Applications of FEM with Macro-Elements: Discrete Elasticity Sequences and Convergence of Lagrange Elements for a Maxwell Eigenvalue Problem

by

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Abstract of "Applications of FEM with Macro-Elements: Discrete Elasticity Sequences and Convergence of Lagrange Elements for a Maxwell Eigenvalue Problem", by Sining Gong, Ph.D., Brown University, May 2023

In this thesis, we investigate two applications of finite element methods that employ macro-elements: discrete elasticity sequences and convergence of Lagrange elements for a Maxwell eigenvalue problem. We employ the combination of finite element exterior calculus and spline theories as our primary tool.

Specifically, we prove that Worsey-Farin refinements inherit the shape regularity of their parent triangulations, and due to the special structure of Worsey-Farin refinements and the existence of smoother differential sequences, we investigate the convergence of the Maxwell eigenvalue problem using quadratic or higher Lagrange finite elements on Worsey-Farin splits. To this end, we construct two Fortin-like operators to demonstrate uniform convergence of the corresponding source problem. We provide numerical experiments to validate our theoretical results.

Moreover, we develop conforming finite element elasticity complexes on Worsey-Farin splits in three dimensions. These complexes connect spaces for displacement, strain, stress, and load through differential operators that represent deformation, incompatibility, and divergence. We exhibit corresponding finite element spaces on Worsey-Farin meshes and develop unisolvent degrees of freedom for these finite elements. This also yields commuting (cochain) projections on smooth functions. Notably, these spaces lack extrinsic supersmoothness at subsimplices of the mesh, yet they yield the first (strongly) symmetric stress element with no vertex or edge degrees of freedom in three dimensions. Lastly, we show that the lowest order stress space uses only piecewise linear functions, which is the lowest feasible polynomial degree for the stress space.

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Chapter One

Introduction

Finite Element Exterior Calculus (FEEC) [7, 8, 10] and splines theories [53] are two important topics in scientific computing. One of the main interest within the spline community is to build splines of various smoothness on some structured triangulations. The spaces of those splines are called macro-elements and they inspire the work in FEEC, making it possible to construct smoother piecewise polynomial spaces fit into discrete de Rham sequences; [28,40,43,44,48]. In this thesis, we introduced some new applications of finite element methods (FEM) with macro-elements, including the convergence analysis for a Maxwell eigenvalue problem and the construction of discrete elasticity elements with the combination of FEEC and splines theories.

Starting from the 1960s, the theory of univariate splines as well as multivariate splines experienced significant growth and later, many engineers brought these results into finite element methods [53]. Several interesting triangulations with splines defined on them will be used through this thesis, including splines on three-dimensional Alfeld splits [2] (or Clough–Tocher splits in two dimensions [31]), three-dimensional Worsey-Farin splits [64] and two-dimensional Powell–Sabin splits [59]. They have been extensively studied since those splits firstly came out; see for example [53]. In particular, throughout this thesis, we will be continuously working with three-dimensional Worsey-Farin splits. In 1987, A.J. Worsey and G. Farin constructed low-order C^1 splines on the composite triangulation which now bears their names. Worsey-Farin splits have very special structure such that the interior faces of the sub-tetrahedra derived by the Alfeld splits are coplanar to the corresponding interior faces of the adjacent sub-tetrahedron; see Section 2.2.1.2 for more details. Such special structure leads to lower-order polynomials and various macro-elements without vertex degrees of freedom.

1.1 Shape regularity

The first topic covered in Chapter 3 is based on splines theories and will be applied to stability analysis in Chapter 4 in combination with FEEC. Specifically, we aim to discuss the properties of various refinements introduced by the spline theory community, particularly in the context of approximation and stability properties of the corresponding discrete spaces. A critical geometric property for approximation theory is the shape regularity of the underlying mesh.

Our focus is on the shape regularity of Worsey-Farin splits, as it is necessary to ensure optimal-order and uniform interpolation estimates in [53, Theorem 18.15], [3, Theorem 6.3], [61, Theorem 6.2], and [56, Theorem 8.14]. Additionally, stability estimates of a finite element method defined on Worsey-Farin splits in [38] also require regularity of the refined triangulation. In Chapter 4, the shape regularity of Worsey-Farin splits is also crucial for achieving uniform bounds in the convergence analysis of the Maxwell eigenvalue problem.

The references [53, Page 515], [3, Remark 14], and [4, Page 54] explicitly suggest that Worsey-Farin splits of a family of shape regular meshes remain shape regular. However, to the best of our knowledge, a proof of this result has not yet been published in the literature. In Chapter 3 of this thesis, we provide a proof to fill this gap.

In [53, Lemma 4.20] and [52, Lemma 2.6], they provide a proof of the relationship between the shape regularity constant of Powell-Sabin splits (see [59] and Section 2.2.1.1) and the parent triangulations. This is achieved by bounding the angles of each macro triangle. For the three-dimensional analogue Worsey-Farin case, we focus on the dihedral angles. In Chapter 3, we prove that these angles are bounded by quantities that only depend on the shape regularity of the original mesh (refer to Lemma 3.1.3). Using this result, we establish that the split points of each face F in the triangulation $\mathcal{T}_h^{wf}$ are uniformly bounded away from ∂F (refer to Lemma 3.1.4), which leads to the proof of the shape regularity of Worsey-Farin refinements.

1.2 A Maxwell Eigenvalue Problem

Finite elements (see Section 2.2) have been a staple in scientific computing for over half a century and continue to be widely used for many important simulations. The simplest form of finite elements is the lowest order Lagrange finite elements (see Section 2.5.1), where a domain is first triangulated using triangles or tetrahedra (see Section 2.2.1), and piecewise linear globally continuous functions are used with respect to the triangulation. These finite elements are utilized as basis functions and applied to the PDE's natural variational formulation, as demonstrated below in (1.2) (for instance). However, for problems in electro-magnetism, these finite elements on general triangulations are problematic, as they fail (as shown in Table 4.1) to provide accurate solutions for the Maxwell eigenvalue problem on a domain Ω. To be more precise, consider the following problem: find $u \in \mathring{H}(\mathrm{curl}\,, \Omega)$ such that

$$(\mathrm{curl}\, u, \mathrm{curl}\, v) = \eta^2 (u, v), \quad \forall v \in \mathring{H}(\mathrm{curl}, \Omega), \tag{1.1}$$

where $(\cdot, \cdot)$ is the $L^2(\Omega)$ inner product over Ω and

$$\mathring{H}(\mathrm{curl}, \Omega) := \{v \in [L^2(\Omega)]^3 : \mathrm{curl}\, v \in [L^2(\Omega)]^3 \text{ and } v \times n = 0 \text{ on } \partial\Omega\}.$$

Accordingly, a canonical finite element method with respect to a given finite-dimensional space $\mathscr{V}_h \subset \mathring{H}(\mathrm{curl}, \Omega)$ is to find $u_h \in \mathscr{V}_h \backslash \{0\}$ and $\eta_h \in \mathbb{R}$ such that

$$(\mathrm{curl}\, u_h, \mathrm{curl}\, v_h) = \eta_h^2 (u_h, v_h), \quad \forall v_h \in \mathscr{V}_h. \tag{1.2}$$

This lead to the celebrated *edge* finite elements of Nédélec in 1980 [58]. There he took a subset of piece-wise linear functions that were no longer globally continuous but instead had only tangential continuity across the inter-element edges/faces. These finite elements are also called Nédélec spaces; see Section 2.5.1.

Since the work of Nédélec there has been an explosion of methods that work for Maxwell eigenvalue problems. But, the question remained: why don't Lagrange elements work? In fact, the survey paper by D. Boffi [17] discuss this question in detail. He points out that an important requirement for the success of a finite element space is that enough gradients are given, which contributes to accurately approximate the zero frequency by vanishing discrete eigenvalues. A common approach in finite element theory that several researchers have taken is using Lagrange elements and modifying the discrete variational form by adding penalization or regularization terms; see [14, 21, 23, 34–36]. On the other hand, although we mentioned that Lagrange elements on generic triangulations without modifying the variational form fail, D. Boffi mentions in [17] that the performance of Lagrange elements is sensitive to the mesh, and it was observed computationally on certain meshes that seem to work! In particular, it was first observed by Wong and Cendes [63] that the use of linear Lagrange finite element spaces defined on Powell-Sabin (see [59] and Section 2.2.1.1) meshes in two dimensions and the use of quadratic Lagrange finite element spaces on "consistent tetrahedral meshes" in three dimensions lead to accurate approximations. Although Wong and Cendes do not explicitly define "consistent tetrahedral meshes", it is reasonable to assume they are referring to the three-dimensional analogue of Powell-Sabin triangulations, in particular, Worsey-Farin meshes [64] which we recall in Section 2.2.1.2.

The theoretical justification of this remained elusive until recently where we give the first analysis of three dimensions in Chapter 4 and the two-dimensional results were carried out by my collaborators [20]. The key to our result is that the space of Lagrange elements on Worsey-Farin splits fit nicely into a discrete de Rham sequence (2.26a). The space $\mathcal{L}_{r-1}^1(\mathcal{T}_h^{wf})$ is the Lagrange space on Worsey-Farin splits and the preceding space $\mathcal{S}_r^0(\mathcal{T}_h^{wf})$ is the C^1 spline space on Worsey-Farin splits used in the spline community.

The work in Chapter 4 can be considered a continuation of paper [20], where we consider Lagrange elements in three dimensions on Worsey-Farin triangulations on a contractible, polyhedral domain. We mathematically justify the numerical experiments of Wong and Cendes [63] and prove that the use of quadratic (or higher) Lagrange elements on Worsey-Farin splits lead to accurate approximations to Maxwell's eigenvalue problem. In order to establish the convergence of a finite element method applied to a Maxwell eigenvalue problem, it is sufficient to prove the uniform convergence of the corresponding source problems. This can be achieved by the construction of Fortin-like operators; see Section 4.1. The present analysis in three dimension is more involved than the two-dimensional analysis given in [20]. In particular, we need to develop two Fortin-like operators whereas in [20] only one was needed. To do this, we exploit that Lagrange elements on Worsey-Farin splits also fit into a discrete de Rham sequence [44]. Then, using the degrees of freedom given in [44] we construct a Fortin-like operators for both curl and divergence operators. In order to prove that these operators are bounded, we use certain embeddings that hold on Lipshitz polyhedral domains; see Section 2.1.1. Note that instead of using the well-established theory of finite element methods for the Maxwell eigenvalue problem (e.g., [10, 17, 47]), we define the projection on the semi-discrete space $\mathscr{V}^t(\mathscr{Q}_h) := \{\tau \in \mathscr{V}^t : \operatorname{curl}\tau \in \mathscr{Q}_h\}$, where $\mathscr{V}^t := \mathring{H}(\operatorname{curl}, \Omega) \cap H(\operatorname{div}, \Omega)$; see Chapter 4 for more details. Defining the projection on this smaller space allows us the flexibility to prove the convergence of the Lagrange finite element methods on Worsey-Farin splits. We also point out that the existence of bounded Fortin-type operators is equivalent to the discrete compactness property [16], which is commonly used in the convergence analysis of the Maxwell eigenvalue problem.

To the best of our knowledge, this seems to be the first work theoretically justifying convergence of Lagrange elements on simplicial meshes in three dimensions without modifying the bilinear form. Parallel work by Hu et al. [48] also develop finite elements on different splits with partially discontinuous elements while leaving the bilinear forms unchanged. They demonstrate convergent results for Worsey-Farin splits numerically but theoretical analysis has not been provided.

We provide numerical results confirming our theoretical findings. In particular, we show that one must use at least quadratic Lagrange elements for Worsey Farin splits to get convergence. That is, the use of linear Lagrange elements on Worsey-Farin refinements does not yield convergent approximations.

1.3 Elasticity Complexes

Let Ω is a bounded domain in $\mathbb{R}^d$, $d = 2, 3$, occupied by an elastic body. The isotropic, homogeneous, linearly elasticity equations can be described as the following:

$$\sigma = 2\mu\varepsilon(u) + \lambda \operatorname{tr}(\varepsilon(u))\boldsymbol{I} \quad \text{in } \Omega, \tag{1.3}$$

$$\operatorname{div} \sigma = f \quad \text{in } \Omega, \tag{1.4}$$

where the function $u : \Omega \to \mathbb{R}^d$ is the displacement field, $\sigma : \Omega \to \mathbb{R}^{d \times d}$ is the stress tensor, and $\varepsilon(u)$ is the deformation operator (or strain tensor). We shall mention here the vector f is the forces applied to the body and the constants $\mu > 0$ and $\lambda \geq 0$ are called Lamé constants specifying the material properties [9]. There are a lot of methods for elasticity problems and mixed formulations are one type of methods particularly robust with respect to Lamé parameters, so we will focus on them exclusively here. In order to create a stable finite element method using mixed formulations, we need to construct two discrete spaces that satisfy the inf-sup condition. Researchers have produced fruitful results for elasticity problems since the 1960s. However, it wasn't until 2002 that Arnold and Winther constructed the first conforming mixed elasticity elements on two-dimensional general triangular meshes in their paper [13]. And later the first three-dimensional (3D) elasticity elements were constructed in [1, 7], which inspired many other similar elements, as mentioned in [49]. In particular, for the two-dimensional spaces in [13], they constructed an entire discrete elasticity sequence and showed how the last two spaces there are relevant for discretizing the Hellinger-Reissner principle in elasticity. If we could construct an exact discrete elasticity sequence with certain properties, including bounded commuting operators, then the last two spaces in the sequence are possible candidates for a stable finite element method to the elasticity equations. Furthermore, the framework of the stability analysis in FEEC could potentially be applied to the corresponding elasticity equations. However, the generalization of two-dimensional work to three dimensional is not that trivial even if the work in [7] laid the foundation.

Luckily, in 1970s, an algebraic technique called the Bernstein-Bernstein-Gelfand

(BGG) resolution came out [15] and later it was applied to derive a new differential sequence from simpler sequences which in particular could be used to construct the elasticity sequences from two de Rham sequences; see [11,24,37]. With BGG method, we could have elasticity sequences with smooth function spaces.

$$0 \longrightarrow \mathsf{R} \xrightarrow{\subset} C^\infty \otimes \mathbb{V} \xrightarrow{\varepsilon} C^\infty \otimes \mathbb{S} \xrightarrow{\text{inc}} C^\infty \otimes \mathbb{S} \xrightarrow{\text{div}} C^\infty \otimes \mathbb{V} \longrightarrow 0. \tag{1.5}$$

Here, $\mathbb{V} = \mathbb{R}^3$, $\mathbb{M} = \mathbb{R}^{3\times3}$, $\mathsf{R} = \{a + b \times x : a, b \in \mathbb{R}^3\}$ denotes rigid displacements, $\text{inc} = \text{curl} \circ \tau \circ \text{curl}$ with τ denoting the transpose, curl and divergence operators are applied row by row on matrix fields, $\mathbb{S} = \text{sym}(\mathbb{M})$, and $\varepsilon = \text{sym} \circ \text{grad}$ denotes the deformation operator (already introduced in the elasticity equations (1.3)). The sequence (1.5) is exact on a three dimensional contractible domain. We assume throughout that our domain Ω is contractible. In addition, with the BGG procedure and de Rham sequences of Sobolev spaces $H^s \equiv H^s(\Omega)$, the authors of [11] extended the elasticity sequence of smooth spaces to the following elasticity sequence of Sobolev spaces:

$$\mathsf{R} \xrightarrow{\subset} H^s \otimes \mathbb{V} \xrightarrow{\varepsilon} H^{s-1} \otimes \mathbb{S} \xrightarrow{\text{inc}} H^{s-3} \otimes \mathbb{S} \xrightarrow{\text{div}} H^{s-4} \otimes \mathbb{V} \longrightarrow 0. \tag{1.6}$$

However, one of the problems in constructing finite element sub-sequences of (1.6) is the increase of four orders of smoothness from the last space (H^{s-4}) to the first space (H^s). A search for finite element sub-sequences of elasticity sequences with different Sobolev spaces seemed to hold more promise [7].

It was not until 2020 that the first 3D discrete elasticity sub-complex was established in [27]. They use a simple version of BGG method to construct a sequence of the following form:

$$\mathcal{R} \xrightarrow{\subset} U^0 \xrightarrow{\varepsilon} U^1 \xrightarrow{\text{inc}} U^2 \xrightarrow{\text{div}} U^3 \longrightarrow 0, \tag{1.7}$$

which is a discrete sub-sequence of

$$\mathcal{R} \xrightarrow{\subset} H^2 \otimes \mathbb{V} \xrightarrow{\varepsilon} H(\text{inc}) \xrightarrow{\text{inc}} H(\text{div}, \mathbb{S}) \xrightarrow{\text{div}} L^2 \otimes \mathbb{V} \longrightarrow 0. \tag{1.8}$$

Some previous work is summarized as follows. Note that they need extra smoothness

Table 1.1: Comparison of previous work in 3D elasticity sequences

	Lowest order of U^2	Vertex/edge DOFs in U^2	Extra vertex smoothness
Alfeld splits [27]	2	required	required
Generic meshes [26]	4	required	required
Work in this thesis	1	not required	not required

on the vertices, need vertex/edge DOFs for the stress spaces and require at least either quadratics [27] or quartics [26] as the lowest polynomial order for the stress spaces. In this thesis, we apply the methodology presented in [27] to construct a new discrete elasticity sequence on Worsey-Farin splits [64]. One of the expected benefits of using triangulations of macro-elements is the potential reduction of polynomial degree and the potential escape from the unavoidability [7] of vertex degrees of freedom in stress elements. We will see that Worsey-Farin splits offer structures where these benefits can be reaped easier than on Alfeld splits. Unlike Afleld splits, which divide each tetrahedron into four sub-tetrahedra, Worsey-Farin triangulations split each tetrahedron into twelve sub-tetrahedra. Using the Worsey-Farin split, we are able to reduce the polynomial degree and also eliminate vertex and edge DOFs of stress elements. In particular, our approach results in stress spaces that are piece-wise linear, which is the lowest possible polynomial degree. Furthermore, this approach does not require vertex or edge DOFs for the stress spaces, making it the first 3D conforming symmetric stress element without vertex DOFs. This is comparable to the 2D elasticity element without vertex DOFs constructed in [9, 45, 50].

Although we have the framework in [27] to guide the construction of the discrete sequence on Worsey-Farin splits, as we shall see, we face significant new difficulties peculiar to Worsey-Farin splits. The most troublesome of these arises in the construction of DOFs and corresponding commuting projections. Unlike Alfeld splits, Worsey-Farin triangulations induce a Clough-Tocher split on each face of the original, unrefined triangulation. As a result, discrete 2D elasticity sequences with respect to Clough-Tocher splits play an essential role in our construction and proofs. These 2D sequences are more complicated than their analogues on Alfeld splits (where the faces are not split). The resulting difficulties are most evident in the design of DOFs for the space before the stress space (named U_r^1 later) in the sequence, as we shall see in Lemma 6.2.9.

1.4 Organization of the dissertation

The rest of the dissertation is organized as follows: In Chapter 2, we introduce some notations and fundamental results needed to understand the proceeding chapters, including the construction of macro-elements on different splits, various inequalities and definitions of differential operators. In Chapter 3, we prove the useful results in spline theories: shape regularity of Worsey-Farin splits that will also be used in the Chapter 4. In Chapter 4, we focus on a Maxwell eigenvalue problem and use de-Rham sequences to prove the convergence of Lagrange elements on Worsey-Farin splits for the problem. Rigrous proofs and numerical experiments are provided in this chapter. Finally, we summarize the discrete elasticity sequences for two-dimensional case and use the BGG tool to construct sequences for three dimensional case, in Chapter 5 and Chapter 6, respectively. Those sequences are provided with the local spaces, degrees of freedom and global spaces.

Chapter Two

Notations and preliminaries

In this chapter, we will introduce some basic notations and terminologies that will apply to the remaining parts of thesis, unless specified.

2.1 Sobolev spaces

Let $\Omega \subset \mathbb{R}^d$ be an open set and for simplicity, we only consider real-valued functions on Ω. Then for $1 \leq p \leq \infty$, we define the *Lebesgue spaces* as follows:

$$L^p(\Omega) := \{u : \|u\|_{L^p(\Omega)} < \infty\}, \tag{2.1}$$

where for $1 \leq p < \infty$,

$$\|u\|_{L^p(\Omega)} := (\int_\Omega |u(x)|^p \, dx)^{1/p},$$

and for $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} := \text{ess sup}\,\{|u(x)| : x \in \Omega\}.$$

With weak derivative, we can define so-called *Sobolev spaces* for non-negative integer k. Let $v \in L^1_{loc}(\Omega)$, the set of locally integrable functions over Ω the *Sobolev spaces* is defined via

$$W^{k,p}(\Omega) := \{u \in L^p_{loc}(\Omega) : \|u\|_{W^{k,p}(\Omega)} < \infty\},$$

where for $1 \leq p < \infty$,

$$\|u\|_{L^p(\Omega)} := (\int_\Omega |u(x)|^p \, dx)^{1/p},$$

and for $p = \infty$,

$$\|u\|_{L^\infty(\Omega)} := \text{ess sup}\,\{|u(x)| : x \in \Omega\}.$$

In particular, we use the notation $H^k(\Omega) := W^{k,2}(\Omega)$.

We also recall the definition of fractional-order Sobolev spaces and their accompanying (semi-) norm; see for example [42].

Definition 2.1.1. For $0 < s < 1$ and $1 \leq p < \infty$ define

$$W^{s,p}(\Omega) := \{u \in L^p(\Omega) : \int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy < \infty\}.$$

The accompanying semi-norm and norm on $W^{s,p}(\Omega)$ are given respectively by

$$|u|_{W^{s,p}(\Omega)} := (\int_\Omega \int_\Omega \frac{|u(x) - u(y)|^p}{|x-y|^{d+sp}} \, dx \, dy)^{1/p};$$

$$\|u\|_{W^{s,p}(\Omega)} := (\|u\|^p_{L^p(\Omega)} + |u|^p_{W^{s,p}(\Omega)})^{1/p}.$$

Finally, we use the notation $H^s(\Omega) := W^{s,2}(\Omega)$.

We could extend the definition of fractional-order Sobolev norms in the case $\Omega = \partial S$, where S is a simplex. We take the same definition as above, where we view Ω as a d-dimensional manifold; see [42].

Throughout this thesis, we commonly use $\mathring{(\cdot)}$ to denote the corresponding spaces with vanishing traces; see the following two examples:

$$\mathring{H}(\mathrm{curl}, \Omega) := \{v \in [L^2(\Omega)]^3 : \mathrm{curl}\, v \in [L^2(\Omega)]^3 \text{ and } v \times n = 0 \text{ on } \partial\Omega\}.$$

$$\mathring{H}(\mathrm{div}, \Omega) := \{v \in [L^2(\Omega)]^3 : \mathrm{div}\, v \in [L^2(\Omega)]^3 \text{ and } v \cdot n = 0 \text{ on } \partial\Omega\}.$$

Here, n is the exterior unit normal vector on $\partial\Omega$ on $\partial\Omega$.

To better describe the analysis later, we introduce the following space notation:

$$\begin{aligned}\mathscr{V} &:= H(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega),\\ \mathscr{V}^t &:= \mathring{H}(\mathrm{curl}, \Omega) \cap H(\mathrm{div}, \Omega),\\ \mathscr{V}^n &:= H(\mathrm{curl}, \Omega) \cap \mathring{H}(\mathrm{div}, \Omega).\end{aligned}$$

2.1.1 Trace Theorems, Inverse inequalities and Embeddings

We will use the following basic Sobolev embedding result; see [42, Theorem 1.4.4.1]. We restrict ourselves to a simplex for simplicity.

Lemma 2.1.2. *Let K be a d-dimensional simplex. Then,*

$$\|w\|_{W^{t,q}(K)} \le C\|w\|_{W^{s,p}(K)}, \quad 0 \le t \le s \le 1,\ 1 \le p \le q < \infty, \tag{2.2}$$

with $s - d/p = t - d/q$. The constant $C > 0$ depends on K.

We will also need a trace inequality; see [42, Theorem 1.5.1.2]

Lemma 2.1.3. *Let K be a d-dimensional simplex. Let $0 < s - 1/p < 1, 0 \le s \le 1$. If $w \in W^{s,p}(K)$, then*

$$\|w\|_{W^{s-1/p,p}(\partial K)} \le C\|w\|_{W^{s,p}(K)}. \tag{2.3}$$

Moreover, if $v \in W^{s-1/p,p}(\partial K)$ there exists $w \in W^{s,p}(K)$ such that $w|_{\partial K} = v$ and

$$\|w\|_{W^{s,p}(K)} \le C\|v\|_{W^{s-1/p,p}(\partial K)}. \tag{2.4}$$

The constant $C > 0$ depends on K.

