

Appendix D

Material parameter estimation procedure – viscoelastic model

D.1 Equilibrium branch

The process for estimating the material parameters of the viscoelastic model is outlined in Fig. D.1, showing the overall flow of the procedure. The first step is to fit the equilibrium response. The material parameters appearing in the equilibrium, hyperelastic response are estimated from the experimental data from simple compression and tension at the lowest strain rate ($2.0 \times 10^{-4} \text{ s}^{-1}$). Our estimate of the experimental hyperelastic response is the average of the loading and unloading curves. Following the procedure discussed in Appendix A, we obtain the material parameters in Table 4.1, and the fit for the equilibrium hyperelastic response shown by the dashed line in Fig. D.2(a).

D.2 Non-equilibrium branches

Once the equilibrium branch has been fitted, the non-equilibrium branches are to be fitted. Several heuristics guide this process:

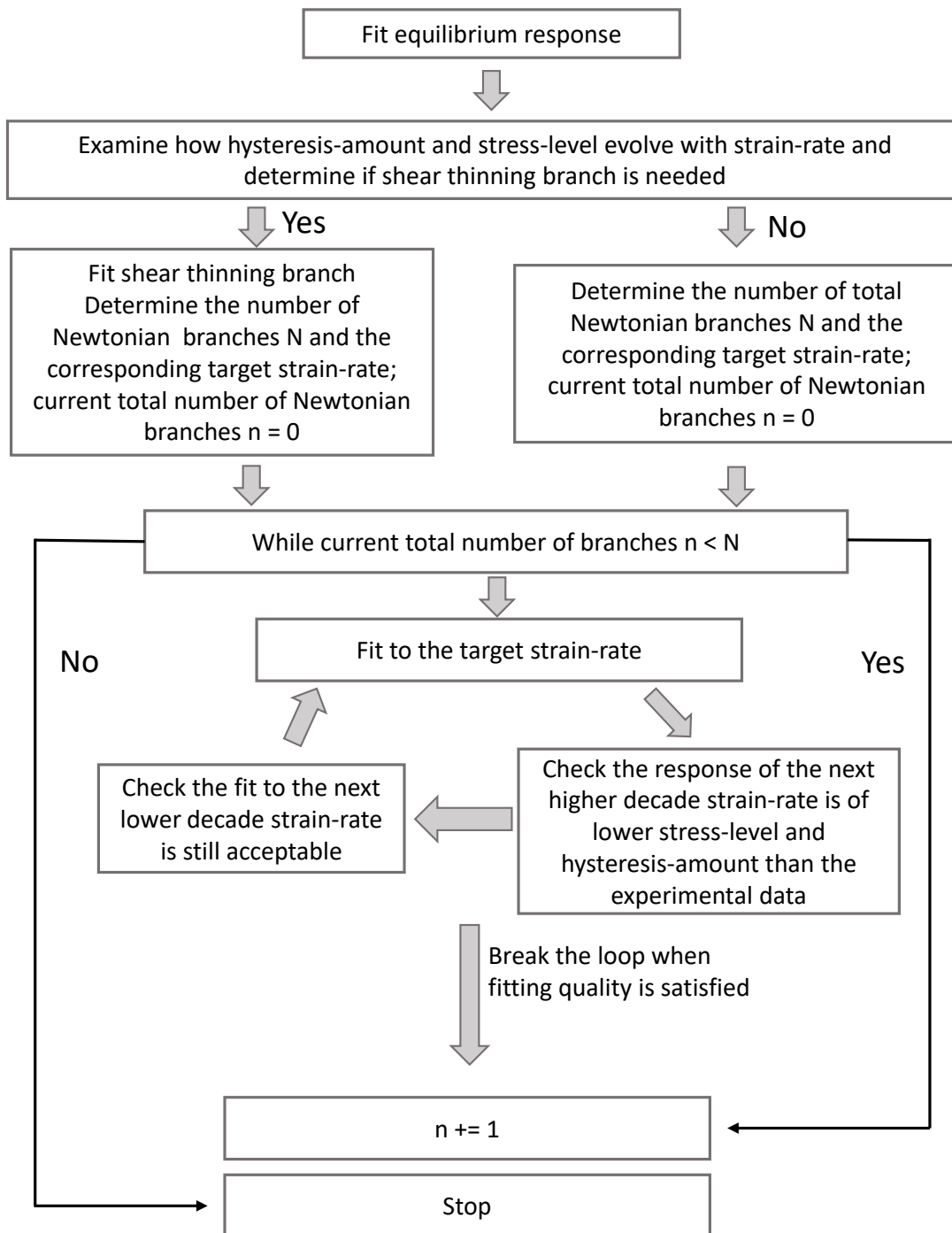


Figure D.1: Outline of the procedure for estimating the material parameters of the viscoelastic model from experimental data for simple compression and tension at different strain rates.

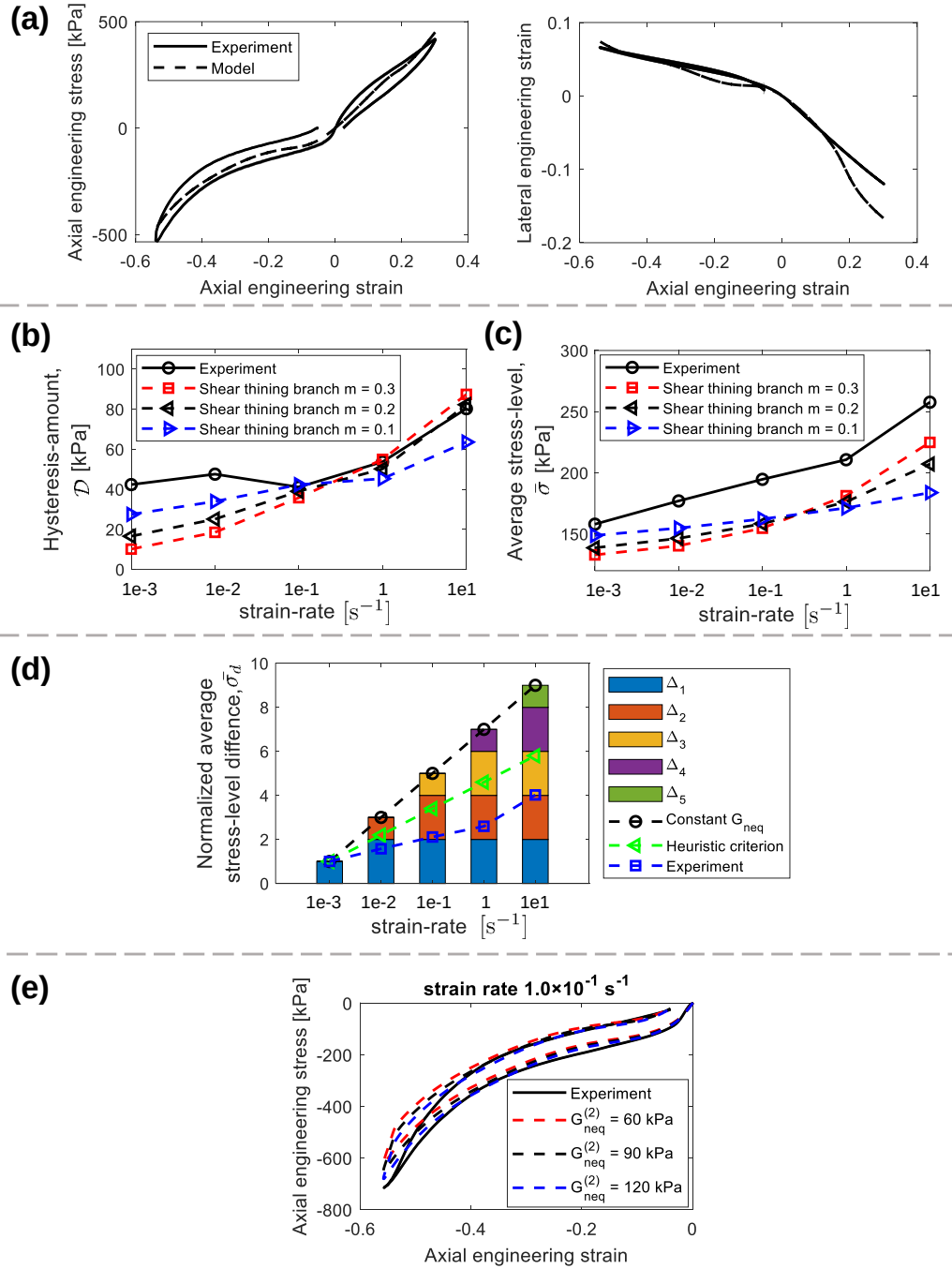


Figure D.2: (a) Fit of the equilibrium branch to experimental data at a low strain rate of $2.0 \times 10^{-4} s^{-1}$. (b) Amount of hysteresis versus strain rate. (c) Average stress-level versus strain rate. (d) Normalized average stress-level difference versus strain rate. Black symbols and bars represent model estimates when the non-equilibrium shear moduli of all branches $G_{neq}^{(\alpha)}$ are equal; blue symbols represent experimental data; and green symbols represent the heuristic criteria for determining whether a shear thinning branch is required. (e) Effect of different values of $G_{neq}^{(2)}$ on the stress-strain curve in compression when the parameter $t^{(2)}$ is fixed at $t_0^{(2)} = 50$ s.

1. To reduce model complexity, we constrain the number of shear thinning branches to be at most one. We examine the dependence of the average stress-level in compression on strain rate to determine if a shear thinning branch is required.
2. The total number of non-equilibrium branches should be chosen to be around the number of decades of strain-rate that the model is to be fitted to. Each Newtonian branch is targeted at a different strain-rate. If no shear thinning branch is invoked, the total number of Newtonian branches should be equal to the number of decades of strain-rate.
3. For the Newtonian branches, the parameters $G_{\text{neq}}^{(\alpha)}$ and $t_0^{(\alpha)}$ are fit to the compression data, and the parameters $q_{\text{neq}}^{(\alpha)}$ and $C_{\text{1neq}}^{(\alpha)}$ are fit to the tension data. To minimize the number of material parameters utilized, we always start by taking $C_{\text{1neq}}^{(\alpha)} = 0$, i.e., taking the non-equilibrium energy to be independent of $K_3^{(\alpha)}$, and we adjust $C_{\text{1neq}}^{(\alpha)}$ and $q_{\text{neq}}^{(\alpha)}$ to be nonzero only when necessary.
4. Depending on what features of the stress-strain curve are prioritized, we may define different quantitative measures of fitting quality, such as the percentage difference of the maximum stress or the percentage difference of the amount of hysteresis. Here, we prioritize the overall fitting quality and define a scalar error as follows:

$$\text{error} = \frac{\int_0^T |\sigma_{\text{exp}}(t) - \sigma_{\text{model}}(t)| dt}{\int_0^T |\sigma_{\text{exp}}(t)| dt}, \quad (\text{D.1})$$

where $\sigma_{\text{exp}}(t)$ is the experimentally-measured stress history, $\sigma_{\text{model}}(t)$ is the stress history predicted by the model, and T is the total time of the experiment. We take an error of less than 0.1 to be acceptable when determining when to end our iterative fitting process.

Determining whether a shear thinning branch is necessary First, we determine whether a shear thinning branch is necessary. To reduce model complexity, in application, we constrain the number of shear thinning branches to be at most one. We then examine plots of the amount of hysteresis $\mathcal{D}$ (defined in (3.2)) and the normalized average stress-level difference $\bar{\sigma}_d$ ¹ as a function of strain rate. Figure D.2(b) shows how the hysteresis-amount changes with strain rate, and Figs. D.2(c) and (d) show how the average stress-level and the normalized average stress-level difference vary as a function of strain rate, respectively. It is expected that the amount of hysteresis should increase as a function of strain rate. In the present case, $\mathcal{D}$ shows slight fluctuations over the first three strain rates, but we regard $\mathcal{D}$ as nearly uniform over this strain-rate range given the fact that the maximum strain magnitude in compression varies slightly between experiments. Overall, here we have that the amount of hysteresis is nearly uniform before increasing slowly with strain rate. In the case of a monotonically-increasing amount of hysteresis, we propose a straightforward way to determine whether a shear thinning branch is required. We compare experimental data for the normalized average stress-level difference as a function of strain rate (blue symbols in Fig. D.2(d)) to the heuristic criterion shown as green symbols in Fig. D.2(d). The criterion we adopt is if the experimental data is below 60% of the constant G_{neq} case (discussed further below) shown as black symbols in Fig. D.2(d), this indicates that a shear thinning branch is required, which is the case for the experimental data considered here.