We will often require particular inverse estimates for polynomial spaces, which follow from equivalence of norms on finite dimensional spaces; for more general inverse estimates consult, for example, [22, Section 1.6 and Section 4.5].

Lemma 2.1.4. *Let K be a d-dimensional simplex. For any $v \in \mathcal{P}_r(K)$ the following bounds hold*

$$|v|_{W^{k,p}(K)} \leq C h_K^{\ell-k+d/p-d/q} |v|_{W^{\ell,q}(K)} \quad \forall 1 \leq p, q < \infty,\ 0 \leq \ell \leq k \leq 1, \tag{2.5a}$$

$$\|v\|_{L^p(\partial K)} \leq C h_K^{-1/p} \|v\|_{L^p(K)} \quad \forall 1 \leq p \leq \infty, \tag{2.5b}$$

where $C > 0$ depends on the shape-regularity of K, r, and d, but is independent of h_K. In addition, when $p = \infty$ and $k = 0$, (2.5a) still holds.

We also need the following two embeddings of vector-valued functions for $\Omega \in \mathbb{R}^3$. The first embedding result is given in [6, Proposition 3.7].

Proposition 2.1.5. *If Ω is a Lipschitz polyhedron, there exists $\delta \in (0, \frac{1}{2}]$ and $C > 0$ such that:*

$$\|v\|_{H^{1/2+\delta}(\Omega)} \leq C(\|v\|_{L^2(\Omega)} + \|\operatorname{curl} v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)}) \qquad \forall v \in \mathscr{V}^t \cup \mathscr{V}^n. \tag{2.6}$$

Here, the constant δ in (2.6) may differ for $v \in \mathscr{V}^t$ and $v \in \mathscr{V}^n$. Thus, we choose the smaller constant δ of these two embeddings. Next, we use a result given in [10, Theorem 2.2] where we use that Ω is a contractible, Lipschitz polyhedron.

Lemma 2.1.6 (Poincaré Inequality)**.** *There exists a positive constant C such that for all $v \in \mathscr{V}^t \cup \mathscr{V}^n$,*

$$\|v\|_{L^2(\Omega)} \leq C(\|\operatorname{curl} v\|_{L^2(\Omega)} + \|\operatorname{div} v\|_{L^2(\Omega)}).$$

2.2 Finite element methods

Finite element method is an important method using variation forms and finite dimensional function spaces to approximate the solutions of different PDEs. To be more precise, it could be defined as follows [22]:

Definition 2.2.1. Let

- $T \subset \mathbb{R}^d$ be a bounded closed set with nonempty interior and piecewise smooth boundary (**the element domain**),
- V be a finite-dimensional space of functions on T (**the space of shape functions**) and
- $\mathcal{N} = \{N_1, ..., N_k\}$ be a basis for V' (**the set of nodal variables**).

Then $(T, V, \mathcal{N})$ is called a finite element.

Some additional definitions from [22] are also needed here.

Definition 2.2.2. Let $(T, V, \mathcal{N})$ be a finite element. The basis $\{\phi_1, \phi_2, ..., \phi_k\}$ of V satisfying $N_i(\phi_j) = \delta_{ij}$ is called the nodal basis of V.

In particular, we need the following lemma to find $\mathcal{N}$, the basis for V'. This is also called *unisolvence*.

Definition 2.2.3. Let V be a k-dimensional vector space and let $\{N_1, ..., N_k\}$ be a subset of the dual space V'. Then the following two statements are equivalent:

- $\{N_1, ..., N_k\}$ is a basis for V'.

- Given $v \in V$ with $N_i v = 0$ for $i = 1, 2, ..., k$, then $v \equiv 0$.

2.2.1 Simplicial meshes

Throughout the thesis, we need to use simplicial triangulations of Ω, a polygonal subset of $\mathbb{R}^2$ (or $\mathbb{R}^3$) as our mesh. To be more specific, let $\{\mathcal{T}_h\}$ be family of shape-regular and simplicial triangulations of Ω in $\mathbb{R}^2$ (resp. $\mathbb{R}^3$) and each element $T \in \mathcal{T}_h$ is a triangle (resp. tetrahedron).

For a set of simplices $\mathcal{S}$, we use $\Delta_s(\mathcal{S})$ to denote the set of s-dimensional simplices (s-simplices for short) in $\mathcal{S}$. If $\mathcal{S}$ is a simplicial triangulation of a domain D with boundary, then $\Delta_s^I(\mathcal{S})$ denotes the subset of $\Delta_s(\mathcal{S})$ that does not belong to the boundary of the domain. If S is a simplex, then we use the convention $\Delta_s(S) = \Delta_s(\{S\})$. For a non-negative integer r, we use $\mathcal{P}_r(S)$ to denote the space of piecewise polynomials of degree $\leq r$ on S, and we define

$$\mathcal{P}_r(\mathcal{S}) = \prod_{S \in \mathcal{S}} \mathcal{P}_r(S).$$

Note that for any face $F \in \Delta_2(T)$ with $T \in \mathcal{T}_h$, F is a triangle and therefore, any results work for F (including splits and sequences) could be applied to $T \in \mathcal{T}_h$ in two-dimensional case locally, same vice verse.

We also introduce the following definition for both two-dimensional mesh and three-dimensional mesh:

Definition 2.2.4. We define the shape regularity constant c_0 of the triangulation

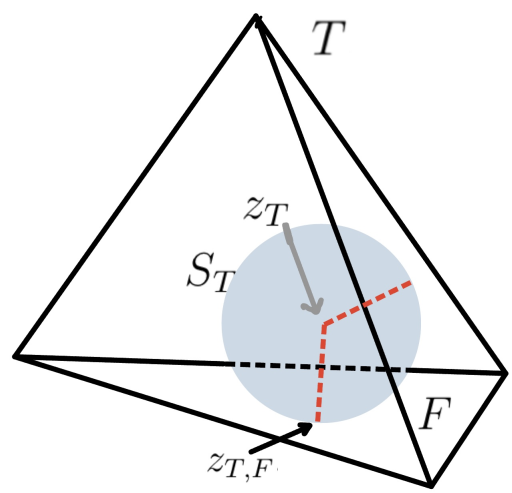


Figure 2.1: The inscribed sphere of a tetrhedron.

$\mathcal{T}_h$ as

$$c_0 = \max_{T \in \mathcal{T}_h} \frac{h_T}{\rho_T},$$

where ρ_T is the diameter of the inscribed sphere of T, which is the largest sphere contained in T and $h_T := \mathrm{diam}(T)$.

In three dimensions, in order to investigate the properties of our mesh, some basic definitions regarding the geometric properties of a tetrahedron need to be introduced; see [53, Definition 16.1-16.2] for more details.

We call the center of the inscribed sphere S_T the incenter of T, denoted by z_T, and call the radius of S_T the inradius of T, equal to $\rho_T/2$. The sphere S_T intersects each face F of T at a unique point, $z_{T,F}$. We note that $z_{T,F}$ is the orthogonal projection of the point z_T to the plane that contains F (i.e., the vector $z_T - z_{T,F}$ is normal to F); see Figure 2.1.

The following two propositions are well-known results of tetrahedra. To be self-contained we provide their proofs.

Proposition 2.2.5. *For a tetrahedron T, there holds*

$$\rho_T = 6|T|/(\sum_{F\in\Delta_2(T)}|F|).$$

Proof. Consider the refinement of T obtained by connecting the incenter of T to its vertices. The resulting four subtetrahedra fill the volume of T, and thus,

$$|T| = \sum_{F\in\Delta_2(T)} \frac{1}{3}|F|\frac{\rho_T}{2},$$

which gives the result. ∎

Proposition 2.2.6. *Given a tetrahedron T, let x be any vertex of T and F_x be the face of T which is opposite to x. Let P_x be the plane containing F_x, then for any point $a \in \mathrm{int}(T)$, we have*

$$\mathrm{dist}(x, P_x) > \mathrm{dist}(a, P_x). \tag{2.7}$$

In particular,

$$\mathrm{dist}(x, P_x) > \rho_T. \tag{2.8}$$

Proof. Since the point $a \in \mathrm{int}(T)$ and F_x is a face of T, a and F_x form a tetrahedron $T' \subset T$. Therefore,

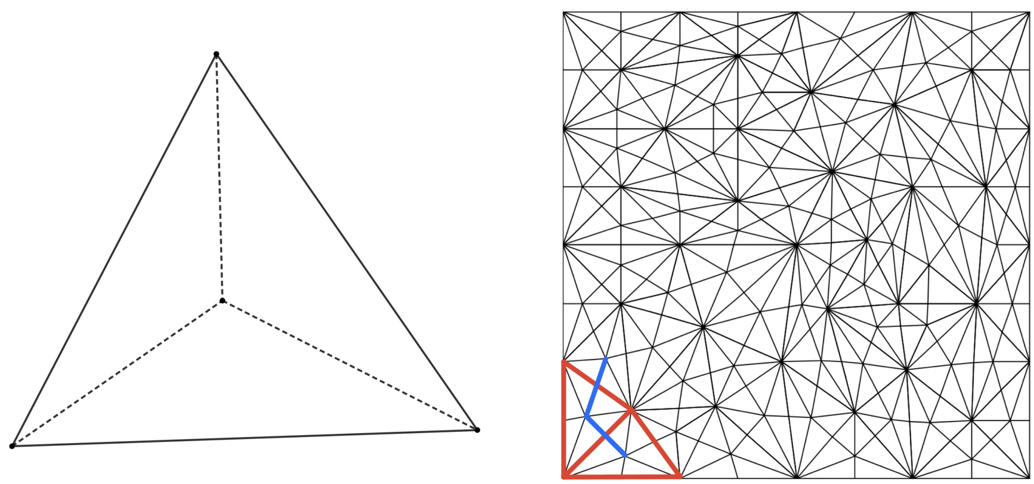
$$\frac{1}{3}|F_x|\mathrm{dist}(x, P_x) = |T| > |T'| = \frac{1}{3}|F_x|\mathrm{dist}(a, P_x),$$

which immediately gives (2.7). Let ℓ be the line containing z_T and z_{T,F_x}. Let ℓ intersect S_T at $a \neq z_{T,F_x}$. Then $a \in \mathrm{int}(T)$ and $\mathrm{dist}(a, P_x) = |[a, z_{T,F_x}]| = \rho_T$. Hence, (2.8) follows from (2.7). ∎

In the following parts, we introduce several important splits which will be used in the remaining parts of the thesis.

2.2.1.1 Clough-Tocher splits and Powell-Sabin splits

We introduce two kinds of splits for triangles and the results for two dimensional cases inspire the work in three dimensions. For a given triangle T, we split T to three triangles with connecting a fixed interior point to three vertices of T and denote the set of these three triangles by T^{ct}. The Clough-Tocher refinement of two dimensional mesh $\mathcal{T}_h$ is denoted by $\mathcal{T}_h^{\text{ct}}$ (cf. [53, Section Chapter 4] and Figure 2.2a).



(a) The Clough-Tocher splits locally.

(b) The Powell-Sabin splits.

Figure 2.2: Two kinds of refinement in 2D.

The Powell-Sabin refinement of $\mathcal{T}_h$, denoted by $\mathcal{T}_h^{\text{ps}}$, is obtained by splitting each $T \in \mathcal{T}_R$ by the following two steps (cf. [53, Definition 4.18] and Figure 2.2b):

(a) Connect the incenter z_T of T to its (three) vertices.

(b) Connect z_T and $z_{T'_R}$ whenever the triangles T and T'_R share a common edge.

(c) Connect the middle of each boundary edge to the incenter of the associated triangle.

2.2.1.2 Alfeld splits and Worsey-Farin splits in three dimensions

The Alfeld refinement of $\mathcal{T}_h$, denoted by $\mathcal{T}_h^a$, is obtained by picking an interior point of any element $T \in \mathcal{T}_h$ and connect that point with the four vertices of T. We denote the local mesh of the Alfeld-type refinement by T^a, which consists of four tetrahedra; see Figure 2.3.

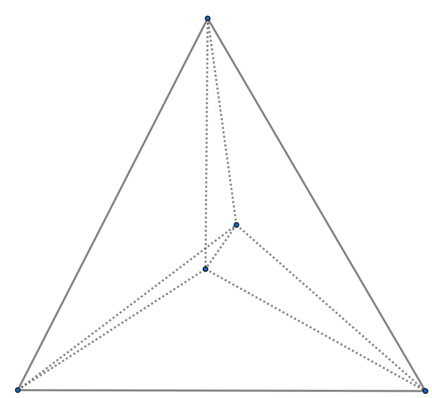


Figure 2.3: The Alfeld splits locally.

The Worsey-Farin refinement of $\mathcal{T}_h$, denoted by $\mathcal{T}_h^{wf}$, is obtained by splitting each $T \in \mathcal{T}_h$ by the following two steps (cf. [44, Section 2]):

(a) Connect the incenter z_T of T to its (four) vertices.

(b) For each face F of T choose $m_F \in \text{int}(F)$. We then connect m_F to the three vertices of F and to the incenter z_T.

Note that the first step is an Alfeld-type refinement of T with respect to the incenter

[27].

The choice of the point m_F in the second step needs to follow specific rules: for each interior face $F = \overline{T_1} \cap \overline{T_2}$ with $T_1, T_2 \in \mathcal{T}_h$, let $m_F = L \cap F$ where $L = [z_{T_1}, z_{T_2}]$, the line segment connecting the incenters of T_1 and T_2; for a boundary face F with $F = \overline{T} \cap \partial\Omega$ with $T \in \mathcal{T}_h$, let m_F be the barycenter of F. The fact that such a m_F exists is established in [53, Lemma 16.24].

For $T \in \mathcal{T}_h$, we denote by T^{wf} the local Worsey-Farin mesh induced by the global refinement $\mathcal{T}_h^{wf}$, i.e.,

$$T^{wf} = \{K \in \mathcal{T}_h^{wf} : \ \bar{K} \subset \bar{T}\}.$$

For any face $F \in \Delta_2(\mathcal{T}_h)$, the refinement $\mathcal{T}_h^{wf}$ induces a Clough-Tocher triangulation of F and the resulting set of three triangles with the common vertex m_F is denoted by F^{ct}. We then define

$$\mathcal{E}(\mathcal{T}_h^{wf}) = \{e \in \Delta_1^I(F^{\mathrm{ct}}) : \forall F \in \Delta_2^I(\mathcal{T}_h)\}$$

to be the set of all interior edges of the Clough-Tocher refinements in the global mesh.

For a tetrahedron $T \in \mathcal{T}_h$ and face $F \in \Delta_2(T)$, we denote by $n_F := n|_F$ the outward unit normal of ∂T restricted to F. Consider the triangulation F^{ct} of F with three triangles labeled as Q_i, $i = 1, 2, 3$. Let $e = \partial Q_1 \cap \partial Q_2$ and t_e be the unit vector tangent to e pointing away from m_F. Then the jump of $p \in \mathcal{P}_r(T^{wf})$ across e is defined as

$$[\![p]\!]_e = (p|_{Q_1} - p|_{Q_2}) s_e,$$

where $s_e = n_F \times t_e$ is a unit vector orthogonal to t_e and n_F. In addition, let f be the internal face of T^{wf} that has e as an edge. Now let n_f be a unit-normal to f and set $t_s = n_f \times t_e$ to be a tangential unit vector on the internal face f.

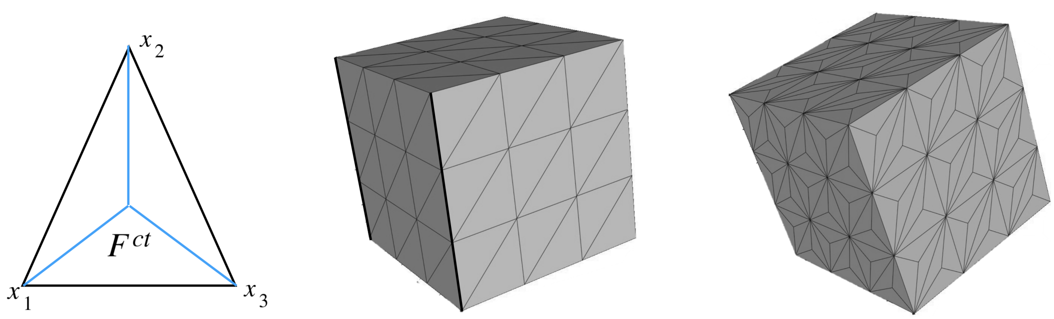
Let T_1 and T_2 be two adjacent tetrahedra in $\mathcal{T}_h$ that share a face F, and let Q_i, $i = 1, 2, 3$ denote three triangles in the set F^{ct}. Let $e = \partial Q_1 \cap \partial Q_2$, and for a piecewise smooth function defined on $T_1 \cup T_2$, we define

$$\theta_e(p) = p|_{\partial T_1 \cap Q_1} - p|_{\partial T_1 \cap Q_2} + p|_{\partial T_2 \cap Q_2} - p|_{\partial T_2 \cap Q_1}, \qquad \text{on } e. \tag{2.9}$$

Note that $\theta_e(p) = 0$ if and only if $[\![p|_{T_1}]\!]_e = [\![p|_{T_2}]\!]_e$.

Similar to Definition 2.2.4, we define the shape constant c_1 of the triangulation $\mathcal{T}_h^{wf}$.

Remark 2.2.7. The Worsey-Farin split's structure facilitates the construction of smooth, low-order, and localized piecewise polynomials [44].



(a) A representation of F^{ct} and $\Delta_1^I(F^{\text{ct}})$ (indicated in blue). (b) The original triangulation (c) Worsey-Farin refinement

Figure 2.4: The Worsey-Farin Splits

2.3 The BGG Method

As discussed in the Introduction, our strategy to obtain an elasticity sequence in two dimensions or three dimensions uses the framework in [11] and utilizes two auxiliary de Rham sequences which is so-called BGG method. In particular, we will use a simplified version of their results found in [27].

Suppose A_i, B_i are Banach spaces, $r_i : A_i \to A_{i+1}$, $t_i : B_i \to B_{i+1}$, and $s_i : B_i \to A_{i+1}$ are bounded linear operators such that the following diagram commutes:

$$\begin{tikzcd} A_0 \arrow[r, "r_0"] & A_1 \arrow[r, "r_1"] & A_2 \arrow[r, "r_2"] & A_3 \\ B_0 \arrow[r, "t_0"] \arrow[ur, "s_0"'] & B_1 \arrow[r, "t_1"] \arrow[ur, "s_1"'] & B_2 \arrow[r, "t_2"] \arrow[ur, "s_2"'] & B_3 \end{tikzcd} \tag{2.10}$$

The following recipe for a derived sequence, borrowed from [27, Proposition 2.3], guides the gathering of ingredients for our construction of the elasticity sequence on Worsey-Farin splits.

Proposition 2.3.1. *Suppose $s_1 : B_1 \to A_2$ is a bijection.*

(a) If A_i and B_i are exact sequences and the diagram (2.10) commutes, then the following is an exact sequence:

$$\begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \xrightarrow{[r_0\ \ s_0]} A_1 \xrightarrow{t_1 \circ s_1^{-1} \circ r_1} B_2 \xrightarrow{\left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]} \begin{bmatrix} A_3 \\ B_3 \end{bmatrix}. \tag{2.11}$$

Here the operators $[r_0\ \ s_0] : \begin{bmatrix} A_0 \\ B_0 \end{bmatrix} \to A_1$ *and* $\begin{bmatrix} s_2 \\ t_2 \end{bmatrix} : B_2 \to \begin{bmatrix} A_3 \\ B_3 \end{bmatrix}$ *are defined,* *respectively, as*

$$[r_0\ z_0]\begin{bmatrix} a \\ b \end{bmatrix} = r_0 a + z_0 b, \qquad \begin{bmatrix} s_2 \\ t_2 \end{bmatrix} b = \begin{bmatrix} s_2 b \\ t_2 b \end{bmatrix}.$$

(b) For the surjectivity of the last map in (2.11)*, namely* $\left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$*, it is sufficient that* r_2 *and* t_2 *are surjective,* $t_1 \circ t_2 = 0$*, and* $s_2 t_1 = r_2 s_1$*.*

The proof of this proposition is also included in the Appendix B to be self-contained. In addition, we share some candidates of those bounded linear operators and their properties in the next section.

2.4 Differential identities involving matrix and vector fields

We introduce the notation used in [27] and consider the three-dimensional mesh $\mathcal{T}_h$. Let $F \in \Delta_2(T)$, and recall n_F is the unit normal vector pointing out of T. Fix two tangent vectors t_1, t_2 such that the ordered set $(b_1, b_2, b_3) = (t_1, t_2, n_F)$ is an orthonormal right-handed basis of $\mathbb{R}^3$. Any matrix field $u : T \to \mathbb{R}^{3\times 3}$ can be written as $\sum_{i,j=1}^{3} u_{ij} b_i b_j'$ with scalar components $u_{ij} : T \to \mathbb{R}$. Let $u_{nn} = n_F' u n_F$ and $\mathrm{tr}_F u = \sum_{i=1}^{2} t_i' u t_i$. With $s \in \mathbb{R}^3$, let

$$u_{FF} = \sum_{i,j=1}^{2} u_{ij} t_i t_j', \qquad u_{Fs} = \sum_{i=1}^{2} (s' u t_i) t_i', \qquad u_{sF} = \sum_{i=1}^{2} (t_i' u s) t_i, . \tag{2.12}$$

Equivalently, $u_{FF} = QuQ$, $u_{Fs} = s'uQ$, and $u_{sF} = Qus$, where $P = n_F n_F'$ and $Q = I - P$. Next, for scalar-valued (component) functions ϕ, w_i, q_i and u_{ij}, we write the standard surface operators as

$$\operatorname{grad}_F\phi = (\partial_{t_1}\phi)t_1 + (\partial_{t_2}\phi)t_2, \qquad \operatorname{rot}_F\phi = (\partial_{t_2}\phi)t_1 - (\partial_{t_1}\phi)t_2,$$

$$\operatorname{grad}_F(w_1t_1 + w_2t_2) = t_1(\operatorname{grad}_F w_1)' + t_2(\operatorname{grad}_F w_2)',$$

$$\operatorname{rot}_F(q_1t_1' + q_2t_2') = t_1(\operatorname{rot}_F q_1)' + t_2(\operatorname{rot}_F q_2)',$$

$$\operatorname{curl}_F(w_1t_1 + w_2t_2) = \partial_{t_1}w_2 - \partial_{t_2}w_1, \qquad \operatorname{curl}_F u_{FF} = t_1' \operatorname{curl}_F(u_{Ft_1})' + t_2' \operatorname{curl}_F(u_{Ft_2})',$$

$$\operatorname{div}_F(w_1t_1 + w_2t_2) = \partial_{t_1}w_1 + \partial_{t_2}w_2, \qquad \operatorname{div}_F u_{FF} = t_1' \operatorname{div}_F(u_{Ft_1})' + t_2' \operatorname{div}_F(u_{Ft_2})'.$$

All those standard surface operators are well-defined for functions with only tangential components so we could abuse the notations for functions defined on any two-dimensional element $T \in \mathcal{T}_h$.

For a vector function v, denote $v_F = Qv = n_F \times (v \times n_F)$. It is easy to see that

$$\begin{aligned} n_F \cdot \operatorname{curl} v = \operatorname{curl}_F v_F, \qquad (\operatorname{grad} v)_{FF} = \operatorname{grad}_F v_F, \\ n_F \times \operatorname{rot}_F\phi = \operatorname{grad}_F\phi, \qquad \operatorname{div} v_F = \operatorname{div}_F v_F. \end{aligned} \tag{2.13}$$

In addition, we have:

Definition 2.4.1. For a tangential vector function v on the face $F \in \Delta_2(T)$, write $v = \sum_{i=1}^{2} v_i t_i'$ with $v_i = v \cdot t_i$. We define the orthogonal complement of v as

$$v^\perp := v \times n_F = v_2 t_1 - v_1 t_2.$$

Using this definition and the standard surface operators introduced above, it is easy to see the following identities:

$$\operatorname{div}_F v^\perp = \operatorname{curl}_F v', \quad v^\perp \cdot t_e = v \cdot s_e. \tag{2.14}$$

We also define the space of rigid body displacements within $\mathbb{R}^3$ and $\mathbb{R}^2$:

$$\mathsf{R} = \{a + b \times x : a, b \in \mathbb{R}^3\} \tag{2.15}$$

$$\mathsf{R}_2 = \{at_1 + bt_2 + c((x \cdot t_1)t_2 - (x \cdot t_2)t_1) : a, b, c \in \mathbb{R}\}. \tag{2.16}$$

Definition 2.4.2. Set $\mathbb{V} = \mathbb{R}^3$, $\mathbb{V}_2 = \mathbb{R}^2$ and $\mathbb{M}_{k\times k} = \mathbb{R}^{k\times k}$.