The reasoning behind shear thinning branch criterion Suppose that all non-equilibrium branches are Newtonian. Then one non-equilibrium branch is invoked for each decade of

¹The normalized average stress-level difference $\bar{\sigma}_d$ is defined as follows. First, the average stress-level $\bar{\sigma}$ is defined as in (3.1). The stress level difference is then defined as the difference between the average stress-level for a certain strain-rate $\bar{\sigma}$ and the stress-level for the equilibrium response $\bar{\sigma}_{\text{eq}}$. Finally, we normalize the stress-level difference by dividing $(\bar{\sigma} - \bar{\sigma}_{\text{eq}})$ by the stress-level difference of the lowest strain-rate, which is $1.0 \times 10^{-3} \text{ s}^{-1}$ for the present case:

$$\bar{\sigma}_d = \frac{\bar{\sigma} - \bar{\sigma}_{\text{eq}}}{\bar{\sigma}(\dot{\epsilon} = 10^{-3} \text{ s}^{-1}) - \bar{\sigma}_{\text{eq}}} \quad (\text{D.2})$$

Therefore, the normalized average stress-level difference $\bar{\sigma}_d$ has a value of 1 at the lowest strain rate of $1.0 \times 10^{-3} \text{ s}^{-1}$ and increases with increasing strain rate.

strain rate under consideration. A heuristic standard for determining whether such a purely Newtonian viscoelastic model can capture a set of experimental data is explained as follows. Let us assume that we have several decades of strain-rate to fit. As a specific example, say we consider five decades of strain rate beginning at $1.0 \times 10^{-3} \text{ s}^{-1}$, which is the case for the experimental data in the present work. For the lowest strain rate, the first Newtonian non-equilibrium branch is fit to the stress-level difference (denoted as $\Delta^{(1)}$) and the amount of hysteresis at that strain rate. A reasonable estimate is that the viscous behavior of this non-equilibrium branch will be nearly frozen for the next highest decade of strain rate ($1.0 \times 10^{-2} \text{ s}^{-1}$), and therefore, that this non-equilibrium branch will contribute an additional $\Delta^{(1)}$ to the stress-level difference for all higher strain rates. For a strain rate of $1.0 \times 10^{-2} \text{ s}^{-1}$, we introduce the second Newtonian non-equilibrium branch to fit to the stress-level and the amount of hysteresis at that strain rate and denote the contribution of the second non-equilibrium branch to the normalized average stress-level difference as $\Delta^{(2)}$. We estimate that the second non-equilibrium branch also contributes an additional $\Delta^{(2)}$ to the stress-level difference of all strain rates greater than $1.0 \times 10^{-2} \text{ s}^{-1}$ for the reasons discussed above. We repeat this procedure for each decade of strain rate. In the end, the normalized average stress-level difference may be decomposed into contributions due to each branch $\Delta^{(\alpha)}$, and each $\Delta^{(\alpha)}$ should be proportional to the non-equilibrium shear modulus for that branch $G_{\text{neq}}^{(\alpha)}$. Figure D.2(d) shows the estimated decomposition of the normalized average stress-level difference as colored bars when all $\Delta^{(\alpha)}$ are equal, which is the case when all $G_{\text{neq}}^{(\alpha)}$ are equal, and the black symbols in Fig. D.2(d) show the estimated dependence of the total $\bar{\sigma}_d$ on strain rate, which appears linear in the semi-logarithmic scale. From our experience, if the slope of the experimentally-measured normalized average stress-level versus strain rate data is below 60% of the estimated constant $G_{\text{neq}}^{(\alpha)}$ slope, which is the heuristic criterion shown as green symbols in Fig. D.2(d), we reach the conclusion that the data cannot be adequately fitted by a model with purely Newtonian non-equilibrium branches.

Fitting the shear thinning branch The shear thinning branch involves five material parameters $\{m^{(1)}, G_{\text{neq}}^{(1)}, \dot{\gamma}_0^{(1)}, q_{\text{neq}}^{(1)}, C_{\text{1neq}}^{(1)}\}$. For a single shear thinning branch, the amount of hysteresis versus strain-rate relation is always a bell-shaped curve – first increasing then decreasing as the strain rate is increased – while the average stress-level versus strain-rate relation is monotonically increasing. The strain-rate sensitivity exponent $m^{(1)}$ is a crucial material parameter since $m^{(1)}$ controls how spread out the amount of hysteresis curve is and the overall slope of the average stress-level relation. The parameter $G_{\text{neq}}^{(1)}$ controls the peak value of the amount of hysteresis curve and the upper limit value of the average stress-level relation, and the parameter $\dot{\gamma}_0^{(1)}$ controls the shift position of the curves. We adjust the parameter $m^{(1)}$ to fit to the overall slope of the amount of hysteresis curve, and we adjust the parameters $G_{\text{neq}}^{(1)}$ and $\dot{\gamma}_0^{(1)}$ to ensure that both the amount of hysteresis curve and the average stress-level relation of the shear thinning branch are below the experimental data (anticipating that additional non-equilibrium mechanisms will be invoked that will increase the model predictions to be in line with the experimental data). We compare the effect of different choices of $m^{(1)}$ on these two relations in Figs. D.2(b) and (c). (Note that only the increasing portion of the amount of hysteresis curve appears in Fig. D.2(b) with the decreasing portion occurring for strain rates greater than the fitted range.) We estimate that $m^{(1)} = 0.2$, $G_{\text{neq}}^{(1)} = 500 \text{ kPa}$, and $\dot{\gamma}_0^{(1)} = 500 \text{ s}^{-1}$. It is also necessary to determine the parameters $q_{\text{neq}}^{(1)}$ and $C_{\text{1neq}}^{(1)}$ from experimental data in tension. When we plot the model prediction for a strain rate of $1.0 \times 10^{-3} \text{ s}^{-1}$ using only the equilibrium branch and the first non-equilibrium (shear thinning) branch with $C_{\text{1neq}}^{(1)} = 0$, we find that the predicted axial stress in tension is nearly consistent with the experimental data, so for simplicity we take $C_{\text{1neq}}^{(1)} = 0$. After the parameters of the shear thinning branch have been chosen, we use Newtonian non-equilibrium branches to fit the remaining features in the amount of hysteresis and average stress-level versus strain rate relations. We observe that for the first three strain rates ($1.0 \times 10^{-3} \text{ s}^{-1}$ to $1.0 \times 10^{-1} \text{ s}^{-1}$), the amount of hysteresis is underestimated for the first two strain rates while the stress-level is underestimated the most for the third strain

rate. This motivates adding a second non-equilibrium branch which is Newtonian and in which the viscous behavior is nearly frozen at a strain rate of $1.0 \times 10^{-1} \text{ s}^{-1}$. We also observe that at strain rates of 1.0 s^{-1} and $1.0 \times 10^1 \text{ s}^{-1}$, the amount of hysteresis is close to the experimental data while the stress-level is underestimated. This motivates adding third and fourth non-equilibrium Newtonian branches in which the viscous responses are nearly frozen at the corresponding strain rates. Accordingly, for the experimental data under consideration here, we invoke one shear thinning branch and three Newtonian branches in total.

Fitting the second non-equilibrium branch (Newtonian) Four material parameters are involved in a Newtonian branch $\{G_{\text{neq}}^{(2)}, t_0^{(2)}, q_{\text{neq}}^{(2)}, C_{1\text{neq}}^{(2)}\}$. We seek a Newtonian non-equilibrium branch, in which the viscous behavior is nearly frozen (i.e., the branch behaves in a nearly elastic manner) at a strain rate of $1.0 \times 10^{-1} \text{ s}^{-1}$ and contributes a fair amount of hysteresis at lower strain rates of $1.0 \times 10^{-3} \text{ s}^{-1}$ and $1.0 \times 10^{-2} \text{ s}^{-1}$. Therefore, the initial estimate of the viscous timescale $t_0^{(2)}$ is taken to be $1/0.1 \text{ s}^{-1} = 10 \text{ s}$. (This is typically an underestimate of the final fitting value of the viscous timescale.) We estimate an initial guess for $G_{\text{neq}}^{(2)}$ as the difference between the ground shear moduli at strain rates of $1.0 \times 10^{-1} \text{ s}^{-1}$ and $1.0 \times 10^{-3} \text{ s}^{-1}$. Next, while fixing $t_0^{(2)}$, we adjust the value of $G_{\text{neq}}^{(2)}$ to fit to the stress-strain data at the target strain rate of $1.0 \times 10^{-1} \text{ s}^{-1}$. Figure D.2(e) illustrates the effect of different choices of the parameter $G_{\text{neq}}^{(2)}$ at a fixed $t_0^{(2)}$. Then, we check if the model predictions for the next two lower strain rates ($1.0 \times 10^{-2} \text{ s}^{-1}$ and $1.0 \times 10^{-3} \text{ s}^{-1}$) remain well fit to the experimental data. If stress-levels predicted by the model are much lower than the experimental data, we increase $t_0^{(2)}$. We also check whether the model prediction for the next highest strain rate (1.0 s^{-1}) is below the experimental data (for both the amount of hysteresis and the stress-level). If not, we increase $t_0^{(2)}$. Whenever $t_0^{(2)}$ is changed, we return to the target strain rate of $1.0 \times 10^{-1} \text{ s}^{-1}$ and adjust $G_{\text{neq}}^{(2)}$ to fit the experimental stress-strain data. We continue this iterative process until the parameters $G_{\text{neq}}^{(2)}$ and $t_0^{(2)}$ converge to