(a) The skew-symmetric operator $\operatorname{skw} : \mathbb{M}_{k\times k} \to \mathbb{M}_{k\times k}$ and the symmetric operator $\operatorname{sym} : \mathbb{M}_{k\times k} \to \mathbb{M}_{k\times k}$ are defined as follows: for any $M \in \mathbb{M}_{k\times k}$,

$$\operatorname{skw}(M) = \frac{1}{2}(M - M'); \quad \operatorname{sym}(M) = \frac{1}{2}(M + M').$$

Denote the range of skw and sym as $\mathbb{K}_k = \operatorname{skw}(\mathbb{M}_{k\times k})$ and $\mathbb{S}_k = \operatorname{sym}(\mathbb{M}_{k\times k})$, respectively.

(b) Define the operator $\Xi : \mathbb{M}_{3\times 3} \to \mathbb{M}_{3\times 3}$ by $\Xi M = M' - \operatorname{tr}(M)\mathbb{I}$, where $\mathbb{I}$ is the 3×3 identity matrix.

(c) The three-dimensional symmetric gradient and incompatibility operators are given, respectively, by:

$$\varepsilon = \operatorname{sym}\operatorname{grad}, \quad \operatorname{inc} = \operatorname{curl}(\operatorname{curl})'.$$

(d) The operators $\operatorname{mskw} : \mathbb{V} \to \mathbb{K}_3$ and $\operatorname{vskw} : \mathbb{M}_{3\times 3} \to \mathbb{V}$ are given by

$$\operatorname{mskw}\begin{pmatrix} v_1 \\ v_2 \\ v_3 \end{pmatrix} = \begin{pmatrix} 0 & -v_3 & v_2 \\ v_3 & 0 & -v_1 \\ -v_2 & v_1 & 0 \end{pmatrix}, \qquad \operatorname{vskw} := \operatorname{mskw}^{-1} \circ \operatorname{skw}.$$

(e) The two-dimensional surface differential operators on a face F are given by

$$\varepsilon_F = \operatorname{sym}\operatorname{grad}_F, \quad \operatorname{airy}_F = \operatorname{rot}_F(\operatorname{rot}_F)', \quad \operatorname{inc}_F := \operatorname{curl}_F(\operatorname{curl}_F)'.$$

(f) The two-dimensional skew operator defined on either a scalar or matrix-valued function is defined, respectively, as

$$\operatorname{skew} u = \begin{bmatrix} 0 & u \\ -u & 0 \end{bmatrix}; \quad \operatorname{skew} \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix} = u_{21} - u_{12}.$$

(g) The transpose operator τ is defined as: $\tau\, u = u'$.

It is simple to see that Ξ is invertible with $\Xi^{-1}M = M' - \frac{1}{2}\operatorname{tr}(M)\mathbb{I}$. Furthermore, there holds the following fundamental identities:

$$\operatorname{div}\Xi = 2\operatorname{vskw}\operatorname{curl}, \tag{2.17a}$$

$$\Xi\operatorname{grad} = -\operatorname{curl}\ \operatorname{mskw}, \tag{2.17b}$$

$$\operatorname{curl}\Xi^{-1}\operatorname{curl}\operatorname{mskw} = -\operatorname{curl}\Xi^{-1}\Xi\operatorname{grad} = -\operatorname{curl}\operatorname{grad} = 0, \tag{2.17c}$$

$$2\operatorname{vskw}\operatorname{curl}\Xi^{-1}\operatorname{curl} = \operatorname{div}\Xi\Xi^{-1}\operatorname{curl} = \operatorname{div}\operatorname{curl} = 0, \tag{2.17d}$$

$$\operatorname{tr}(\operatorname{curl}\operatorname{sym}) = 0, \quad \operatorname{curl}\Xi^{-1}\operatorname{curl}\operatorname{sym} = \operatorname{curl}(\operatorname{curl}\operatorname{sym})' = \operatorname{inc}\operatorname{sym}. \tag{2.17e}$$

On a two-dimensional face F, there also holds

$$\operatorname{div}_F \operatorname{airy}_F = \operatorname{div}_F \operatorname{rot}_F \tau\,(\operatorname{rot}_F) = 0, \tag{2.18a}$$

$$\operatorname{inc}_F \operatorname{sym} = \operatorname{inc}_F, \quad \operatorname{inc}_F \varepsilon_F = \operatorname{curl}_F\ \tau\ \operatorname{curl}_F \operatorname{grad}_F = 0, \tag{2.18b}$$

$$\operatorname{curl}_F\ \operatorname{skew} = \tau\operatorname{grad}_F. \tag{2.18c}$$

The following lemma states additional identities used throughout the paper. It has been proven in [27, Lemma 5.7] and to be self-contained, we include this in the appendix.

Lemma 2.4.3. *For a sufficiently smooth matrix-valued function u,*

$$s'\,(\operatorname{curl} u)\, n_F \; = \operatorname{curl}_F (u_{Fs})', \; \textit{for any } s \in \mathbb{R}^3, \tag{2.19a}$$

$$\big[(\operatorname{curl} u)'\big]_{Fn} = \operatorname{curl}_F u_{FF}. \tag{2.19b}$$

If in addition u is symmetric, then

$$(\operatorname{inc} u)_{nn} = \operatorname{inc}_F u_{FF}, \tag{2.19c}$$

$$(\operatorname{inc} u)_{Fn} = \operatorname{curl}_F \big[(\operatorname{curl} u)'\big]_{FF}, \tag{2.19d}$$

$$\operatorname{tr}_F \operatorname{curl} u = -\operatorname{curl}_F (u_{Fn})'. \tag{2.19e}$$

For a sufficiently smooth vector-valued function v,

$$2(\operatorname{curl} \varepsilon(v))' = \operatorname{grad} \operatorname{curl} v, \tag{2.19f}$$

$$2\,[(\operatorname{curl} \varepsilon(v))']_{FF} = \operatorname{grad}_F (\operatorname{curl} v)_F, \tag{2.19g}$$

$$\operatorname{curl} v = n_F(\operatorname{curl}_F v_F) + \operatorname{rot}_F(v \cdot n_F) + n_F \times \partial_n v, \tag{2.19h}$$

$$2[\varepsilon(v)]_{nF} = 2[\varepsilon(v)_{Fn}]' = \operatorname{grad}_F(v \cdot n_F) + \partial_n v_F, \tag{2.19i}$$

$$\operatorname{tr}_F(\operatorname{rot}_F v_F') = \operatorname{curl}_F v_F. \tag{2.19j}$$

2.5 Exact sequences on simplicial meshes

2.5.1 Function spaces

We summarize all the Hilbert spaces and discrete spaces needed to define the discrete sequences. All the notations can also be found in [43, 44].

2.5.1.1 Two dimensional case

For any $T \in \mathcal{T}_h$ in two dimensions (triangles), we define the following spaces by using surface operators in Section 2.4:

$$H(\delta, T) := \{v \in [L^2(T)]^2 : \delta\, v \in L^2(T)\}, \mathring{H}(\delta, T) := \{v \in H(\delta, T) : \operatorname{tr} v|_{\partial T} = 0\},$$

where δ is grad_F (or div_F or curl_F) and $\operatorname{tr} v$ is v (*resp.* $v \cdot s$, $v \cdot t$) with s (or t) denotes the outward unit normal (*resp.* the unit tangential) of ∂T.

Next, we define the Nédélec spaces (without and with boundary conditions) with respect to the two-dimensional splits T^* (with choices within T^{ct} or T^{ps}):

$$\begin{aligned}
V^1_{\mathrm{div},r}(T^*) &:= \{v \in H(\operatorname{div}_F, T) : v|_Q \in [\mathcal{P}_r(T)]^2, \forall Q \in T^*\},\\
\mathring{V}^1_{\mathrm{div},r}(T^*) &:= V^1_{\mathrm{div},r}(T^*) \cap \mathring{H}(\operatorname{div}_F, T)\\
V^1_{\mathrm{curl},r}(T^*) &:= \{v \in H(\operatorname{curl}_F, T) : v|_Q \in [\mathcal{P}_r(T)]^2, \forall Q \in T^*\},\\
\mathring{V}^1_{\mathrm{curl},r}(T^*) &:= V^1_{\mathrm{curl,r}}(T^*) \cap \mathring{H}(\operatorname{curl}_F, T),\\
V^2_r(T^*) &:= \{v \in L^2(T) : v|_Q \in \mathcal{P}_r(T), \forall Q \in T^*\},
\end{aligned}$$

$$\mathring{V}_r^2(T^*) := V_r^2(T^*) \cap L_0^2(T),$$

the Lagrange spaces,

$$\begin{aligned}
&\mathsf{L}_r^0(T^*) := V_r^2(T^*) \cap H(\mathrm{grad}_F, T), && \mathring{\mathsf{L}}_r^0(T^*) := \mathsf{L}_r^0(T^*) \cap \mathring{H}(\mathrm{grad}_F, T),\\
&\mathsf{L}_r^1(T^*) := [\mathsf{L}_r^0(T^*)]^2, && \mathring{\mathsf{L}}_r^1(T^*) := [\mathring{\mathsf{L}}_r^0(T^*)]^2,\\
&\mathsf{L}_r^2(T^*) := \mathsf{L}_r^0(T^*), && \mathring{\mathsf{L}}_r^2(T^*) := \mathring{\mathsf{L}}_r^0(T^*) \cap L_0^2(T),
\end{aligned}$$

and finally the (smooth) piece-wise polynomial subspaces with additional continuity:

$$\begin{aligned}
&S_r^0(T^*) := \{v \in \mathsf{L}_r^0(T^*) : \ \mathrm{grad}_F\, v \in \mathsf{L}_{r-1}^1(T^*)\},\\
&\mathring{S}_r^0(T^*) := \{v \in \mathring{\mathsf{L}}_r^0(T^*) : \ \mathrm{grad}_F\, v \in \mathring{\mathsf{L}}_{r-1}^1(T^*)\},\\
&S_{\mathrm{div},r}^1(T^*) := \{v \in \mathsf{L}_r^1(T^*) : \ \mathrm{div}_F\, v \in \mathsf{L}_{r-1}^2(T^*)\},\\
&\mathring{S}_{\mathrm{div},r}^1(T^*) := \{v \in \mathring{\mathsf{L}}_r^1(T^*) : \ \mathrm{div}_F\, v \in \mathring{\mathsf{L}}_{r-1}^2(T^*)\},\\
&S_{\mathrm{curl},r}^1(T^*) := \{v \in \mathsf{L}_r^1(T^*) : \ \mathrm{curl}_F\, v \in \mathsf{L}_{r-1}^2(T^*)\},\\
&\mathring{S}_{\mathrm{curl},r}^1(T^*) := \{v \in \mathring{\mathsf{L}}_r^1(T^*) : \ \mathrm{curl}_F\, v \in \mathring{\mathsf{L}}_{r-1}^2(T^*)\},\\
&\mathcal{R}_r^0(T^*) := \{v \in S_r^0(T^*) : v|_{\partial T} = 0\}.
\end{aligned}$$

In addition, for Powell-Sabin splits, two intermediate spaces need to be introduced:

$$\begin{aligned}
&\mathcal{V}_r^2(T^{\mathrm{ps}}) := \{v \in V_r^2(T^{\mathrm{ps}}), v \text{ is continuous at } z_e, \forall e \in \Delta_1(T)\},\\
&\mathring{\mathcal{V}}_r^2(T^{\mathrm{ct}}) := \mathcal{V}_r^2(T^{\mathrm{ps}}) \cap L_0^2(T).
\end{aligned}$$

2.5.1.2 Three-dimensional case

The Nédélec spaces with respect to the local Worsey-Farin split T^{wf} are given as

$$
\begin{aligned}
V_r^1(T^{wf}) &:= [\mathcal{P}_r(T^{wf})]^3 \cap H(\mathrm{curl}, T), &\quad \mathring{V}_r^1(T^{wf}) &:= V_r^1(T^{wf}) \cap \mathring{H}(\mathrm{curl}, T),\\
V_r^2(T^{wf}) &:= [\mathcal{P}_r(T^{wf})]^3 \cap H(\mathrm{div}, T), &\quad \mathring{V}_r^2(T^{wf}) &:= V_r^2(T^{wf}) \cap \mathring{H}(\mathrm{div}, T),\\
V_r^3(T^{wf}) &:= \mathcal{P}_r(T^{wf}), &\quad \mathring{V}_r^3(T^{wf}) &:= V_r^3(T^{wf}) \cap L_0^2(T).
\end{aligned}
$$

The Lagrange spaces on T^{wf} are defined by

$$
\begin{aligned}
\mathsf{L}_r^0(T^{wf}) &:= \mathcal{P}_r(T^{wf}) \cap H^1(T), &\quad \mathring{\mathsf{L}}_r^0(T^{wf}) &:= \mathsf{L}_r^0(T^{wf}) \cap \mathring{H}^1(T),\\
\mathsf{L}_r^1(T^{wf}) &:= [\mathsf{L}_r^0(T^{wf})]^3, &\quad \mathring{\mathsf{L}}_r^1(T^{wf}) &:= [\mathring{\mathsf{L}}_r^0(T^{wf})]^3,
\end{aligned}
$$

and the discrete spaces with additional smoothness are

$$
\begin{aligned}
S_r^0(T^{wf}) &:= \{u \in \mathsf{L}_r^0(T^{wf}) : \mathrm{grad}\, v \in \mathsf{L}_{r-1}^1(T^{wf})\},\\
\mathring{S}_r^0(T^{wf}) &:= \{u \in \mathring{\mathsf{L}}_r^0(T^{wf}) : \mathrm{grad}\, v \in \mathring{\mathsf{L}}_{r-1}^1(T^{wf})\},\\
S_r^1(T^{wf}) &:= \{u \in \mathsf{L}_r^1(T^{wf}) : \mathrm{curl}\, v \in \mathsf{L}_{r-1}^1(T^{wf})\},\\
\mathring{S}_r^1(T^{wf}) &:= \{u \in \mathring{\mathsf{L}}_r^1(T^{wf}) : \mathrm{curl}\, v \in \mathring{\mathsf{L}}_{r-1}^1(T^{wf})\}.
\end{aligned}
$$

We also define the intermediate spaces

$$
\begin{aligned}
\mathcal{V}_r^2(T^{wf}) &:= \{v \in V_r^2(T^{wf}) : v \times n \text{ is continuous on each } F \in \Delta_2(T)\},\\
\mathring{\mathcal{V}}_r^2(T^{wf}) &:= \{v \in \mathcal{V}_r^2(T^{wf}) : v \cdot n = 0 \text{ on each } F \in \Delta_2(T)\},\\
\mathcal{V}_r^3(T^{wf}) &:= \{q \in V_r^3(T^{wf}) : q \text{ is continuous on each } F \in \Delta_2(T)\},\\
\mathring{\mathcal{V}}_r^3 &:= \mathcal{V}_r^3(T^{wf}) \cap L_0^2(T),
\end{aligned}
$$

and note that

$$\begin{aligned} &S_r^0(T^{wf}) \subset \mathsf{L}_r^0(T^{wf}), && S_r^1(T^{wf}) \subset \mathsf{L}_r^1(T^{wf}) \subset V_r^1(T^{wf}), \\ &\mathsf{L}_r^2(T^{wf}) \subset \mathcal{V}_r^2(T^{wf}) \subset V_r^2(T^{wf}), && \mathcal{V}_r^3(T^{wf}) \subset V_r^3(T^{wf}), \end{aligned}$$

with similar inclusions holding for the analogous spaces with boundary conditions.

2.5.2 Local discrete sequences with respect 2D splits

In this section, we state several local discrete de Rham sequences on Clough-Tocher splits and Powell-Sabin splits with various levels of smoothness. These sequences play an important role not only in the two-dimensional elasticity sequences, but also in the analysis of the results related to Worsey-Farin splits. For any $T \in \mathcal{T}_h$ (triangles), T^* could be chosen within T^{ct} or T^{ps}.

Theorem 2.5.1. *Let $r \geq 2$. The following sequences are exact [12, 41, 43].*

$$\mathbb{R} \longrightarrow \mathsf{L}_r^0(T^*) \xrightarrow{\mathrm{grad}_F} V_{\mathrm{curl},r-1}^1(T^*) \xrightarrow{\mathrm{curl}_F} V_{r-2}^2(T^*) \longrightarrow 0, \tag{2.20a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^*) \xrightarrow{\mathrm{grad}_F} S_{\mathrm{curl},r-1}^1(T^*) \xrightarrow{\mathrm{curl}_F} \mathsf{L}_{r-2}^2(T^*) \longrightarrow 0, \tag{2.20b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^*) \xrightarrow{\mathrm{grad}_F} \mathsf{L}_{r-1}^1(T^*) \xrightarrow{\mathrm{curl}_F} V_{r-2}^2(T^*) \longrightarrow 0, \tag{2.20c}$$

$$0 \longrightarrow \mathring{\mathsf{L}}_r^0(T^*) \xrightarrow{\mathrm{grad}_F} \mathring{V}_{\mathrm{curl},r-1}^1(T^*) \xrightarrow{\mathrm{curl}_F} \mathring{V}_{r-2}^2(T^*) \longrightarrow 0, \tag{2.20d}$$

$$0 \longrightarrow \mathring{S}_r^0(T^*) \xrightarrow{\mathrm{grad}_F} \mathring{S}_{\mathrm{curl},r-1}^1(T^*) \xrightarrow{\mathrm{curl}_F} \mathring{\mathsf{L}}_{r-2}^2(T^*) \longrightarrow 0. \tag{2.20e}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{ct}}) \xrightarrow{\mathrm{grad}_F} \mathring{\mathsf{L}}_{r-1}^1(T^{\mathrm{ct}}) \xrightarrow{\mathrm{curl}_F} \mathring{V}_{r-2}^2(T^{\mathrm{ct}}) \longrightarrow 0. \tag{2.20f}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\mathrm{ps}}) \xrightarrow{\mathrm{grad}_F} \mathring{\mathsf{L}}_{r-1}^1(T^{\mathrm{ps}}) \xrightarrow{\mathrm{curl}_F} \mathring{\mathcal{V}}_{r-2}^2(T^{\mathrm{ps}}) \longrightarrow 0. \tag{2.20g}$$

Theorem 2.5.1 has an alternate form that follows from a rotation of the coordinate axes, where the operators grad_F and curl_F are replaced by rot_F and div_F, respectively.

Corollary 2.5.2. *Let $r \geq 2$. The following sequences are exact [12, 41, 43].*

$$\mathbb{R} \longrightarrow \mathsf{L}_r^0(T^*) \xrightarrow{\text{rot}_F} V_{\text{div},r-1}^1(T^*) \xrightarrow{\text{div}_F} V_{r-2}^2(T^*) \longrightarrow 0, \tag{2.21a}$$

$$\mathbb{R} \longrightarrow S_r^0(T^*) \xrightarrow{\text{rot}_F} S_{\text{div},r-1}^1(T^*) \xrightarrow{\text{div}_F} \mathsf{L}_{r-2}^2(T^*) \longrightarrow 0, \tag{2.21b}$$

$$\mathbb{R} \longrightarrow S_r^0(T^*) \xrightarrow{\text{rot}_F} \mathsf{L}_{r-1}^1(T^*) \xrightarrow{\text{div}_F} V_{r-2}^2(T^*) \longrightarrow 0, \tag{2.21c}$$

$$0 \longrightarrow \mathring{\mathsf{L}}_r^0(T^*) \xrightarrow{\text{rot}_F} \mathring{V}_{\text{div},r-1}^1(T^*) \xrightarrow{\text{div}_F} \mathring{V}_{r-2}^2(T^*) \longrightarrow 0, \tag{2.21d}$$

$$0 \longrightarrow \mathring{S}_r^0(T^*) \xrightarrow{\text{rot}_F} \mathring{S}_{\text{div},r-1}^1(T^*) \xrightarrow{\text{div}_F} \mathring{\mathsf{L}}_{r-2}^2(T^*) \longrightarrow 0. \tag{2.21e}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\text{ct}}) \xrightarrow{\text{rot}_F} \mathring{\mathsf{L}}_{r-1}^1(T^{\text{ct}}) \xrightarrow{\text{div}_F} \mathring{V}_{r-2}^2(T^{\text{ct}}) \longrightarrow 0. \tag{2.21f}$$

$$0 \longrightarrow \mathring{S}_r^0(T^{\text{ps}}) \xrightarrow{\text{rot}_F} \mathring{\mathsf{L}}_{r-1}^1(T^{\text{ps}}) \xrightarrow{\text{div}_F} \mathring{\mathcal{V}}_{r-2}^2(T^{\text{ps}}) \longrightarrow 0. \tag{2.21g}$$

2.5.3 Local discrete sequences on Worsey-Farin splits

The next lemma summarizes the exactness properties of several (local) sequences using these spaces. Its proof is found in [44, Theorem 3.1-3.2].

Lemma 2.5.3. *The following sequences are exact for any $r \geq 3$.*

$$\mathbb{R} \xrightarrow{\subset} \mathsf{L}_r^0(T^{wf}) \xrightarrow{\text{grad}} V_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} V_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} V_{r-3}^3(T^{wf}) \to 0, \tag{2.22a}$$

$$0 \to \mathring{\mathsf{L}}_r^0(T^{wf}) \xrightarrow{\text{grad}} \mathring{V}_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} \mathring{V}_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{wf}) \to 0, \tag{2.22b}$$

$$\mathbb{R} \xrightarrow{\subset} S_r^0(T^{wf}) \xrightarrow{\text{grad}} \mathsf{L}_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} V_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} V_{r-3}^3(T^{wf}) \to 0, \tag{2.22c}$$

$$0 \to \mathring{S}_r^0(T^{wf}) \xrightarrow{\text{grad}} \mathring{\mathsf{L}}_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} \mathring{\mathcal{V}}_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(T^{wf}) \to 0. \tag{2.22d}$$

$$\mathbb{R} \xrightarrow{\subset} S_r^0(T^{wf}) \xrightarrow{\text{grad}} S_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} \mathsf{L}_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} V_{r-3}^3(T^{wf}) \to 0. \tag{2.22e}$$

$$0 \to \mathring{S}_r^0(T^{wf}) \xrightarrow{\text{grad}} \mathring{S}_{r-1}^1(T^{wf}) \xrightarrow{\text{curl}} \mathring{\mathsf{L}}_{r-2}^2(T^{wf}) \xrightarrow{\text{div}} \mathring{\mathcal{V}}_{r-3}^3(T^{wf}) \to 0. \tag{2.22f}$$

The following lemma summarizes the degrees of freedom (dofs) for L_{r-1}^1, V_{r-2}^2, and V_{r-3}^3 firstly given in [44, Lemmas 5.4–5.6].