satisfactory values. We have estimated $G_{\text{neq}}^{(2)} = 90 \text{ kPa}$ and $t_0^{(2)} = 50 \text{ s}$. Finally, we check the response in tension and observe that the parameters $C_{\text{ineq}}^{(2)}$ and $q_{\text{neq}}^{(2)}$ are not required to capture the tension data, so we take $C_{\text{ineq}}^{(2)} = 0$.

Fitting the third non-equilibrium branch (Newtonian) We seek a Newtonian non-equilibrium branch in which the viscous response is nearly frozen at a strain-rate of 1.0 s^{-1} , so the initial estimate of the viscous timescale $t_0^{(3)}$ is taken to be $1/1.0 \text{ s}^{-1} = 1 \text{ s}$. We take the difference between the ground shear moduli at strain rates of 1.0 s^{-1} and $1.0 \times 10^{-1} \text{ s}^{-1}$ as the initial estimate for the parameter $G_{\text{neq}}^{(3)}$. For a fixed value of $t_0^{(3)}$, we adjust $G_{\text{neq}}^{(3)}$ to fit to the experimental data in compression for the target strain-rate of 1.0 s^{-1} . Then, we check if the model predictions for the next lower rate ($1.0 \times 10^{-1} \text{ s}^{-1}$) remain well fit to the experimental data. If the stress-level predicted by the model is much lower than the experimental data, we increase $t_0^{(3)}$. We also check whether the model prediction for the next higher rate ($1.0 \times 10^1 \text{ s}^{-1}$) is below the experimental data. If not, we increase $t_0^{(3)}$. Whenever $t_0^{(3)}$ is modified, we return the target strain-rate of 1.0 s^{-1} and adjust $G_{\text{neq}}^{(3)}$ to fit the experimental stress-strain data. This process is repeated until $G_{\text{neq}}^{(3)}$ and $t_0^{(3)}$ converge to satisfactory values. Finally, we check the response in tension. For the present closed-cell foam material, we note that there is a jump in axial stress magnitude in tension for the target strain rate of 1.0 s^{-1} , so we adjust the values of $C_{\text{ineq}}^{(3)}$ and $q_{\text{neq}}^{(3)}$ to fit to the experimental data in tension. The parameter $q_{\text{neq}}^{(3)}$ controls the shape of stress-strain curve, while $C_{\text{ineq}}^{(3)}$ controls the stress-level. We have estimated $G_{\text{neq}}^{(3)} = 80 \text{ kPa}$, $t_0^{(3)} = 2 \text{ s}$, $C_{\text{ineq}}^{(3)} = 2.5$, and $q_{\text{neq}}^{(3)} = 2.1$.

Fitting the fourth non-equilibrium branch (Newtonian) We seek a Newtonian non-equilibrium branch in which the viscous response is nearly frozen at a strain rate of $1.0 \times 10^1 \text{ s}^{-1}$ and follow a similar process as described above for a strain rate of 1.0 s^{-1} . The parameters $G_{\text{neq}}^{(4)}$ and $t_0^{(4)}$ are fit to the compression data, and we adjust the parameters $C_{\text{ineq}}^{(4)}$ and $q_{\text{neq}}^{(4)}$ to fit to the tension data. We have estimated $G_{\text{neq}}^{(4)} = 50 \text{ kPa}$ and $t_0^{(4)} = 0.1 \text{ s}$ and observed that the parameters $C_{\text{ineq}}^{(4)}$ and $q_{\text{neq}}^{(4)}$ are not required to fit to the tension data, so

that $C_{\text{ineq}}^{(4)} = 0$.

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