Lemma 2.5.4. *Let $r \geq 3$.*

(a) A function $\tau \in \mathsf{L}_{r-1}^1(T^{wf})$ is uniquely defined by the following conditions:

$$\tau(a) \qquad \forall a \in \Delta_0(T) \tag{2.23a}$$

$$\int_e \tau \cdot \kappa \, ds \qquad \forall \kappa \in [\mathcal{P}_{r-3}(e)]^3, \ \forall e \in \Delta_1(T), \tag{2.23b}$$

$$\int_e [\![\text{curl } \tau \cdot t_e]\!]_e \kappa \, ds \qquad \forall \kappa \in \mathcal{P}_{r-3}(e), \ \forall e \in \Delta_1^I(F^{\text{ct}}) \backslash \{e_F\}, \ \forall F \in \Delta_2(T), \tag{2.23c}$$

$$\int_{e_F} [\![\text{curl } \tau \cdot t_{e_F}]\!]_{e_F} \kappa \, ds \qquad \forall \kappa \in \mathcal{P}_{r-2}(e_F), \ \forall F \in \Delta_2(T), \tag{2.23d}$$

$$\int_F (\tau \cdot n_F) \kappa \, dA \qquad \forall \kappa \in \mathcal{R}_{r-1}^0(F^{\text{ct}}), \ \forall F \in \Delta_2(T), \tag{2.23e}$$

$$\int_F (\text{curl}_F \, \tau_F) \kappa \, dA \qquad \forall \kappa \in \mathring{V}_{r-2}^2(F^{\text{ct}}), \ \forall F \in \Delta_2(T), \tag{2.23f}$$

$$\int_F \tau_F \cdot \kappa \, dA \qquad \forall \kappa \in \text{grad}_F \, \mathring{S}_r^0(F^{\text{ct}}), \ \forall F \in \Delta_2(T), \tag{2.23g}$$

$$\int_T \text{curl} \, \tau \cdot \kappa \, dx \qquad \forall \kappa \in \text{curl} \, \mathring{\mathsf{L}}_{r-1}^1(T^{wf}), \tag{2.23h}$$

$$\int_T \tau \cdot \kappa \, dx \qquad \forall \kappa \in \text{grad} \, \mathring{S}_r^0(T^{wf}). \tag{2.23i}$$

(b) A function $p \in V_{r-2}^2(T^{wf})$ is uniquely defined by the following conditions:

$$\int_e [\![p \cdot t_e]\!]_e \kappa \, ds \qquad \forall \kappa \in \mathcal{P}_{r-3}(e), \ \forall e \in \Delta_1^I(F^{\text{ct}}) \backslash \{e_F\}, \ \forall F \in \Delta_2(T), \tag{2.24a}$$

$$\int_{e_F} [\![p \cdot t_{e_F}]\!]_{e_F} \kappa \, ds \qquad \forall \kappa \in \mathcal{P}_{r-2}(e_F), \ \forall F \in \Delta_2(T), \tag{2.24b}$$

$$\int_F (p \cdot n_F)\kappa \, dA \qquad \forall \kappa \in V_{r-2}^2(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{2.24c}$$

$$\int_T (\operatorname{div} p)\kappa \, dx \qquad \forall \kappa \in \mathring{V}_{r-3}^3(T^{wf}), \tag{2.24d}$$

$$\int_T p \cdot \kappa \, dx \qquad \forall \kappa \in \operatorname{curl} \mathring{\mathsf{L}}_{r-1}^1(T^{wf}). \tag{2.24e}$$

(c) A function $v \in V_{r-3}^3(T^{wf})$ is uniquely defined by the following conditions:

$$\int_T v \, dx \tag{2.25a}$$

$$\int_T v \cdot \kappa \, dx \qquad \forall \kappa \in \mathring{V}_{r-3}^3(T^{wf}). \tag{2.25b}$$

2.5.4 Dimension counts

The dimensions of the spaces in Section 2.5.1 are summarized in Table 2.1 and 2.3. These counts essentially from Lemma 2.5.3 and Theorem 2.5.1 and the rank-nullity theorem; see [43, 44, 44] for details.

Table 2.1: Dimension counts of the canonical (two–dimensional) Nédélec, Lagrange, and smooth spaces with respect to the Clough–Tocher split. Here, $\dim V_{\mathrm{div},r}^1(T^{\mathrm{ct}}) = \dim V_{\mathrm{curl},r}^1(T^{\mathrm{ct}}) =: \dim V_r^1(T^{\mathrm{ct}})$

	$k=0$	$k=1$	$k=2$
$\dim V_r^k(T^{\mathrm{ct}})$	$\frac{1}{2}(3r^2+3r+2)$	$3(r+1)^2$	$\frac{3}{2}(r+1)(r+2)$
$\dim \mathring{V}_r^k(T^{\mathrm{ct}})$	$\frac{1}{2}(3r^2-3r+2)$	$3r(r+1)$	$\frac{3}{2}(r+1)(r+2)-1$
$\dim \mathsf{L}_r^k(T^{\mathrm{ct}})$	$\frac{1}{2}(3r^2+3r+2)$	$3r^2+3r+2$	$\frac{1}{2}(3r^2+3r+2)$
$\dim \mathring{\mathsf{L}}_r^k(T^{\mathrm{ct}})$	$\frac{1}{2}(3r^2-3r+2)$	$3r^2-3r+2$	$\frac{3}{2}r(r-1)$
$\dim S_r^k(T^{\mathrm{ct}})$	$\frac{3}{2}(r^2-r+2)$	$3r^2+3$	$\frac{1}{2}(3r^2+3r+2)$
$\dim \mathring{S}_r^k(T^{\mathrm{ct}})$	$\frac{3}{2}(r^2-5r+6)$	$3r^2-9r+6$	$\frac{3}{2}r(r-1)$
$\dim \mathcal{R}_r^k(T^{\mathrm{ct}})$	$\frac{3}{2}(r-1)(r-2)$ [55]	—	—

We need the dimensions of these spaces to compute the dimensions of elasticity spaces derived later.

Table 2.2: Dimension counts of the canonical (two–dimensional) Nédélec, Lagrange, and smooth spaces with respect to the Powell–Sabin split. Here, $\dim V^1_{\mathrm{div},r}(T^{\mathrm{ps}}) = \dim V^1_{\mathrm{curl},r}(T^{\mathrm{ps}}) =: \dim V^1_r(T^{\mathrm{ps}})$

	$k=0$	$k=1$	$k=2$
$\dim V^k_r(T^{\mathrm{ps}})$	$3r^2+3r+1$	$6(r+1)^2$	$3(r+1)(r+2)$
$\dim \mathring{V}^k_r(T^{\mathrm{ps}})$	$3r^2-3r+3$	$6r^2+6r+2$	$3(r+1)(r+2)-1$
$\dim \mathsf{L}^k_r(T^{\mathrm{ps}})$	$3r^2+3r+1$	$2(3r^2+3r+1)$	$3r^2+3r+1$
$\dim \mathring{\mathsf{L}}^k_r(T^{\mathrm{ps}})$	$3r^2-3r+3$	$6r^2-6r+6$	$3r^2-3r+2$
$\dim S^k_r(T^{\mathrm{ps}})$	$3(r^2-r+1)$	$6r^2+3$	$3r^2+3r+1$
$\dim \mathring{S}^k_r(T^{\mathrm{ps}})$	$3(r^2-5r+6)$	$6(r-1)(r-2)$	$3r(r-1)$
$\dim \mathring{\mathcal{V}}^k_r(T^{\mathrm{ps}})$	—	—	$3(r+1)(r+2)-4$

2.5.5 Global de-Rham squences on different refinements

We present below the global exact de Rham sequences on Worsey-Farin splits which are needed to construct elasticity sequences; for more details, see [44, Section 6]:

$$0 \to \mathcal{S}^0_r(\mathcal{T}_h^{wf}) \xrightarrow{\text{grad}} \mathcal{L}^1_{r-1}(\mathcal{T}_h^{wf}) \xrightarrow{\text{curl}} \mathscr{V}^2_{r-2}(\mathcal{T}_h^{wf}) \xrightarrow{\text{div}} V^3_{r-3}(\mathcal{T}_h^{wf}) \to 0, \tag{2.26a}$$

$$0 \to \mathcal{S}^0_r(\mathcal{T}_h^{wf}) \xrightarrow{\text{grad}} \mathcal{S}^1_{r-1}(\mathcal{T}_h^{wf}) \xrightarrow{\text{curl}} \mathcal{L}^2_{r-2}(\mathcal{T}_h^{wf}) \xrightarrow{\text{div}} \mathscr{V}^3_{r-3}(\mathcal{T}_h^{wf}) \to 0, \tag{2.26b}$$

where the spaces involved are defined as follows:

$$\begin{aligned}
\mathcal{S}^0_r(\mathcal{T}_h^{wf}) &= \{q \in C^1(\Omega) : q|_T \in S^0_r(T^{wf}),\, \forall T \in \mathcal{T}_h\},\\
\mathcal{S}^1_{r-1}(\mathcal{T}_h^{wf}) &= \{v \in [C(\Omega)]^3 : \operatorname{curl} v \in [C(\Omega)]^3,\, v|_T \in S^1_{r-1}(T^{wf})\, \forall T \in \mathcal{T}_h\},\\
\mathcal{L}^1_{r-1}(\mathcal{T}_h^{wf}) &= \{v \in [C(\Omega)]^3 : v|_T \in \mathcal{L}^1_{r-1}(T^{wf}),\, \forall T \in \mathcal{T}_h\},\\
\mathcal{L}^2_{r-2}(\mathcal{T}_h^{wf}) &= \{w \in [C(\Omega)]^3 : w|_T \in \mathcal{L}^2_{r-2}(T^{wf}),\, \forall T \in \mathcal{T}_h\},\\
\mathscr{V}^2_{r-2}(\mathcal{T}_h^{wf}) &= \{w \in H(\mathrm{div};\Omega) : w|_T \in V^2_{r-2}(T^{wf}),\, \forall T \in \mathcal{T}_h,\\
&\qquad\qquad \theta_e(w\cdot t) = 0,\, \forall e \in \mathcal{E}(\mathcal{T}_h^{wf})\},\\
\mathscr{V}^3_{r-3}(\mathcal{T}_h^{wf}) &= \{p \in L^2(\Omega) : p|_T \in V^3_{r-3}(T^{wf}),\, \forall T \in \mathcal{T}_h,\, \theta_e(p) = 0,\, \forall e \in \mathcal{E}(\mathcal{T}_h^{wf})\},\\
V^3_{r-3}(\mathcal{T}_h^{wf}) &= \mathcal{P}_{r-3}(\mathcal{T}_h^{wf}),
\end{aligned}$$

Table 2.3: Dimension counts of the canonical Nédélec, Lagrange spaces and smoother spaces on a WF split. Here, the superscript + indicates the positive part of the number.

	$k=0$	$k=1$
$V_r^k(T^{wf})$	$(2r+1)(r^2+r+1)$	$2(r+1)(3r^2+6r+4)$
$\mathring{V}_r^k(T^{wf})$	$(2r-1)(r^2-r+1)$	$2(r+1)(3r^2+1)$
$\mathsf{L}_r^k(T^{wf})$	$(2r+1)(r^2+r+1)$	$3(2r+1)(r^2+r+1)$
$\mathring{\mathsf{L}}_r^k(T^{wf})$	$(2r-1)(r^2-r+1)$	$3(2r-1)(r^2-r+1)$
$\mathring{\mathcal{V}}_r^k(T^{wf})$	—	—
$S_r^k(T^{wf})$	$2r^3-6r^2+10r-2$	$3r(2r^2-3r+5)$
$\mathring{S}_r^0(T^{wf})$	$\big(2(r-2)(r-3)(r-4)\big)^+$	$\big(3(2r-3)(r-2)(r-3)\big)^+$
	$k=2$	$k=3$
$V_r^k(T^{wf})$	$3(r+1)(r+2)(2r+3)$	$2(r+1)(r+2)(r+3)$
$\mathring{V}_r^k(T^{wf})$	$3(r+1)(r+2)(2r+1)$	$2r^3+12r^2+22r+11$
$\mathsf{L}_r^k(T^{wf})$	$3(2r+1)(r^2+r+1)$	$(2r+1)(r^2+r+1)$
$\mathring{\mathsf{L}}_r^k(T^{wf})$	$3(2r-1)(r^2-r+1)$	$(r-1)(2r^2-r+2)$
$\mathring{\mathcal{V}}_r^k(T^{wf})$	$6r^3+21r^2+9r+2$	$2r^3+12r^2+10r+3$
$S_r^k(T^{wf})$	$6r^3+8r+2$	$(2r+1)(r^2+r+1)$
$\mathring{S}_r^0(T^{wf})$	$\big(2(r-2)(3r^2-6r+4)\big)^+$	$(r-1)(2r^2-r+2)$

and we recall $\theta_e(\cdot)$ is defined in (2.9).

Chapter Three

Shape regularity of Worsey-Farin Splits

The goal of the chapter is to derive the shape regularity of three dimensional Worsey-Farin splits. For the meshes used in the finite element methods, the shape regularity of the underlying mesh is a critical geometric property for approximation theory. To be more precise, as mentioned in the introduction, it is necessary to ensure optimal-order and uniform interpolation estimates and to do the stability analysis of a finite element method. For example, in Chapter 4, the shape regularity of Worsey-Farin splits is needed for achieving uniform bounds in the convergence analysis of the Maxwell eigenvalue problem.

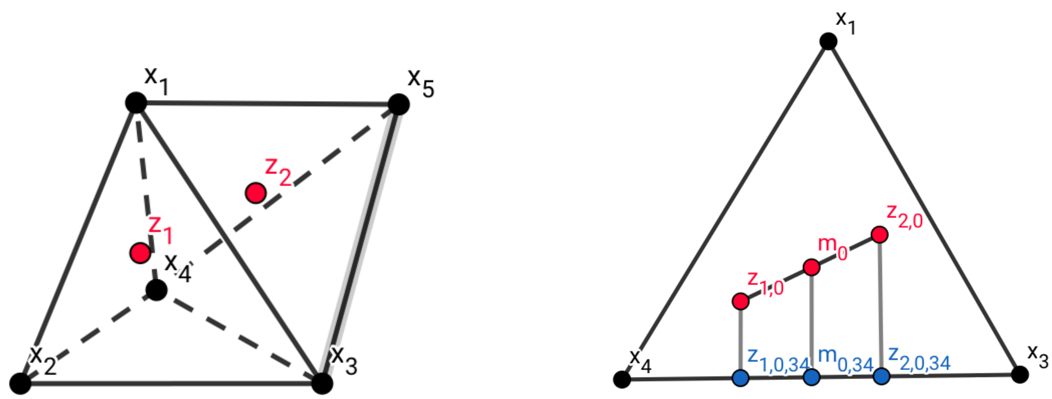
It was conjectured in [53, Page 515], [3, Remark 14], and [4, Page 54] that Worsey-Farin splits of a family of shape regular meshes remain shape regular. However, to the best of our knowledge, a proof of this result has not yet been published in the literature. The goal of the chapter is to fill this gap and we prove that three dimensional Worsey-Farin split inherit their parent triangulations' shape regularity. To be more precise, the following theorem will be proved:

Theorem 3.0.1. *There exists a constant $c_1 > 0$ only depending on c_0, the shape regularity constant of $\mathcal{T}_h$ given in Definition 2.2.4 such that*

$$\max_{K \in \mathcal{T}_h^{wf}} \frac{h_K}{\rho_K} \le c_1.$$

For an explicit formula of c_1, see (3.8) and (3.5).

This chapter is organized as follows. In Section 3.1, we show the shape regularity of Worsey-Farin splits is solely determined by the shape regularity of the parent mesh. We first prove the dihedral angles are bounded by quantities that only depend on the shape regularity of the original mesh (see Lemma 3.1.3 below). Using this result we prove the crucial result that the split points of each face F in the triangulation is uniformly bounded away from ∂F; see Lemma 3.1.4. From this result, the shape



(a) Two adjacent elements of the mesh. (b) The common face

Figure 3.1: A representation of the Worsey-Farin splits.

regularity of Worsey-Farin refinements is then shown. In Section 3.2,we present some numerical experiments to demonstrate the relationship between the shape regularity of Worsey-Farin splits and the original mesh.

3.1 Analysis of the shape regularity of Worsey-Farin splits

To prove Theorem 3.0.1, we need to consider two cases: interior and boundary faces of $\mathcal{T}_h$. The case of boundary faces is simpler, so we first focus on the interior faces. For that case, it is sufficient to consider two adjacent elements of the mesh $\mathcal{T}_h$. To this end, let $T_1, T_2 \in \mathcal{T}_h$ be two tetrahedra that share a common face F_0. We write $T_1 = [x_1, x_2, x_3, x_4]$, $T_2 = [x_1, x_3, x_4, x_5]$, so that the common face is $F_0 = [x_1, x_3, x_4]$. We further set $F_1 = [x_2, x_3, x_4]$, and let z_i be the incenter of T_i, $i = 1, 2$ (see Figure 3.1a). For $i = 1, 2$, we denote by $z_{i,0}$ the orthogonal projections of z_i onto the plane containing the face F_0 (see Figure 3.1b). Likewise the orthogonal projection of z_1 onto the plane containing the face F_1 is denoted by $z_{1,1}$ (see Figure 3.2a). We denote the split point of the face F_0 by m_0, i.e., m_0 is the intersection of the line $[z_1, z_2]$ and F_0.

3.1.1 The position of split points and bounded dihedral angles

The following proposition shows the relation between the split point m_0 and the projections $z_{i,0}$, $i = 1, 2$ of the incenter on the face F_0.

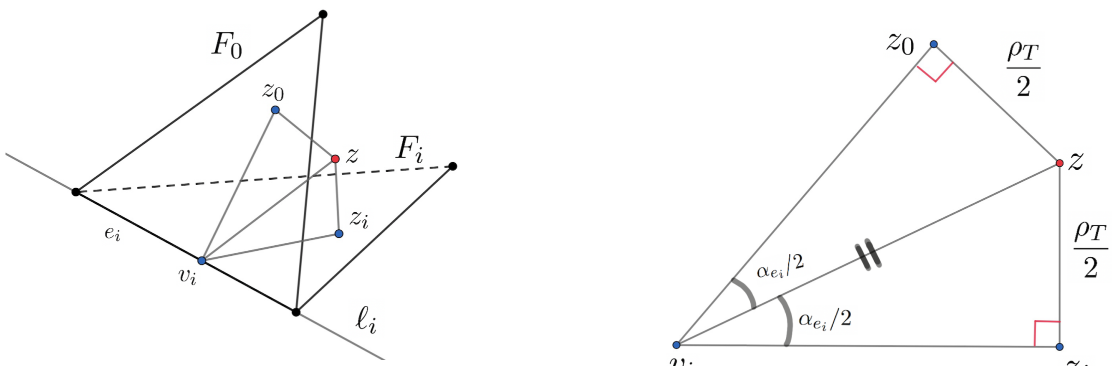
Proposition 3.1.1. *The orthogonal projections $z_{i,0}$ $(i = 1, 2)$ lie in the interior of F_0, and the split point m_0 lies on the line segment $[z_{1,0}, z_{2,0}]$. Furthermore, we have*

$$\mathrm{dist}(m_0, \partial F_0) \geq \min_{i=1,2} \mathrm{dist}(z_{i,0}, \partial F_0).$$

Proof. The proof of [53, Lemma 16.24] shows that m_0 lies on the line segment $[z_{1,0}, z_{2,0}]$ and that $z_{i,0}$ $(i = 1, 2)$ lie in the interior of F_0.

Let ℓ_i, $i = 1, 2, 3$ denote the lines that contain the three edges of F_0. Because m_0 lies on the interior of the line segment $[z_{1,0}, z_{2,0}]$, there exists a constant $\theta \in (0, 1)$ such that $m_0 = \theta z_{1,0} + (1 - \theta) z_{2,0}$. Then by constructing similar triangles, we have

$$\begin{aligned}
\mathrm{dist}(m_0, \partial F_0) &= \min_{1 \leq i \leq 3} \mathrm{dist}(m_0, \ell_i) \\
&= \min_{1 \leq i \leq 3} \Big(\theta \mathrm{dist}(z_{1,0}, \ell_i) + (1 - \theta) \mathrm{dist}(z_{2,0}, \ell_i) \Big) \\
&\geq \theta \min_{1 \leq i \leq 3} \mathrm{dist}(z_{1,0}, \ell_i) + (1 - \theta) \min_{1 \leq i \leq 3} \mathrm{dist}(z_{2,0}, \ell_i) \\
&= \theta \mathrm{dist}(z_{1,0}, \partial F_0) + (1 - \theta) \mathrm{dist}(z_{2,0}, \partial F_0) \\
&\geq \min_{i=1,2} \mathrm{dist}(z_{i,0}, \partial F_0).
\end{aligned}$$



(a) The projection of z_1 on face F_0 and F_1.

(b) The plane through the incenter.

Figure 3.2: A representation of the dihedral angle.

■

The following lemma tells us that the dihedral angles of tetrahedra determine the distance of $z_{i,0}$, $i = 1, 2$ to ∂F_0.

Lemma 3.1.2. *Let T be a tetrahedron, and for each face $F \in \Delta_2(T)$, let $z_{T,F}$ denote the orthogonal projection of the incenter of T onto F. Let α_e be the dihedral angle of T with respect to $e \in \Delta_1(T)$. We have*

$$\min_{F\in\Delta_2(T)} \operatorname{dist}(z_{T,F}, \partial F) \geq \min_{e\in\Delta_1(T)} \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_e)}{1-\cos(\alpha_e)}}. \tag{3.1}$$

Proof. We prove the lemma using the labeling depicted in Figure 3.2. Let z be the incenter of T and denote the four faces $F_i \in \Delta_2(T)$ for $i = 0, \ldots, 3$. Let z_i be the orthogonal projection of z onto the plane containing F_i and note that $|[z, z_i]| = \rho_T/2$. We need to find a lower bound for $\operatorname{dist}(z_\ell, \partial F_\ell)$ ($\ell = 0, \ldots, 3$) and without loss of generality we consider the case $\ell = 0$. To this end, let $e_i = \partial F_0 \cap \partial F_i$, $i = 1, 2, 3$ and furthermore let ℓ_i be the line containing e_i.

Let γ_i be the plane determined by the points z, z_0, and z_i and let $v_i = \ell_i \cap \gamma_i$.

The line ℓ_i is perpendicular to the plane γ_i, and thus $\ell_i \perp [v_i, z_j]$ for $j = 0, i$. This implies

$$\operatorname{dist}(z_j, \ell_i) = |[z_j, v_i]|, \; j = 0, i, \quad \text{and} \quad \alpha_{e_i} := \angle z_0 v_i z_i = \angle z v_i z_0 + \angle z v_i z_i.$$

Next, note the properties $[z, z_j] \perp [z_j, v_i]$ for $j = 0, i$ and $|[z, z_j]| = \rho_T/2$ imply that the triangles $[z, v_i, z_0]$ and $[z, v_i, z_i]$ are congruent (see Figure 3.2b). Consequently, $\angle z v_i z_0 = \angle z v_i z_i = \alpha_{e_i}/2$ and so

$$\operatorname{dist}(z_0, \ell_i) = |[z_0, v_i]| = \frac{\rho_T/2}{\tan(\alpha_{e_i}/2)} = \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_{e_i})}{1-\cos(\alpha_{e_i})}}. \tag{3.2}$$

Thus, we have

$$\operatorname{dist}(z_0, \partial F_0) \geq \min_{1\leq i\leq 3} \operatorname{dist}(z_0, \ell_i) = \min_{1\leq i\leq 3} \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_{e_i})}{1-\cos(\alpha_{e_i})}}. \tag{3.3}$$

■

It is well-known that shape regularity of a mesh leads to bounded dihedral angles. To be self-contained, we present a proof here.

Lemma 3.1.3. *Recall the shape regularity constant c_0 of $\mathcal{T}_h$ given in Definition 2.2.4. Fix $T \in \mathcal{T}_h$, and let α_e denote the dihedral angle of T with respect to $e \in \Delta_1(T)$. We then have*

$$|\cos(\alpha_e)| \leq \sqrt{1 - c_0^{-2}} \qquad \forall e \in \Delta_1(T). \tag{3.4}$$

Proof. Write $T = [x_1, x_2, x_3, x_4]$, consider the edge $e = [x_3, x_4]$, and let ℓ be the line containing e; see Figure 3.3. Let A be the orthogonal projection of x_1 onto the plane γ containing the face $[x_2, x_3, x_4]$, and let B be the point on ℓ such that $[x_1, B] \perp \ell$.

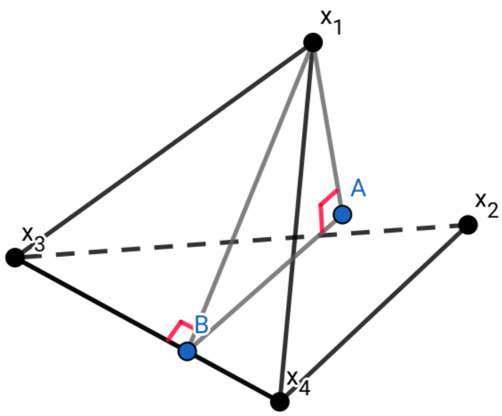


Figure 3.3: Computing dihedral angles.

Note that $[x_1, B] \perp \ell$ and $[x_1, A] \perp \ell$ implies $[A, B] \perp \ell$. Since $|[x_1, A]| \geq \rho_T$ by Proposition 2.2.6 and $|[x_1, B]| \leq h_T$, the dihedral angle of e satisfies

$$\sin(\alpha_e) = \frac{|[x_1, A]|}{|[x_1, B]|} \geq \frac{\rho_T}{h_T} \geq c_0^{-1}.$$

Therefore, we have $|\cos(\alpha_e)| = \sqrt{1 - \sin^2(\alpha_e)} \leq \sqrt{1 - c_0^{-2}}$. ■

Combining Proposition 3.1.1, Lemma 3.1.2 and Lemma 3.1.3, we have the following lemma which describes the position of split points. We also include the case for boundary faces.

Lemma 3.1.4. *Recall that m_F is the split point of F constructed by the Worsey-Farin split. For any face F of $\mathcal{T}_h$,*

$$\operatorname{dist}(m_F, \partial F) \geq c_2 \min_{\substack{T \in \mathcal{T}_h \\ F \in \Delta_2(T)}} h_T,$$

where

$$c_2 := \min\{\mathfrak{c}_2, (3c_0)^{-1}\}, \quad \mathfrak{c}_2 := (2c_0)^{-1}\sqrt{-1 + \frac{2}{1 + \sqrt{1 - c_0^{-2}}}}. \tag{3.5}$$

Proof. (i) F **is an interior face.** In this case $F \in \Delta_2(T)$ and $F \in \Delta_2(T')$ for some $T, T' \in \mathcal{T}_h$. Without loss of generality, we assume $\mathrm{dist}(z_{T',F}, \partial F) \geq \mathrm{dist}(z_{T,F}, \partial F)$. Proposition 3.1.1 and Lemma 3.1.2 tell us that

$$\begin{aligned}\mathrm{dist}(m_F, \partial F) &\geq \mathrm{dist}(z_{T,F}, \partial F)\\ &\geq \min_{e\in\Delta_1(T)} \frac{\rho_T}{2}\sqrt{\frac{1+\cos(\alpha_e)}{1-\cos(\alpha_e)}} = \min_{e\in\Delta_1(T)} \frac{\rho_T}{2}\sqrt{-1+\frac{2}{1-\cos(\alpha_e)}}.\end{aligned}$$

If $\cos(\alpha_e) \geq 0$, then $\frac{2}{1-\cos(\alpha_e)} \geq 2$, and if $\cos(\alpha_e) \leq 0$, then $\frac{2}{1-\cos(\alpha_e)} = \frac{2}{1+|\cos(\alpha_e)|} \geq \frac{2}{1+\sqrt{1-c_0^{-2}}}$ by Lemma 3.1.3. Consequently,

$$\mathrm{dist}(m_F, \partial F) \geq \min_{e\in\Delta_1(T)} \frac{\rho_T}{2}\sqrt{-1+\frac{2}{1-\cos(\alpha_e)}} \geq \mathfrak{c}_2 h_T.$$

(ii) F **is a boundary face.** Without loss of generality assume that

$$\mathrm{dist}(m_F, \partial F) = \mathrm{dist}(m_F, \ell)$$

where $F = [x_1, x_3, x_4]$, $e = [x_3, x_4]$, and ℓ is the line containing e. We further adopt the notations in the proof of Lemma 3.1.3; see Figure 3.3. Because m_F is the barycenter of F, we have

$$\frac{1}{3}|F| = \frac{1}{2}\mathrm{dist}(m_F, \ell)|e|.$$

Moreover, clearly

$$|F| = \frac{1}{2}|e||[x_1, B]|.$$

And therefore, since $|[x_1, B]| \geq |[x_1, A]| > \rho_T$, (where we used (2.8) and the right triangle $[x_1, A, B]$) we get

$$\operatorname{dist}(m_F, \ell) = \frac{1}{3}|[x_1, B]| \geq \frac{1}{3}\rho_T \geq (3c_0)^{-1}h_T.$$

■

3.1.2 Proof of Theorem 3.0.1

Now we are ready to use Lemma 3.1.4 to prove Theorem 3.0.1.

Proof of Theorem 3.0.1. Let $K \in \mathcal{T}_h^{wf}$, and let $T \in \mathcal{T}_h$ such that $K \in T^{wf}$. We write $T = [x_1, x_2, x_3, x_4]$, and assume, without loss of generality, that $e := [x_1, x_2]$ is an edge of both T and K. Let $F \in \Delta_2(T)$ such that the split point m_F is a vertex of K. In particular, $e \in \Delta_1(F)$ and $K = [x_1, x_2, m_F, z_T]$, where z_T is the incenter of T. We futher denote by ℓ, the line containing the edge e.

We again adopt the notations in the proof of Lemma 3.1.3 and refer to Figure 3.3. Note that $[x_1, A]$ is normal to the plane γ containing $[x_2, x_3, x_4]$, in particular, $[x_1, A] \perp [A, x_2]$. Thus $|e| = |[x_1, x_2]| > |[x_1, A]| > \rho_T$ by (2.8). Now we have $h_K \leq h_T$, $\rho_T < |e| \leq h_T$ and, by Lemma 3.1.4, the volume of K is

$$\begin{aligned} |K| &= \frac{1}{3}\frac{\rho_T}{2} \times |[x_1, x_2, m_F]| = \frac{1}{12}\rho_T|e|\operatorname{dist}(m_F, \ell) \\ &\geq \frac{1}{12}\rho_T^2\operatorname{dist}(m_F, \partial F) \geq \frac{c_2}{12}\rho_T^2\big(\min_{\substack{T' \in \mathcal{T}_h \\ F \in \Delta_2(T')}} h_{T'}\big) \geq \frac{c_2}{12}\rho_T^3 \geq \frac{c_2}{12c_0^3}h_T^3. \end{aligned} \tag{3.6}$$

Here we also used

$$\min_{\substack{T' \in \mathcal{T}_h \\ F \in \Delta_2(T')}} h_{T'} \geq |e| > \rho_T.$$

Additionally, each face of K is contained in a circle with radius $h_K/2$, and thus we have

$$\sum_{F\in\Delta_2(K)} |F| \le \sum_{F\in\Delta_2(K)} \frac{\pi h_K^2}{4} = \pi h_K^2. \tag{3.7}$$

Consequently, with Proposition 2.2.5, (3.6) and (3.7), we have

$$\rho_K = \frac{6|K|}{\sum\limits_{F\in\Delta_2(K)} |F|} \ge \frac{c_2 h_T^3}{2\pi c_0^3 h_K^2} \ge \frac{c_2 h_K}{2\pi c_0^3}.$$

Thus, setting

$$c_1 = \frac{2\pi c_0^3}{c_2}, \tag{3.8}$$

we have $\frac{h_K}{\rho_K} \le c_1$. Because $K \in \mathcal{T}_h^{wf}$ was arbitrary, we conclude $\max\limits_{K\in\mathcal{T}_h^{wf}} \frac{h_K}{\rho_K} \le c_1$. ■

3.2 Numerical experiment

The bound c_1 in (3.8) is not an optimal estimate and we use the following numerical experiment to show this. We compute the shape regularity constants c_0 and $\tilde{c}_1$ for the triangulation of a unit cube and its Worsey-Farin refinements (see Figure 2.4b and Figure 2.4c) and see the relationship between them. To be more precise, we firstly split the unit cube into multiple sub-cubes with size $\frac{1}{2} \times \frac{1}{5} \times \frac{1}{2^j}$, $j = 5, ..., 14$ and then we split each sub-cube into 6 tetrahedra which is the original mesh $\mathcal{T}_h$. After we get the the shape regularity constants c_0 and $\tilde{c}_1$ for $\mathcal{T}_h$ and its Worsey-Farin refinements $\mathcal{T}_h^{wf}$, we want to know if we can find an α such that $c_0 \le C\tilde{c}_1^{\alpha}$, where C is a constant.

Note here for this specific example, the numerical approximation of α is 2. We could see that there do exist some relationship between the shape regularity of a

j	c_0	$\tilde{c}_1$	α
5	24.380	3802.276	–
6	46.984	15763.493	2.1677
7	92.228	64260.478	2.0835
8	182.732	259557.684	2.0417
9	363.749	1043367.702	2.0208
10	725.786	4183851.537	2.0104
11	1449.862	16756275.107	2.0052
12	2898.017	67066946.197	2.0026
13	5794.326	268351584.352	2.0013

Table 3.1: The relationship between the shape regularity of a Worsey-Farin refinement with its parent triangulation.

Worsey-Farin refinement with its parent triangulation even though our theoretical result is not optimal.

3.3 Conclusion

We have settled a conjecture concerning the shape regularity of a Worsey-Farin refinement of a parent triangulation. As described in the introduction, this is a crucial bound to obtain approximation results for splines; see for example [53, Theorem 18.15]. However, based on initial numerical calculations, the constant c_1 in Theorem 3.0.1 that relates the shape regularity of the parent triangulation (i.e., c_0) and its Worsey-Farin refinement is most likely not sharp. In particular, the theorem suggest that c_1 scales like c_0^5 which could be quite large even for a good quality parent triangulation. We hope that this work leads to further investigations and sharper estimates will emerge.

Chapter Four

Convergence of Lagrange Finite Element Methods for Maxwell Eigenvalue Problem in 3D

As we mentioned in the introduction, it is well known that using Lagrange finite elements to approximate Maxwell's eigengvalue problem leads to erroneous approximations on general triangulations (see for example [10, 17]) while Nédélec edge elements give reliable approximations to this problem. However, Wong and Cendes [63] showed numerically Lagrange FEM give the correct approximations on certain meshes: Powell-Sabin meshes in two dimensions and Worsey-Farin meshes in three dimensions.

The work here for three dimension is more involved than the two dimensional analysis given in [20]. In particular, we need to develop two Fortin-like operators whereas in [20] only one was needed. To do this, we exploit that Lagrange elements on Worsey-Farin splits also fit into an a discrete de Rham sequence [44]. Then, using the degrees of freedom given in [44] we construct a Fortin-like operators for both curl and divergence operators and use certain embeddings to prove that these operators are bounded.

This chapter is organized as follows. In the next section, we state the Maxwell eigenvalue problem and its mixed formulation. We also introduce general primal and mixed finite element methods for the eigenvalue problem and present a convergence framework. Section 4.2 summarizes some exactness properties of finite element spaces on such meshes, and constructs a Scott-Zhang-type interpolant. In Section 4.3, we construct two Fortin-like operators and show the stability estimates for those two operators. As a byproduct, in Section 4.4, we show convergence of Lagrange finite element methods for the Maxwell eigenvalue problem on Worsey-Farin triangulations provided the polynomial degree is at least two. Finally, in Section 4.5, we present some numerical experiments illustrating that continuous piecewise polynomials can be applied to three dimensional Maxwell eigenvalue problem after Worsey-Farin refinement.

4.1 The Maxwell's eigenvalue problem, its discretization, and convergence framework

Throughout this chapter, let $\Omega \subset \mathbb{R}^3$ be a contractible, Lipschitz polyhedral domain. Recall the Maxwell's eigenvalue problem (1.1) mentioned in the Introduction. When $\eta \neq 0$, it is equivalent to a mixed formulation given by Boffi et al. [18]: find $\lambda \in \mathbb{R}\backslash\{0\}$ and $0 \neq p \in \mathring{H}(\mathrm{div}^0, \Omega)$, $\sigma \in \mathring{H}(\mathrm{curl}, \Omega)$ such that:

$$\begin{aligned} (\sigma, \tau) + (p, \mathrm{curl}\, \tau) &= 0 && \forall \tau \in \mathring{H}(\mathrm{curl}, \Omega), \\ (\mathrm{curl}\, \sigma, q) &= -\lambda(p, q) && \forall q \in \mathring{H}(\mathrm{div}^0, \Omega), \end{aligned} \tag{4.1}$$

where

$$\mathring{H}(\mathrm{div}^0, \Omega) := \{v \in \mathring{H}(\mathrm{div}, \Omega) : \mathrm{div}\, v = 0\}.$$

We note that $\mathring{H}(\mathrm{div}^0, \Omega) = \mathrm{curl}\, \mathring{H}(\mathrm{curl}\,, \Omega)$, and also $\lambda = \eta^2$, $\sigma = u$ and $p = -\frac{\mathrm{curl}\, u}{\lambda}$. Similarly, an equivalent mixed formulation of the discrete problem (1.2) (when $\eta_h \neq 0$) is given by: find $\lambda_h \in \mathbb{R}\backslash\{0\}$ and $0 \neq p_h \in \mathscr{Q}_h, \sigma_h \in \mathscr{V}_h$ such that:

$$\begin{aligned} (\sigma_h, \tau_h) + (p_h, \mathrm{curl}\, \tau_h) &= 0 && \forall \tau_h \in \mathscr{V}_h, \\ (\mathrm{curl}\, \sigma_h, q_h) &= -\lambda_h(p_h, q_h) && \forall q_h \in \mathscr{Q}_h, \end{aligned} \tag{4.2}$$

where $\mathscr{Q}_h := \mathrm{curl}\, \mathscr{V}_h$. Analogous to the continuous setting, we have $\lambda_h = \eta_h^2$, $\sigma_h = u_h$ and $p_h = -\frac{\mathrm{curl}\, u_h}{\lambda_h}$.

We follow the classical theory (e.g., [17, Section 14]) and analyze the mixed finite element method (4.2) by considering the corresponding source problem. Define the solution operators $\boldsymbol{A} : [L^2(\Omega)]^3 \to \mathring{H}(\mathrm{curl}, \Omega)$ and $\boldsymbol{T} : [L^2(\Omega)]^3 \to \mathring{H}(\mathrm{div}^0, \Omega)$ such that for given $f \in [L^2(\Omega)]^3$, there holds

$$\begin{aligned} (\boldsymbol{A} f, \tau) + (\boldsymbol{T} f, \operatorname{curl} \tau) &= 0 \qquad && \forall \tau \in \mathring{H}(\operatorname{curl}, \Omega), \\ (\operatorname{curl} \boldsymbol{A} f, q) &= (f, q) && \forall q \in \mathring{H}(\operatorname{div}^0, \Omega). \end{aligned} \tag{4.3}$$

Likewise, the discrete solution operators $\boldsymbol{A}_h : [L^2(\Omega)]^3 \to \mathscr{V}_h$ and $\boldsymbol{T}_h : [L^2(\Omega)]^3 \to \mathscr{Q}_h$ are defined as

$$\begin{aligned} (\boldsymbol{A}_h f, \tau_h) + (\boldsymbol{T}_h f, \operatorname{curl} \tau_h) &= 0 \qquad && \forall \tau_h \in \mathscr{V}_h, \\ (\operatorname{curl} \boldsymbol{A}_h f, q_h) &= (f, q_h) && \forall q_h \in \mathscr{Q}_h. \end{aligned} \tag{4.4}$$

4.1.1 Convergence theory

It is well known that the convergence of the eigenvalues to the discrete problem (4.2) converge to the exact eigenvalues (given in problem (4.1)) provided the source problem converges uniformly (see for example [17, Section 7]). In the next proposition, the operator norm is defined as

$$\|\boldsymbol{T}\| := \sup_{f \in L^2(\Omega)\backslash\{0\}} \frac{\|\boldsymbol{T} f\|_{L^2(\Omega)}}{\|f\|_{L^2(\Omega)}}. \tag{4.5}$$

Proposition 4.1.1. *Let $\boldsymbol{T}$ and $\boldsymbol{T}_h$ be defined from (4.3) and (4.4), respectively, and suppose that $\|\boldsymbol{T} - \boldsymbol{T}_h\| \to 0$ as $h \to 0$. Consider the problem (4.1) with the nonzero eigenvalues $0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \cdots$ and the problem (4.2) with the nonzero eigenvalues $0 < \lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots$. Then, for any fixed i, $\lim_{h\to 0} \lambda_h^{(i)} = \lambda^{(i)}$.*

It will suffice to verify one assumption of our discrete spaces to guarantee $\|\boldsymbol{T} -$

$\boldsymbol{T}_h\| \to 0$. To describe this assumption, we introduce the space

$$\mathscr{V}^t(\mathscr{Q}_h) = \{\tau \in \mathscr{V}^t : \operatorname{curl}\tau \in \mathscr{Q}_h\}. \tag{4.6}$$

Assumption 4.1.2. We assume the existence of a projection $\mathbf{\Pi}_{\mathscr{V}} : \mathscr{V}^t(\mathscr{Q}_h) \to \mathscr{V}_h$ such that

$$\begin{aligned}
\operatorname{curl}\mathbf{\Pi}_{\mathscr{V}}\,\tau &= \operatorname{curl}\tau && \forall \tau \in \mathscr{V}^t(\mathscr{Q}_h),\\
\|\mathbf{\Pi}_{\mathscr{V}}\tau - \tau\| &\le \omega_0(h)(\|\tau\|_{H^{1/2+\delta}(\Omega)} + \|\operatorname{curl}\tau\|_{L^2(\Omega)}) && \forall \tau \in \mathscr{V}^t(\mathscr{Q}_h).
\end{aligned}$$

Furthermore, we assume that the L^2-orthogonal projection $\mathbb{P}_Q : [L^2(\Omega)]^3 \to \mathscr{Q}_h$ satisfies

$$\|\mathbb{P}_Q p - p\|_{L^2(\Omega)} \le \omega_1(h)\|\operatorname{curl} p\|_{L^2(\Omega)} \qquad \forall p \in H(\operatorname{curl},\Omega) \cap \mathring{H}(\operatorname{div}^0,\Omega). \tag{4.7}$$

Here, $\omega_i > 0$ satisfies $\lim_{h\to 0^+}\omega_i(h) = 0$ for $i = 0, 1$.

With this assumption we have the following result. We give the proof of this result in the Appendix B; it is similar to the argument in [20].

Theorem 4.1.3. *Suppose that $(\mathscr{V}_h, \mathscr{Q}_h)$ satisfy Assumption 4.1.2 and let $\boldsymbol{T}$ and $\boldsymbol{T}_h$ defined as in the equations (4.3) and (4.4), respectively. Then we have*

$$\|\boldsymbol{T} - \boldsymbol{T}_h\| \le C(\omega_0(h) + \omega_1(h)). \tag{4.8}$$

The following corollary is a consequence of the above theorem and Proposition 4.1.1.

Corollary 4.1.4. *Consider the problem (4.1) with the nonzero eigenvalues* $0 <$

$\lambda^{(1)} \leq \lambda^{(2)} \leq \cdots$ *and the problem (4.2) with the nonzero eigenvalues* $0 < \lambda_h^{(1)} \leq \lambda_h^{(2)} \leq \cdots$. *Suppose that* $(\mathscr{V}_h, \mathscr{Q}_h)$ *satisfy Assumption 4.1.2. Then, for any fixed* i, $\lim_{h\to 0} \lambda_h^{(i)} = \lambda^{(i)}$.

4.2 Finite element spaces on Worsey-Farin splits

The global sequence without boundary conditions (2.26a) is defined in Section 2.5.5. We also consider the sequence with appropriate boundary conditions:

$$0 \to \mathring{\mathcal{S}}_r^0(\mathcal{T}_h^{wf}) \xrightarrow{\text{grad}} \mathcal{L}_{r-1}^t \xrightarrow{\text{curl}} \mathscr{V}_{r-2}^n \xrightarrow{\text{div}} \mathring{V}_{r-3}^3(\mathcal{T}_h^{wf}) \to 0, \tag{4.9}$$

where

$$\mathcal{L}_{r-1}^t = \mathcal{L}_{r-1}^1(\mathcal{T}_h^{wf}) \cap \mathring{H}(\text{curl}, \Omega), \quad \mathscr{V}_{r-2}^n = \mathscr{V}_{r-2}^2(\mathcal{T}_h^{wf}) \cap \mathring{H}(\text{div}, \Omega). \tag{4.10}$$

Although the sequence (2.26a) is only considered in [44], the exactness of the sequence with boundary conditions (4.9) holds by a simple counting argument.

4.2.1 Equivalence norms assumptions

We will use the dofs defined in (2.24) to build Fortin-like projections. Unfortunately, the dofs are not preserved with a covariant Piola transform, as is the case for the Nédélec elements [57]. Rather, our strategy to prove estimates of the projections is to map a tetrahedra to a unit size tetrahedra via dilation. To this end, we define scaled tetrahedra.

Definition 4.2.1. For a tetrahedron $T \in \mathcal{T}_h$, define its dilation and induced Worsey-

Farin split

$$\hat{T} = \frac{1}{h_T} T := \{\frac{x}{h_T} : \ x \in T\}, \qquad \hat{T}^{wf} = \{\frac{1}{h_T} K : \ K \in T^{wf}\}. \tag{4.11}$$

Note the scaled tetrahedra $\hat{T}$ are of unit size, and $\hat{T}^{wf}$ inherit the shape-regular properties of T^{wf}. From now on, unless otherwise specified, we denote $\hat{v}(\hat{x}) = v(h_T \hat{x})$ ($\hat{x} \in \hat{T}$) where v is either a scalar function or vector-valued function.

As a first step, we note that Lemma 2.5.4 and equivalence of norms in finite dimensional spaces immediately give the following lemma.

Lemma 4.2.2. *Consider $T \in \mathcal{T}_h$.*

(a) Let G_i^L, $i = 1, ..., m_L$ be the functional given by dofs (2.23) on $\hat{T}$, where m_L is the dimension of $\mathsf{L}_{r-1}^1(\hat{T}^{wf})$. Then, there exists a constant $\beta(\hat{T}^{wf}) > 0$ such that

$$\|\hat{v}\|_{L^2(\hat{T})} \le \beta(\hat{T}^{wf}) \sum_{i=1}^{m_L} |G_i^L(\hat{v})|, \quad \forall \hat{v} \in \mathsf{L}_{r-1}^1(\hat{T}^{wf}).$$

(b) Let G_i^N, $i = 1, ..., m_N$ be the functional given by dofs (2.24) on $\hat{T}$, where m_N is the dimension of $V_{r-1}^2(\hat{T}^{wf})$. Then, there exists a constant $\theta(\hat{T}^{wf}) > 0$ such that

$$\|\widehat{v}\|_{L^2(\hat{T})} \le \theta(\hat{T}^{wf}) \sum_{i=1}^{m_L} |G_i^N(\widehat{v})|, \quad \forall \widehat{v} \in V_{r-1}^2(\hat{T}^{wf}).$$

In the following, we make an assumption concerning the uniform bound of the constants appearing in Lemma 4.2.2.

Assumption 4.2.3. There exists constants $\theta, \beta > 0$ that depend only on the shape regularity of $\{\mathcal{T}_h\}$ and polynomial order r such that

$$\begin{aligned}\theta(\hat{T}^{wf}) \leq& \theta, \quad \forall T \in \mathcal{T}_h,\\ \beta(\hat{T}^{wf}) \leq& \beta, \quad \forall T \in \mathcal{T}_h.\end{aligned}$$

Remark 4.2.4. One common approach (see for example [62]) to verify this assumption is to argue that the constants $\theta(\hat{T}^{wf})$ and $\beta(\hat{T}^{wf})$ are continuous functions of the vertices of $\hat{T}^{wf}$. If so, since the vertices live on a compact set, such a result implies the uniform bounds stated in Assumption 4.2.3.

4.2.2 The modified Scott-Zhang interpolants

The Fortin-like projections utilize Scott-Zhang-type interpolants onto piecewise linear polynomials that preserve zero tangential or normal boundary conditions. The proof of the following lemma can be found in the appendix. To state the lemma we denote the patch around T:

$$\omega(T) = \bigcup_{\substack{T' \in \mathcal{T}_h \\ \bar{T} \cap \bar{T}' \neq \emptyset}} T'.$$

Lemma 4.2.5. *Let $0 < \delta \leq \frac{1}{2}$. There exists a projection $\boldsymbol{I}_h^{curl} : [H^{1/2+\delta}(\Omega)]^3 \longrightarrow \mathcal{L}_1^1(\mathcal{T}_h)$ with the following bounds:*

$$h_T^{-1/2-\delta} \|\tau - \boldsymbol{I}_h^{curl}\tau\|_{L^2(T)} + \|\boldsymbol{I}_h^{curl}\tau\|_{H^{1/2+\delta}(T)} \leq C\|\tau\|_{H^{1/2+\delta}(\omega(T))} \qquad \forall \tau \in [H^{1/2+\delta}(\Omega)]^3 \tag{4.12}$$

for all $T \in \mathcal{T}_h$. Moreover, if $\tau \in [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl}, \Omega)$ then $\boldsymbol{I}_h^{curl}\tau \in \mathcal{L}_1^1(\mathcal{T}_h) \cap \mathring{H}(\mathrm{curl}, \Omega)$.

There also exists a projection $\boldsymbol{I}_h^{div} : [H^{1/2+\delta}(\Omega)]^3 \longrightarrow \mathcal{L}_1^2(\mathcal{T}_h)$ with the following *bounds:*

$$h_T^{-1/2-\delta}\|\tau - \boldsymbol{I}_h^{div}\tau\|_{L^2(T)} + \|\boldsymbol{I}_h^{div}\tau\|_{H^{1/2+\delta}(T)} \leq C\|\tau\|_{H^{1/2+\delta}(\omega(T))} \qquad \forall \tau \in [H^{1/2+\delta}(\Omega)]^3 \tag{4.13}$$

for all $T \in \mathcal{T}_h$. Moreover, if $\tau \in [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{div}\,,\Omega)$, then $\boldsymbol{I}_h^{div}\tau \in \mathcal{L}_1^2(\mathcal{T}_h) \cap \mathring{H}(\mathrm{div}\,,\Omega)$.

4.3 Construction of two Fortin-like projections

With all those preparation, in this section, we work on constructing projections that conform to the framework given in Section 4.1. Firstly, let

$$H^D(\mathrm{curl}\,,\Omega) := \{v \in [L^2(\Omega)]^3 : \mathrm{curl}\, v \in \mathscr{V}_{r-2}^2(\mathcal{T}_h^{wf})\} \subset H(\mathrm{curl}\,,\Omega).$$

4.3.1 Definition of Fortin-like operators

The following operators are defined through the use of Lemma 2.5.4. We separate those degrees of freedom into two parts with one part affecting commuting properties and the second one not.

Definition 4.3.1. Define the operator $\boldsymbol{\Pi}_L : H^D(\mathrm{curl}\,,\Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \to \mathcal{L}_{r-1}^1(\mathcal{T}_h^{wf})$ such that on each $T \in \mathcal{T}_h$,

$$\int_e (\boldsymbol{\Pi}_L v \cdot t_e)\kappa\, ds = \int_e (v \cdot t_e)\kappa\, ds \qquad \forall \kappa \in \mathcal{P}_{r-3}(e),\ \forall e \in \Delta_1(T), \tag{4.14a}$$

$$\int_e [\![\mathrm{curl}\,\boldsymbol{\Pi}_L v \cdot t_e]\!]_e \kappa\, ds = \int_e [\![\mathrm{curl}\, v \cdot t_e]\!]_e \kappa\, ds \qquad \forall \kappa \in \mathcal{P}_{r-3}(e),\ \forall e \in \Delta_1^I(F^{\mathrm{ct}})\backslash\{e_F\},\ \forall F \in \Delta_2(T), \tag{4.14b}$$

$$\int_{e_F} [\![\operatorname{curl} \mathbf{\Pi}_L v \cdot t_{e_F}]\!]_{e_F} \kappa \, ds = \int_{e_F} [\![\operatorname{curl} v \cdot t_{e_F}]\!]_{e_F} \kappa \, ds \quad \forall \kappa \in \mathcal{P}_{r-2}(e_F),\ \forall F \in \Delta_2(T), \tag{4.14c}$$

$$\int_F (\operatorname{curl}_F (\mathbf{\Pi}_L v)_F) \kappa \, dA = \int_F (\operatorname{curl}_F v_F) \kappa \, dA \qquad \forall \kappa \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{4.14d}$$

$$\int_T \operatorname{curl} \mathbf{\Pi}_L v \cdot \kappa \, dx = \int_T \operatorname{curl} v \cdot \kappa \, dx \qquad \forall \kappa \in \operatorname{curl} \mathring{\mathsf{L}}^1_{r-1}(T^{wf}), \tag{4.14e}$$

$$(\mathbf{\Pi}_L v)(a) = 0 \qquad \forall a \in \Delta_0(T), \tag{4.14f}$$

$$\int_e (\mathbf{\Pi}_L v \times t_e) \cdot \kappa \, ds = 0 \qquad \forall \kappa \in [\mathcal{P}_{r-3}(e)]^2,\ \forall e \in \Delta_1(T), \tag{4.14g}$$

$$\int_F (\mathbf{\Pi}_L v \cdot n_F) \kappa \, dA = 0 \qquad \forall \kappa \in \mathcal{R}^0_{r-1}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{4.14h}$$

$$\int_F \mathbf{\Pi}_L v_F \cdot \kappa \, dA = 0 \qquad \forall \kappa \in \operatorname{grad}_F \mathring{S}^0_r(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{4.14i}$$

$$\int_T \mathbf{\Pi}_L v \cdot \kappa \, dx = 0 \qquad \forall \kappa \in \operatorname{grad} \mathring{S}^0_r(T^{wf}). \tag{4.14j}$$

Definition 4.3.2. The operator $\mathbf{\Pi}_N : [H^{1/2+\delta}(\Omega)]^3 \cap H(\operatorname{div}, \Omega) \to \mathscr{V}^2_{r-2}(\mathcal{T}^{wf}_h)$ is defined such that, on each $T \in \mathcal{T}_h$,

$$\int_e [\![\mathbf{\Pi}_N p \cdot t_e]\!]_e \kappa \, ds = 0 \qquad \forall \kappa \in \mathcal{P}_{r-3}(e),\ \forall e \in \Delta^I_1(F^{\mathrm{ct}}) \backslash \{e_F\},\ \forall F \in \Delta_2(T), \tag{4.15a}$$

$$\int_{e_F} [\![\mathbf{\Pi}_N p \cdot t_{e_F}]\!]_{e_F} \kappa \, ds = 0 \qquad \forall \kappa \in \mathcal{P}_{r-2}(e_F),\ \forall F \in \Delta_2(T), \tag{4.15b}$$

$$\int_F (\mathbf{\Pi}_N p \cdot n_F) \kappa \, dA = \int_F (p \cdot n_F) \kappa \, dA \qquad \forall \kappa \in V^2_{r-2}(F^{\mathrm{ct}}),\ \forall F \in \Delta_2(T), \tag{4.15c}$$

$$\int_T (\operatorname{div} \mathbf{\Pi}_N p) \kappa \, dx = \int_T (\operatorname{div} p) \kappa \, dx \qquad \forall \kappa \in \mathring{V}^3_{r-3}(T^{wf}), \tag{4.15d}$$

$$\int_T \mathbf{\Pi}_N p \cdot \kappa \, dx = \int_T p \cdot \kappa \, dx \qquad \forall \kappa \in \operatorname{curl} \mathring{\mathsf{L}}^1_{r-1}(T^{wf}). \tag{4.15e}$$

Definition 4.3.3. The operator $\Pi_W : L^2(\Omega) \to V^3_{r-3}(T^{wf})$ is defined such that, on each $T \in \mathcal{T}_h$,

$$\int_T \Pi_W v \, dx = \int_T v \, dx, \tag{4.16a}$$

$$\int_T (\Pi_W v)\kappa \, dx = \int_T v\kappa \, dx \qquad \forall \kappa \in \mathring{V}^3_{r-3}(T^{wf}). \tag{4.16b}$$

We modify the operator $\mathbf{\Pi}_L$ to obtain an Fortin-like projection that inherits the commuting properties of $\mathbf{\Pi}_L$ and approximation properties of the Scott-Zhang interpolant $\boldsymbol{I}_h^{curl}$. To this end, we maintain the dofs that affect the commuting properties and modify the rest parts with the Scott-Zhang interpolant $\boldsymbol{I}_h^{curl}$ to find some uniform bounds.

Definition 4.3.4. The operator $\mathbf{\Pi}_V : H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \to \mathcal{L}^1_{r-1}(\mathcal{T}_h^{wf})$ is defined as

$$\mathbf{\Pi}_V = \boldsymbol{I}_h^{curl} + \mathbf{\Pi}_L(\mathbf{1} - \boldsymbol{I}_h^{curl}),$$

where $\mathbf{1}$ is the identity operator and $\mathbf{\Pi}_L$ is defined in Definition 4.3.1.

The next two theorems state the commuting properties of $\mathbf{\Pi}_V$ and $\mathbf{\Pi}_N$ and their approximation properties in the L^2-norm. The proof of these theorems are postponed to Section 4.3.3.

Theorem 4.3.5. *The operator $\mathbf{\Pi}_V : H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \to \mathcal{L}^1_{r-1}(\mathcal{T}_h^{wf})$ defined in Definition 4.3.4 satisfies*

$$\mathrm{curl}\ \mathbf{\Pi}_V \tau = \mathrm{curl}\ \tau, \quad \forall \tau \in H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3. \tag{4.17}$$

Moreover, under Assumption 4.2.3, the following bound holds:

$$\|\mathbf{\Pi}_V \tau - \tau\|_{L^2(\Omega)} \le C\big(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\,\mathrm{curl}\,\tau\|_{L^2(\Omega)}\big), \tag{4.18}$$

for any $\tau \in H^D(\mathrm{curl}\,,\Omega) \cap [H^{1/2+\delta}(\Omega)]^3$. *Finally, if*

$$\tau \in H^D(\mathrm{curl}\,,\Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl},\Omega),$$

then $\mathbf{\Pi}_V \tau \in \mathcal{L}^t_{r-1}$.

We also state the analogous result for $\mathbf{\Pi}_N$.

Theorem 4.3.6. *The operator* $\mathbf{\Pi}_N : [H^{1/2+\delta}(\Omega)]^3 \cap H(\mathrm{div}\,,\Omega) \to \mathscr{V}^2_{r-2}(\mathcal{T}^{wf}_h)$ *defined in Definition 4.3.1 satisfies*

$$\mathrm{div}\,\mathbf{\Pi}_N p = \Pi_W \mathrm{div}\, p \qquad \forall p \in [H^{1/2+\delta}(\Omega)]^3 \cap H(\mathrm{div},\Omega). \tag{4.19}$$

Moreover, under Assumption 4.2.3, the following bound holds $\forall p \in [H^{1/2+\delta}(\Omega)]^3 \cap H(\mathrm{div},\Omega)$*:*

$$||\mathbf{\Pi}_N p - p||_{L^2(\Omega)} \leq C\big(h^{1/2+\delta}||p||_{[H^{1/2+\delta}(\Omega)]^3} + h\|\mathrm{div}\, p\|_{L^2(\Omega)}\big), \\ \forall p \in [H^{1/2+\delta}(\Omega)]^3 \cap H(\mathrm{div},\Omega). \tag{4.20}$$

Finally, if $p \in [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{div},\Omega)$ *then* $\mathbf{\Pi}_N p \in \mathscr{V}^n_{r-2}$.

4.3.2 Bounds after dilation

4.3.2.1 Some inequalities on scaled tetrahedra

We first give several inequalities on the scaled tetrahedra $\hat{T}$ from Definition 4.2.1. The proofs of the next three results are shown in Appendix A.

Proposition 4.3.7. *Under Definition 4.2.1, we have the following results:*

(i) *There holds* $\mathrm{diam}(\hat{T}) = 1 \leq c_0\rho_{\hat{T}}$, *where* c_0 *is the shape-regularity constant given in Definition 2.2.4.*

(ii) $\widehat{\mathrm{div}}\,\hat{v} = h_T \mathrm{div}\, v$ *and* $\widehat{\mathrm{curl}}\,\hat{v} = h_T \mathrm{curl}\, v$.

(iii) *There holds for* $0 \leq s \leq 1$

$$c_0^{-3/2-s} h_T^{-3/2+s} |v|_{H^s(T)} \leq |\hat{v}|_{H^s(\hat{T})} \leq h_T^{-3/2+s} |v|_{H^s(T)} \qquad \forall v \in H^s(T).$$

The next result establishes a trace inequality on $\hat{T}$.

Lemma 4.3.8. *For any* $\hat{v} \in H^{1/2+\delta}(\hat{T})$ $(\delta \in (0, \frac{1}{2}])$, *we have* $\hat{v}|_{\partial\hat{T}} \in L^p(\partial\hat{T})$ *for* $2 \leq p < \frac{2}{1-\delta}$. *In particular,*

$$\|\hat{v}\|_{L^p(\partial\hat{T})} \leq C\|\hat{v}\|_{H^{1/2+\delta}(\hat{T})} \qquad \forall \hat{v} \in H^{1/2+\delta}(\hat{T}),$$

where C *is a uniform constant for all* $\hat{T}$.

The following lemma gives an inverse trace operator with estimate. Its proof is given in [6, Lemma 4.7]; however, for completeness, we provide the proof with additional details in the appendix.

Lemma 4.3.9. *Let* $\hat{e} \in \Delta_1(\hat{T})$, *and let* $\hat{F} \in \Delta_2(\hat{T})$ *be a face that has* $\hat{e}$ *as an edge. Then, there exists an extension operator* $E : \mathcal{P}_{r-3}(\hat{e}) \to W^{1,q}(\hat{T})$ *with* $1 < q < 2$ *such that* $(E\hat{\kappa})|_{\hat{e}} = \hat{\kappa}$, $E\hat{\kappa}|_{\partial\hat{F}\backslash\hat{e}} = 0$ *and* $E\hat{\kappa}|_{\partial\hat{T}\backslash\hat{F}} = 0$. *Moreover, the following estimates hold:*

$$\|E\hat{\kappa}\|_{W^{1,q}(\hat{F})} \leq C_1\|\hat{\kappa}\|_{W^{1-1/q,q}(\hat{e})}, \tag{4.21a}$$

$$\|E\hat{\kappa}\|_{W^{1,q}(\hat{T})} \leq C_2\|\hat{\kappa}\|_{W^{1-1/q,q}(\hat{e})}, \tag{4.21b}$$

where $C_1, C_2 > 0$ *are two constants uniform for all* $\hat{T}$.

4.3.2.2 The estimates

The following two lemmas not only show that the operators $\mathbf{\Pi}_L$ and $\mathbf{\Pi}_N$ are well-defined on their respective domains but also give local estimates on the tetrahedra after dilating.

Lemma 4.3.10. *Let $\hat{\mathbf{\Pi}}_L$ be operator given in Definition 4.3.1 defined on $\hat{T}$ (4.11). Under Assumption 4.2.3, there holds*

$$\|\hat{\mathbf{\Pi}}_L \hat{v}\|_{L^2(\hat{T})} \leq C(\|\widehat{\operatorname{curl}}\, \hat{v}\|^2_{L^2(\hat{T})} + \|\hat{v}\|_{H^{1/2+\delta}(\hat{T})}), \tag{4.22}$$

for all $\hat{v} \in H^{1/2+\delta}(\hat{T})$ with $\widehat{\operatorname{curl}}\, \hat{v} \in [\mathcal{P}_{r-2}(\hat{T}^{wf})]^3 \cap H(\widehat{\operatorname{div}}, \hat{T})$.

Proof. We bound the corresponding non-zero functionals appearing on the right-hand side of Definition 4.3.1. We start with the the following estimates, which follow from Hölder's inequality and inverse estimates (2.5b) that hold since $\widehat{\operatorname{curl}}\, \hat{v}$ is a piecewise polynomial.

$$\begin{aligned}
|\int_{\hat{e}} [\![\widehat{\operatorname{curl}}\, \hat{v} \cdot \hat{t}]\!]_{\hat{e}} \hat{\kappa}\, ds| &\leq C\|\widehat{\operatorname{curl}}\, \hat{v}\|_{L^2(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{e})} && \forall \hat{\kappa} \in L^2(\hat{e}),\ \forall \hat{e} \in \Delta_1(\hat{T}^{wf}), \hat{e} \subset \hat{F} \\
|\int_{\hat{F}} (\widehat{\operatorname{curl}}_{\hat{F}} \hat{v}_{\hat{F}}) \hat{\kappa}\, dA| &\leq C\|\widehat{\operatorname{curl}}\, \hat{v}\|_{L^2(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{F})} && \forall \hat{\kappa} \in L^2(\hat{F}),\ \forall \hat{F} \in \Delta_2(\hat{T}), \\
|\int_{\hat{T}} \widehat{\operatorname{curl}}\, \hat{v} \cdot \hat{\kappa}\, dx| &\leq \|\widehat{\operatorname{curl}}\, \hat{v}\|_{L^2(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{T})} && \forall \hat{\kappa} \in [L^2(\hat{T})]^3.
\end{aligned}$$

We now bound the remaining functionals coming from the right-hand side of (4.14a) by adopting the technique developed in the proof of [6, Lemma 4.7]. Let $\hat{e} \in \Delta_1(\hat{T})$ and let $\hat{F} \in \Delta_2(\hat{T})$ have $\hat{e}$ as an edge. We choose p such that $2 < p < \frac{2}{1-\delta}$ in order to apply Lemma 4.3.8. Let $\hat{\kappa} \in \mathcal{P}_{r-3}(\hat{e})$, and let $E\hat{\kappa} \in W^{1,p'}(\hat{T})$ be as in Lemma 4.3.9 (with $q = p' < 2$, the Hölder conjugate of p). Integration by parts and using $E\hat\kappa|_{\hat e} = \hat\kappa$ and $E\hat\kappa|_{\partial\hat F\backslash\hat e} = 0$ gives

$$\int_{\hat e} (\hat v \cdot \hat t)\hat\kappa \, ds = \int_{\hat F} (\widehat{\operatorname{curl}}\, \hat v) \cdot \hat n \ (E\hat\kappa) \, d\hat A + \int_{\hat F} (\hat v \times \hat n) \cdot (\widehat{\operatorname{grad}}\, E\hat\kappa) \, d\hat A.$$

An additional integration by parts, using $E\hat\kappa|_{\partial\hat T\backslash\hat F} = 0$, Hölders inequality, estimate (4.21b), an inverse estimate (2.5a) on $\hat T^{wf}$ and the shape regularity of $\hat T^{wf}$ gives

$$\begin{aligned} \left| \int_{\hat F} (\widehat{\operatorname{curl}}\, \hat v) \cdot \hat n \ (E\hat\kappa) \, d\hat A \right| &= \left| \int_{\hat T} \widehat{\operatorname{curl}}\, \hat v \cdot \widehat{\operatorname{grad}}\, E\hat\kappa \, d\hat x \right| \\ &\le C \|\widehat{\operatorname{curl}}\, \hat v\|_{L^p(\hat T)} \|\hat\kappa\|_{W^{1-1/p',p'}(\hat e)} \\ &\le C \|\widehat{\operatorname{curl}}\, \hat v\|_{L^2(\hat T)} \|\hat\kappa\|_{W^{1-1/p',p'}(\hat e)}. \end{aligned}$$

Using Hölder's inequality, Lemma 4.3.8, and (4.21b) we obtain

$$\begin{aligned} \left| \int_{\hat F} (\hat v \times \hat n) \cdot (\widehat{\operatorname{grad}} \ E\hat\kappa) \ d\hat A \right| &\le \|\hat v \times \hat n\|_{L^p(\hat F)} \|E\hat\kappa\|_{W^{1,p'}(\hat F)} \\ &\le C \|\hat v\|_{H^{1/2+\delta}(\hat T)} \|\hat\kappa\|_{W^{1-1/p',p'}(\hat e)}. \end{aligned}$$

Combining the above estimates and using Definition 4.3.1 with Assumption 4.2.3 yields the result (4.22). ■

We now derive a similar estimate for $\hat{\mathbf{\Pi}}_N$.

Lemma 4.3.11. *Let $\hat{\mathbf{\Pi}}_N$ be the operator given in Definition 4.3.2 defined on $\hat T$ (4.11). Under Assumption 4.2.3, there holds with a uniform constant C for all $\hat T$,*

$$\|\hat{\mathbf{\Pi}}_N \hat p\|_{L^2(\hat T)} \le C(\|\widehat{\operatorname{div}}\, \hat p\|_{L^2(\hat T)} + \|\hat p\|_{H^{1/2+\delta}(\hat T)}) \quad \forall \hat p \in H^{1/2+\delta}(\hat T) \cap H(\widehat{\operatorname{div}}, \hat T). \quad (4.23)$$

Proof. Using the estimate in Lemma 4.3.8, the Cauchy-Schwarz inequality, along with $\widehat{\mathrm{div}}\,\hat{p} \in L^2(\hat{T})$, we obtain

$$\begin{aligned}
\Big| \int_{\hat{F}} (\hat{p} \cdot \hat{n}_{\hat{F}}) \hat{\kappa}\, d\hat{A} \Big| &\leq C \|\hat{p}\|_{H^{1/2+\delta}(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{F})} && \forall \hat{\kappa} \in L^2(\hat{F}),\ \forall \hat{F} \in \Delta_2(\hat{T}), \\
\Big| \int_{\hat{T}} (\widehat{\mathrm{div}}\,\hat{p}) \hat{\kappa}\, d\hat{x} \Big| &\leq \|\widehat{\mathrm{div}}\,\hat{p}\|_{L^2(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{T})} && \forall \hat{\kappa} \in L^2(\hat{T}), \\
\Big| \int_{\hat{T}} \hat{p} \cdot \hat{\kappa}\, d\hat{x} \Big| &\leq \|\hat{p}\|_{H^{1/2+\delta}(\hat{T})} \|\hat{\kappa}\|_{L^2(\hat{T})} && \forall \hat{\kappa} \in [L^2(\hat{T})]^3.
\end{aligned}$$

These estimates, combined with Lemma 2.5.4, Definition 4.3.2, and Assumption 4.2.3 yield

$$\|\hat{\mathbf{\Pi}}_N \hat{p}\|_{L^2(\hat{T})} \leq C(\|\widehat{\mathrm{div}}\,\hat{p}\|_{L^2(\hat{T})} + \|\hat{p}\|_{H^{1/2+\delta}(\hat{T})}),$$

where C is a uniform constant for all $\hat{T}$. ■

4.3.3 Proofs of Theorems 4.3.5–4.3.6

In order to prove these theorems, we transfer the results for $\hat{T}$ back to T. We start with the proof of Theorem 4.3.6.

Proof of Theorem 4.3.6. We first prove (4.19). Let $p \in [H^{1/2+\delta}(\Omega)]^3 \cap H(\mathrm{div}, \Omega)$ and set $\rho = (\mathrm{div}\, \mathbf{\Pi}_N p - \Pi_W \mathrm{div} p) \in V^3_{r-3}(\mathcal{T}_h^{wf})$. First, by (4.15c), (4.16a) and Stokes theorem, we have on each $T \in \mathcal{T}_h$,

$$\int_T \rho\, dx = \int_T \mathrm{div}(\mathbf{\Pi}_N p - p)\, dx = \int_{\partial T} (\mathbf{\Pi}_N p - p) \cdot n\, dA = 0,$$

where we use that the constant functions are in $\mathcal{P}_{r-2}(F^{\mathrm{ct}})$. Next, for any $\kappa \in$

$\mathring{V}^3_{r-3}(T^{wf})$, we use (4.15d) and (4.16b) to obtain

$$\int_T \rho\kappa\, dx = \int_T \operatorname{div}(\mathbf{\Pi}_N p - p)\kappa\, dx = 0.$$

Thus $\rho = 0$ by (2.25), and so (4.19) holds.

Next we prove the bound (4.20). For any $T \in \mathcal{T}_h$, with $\hat{T}$ defined in (4.11) and $\hat{p}(\hat{x}) = p(h_T\hat{x})$, $\forall \hat{x} \in \hat{T}$, it is easy to check that $\widehat{\mathbf{\Pi}_N p} = \hat{\mathbf{\Pi}}_N \hat{p}$ by Definition 4.3.2 and Lemma 2.5.4. With Lemma 4.3.11 and Proposition 4.3.7, we have:

$$\begin{aligned}
\|\mathbf{\Pi}_N p\|^2_{L^2(T)} &= h_T^3 \|\widehat{\mathbf{\Pi}_N p}\|^2_{L^2(\hat{T})} = h_T^3 \|\hat{\mathbf{\Pi}}_N \hat{p}\|^2_{L^2(\hat{T})} \\
&\le C h_T^3 (\|\widehat{\operatorname{div}}\, \hat{p}\|^2_{L^2(\hat{T})} + \|\hat{p}\|^2_{H^{1/2+\delta}(\hat{T})}) \\
&= C h_T^3 (\|\widehat{\operatorname{div}}\, \hat{p}\|^2_{L^2(\hat{T})} + |\hat{p}|^2_{H^{1/2+\delta}(\hat{T})} + \|\hat{p}\|^2_{L^2(\hat{T})}) \\
&\le C (h_T^2 \|\operatorname{div} p\|^2_{L^2(T)} + h_T^{1+2\delta} |p|^2_{H^{1/2+\delta}(T)} + \|p\|^2_{L^2(T)}),
\end{aligned} \tag{4.24}$$

where the constant C is independent of h_T.

Then by Lemma 4.2.5, (4.24) and the inverse estimate (2.5a) , we have

$$\begin{aligned}
\|\mathbf{\Pi}_N p - p\|^2_{L^2(T)} &\le 2\|\mathbf{\Pi}_N p - \boldsymbol{I}_h^{div} p\|^2_{L^2(T)} + 2\|\boldsymbol{I}_h^{div} p - p\|^2_{L^2(T)} \\
&= 2\|\mathbf{\Pi}_N (\boldsymbol{I}_h^{div} p - p)\|^2_{L^2(T)} + 2\|\boldsymbol{I}_h^{div} p - p\|^2_{L^2(T)} \\
&\le C\big(h_T^2 \|\operatorname{div} \boldsymbol{I}_h^{div} p\|^2_{L^2(T)} + h_T^2 \|\operatorname{div} p\|^2_{L^2(T)} \\
&\qquad + h_T^{1+2\delta} |\boldsymbol{I}_h^{div} p - p|^2_{H^{\frac{1}{2}+\delta}(T)} + \|\boldsymbol{I}_h^{div} p - p\|^2_{L^2(T)}\big) \\
&\le C(h_T^2 \|\nabla \boldsymbol{I}_h^{div} p\|^2_{L^2(T)} + h_T^2 \|\operatorname{div} p\|^2_{L^2(T)} + h_T^{1+2\delta} \|p\|^2_{H^{\frac{1}{2}+\delta}(\omega(T))}) \\
&\le C\big(h_T^{1+2\delta} \|\boldsymbol{I}_h^{div} p\|^2_{H^{\frac{1}{2}+\delta}(T)} + h_T^{1+2\delta} \|p\|^2_{H^{\frac{1}{2}+\delta}(\omega(T))} + h_T^2 \|\operatorname{div} p\|^2_{L^2(T)}\big) \\
&\le C\big(h_T^{1+2\delta} \|p\|^2_{H^{\frac{1}{2}+\delta}(\omega(T))} + h_T^2 \|\operatorname{div} p\|^2_{L^2(T)}\big).
\end{aligned}$$

Summing this result over all tetrahedra $T \in \mathcal{T}_h$ gives (4.20).

Finally, if $p \in [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{div}, \Omega)$, then it easily follows from (4.15c) that $\mathbf{\Pi}_N p \cdot n = 0$ on $\partial\Omega$, which implies $\mathbf{\Pi}_N p \in \mathscr{V}^n_{r-2}$. ■

Next we turn our attention to Theorem 4.3.5. To this end, we first state and prove an intermediate result for the operator $\mathbf{\Pi}_L$.

Lemma 4.3.12. *Under Assumption 4.2.3, the operator* $\mathbf{\Pi}_L : H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \to \mathcal{L}^1_{r-1}(\mathcal{T}_h^{wf})$ *defined in Definition 4.3.1 satisfies*

$$\mathrm{curl}\ \mathbf{\Pi}_L \tau = \mathrm{curl}\ \tau \qquad \forall \tau \in H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3. \tag{4.25}$$

Moreover, the following bound holds for any $\tau \in H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3$,

$$\|\mathbf{\Pi}_L \tau\|_{L^2(\Omega)} \le C\big(\|\tau\|_{L^2(\Omega)} + h^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\Omega)} + h\|\,\mathrm{curl}\,\tau\|_{L^2(\Omega)}\big). \tag{4.26}$$

Finally, if $\tau \in H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl}, \Omega)$ *then* $\mathbf{\Pi}_L \tau \in \mathcal{L}^t_{r-1}$.

Proof. We first prove (4.25). Set $\rho = \mathrm{curl}\ \mathbf{\Pi}_L \tau - \mathrm{curl}\ \tau \in \mathscr{V}^2_{r-2}(\mathcal{T}_h^{wf})$. Then by directly using the definition of $\mathbf{\Pi}_L$, we see that ρ vanishes on the DOFs (2.24a)–(2.24b),(2.24d)–(2.24e), and

$$\int_F (\rho \cdot n_F)\kappa \, dA = 0 \qquad \forall \kappa \in \mathring{V}^2_{r-2}(F^{\mathrm{ct}}),$$

where we used the identity $\mathrm{curl}_F\, \tau_F = \mathrm{curl}\ \tau \cdot n_F$. Next, by Stokes Theorem

$$\begin{aligned}\int_F \rho \cdot n_F \, dA &= \int_F \mathrm{curl}_F (\mathbf{\Pi}_L \tau - \tau)_F \, dA \\ &= \sum_{e \in \Delta_1(F)} \int_e (\mathbf{\Pi}_L \tau - \tau) \cdot t_e ds = 0.\end{aligned}$$

Since the difference between $V^2_{r-2}(F^{\mathrm{ct}})$ and $\mathring{V}^2_{r-2}(F^{\mathrm{ct}})$ is the space of the constant functions on F, we have

$$\int_F (\rho \cdot n_F)\kappa \, dA = 0 \qquad \forall \kappa \in V_{r-2}^2(F^{\mathrm{ct}}),$$

and so ρ vanishes on all the DOFs (2.24). We thus conclude $\rho \equiv 0$ by Lemma 2.5.4, and therefore (4.25) is satisfied.

We now prove (4.26). For any $T \in \mathcal{T}_h$, with $\hat{T}$ defined in (4.11) and $\hat{\tau}(\hat{x}) = \tau(h_T \hat{x})$, $\forall \hat{x} \in \hat{T}$, it is easy to check that $\widehat{\mathbf{\Pi}_L \tau} = \hat{\mathbf{\Pi}}_L \hat{\tau}$ by Definition 4.3.1 and Lemma 2.5.4. With Lemma 4.3.10 and Proposition 4.3.7, for any $T \in \mathcal{T}_h$, we have:

$$\begin{aligned}\|\mathbf{\Pi}_L \tau\|_{L^2(T)}^2 = h_T^3 \|\hat{\mathbf{\Pi}}_L \hat{\tau}\|_{L^2(\hat{T})}^2 &\le C h_T^3 (\|\widehat{\mathrm{curl}}\, \hat{\tau}\|_{L^2(\hat{T})}^2 + \|\hat{\tau}\|_{H^{1/2+\delta}(\hat{T})}^2) \\ &= C h_T^3 (\|\widehat{\mathrm{curl}}\, \hat{\tau}\|_{L^2(\hat{T})}^2 + \|\hat{\tau}\|_{L^2(\hat{T})}^2 + |\hat{\tau}|_{H^{1/2+\delta}(\hat{T})}^2) \\ &\le C (h_T^2 \|\mathrm{curl}\, \tau\|_{L^2(T)}^2 + \|\tau\|_{L^2(T)}^2 + h_T^{1+2\delta} |\tau|_{H^{1/2+\delta}(T)}^2),\end{aligned} \tag{4.27}$$

where the constant C is independent of h_T. Then summing up all the tetrahedra gives the bound (4.26).

Finally, we will show that if $\tau \in H^D(\mathrm{curl}\,, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl}, \Omega)$, then $\mathbf{\Pi}_L \tau \in \mathcal{L}_{r-1}^t$. Since $\mathbf{\Pi}_L \tau \in \mathcal{L}_{r-1}^1(\mathcal{T}_h^{wf})$, we only need to show $\mathbf{\Pi}_L \tau \times n = 0$ on $\partial\Omega$. Note that $\tau \times n_F = 0$ and $\mathrm{curl}\, \tau \cdot n_F = 0$ on F for all $F \in \Delta_2(\mathcal{T}_h)$ with $F \subset \partial\Omega$. Let F be a boundary face and let $e \in \Delta_1(F)$. Then for all $\kappa \in \mathcal{P}_{r-3}(e)$, recalling $E\kappa$ in Lemma 4.3.9, we use integration by parts to get

$$\begin{aligned}\int_e (\mathbf{\Pi}_L \tau \cdot t_e)\kappa \, ds &= \int_e (\tau \cdot t_e)\kappa \; ds \\ &= \int_F (\mathrm{curl}\, \tau) \cdot n_F \, E\kappa \, dA + \int_F (\tau \times n_F) \cdot \mathrm{grad}\; E\kappa \, dA = 0.\end{aligned}$$

Therefore, because $\mathbf{\Pi}_L \tau$ vanishes at the vertices of F, $(\mathbf{\Pi}_L \tau \cdot t_e)|_e = 0$ and hence $\mathrm{curl}_F\, (\mathbf{\Pi}_L \tau)_F \in \mathring{V}_{r-2}^2(F^{\mathrm{ct}})$. By (4.14d) we get $\mathrm{curl}_F\, (\mathbf{\Pi}_L \tau)_F = 0$. Using the exactness on Clough-Tocher splits (2.20f), we know $(\mathbf{\Pi}_L\tau)_F \in \mathrm{grad}_F\ \mathring{S}_r^0(F^{\mathrm{ct}})$ which after applying (4.14i) shows $(\mathbf{\Pi}_L\tau)_F = 0$ on F. Since $F \subset \partial\Omega$ was arbitrary, we conclude that $\mathbf{\Pi}_L\tau \times n = 0$ on $\partial\Omega$ and thus $\mathbf{\Pi}_L\tau \in \mathcal{L}_{r-1}^t$. ■

We finish with the proof of Theorem 4.3.5.

Proof of Theorem 4.3.5. The commuting property (4.17) easily follows from (4.25). Indeed,

$$
\begin{aligned}
\mathrm{curl}\ \mathbf{\Pi}_V\tau &= \mathrm{curl}(\boldsymbol{I}_h^{curl}\tau) + \mathrm{curl}(\mathbf{\Pi}_L(\tau - \boldsymbol{I}_h^{curl}\tau)) \\
&= \mathrm{curl}(\boldsymbol{I}_h^{curl}\tau) + \mathrm{curl}(\tau - \boldsymbol{I}_h^{curl}\tau) \\
&= \mathrm{curl}\ \tau.
\end{aligned}
$$

Since $\boldsymbol{I}_h^{curl}\tau \in \mathcal{L}_{r-1}^t$ if $\tau \in H^D(\mathrm{curl}\,,\Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl},\Omega)$, with Lemma 4.3.12, we have $\mathbf{\Pi}_V\tau \in \mathcal{L}_{r-1}^t$.

To prove (4.18) we use Lemma 4.2.5 and Lemma 4.3.12 to obtain

$$
\begin{aligned}
\|\mathbf{\Pi}_V\tau - \tau\|_{L^2(\Omega)} &\le \|\boldsymbol{I}_h^{curl}\tau - \tau\|_{L^2(\Omega)} + \|\mathbf{\Pi}_L(\boldsymbol{I}_h^{curl}\tau - \tau)\|_{L^2(\Omega)} \\
&\le C\big(\|\boldsymbol{I}_h^{curl}\tau - \tau\|_{L^2(\Omega)} + h^{1/2+\delta}|\boldsymbol{I}_h^{curl}\tau - \tau|_{H^{1/2+\delta}(\Omega)} \\
&\qquad + h\|\,\mathrm{curl}(\boldsymbol{I}_h^{curl}\tau)\|_{L^2(\Omega)} + h\|\,\mathrm{curl}\,\tau\|_{L^2(\Omega)}\big) \\
&\le C\big(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\nabla\boldsymbol{I}_h^{curl}\tau\|_{L^2(\Omega)} + h\|\,\mathrm{curl}\,\tau\|_{L^2(\Omega)}\big) \\
&\le C\big(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\,\mathrm{curl}\,\tau\|_{L^2(\Omega)}\big),
\end{aligned}
\tag{4.28}
$$

where we used the inverse inequality (2.5a).

Finally, we will show that if $\tau \in H^D(\mathrm{curl}\,,\Omega) \cap [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\mathrm{curl},\Omega)$, then

$\boldsymbol{\Pi}_V \tau \in \mathcal{L}^t_{r-1}$. Similar to the proof for $\boldsymbol{\Pi}_L$, we only need to check the boundary conditions. Since $\tau \times n = 0$ on $\partial\Omega$, by Lemma 4.2.5, $\boldsymbol{I}_h^{curl} \tau \times n = 0$ on $\partial\Omega$ and by Lemma 4.3.12, $\boldsymbol{\Pi}_L \tau \times n = 0$ on $\partial\Omega$. Therefore, $\boldsymbol{\Pi}_L \boldsymbol{I}_h^{curl} \tau \times n = 0$ on $\partial\Omega$ and thus,

$$\boldsymbol{\Pi}_V \tau \times n = \boldsymbol{I}_h^{curl} \tau \times n + \boldsymbol{\Pi}_L \tau \times n - \boldsymbol{\Pi}_L \boldsymbol{I}_h^{curl} \tau \times n = 0 \quad \text{on } \partial\Omega.$$

This gives $\boldsymbol{\Pi}_V \tau \in \mathcal{L}^t_{r-1}$. ■

4.4 Application to the Maxwell eigenvalue problem

In this section, we apply the convergence theory established in Section 4.1.1 and the properties of the two Fortin-like projections to show that quadratic (or higher) Lagrange finite element on Worsey-Farin meshes lead to convergent approximations of the Maxwell eigenvalue problem (1.1). First, we require the following proposition.

Proposition 4.4.1. *The domain Ω is contractible, then the sequence* (2.26a) *and* (4.9) *are exact sequences. In particular,*

$$\operatorname{curl} \mathcal{L}^t_{r-1} = \boldsymbol{\Psi}_{r-2} := \mathbf{ker}(\mathscr{V}^n_{r-2}, \operatorname{div}).$$

This result follows from the Bogovskii operator in [32, Theorem 4.9] and the projections $\boldsymbol{\Pi}_L$, $\boldsymbol{\Pi}_N$; see Appendix A for the proof.

In Assumption 4.1.2, set $\mathscr{V}_h = \mathcal{L}^t_{r-1}$, $\mathscr{Q}_h = \boldsymbol{\Psi}_{r-2}$ and

$$\mathscr{V}^t(\boldsymbol{\Psi}_{r-2}) := \{\tau \in \mathscr{V}^t : \operatorname{curl} \tau \in \boldsymbol{\Psi}_{r-2}\}.$$

Then Theorems 4.3.5–4.3.6 lead to the following results:

Corollary 4.4.2. *The projection* $\mathbf{\Pi}_V : \mathscr{V}^t(\mathbf{\Psi}_{r-2}) \to \mathcal{L}^t_{r-1}$ *satisfies*

$$\begin{aligned}
\operatorname{curl} \mathbf{\Pi}_V \tau &= \operatorname{curl} \tau && \forall \tau \in \mathscr{V}^t(\mathbf{\Psi}_{r-2}),\\
\|\mathbf{\Pi}_V \tau - \tau\|_{L^2(\Omega)} &\le C\big(h^{1/2+\delta}\|\tau\|_{H^{1/2+\delta}(\Omega)} + h\|\operatorname{curl} \tau\|_{L^2(\Omega)}\big) && \forall \tau \in \mathscr{V}^t(\mathbf{\Psi}_{r-2}).
\end{aligned}$$

Furthermore, the L^2*-orthogonal projection* $\mathbb{P}_\Psi : [L^2(\Omega)]^3 \to \mathbf{\Psi}_{r-2}$ *satisfies*

$$\|\mathbb{P}_\Psi p - p\|_{L^2(\Omega)} \le Ch^{1/2+\delta}\|\operatorname{curl} p\|_{L^2(\Omega)} \qquad \forall p \in H(\operatorname{curl}, \Omega) \cap \mathring{H}(\operatorname{div}^0, \Omega).$$

Proof. Let $\delta \in (0, \frac{1}{2}]$ be the constant in the embedding result of Proposition 2.1.5. The results for the operator $\mathbf{\Pi}_V$ is then a direct consequence of Theorem 4.3.5 since $\mathscr{V}^t(\mathbf{\Psi}_{r-2}) \subset H^D(\operatorname{curl}, \Omega) \cap [H^{1/2+\delta}(\Omega)]^3$.

Next, we prove the estimate for $\mathbb{P}_\Psi$. Since we have

$$H(\operatorname{curl}, \Omega) \cap \mathring{H}(\operatorname{div}^0, \Omega) \subset [H^{1/2+\delta}(\Omega)]^3 \cap \mathring{H}(\operatorname{div}, \Omega),$$

we use Theorem 4.3.6 to obtain $\operatorname{div} \mathbf{\Pi}_N p = \Pi_W \operatorname{div} p = 0$. Thus, $\mathbf{\Pi}_N p \in \mathbf{\Psi}_{r-2}$. Consequently, we have the estimate for $\mathbb{P}_\Psi$:

$$\begin{aligned}
\|\mathbb{P}_\Psi p - p\|_{L^2(\Omega)} &\le \|\mathbf{\Pi}_N p - p\|_{L^2(\Omega)} \le Ch^{1/2+\delta}\|p\|_{H^{1/2+\delta}(\Omega)}\\
&\le Ch^{1/2+\delta}(\|\operatorname{curl} p\|_{L^2(\Omega)} + \|\operatorname{div} p\|_{L^2(\Omega)}) = Ch^{1/2+\delta}\|\operatorname{curl} p\|_{L^2(\Omega)},
\end{aligned}$$

where we used Proposition 2.1.5 and Lemma 2.1.6. ∎

Corollary 4.4.2 tells us $(\mathcal{L}^t_{r-1}, \mathbf{\Psi}_{r-2})$ satisfy Assumption 4.1.2 and now with Theorem 4.1.3, we have the final result.

Corollary 4.4.3. *Let $\mathscr{V}_h = \mathcal{L}^t_{r-1}$ and $\mathscr{Q}_h = \mathbf{\Psi}_{r-2}$. Consider the problem (4.1) with the nonzero eigenvalues $0 < \lambda^{(1)} \leq \lambda^{(2)} \leq \ldots$ and the problem (4.2) with the nonzero eigenvalues $0 < \lambda^{(1)}_h \leq \lambda^{(2)}_h \leq \ldots$. Then, for any fixed i, $\lim_{h\to 0} \lambda^{(i)}_h = \lambda^{(i)}$.*

4.5 Numerical experiments

In this section we provide numerical experiments which support our theoretical work. All the computations were carried out using FEniCS [5]. We consider the domain $\Omega = (0,\pi)^3$, so that the exact eigenvectors of problem (1.1) (with non-zero eigenvalues) are of the following form:

$$u = \begin{pmatrix} a_1 \cos(k_1 x)\sin(k_2 y)\sin(k_3 z) \\ a_2 \sin(k_1 x)\cos(k_2 y)\sin(k_3 z) \\ a_3 \sin(k_1 x)\sin(k_2 y)\cos(k_3 z) \end{pmatrix}, \quad k_j \in \mathbb{N}\cup\{0\},\ k_1+k_2+k_3 \geq 2, \tag{4.29}$$

where the coefficient a_1, a_2, a_3 are determined by the divergence-free equation: $a_1 k_1 + a_2 k + a_3 k_3 = 0$ and the non-zero eigenvalues are of the form $\eta^2 = \lambda = k_1^2 + k_2^2 + k_3^2$. We can compute the first few eigenvalues explicitly: 2 (with multiplicity 3), 3 (with multiplicity 2), 5 (with multiplicity 6), 6 (with multiplicity 6).

We compute problem (1.2) using three different choices of $\mathscr{V}_h$: (i) linear and quadratic Lagrange finite element on mesh without Worsey-Farin refinement, (ii) linear Lagrange finite element space on Worsey-Farin meshes and (iii) quadratic Lagrange finite element space on Worsey-Farin meshes.

Table 4.1 states the first 13 computed eigenvalues using linear and quadratic elements on a mesh without Worsey-Farin refinement with $h = \pi/8$. The numerics clearly indicate that the discrete eigenvalues are poor approximations with $O(1)$ errors. Likewise, numerical experiments using the linear Lagrange finite element space on Worsey-Farin meshes lead to computed eigenvalues that are far from the exact solution (cf. Table 4.2).

Finally, we report the computed eigenvalues of method (1.2) using quadratic Lagrange elements on Worsey-Farin meshes and report the first 13 non-zero eigenvalues with $h = 1/6$ in Table 4.3. As expected from the theoretical results, this scenario leads to accurate approximate eigenvalues. In addition the right column in Table 4.3 lists the errors of the first computed (non-zero) eigenvalue and indicates converges with at least cubic rate.

The left table provides the result of first 13 non-zero eigenvalues with $h = \pi/6$. The right table provides the error and the rate of the first non-zero eigenvalue with the mesh size from $\pi/5$ to $\pi/9$ with computed rates.

1. P_1 and P_2 Lagrange finite element space on a mesh without Worsey-Farin refinement In this case, both P_1 and P_2 Lagrange finite element space can not provide satisfying results (**Table 4.1**) as expected.

Our theory also works for non-convex domains, and in the two-dimensional case, numerical experiments presented in [19] show convergent eigenvalues for the L-shaped domain with Powell-Sabin splits. In the three-dimensional case, we conducted preliminary numerical experiments on the Fichera corner domain $\Omega = (-1, 1)^3 \backslash [-1, 0]^3$. We observe that the second and subsequent eigenvalues converge well, but due to limited computational resources, with only the coarse mesh used, we cannot see asymptotic convergence rates for the first eigenvalue. As the numerical results are not inclusive, we do not present them here.

i	λ_h	$\|\lambda - \lambda_h\|$
1	2.0610	6.10 $\times 10^{-2}$
2	2.0610	6.10 $\times 10^{-2}$
3	2.0774	7.74 $\times 10^{-2}$
4	2.0900	9.10 $\times 10^{-1}$
5	2.0900	9.10 $\times 10^{-1}$
6	2.1506	2.85
7	2.1506	2.85
8	2.2698	2.73
9	2.2698	2.73
10	2.2910	2.71
11	2.3304	2.67
12	2.3514	3.65
13	2.3514	3.65

i	λ_h	$\|\lambda - \lambda_h\|$
1	2.6699	6.67 $\times 10^{-1}$
2	2.6766	6.77 $\times 10^{-1}$
3	2.7369	7.34 $\times 10^{-1}$
4	2.7510	2.49 $\times 10^{-1}$
5	2.7510	2.49 $\times 10^{-1}$
6	2.7615	2.24
7	2.7615	2.24
8	2.7782	2.22
9	2.7782	2.22
10	2.7941	2.21
11	2.8244	2.18
12	2.8244	3.17
13	2.8855	3.11

Table 4.1: The case of linear (left) and quadratic (right) Lagrange finite element space on a mesh without Worsey-Farin refinement, $h = \pi/8$.

i	λ_h	$\|\lambda - \lambda_h\|$
1	2.6763	6.77 $\times 10^{-1}$
2	2.6763	6.77 $\times 10^{-1}$
3	2.6775	6.78 $\times 10^{-1}$
4	2.6775	3.23 $\times 10^{-1}$
5	2.7112	2.89 $\times 10^{-1}$
6	2.7825	2.22
7	2.7825	2.22
8	2.7860	2.21
9	2.8254	2.17
10	2.8878	2.11
11	2.9440	2.06
12	2.9440	3.56
13	2.9881	3.01

Table 4.2: The case of linear Lagrange finite element space on a Worsey-Farin mesh, $h = \pi/10$.

i	λ_h	$\lvert\lambda - \lambda_h\rvert$
1	2.000121	1.21 $\times 10^{-4}$
2	2.000243	2.43 $\times 10^{-4}$
3	2.000243	2.43 $\times 10^{-4}$
4	3.000741	7.41 $\times 10^{-4}$
5	3.000741	7.41 $\times 10^{-4}$
6	5.001307	1.31 $\times 10^{-3}$
7	5.001307	1.31 $\times 10^{-3}$
8	5.001862	1.86 $\times 10^{-3}$
9	5.002382	2.38 $\times 10^{-3}$
10	5.002913	2.91 $\times 10^{-3}$
11	5.002913	2.91 $\times 10^{-3}$
12	6.001822	1.82 $\times 10^{-3}$
13	6.002854	2.85 $\times 10^{-3}$

h	$\lvert\lambda^{(1)} - \lambda_h^{(1)}\rvert$	Rate
$\pi/5$	1.95 $\times 10^{-4}$	/
$\pi/6$	1.21 $\times 10^{-4}$	2.62
$\pi/7$	7.40 $\times 10^{-5}$	3.19
$\pi/8$	4.64 $\times 10^{-5}$	3.48
$\pi/9$	3.03 $\times 10^{-5}$	3.62

Table 4.3: The case of P_2 finite element space with Worsey-Farin meshes

4.6 Conclusion

In this chapter, we studied and justified the convergence theory of the three-dimensional Maxwell eigenvalue problem using Lagrange finite element spaces on Worsey-Farin splits. Although we only focus on Worsey-Farin splits in this paper, we provide a framework which may be applied to other finite element spaces if they form an exact de Rham sequence (e.g., [25]) and there exists bounded commuting projections. In particular, if such spaces satisfy Assumption 4.1.2, then the framework presented here shows that they form a convergence finite element method for the Maxwell eigenvalue problem. While there are many candidates of such spaces, one would need to construct bounded commuting projections which is typically non-trivial and done on a case-by-case basis.

Chapter Five

Elasticity Exact Sequences in 2D

Elasticity sequences in two dimensions have been discussed widely by many researchers; see [9, 29]. To be self-contained and as an auxiliary chapter to our three-dimensional results in Chapter 6, we do some modifications and summarize the following two-dimensional results here. To be more precise, the results for Clough-Tocher splits and Powell-Sabin splits defined in Section 2.2.1.1 will be covered in this chapter.

Recall that Worsey-Farin splits induce face degrees of freedom on the Clough-Tocher splits and they are also the natural three dimensional analog of Powell-Sabin splits. Investigations on Clough-Tocher splits and Powell-Sabin splits will give us a inspiration of the results on Worsey-Farin splits. The main results of this chapter are very similar to the ones found [29] (with spaces slightly different) and can be proved with the techniques there. However, to be self-contained, we provide the main proofs.

In this chapter, $T \in \mathcal{T}_h$ is a triangle in a two-dimensional triangular mesh. Recall that $\mathcal{T}_h^{\mathrm{ct}}$ and $\mathcal{T}_h^{\mathrm{ps}}$ denote the Clough-Tocher refinement and the Powell-Sabin refinement of $\mathcal{T}_h$, respectively. In addition, given a face $F \in \Delta_2(\mathcal{T}_h)$ of the (unrefined) triangulation $\mathcal{T}_h$ in three-dimensional spaces, F^{ct} denotes its Clough-Tocher refinement with respect to the split point m_F (arising from the Worsey-Farin refinement of $\mathcal{T}_h$). The local results of T^{ct} could be directly applied to F^{ct}.

5.1 Local discrete sequences

With T^* denoting T^{ct} or T^{ps}, the elasticity sequences on Clough-Tocher splits and Powell-Sabin splits utilize the following spaces:

$$\mathring{Q}^1_{\mathrm{inc},r}(T^*) := \{v \in \mathring{\mathsf{L}}^1_r(T^*) \otimes \mathbb{V}_2 : \ \mathrm{curl}_F\, v \in \mathring{V}^1_{\mathrm{curl},r-1}(T^*)\}, \tag{5.1a}$$

$$\mathring{Q}^{1,s}_{\mathrm{inc},r}(T^*) := \{\mathrm{sym}(u) : u \in \mathring{Q}^1_{\mathrm{inc},r}(T^*)\}, \tag{5.1b}$$

$$Q^1_r(T^*) := \{u \in V^1_{\mathrm{div},r}(T^*) \otimes \mathbb{V}_2 : \mathrm{skew}(u) = 0\}, \tag{5.1c}$$

$$\tilde{Q}^1_r(T^*) := \{u \in \mathsf{L}^1_r(T^*) \otimes \mathbb{V}_2 : \mathrm{skew}(u) = 0\} \subset Q^1_r(T^*), \tag{5.1d}$$

$$\mathring{Q}^2_r(T^*) := \{u \in V^2_r(T^*) : u \perp \mathcal{P}_1(T)\}, \tag{5.1e}$$

$$\mathring{Q}^2_{\mathrm{rig},r}(T^{\mathrm{ct}}) := \{u \in V^2_r(T^{\mathrm{ct}}) \otimes \mathbb{V}_2 : u \perp \mathsf{R}_2\}. \tag{5.1f}$$

$$\mathring{Q}^2_{\mathrm{rig},r}(T^{\mathrm{ps}}) := \{u \in \mathcal{V}^2_r(T^{\mathrm{ps}}) \otimes \mathbb{V}_2 : u \perp \mathsf{R}_2\}. \tag{5.1g}$$

The next theorem is the main result of the section, where exact local discrete elasticity sequences are presented on two kinds of two-dimensional splits.

Theorem 5.1.1. *Let $r \geq 3$. The following elasticity sequences are exact.*

$$0 \longrightarrow \mathring{S}^0_{r+1}(T^*) \otimes \mathbb{V}_2 \xrightarrow{\varepsilon_F} \mathring{Q}^{1,s}_{\mathrm{inc},r}(T^*) \xrightarrow{\mathrm{inc}_F} \mathring{Q}^2_{r-2}(T^*) \longrightarrow 0, \tag{5.2a}$$

$$\mathcal{P}_1(T) \xrightarrow{\subset} S^0_r(T^*) \xrightarrow{\mathrm{airy}_F} Q^1_{r-2}(T^*) \xrightarrow{\mathrm{div}_F} V^2_{r-3}(T^*) \otimes \mathbb{V}_2 \longrightarrow 0. \tag{5.2b}$$

$$0 \longrightarrow \mathring{S}^0_r(T^*) \xrightarrow{\mathrm{airy}_F} \mathring{Q}^1_{r-2}(T^*) \xrightarrow{\mathrm{div}_F} \mathring{Q}^2_{\mathrm{rig},r-3}(T^*) \longrightarrow 0. \tag{5.2c}$$

Note that when $T^* = T^{\mathrm{ct}}$, the sequence (5.2a) is exactly the sequence in [9, Page 13]. To prove Theorem 5.1.1, we require a few intermediate results. First, we state a corollary of Theorem 2.5.1.

Corollary 5.1.2. *Let $r \geq 2$. The following sequence is exact.*

$$0 \to \mathring{S}^0_r(T^{\mathrm{ct}}) \otimes \mathbb{V}_2 \xrightarrow{\mathrm{grad}_F} \mathring{Q}^1_{\mathrm{inc},r-1}(T^{\mathrm{ct}})$$
$$\xrightarrow{\mathrm{curl}_F} \mathring{V}^1_{\mathrm{curl},r-2}(T^{\mathrm{ct}}) \cap \left(\mathring{V}^2_{r-2}(T^{\mathrm{ct}}) \otimes \mathbb{V}_2\right) \to 0. \tag{5.3a}$$
$$0 \to \mathring{S}^0_r(T^{\mathrm{ps}}) \otimes \mathbb{V}_2 \xrightarrow{\mathrm{grad}_F} \mathring{Q}^1_{\mathrm{inc},r-1}(T^{\mathrm{ps}})$$
$$\xrightarrow{\mathrm{curl}_F} \mathring{V}^1_{\mathrm{curl},r-2}(T^{\mathrm{ps}}) \cap \left(\mathring{\mathcal{V}}^2_{r-2}(T^{\mathrm{ps}}) \otimes \mathbb{V}_2\right) \to 0. \tag{5.3b}$$

Proof. This directly follows from the exactness of the sequence (2.20f) and (2.20g). ∎

Lemma 5.1.3. *The following sequences are exact for $r \geq 2$:*

$$\begin{bmatrix} \mathring{S}^0_{r+1}(T^*) \otimes \mathbb{V}_2 \\ \mathring{\mathsf{L}}^0_r(T^*) \end{bmatrix} \xrightarrow{\begin{bmatrix}\mathrm{grad}_F & \mathrm{skew}\end{bmatrix}} \mathring{Q}^1_{\mathrm{inc},r}(T^*) \otimes \mathbb{V}_2 \xrightarrow{\mathrm{inc}_F} V^2_{r-2}(T^*) \xrightarrow{\begin{bmatrix}\int_F^{\perp} \\ \int_F\end{bmatrix}} \begin{bmatrix} \mathbb{V}_2 \\ \mathbb{R} \end{bmatrix}, \tag{5.4}$$

$$\begin{bmatrix} \mathbb{R} \\ \mathbb{V}_2 \end{bmatrix} \xrightarrow{\begin{bmatrix}\subset & x^{\perp}\cdot\end{bmatrix}} S^0_{r+1}(T^*) \xrightarrow{\mathrm{airy}_F} V^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2 \xrightarrow{\begin{bmatrix}\mathrm{skew} \\ \mathrm{div}_F\end{bmatrix}} \begin{bmatrix} V^2_{r-1}(T^*) \\ V^2_{r-2}(T^*) \otimes \mathbb{V}_2 \end{bmatrix}, \tag{5.5}$$

$$\mathring{S}^0_{r+1}(T^*) \xrightarrow{\mathrm{airy}_F} \mathring{V}^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2 \xrightarrow{\begin{bmatrix}\mathrm{skew} \\ \mathrm{div}_F\end{bmatrix}} \begin{bmatrix} Z^2_{r-2}(T^*) \\ \mathring{V}^2_{r-1}(T^*) \otimes \mathbb{V}_2 \end{bmatrix}. \tag{5.6}$$

Here, $\int_F^{\perp} u := \int_F x^{\perp} u \, dx$, $\int_F^{\perp} \cdot u := \int_F x^{\perp} \cdot u \, dx$. and

$$Z^2_{r-2}(T^*) = \begin{cases} V^2_{r-2}(T^*), & T^* = T^{\mathrm{ct}} \\ \mathring{\mathcal{V}}^2_{r-2}(T^*), & T^* = T^{\mathrm{ps}} \end{cases}.$$

Proof. Using (2.18c) and the identity $\int_F^{\perp} \operatorname{curl}_F u = \int_F \tau u$ for any $u \in \mathring{V}^1_{\mathrm{curl},r-1}(T^*)$, we find that the following sequence commutes:

$$\begin{tikzcd}
\mathring{S}^0_{r+1}(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{grad}_F"] & \mathring{Q}^1_{\mathrm{inc},r}(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{curl}_F"] & \mathring{V}^1_{\mathrm{curl},r-1}(T^*) \arrow[r, "\int_F"] & \mathbb{V}_2 \\
\mathring{\mathsf{L}}^0_r(T^*) \arrow[r, "\operatorname{grad}_F"] \arrow[ur, "\operatorname{skew}"] & \mathring{V}^1_{\mathrm{curl},r-1}(T^*) \arrow[r, "\operatorname{curl}_F"] \arrow[ur, "\tau"] & V^2_{r-2}(T^*) \arrow[r, "\int_F"] \arrow[ur, "\int_F^{\perp}"] & \mathbb{R},
\end{tikzcd} \tag{5.7}$$

Moreover, the transpose operator τ from $\mathring{V}^1_{\mathrm{curl},r-1}(T^*)$ to $\mathring{V}^1_{\mathrm{curl},r-1}(T^*)$ is a bijection, and the top and bottom sequences in (5.7) are exact by Corollary 5.1.2 and Theorem 2.5.1, respectively. Using the identity $\operatorname{inc}_F = \operatorname{curl}_F \tau \operatorname{curl}_F$ and Proposition 2.3.1, we conclude that 5.4 is exact.

Likewise, using the identity $\operatorname{div}_F \tau u = \operatorname{skew} \operatorname{rot}_F u$ for any $u \in (\mathsf{L}^0_r(T^*) \otimes \mathbb{V}_2)$ and $\operatorname{rot}_F x^{\perp} = \tau$, we find that the following sequence commutes:

$$\begin{tikzcd}
\mathbb{R} \arrow[r, "\subset"] & S^0_{r+1}(T^*) \arrow[r, "\operatorname{rot}_F"] & \mathsf{L}^1_r(T^*) \arrow[r, "\operatorname{div}_F"] & V^2_{r-1}(T^*) \\
\mathbb{V}_2 \arrow[r, "\subset"] \arrow[ur, "x^{\perp}\cdot"] & \mathsf{L}^0_r(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{rot}_F"] \arrow[ur, "\tau"] & V^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{div}_F"] \arrow[ur, "\operatorname{skew}"] & V^2_{r-2}(T^*) \otimes \mathbb{V}_2.
\end{tikzcd} \tag{5.8}$$

The top and bottom sequences in (5.8) are exact by Corollary 2.5.2. We then find that (5.5) is exact by Proposition 2.3.1, using the identity $\operatorname{airy}_F = \operatorname{rot}_F \tau \operatorname{rot}_F$.

For the case with boundary conditions, the proof is similar to the previous two by constructing the following commuting diagrams:

$$\begin{tikzcd}
\mathring{S}^0_{r+1}(T^*) \arrow[r, "\operatorname{rot}_F"] & \mathring{\mathsf{L}}^1_r(T^*) \arrow[r, "\operatorname{div}_F"] & Z^2_{r-2}(T^*) \arrow[r, "\int_F"] & \mathbb{R} \\
\mathring{\mathsf{L}}^0_r(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{rot}_F"] \arrow[ur, "\tau"] & \mathring{V}^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2 \arrow[r, "\operatorname{div}_F"] \arrow[ur, "\operatorname{skew}"] & \mathring{V}^2_{r-1}(T^*) \otimes \mathbb{V}_2 \arrow[r] & 0.
\end{tikzcd} \tag{5.9}$$

■

Now we are ready to prove Theorem 5.1.1:

Proof. (i) Proof of (5.2a): from the definitions of the discrete spaces and operators, we see that (5.2a) is a sequence, so we only need to show exactness.

Let $v \in \mathring{Q}^2_{r-2}(T^*)$. Then since $v \perp \mathcal{P}_1(T)$, we have $\int_F v = 0$ and $\int_F^\perp v = 0$. By the exactness of (5.4), there exists $u \in \mathring{Q}^1_{\mathrm{inc},r}(T^*)$ such that $\mathrm{inc}_F\, u = v$. But by (2.18b), we have $\mathrm{inc}_F \operatorname{sym} u = \mathrm{inc}_F\, u = v$. Thus we found a function $w = \operatorname{sym} u \in \mathring{Q}^{1,s}_{\mathrm{inc},r}(T^*)$ such that $\mathrm{inc}_F\, w = v$.

Next, let $u \in \mathring{Q}^{1,s}_{\mathrm{inc},r}(T^*)$ with $\mathrm{inc}_F\, u = 0$. Then $u = \operatorname{sym}(z)$ for some $z \in \mathring{Q}^1_{\mathrm{inc},r}(T^*)$ and $\mathrm{inc}_F\, z = 0$ due to (2.18b). By exactness of (5.4), we have $z = \mathrm{grad}_F w + \operatorname{skew} s$ for some $w \in \mathring{S}^0_{r+1}(T^*) \otimes \mathbb{V}_2$ and $s \in \mathring{\mathsf{L}}^0_r(T^*)$. Then $u = \operatorname{sym}(z) = \varepsilon_F(w) - \operatorname{sym}(\operatorname{skew} s) = \varepsilon_F(w)$.

(ii) Proof of (5.2b): again, it is easy to see that (5.2b) is a sequence, so we only need to show exactness.

Let $v \in V^2_{r-3}(T^*) \otimes \mathbb{V}_2$. Then by the exactness of (5.5), we have $u \in V^1_{\mathrm{div},r-2}(T^*) \otimes \mathbb{V}_2$ such that $\mathrm{div}_F\, u = v$ and $\operatorname{skew} u = 0$ and thus making $u \in Q^1_{r-2}(T^*)$.

Next, let $u \in Q^1_{r-2}(T^*)$ with $\mathrm{div}_F\, u = 0$. Then again using (5.5) and $\operatorname{skew} u = 0$, there exists $z \in S^0_r(T^*)$ such that $\mathrm{airy}_F\, z = u$.

Finally, for any $u \in S^0_r(T^*)$ with $\mathrm{airy}_F\, u = 0$, we have $u = w + x^\perp \cdot s$ for some $w \in \mathbb{R}$, $s \in \mathbb{V}_2$, and x a point on T. Therefore, $u \in \mathcal{P}_1(T)$.

(iii) Proof of (5.2c): first of all, it is still easy to see that (5.2c) is a sequence, so we only need to show exactness.

Let $v \in \mathring{Q}^2_{\mathrm{rig},r-3}(T^*)$ defined by (5.1f) and (5.1g). Then by the exactness of (5.6), we have a $u \in \mathring{V}^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2$ such that $\mathrm{div}_F\, u = v$. Furthermore, since $v \perp \mathsf{R}_2$, for any $c \in \mathbb{R}$, we have

$$\int_T (\mathrm{skew}\, u)c\; dx = -\int_T cx^\perp \cdot (\mathrm{div}_F u)\; dx = 0.$$

Therefore, $\mathrm{skew}\, u \in \mathring{Z}^2_{r-2}(T^*)$ and by the exactness of (2.21f) and (2.21g), there exists $m \in \mathring{\mathsf{L}}^1_{r-1}(T^*)$ such that $\mathrm{div}_F m = \mathrm{skew}\, u$. Let $w = u - \mathrm{rot}_F(\tau m) \in \mathring{V}^1_{\mathrm{div},r-1}(T^*) \otimes \mathbb{V}_2$, then

$$\mathrm{skew}\, w = \mathrm{skew}\, u - \mathrm{skew}\,\mathrm{rot}_F(\tau^{-1} m) = \mathrm{skew}\, u - \mathrm{div}_F m = 0.$$

■

5.2 Dimension counting

We summarize the dimension counts of the discrete elasticity spaces on the Clough-Tocher split and Powell-Sabin split here. All these dimensions are computed with Corollary 5.1.2, Theorem 5.1.1, Lemma 5.1.3, Table 2.1 and Table 2.2.

Table 5.1: Dimension counts of the elasticity spaces with respect to the Clough–Tocher and the Powell–Sabin split.

	Clough–Tocher	Powell–Sabin
$\mathring{Q}^1_{\mathrm{inc},r}$	$6r^2 - 12r + 4$	$9r^2 - 15r + 6$
$\mathring{Q}^{1,s}_{\mathrm{inc},r}$	$\frac{9}{2}r^2 - \frac{21}{2}r + 3$	$9r^2 - 21r + 9$
Q^1_r	$\frac{3}{2}(3r^2 + 5r + 2)$	$9r^2 - 3r - 3$
$\tilde{Q}^1_r$	$\frac{3}{2}(3r^2 - 3r + 2)$	—
$\mathring{Q}^2_r$	$r^2 + 3r$	$3r^2 + 9r + 3$
$\mathring{Q}^2_{\mathrm{rig},r}$	$3r^2 + 9r + 3$	$6r^2 + 18r + 3$

Chapter Six

Elasticity Exact Sequences on Worsey-Farin Splits

The discrete elasticity sequences in 3D can not only be used to design stable finite element methods with mixed formulation using the last two spaces in the sequence [7, 13, 51], but also be used to characterize the kernel of differential operators, which will be used to design robust solvers and preconditioners in the framework of kernel-capturing subspace correction methods [39, 54]. However, as mentioned in the introduction, it is not easy to construct the entire discrete elasticity sequences in 3D. It was not until 2020 that the first 3D discrete elasticity sub-complex was established in [27] with the Alfeld splits. And then, the case for generic meshes was constructed in [26].

In this chapter, we work on the discrete elasticity sequences for the Worsey-Farin splits in 3D with BGG method and provide the degrees of freedom of the involved spaces. Compared with previous work, there will be three advantages:

- The lowest polynomial order required for stress space U^2 is linear polynomials while quadratics [27] and quartics [25] needed.
- No vertex or edge degrees of freedom need for the stress space U^2.
- No extra continuity for the vertices of the original mesh.

Although the work in [27] provides us with a lot of guidance for constructing the discrete sequence on Worsey-Farin splits, there are significant new difficulties that arise. The main challenge stems from the construction of face degrees of freedom. Unlike Alfeld splits, Worsey-Farin triangulations induce a Clough-Tocher split on each face of the original, unrefined triangulation. Therefore, the discrete two-dimensional elasticity sequences developed in the previous chapter with respect to Clough-Tocher splits will play an essential role here.

This chapter is organized as follows. In the next section, we present the construction of the discrete elasticity sequence locally on Worsey-Farin splits with the dimensions of each spaces involved. This leads to our main contribution in Section 6.2 where we present the degrees of freedom of the discrete spaces in the elasticity sequence with commuting projections in Section 6.3. We finish this chapter with the analogous global discrete elasticity sequence in Section 6.4 and state some conclusions and future directions in Section 6.5.

6.1 The local discrete elasticity sequence in 3D

6.1.1 Elasticity sequence for stresses with weakly imposed symmetry

In this section we will again apply Proposition 2.3.1 to the de-Rham sequences on Worsey-Farin splits. This gives rise to a derived sequence useful for analyzing mixed methods for elasticity with weakly imposed stress symmetry. From this intermediate step, an elasticity sequence with strong symmetry will readily follow. We start with the following definition and lemma.

Definition 6.1.1. Let $\mu \in \mathring{\mathsf{L}}_1^0(T^{wf})$ be the unique continuous, piecewise linear polynomial that vanishes on ∂T and takes the value 1 at the incenter of T.

Lemma 6.1.2.

(a) The map $\Xi : \mathsf{L}_r^1(T^{wf}) \otimes \mathbb{V} \to \mathsf{L}_r^2(T^{wf}) \otimes \mathbb{V}$ *is a bijection.*

(b) The following inclusions hold $\operatorname{vskw}(V_{r-2}^2(T^{wf}) \otimes \mathbb{V}) \subset V_{r-2}^3(T^{wf}) \otimes \mathbb{V}$ *and* $\operatorname{vskw}(\mathring{\mathcal{V}}_{r-2}^2(T^{wf}) \otimes \mathbb{V}) \subset \mathcal{V}_{r-2}^3(T^{wf}) \otimes \mathbb{V}$.

(c) The mappings $\mathrm{vskw}: V^2_{r-2}(T^{wf}) \otimes \mathbb{V} \to V^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ *and* $\mathrm{vskw}: \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} \to \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ *are both surjective.*

Proof. Both (1) and (2) are trivial to verify and hence we only prove (3). Let $v \in V^3_{r-2}(T^{wf}) \otimes \mathbb{V}$. By the exactness of (2.22e), there exists a function $z \in \mathsf{L}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{div} z = v$. Since Ξ is a bijection from $\mathsf{L}^1_{r-2}(T^{wf}) \otimes \mathbb{V}$ to $\mathsf{L}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$, we have $q = \Xi^{-1} z \in \mathsf{L}^1_{r-2}(T^{wf}) \otimes \mathbb{V}$. Thus, by setting $w = \operatorname{curl} q \in V^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ we obtain

$$2\mathrm{vskw}(w) = 2\mathrm{vskw}\,\operatorname{curl}(q) = 2\mathrm{vskw}\,\operatorname{curl}(\Xi^{-1}z) = \operatorname{div}\Xi(\Xi^{-1}z) = v,$$

where we used (2.17a). We conclude $\mathrm{vskw}: V^2_{r-2}(T^{wf}) \otimes \mathbb{V} \to V^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ is a surjection.

We now prove the analogous result with boundary condition. Let $v \in \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$, and let $M \in \mathbb{M}_{3\times 3}$ be a constant matrix such that $\int_T 2\mathrm{vskw} M = \frac{1}{\int_T \mu} \int_T v$. Then, by taking $\tilde{w} = \mu M$, we have $\tilde{w} \in \mathring{\mathcal{V}}^2_1(T^{wf}) \otimes \mathbb{V}$ with $\int_T 2\mathrm{vskw}\tilde{w} = \int_T v$. Therefore, we have $v - 2\,\mathrm{vskw}(\tilde{w}) \in \mathring{\mathcal{V}}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ and the exactness of (2.22f) yields the existence of $z \in \mathring{\mathsf{L}}^2_{r-1}(T^{wf}) \otimes \mathbb{V}$, such that $\operatorname{div} z = v - 2\mathrm{vskw}(\tilde{w})$. Let $q = \Xi^{-1} z \in \mathring{\mathsf{L}}^1_{r-1}(T^{wf}) \otimes \mathbb{V}$, and from (2.22d), we have $w := \operatorname{curl}(q) + \tilde{w} \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$. Finally, using (2.17a)

$$2\mathrm{vskw}(w) = 2\mathrm{vskw}\,\operatorname{curl}(\Xi^{-1}z) + 2\mathrm{vskw}(\tilde{w}) = \operatorname{div} z + 2\mathrm{vskw}(\tilde{w}) = v.$$

This shows the surjectivity of $\mathrm{vskw}: \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} \to \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$, thus completing the proof. ∎

Using the sequences (2.22c)-(2.22f) and the two identities (2.17a)-(2.17b), we construct the following commuting diagrams:

$$
\begin{tikzcd}
S^0_{r+1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & S^1_r(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathsf{L}^2_{r-1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & V^3_{r-2}(T^{wf}) \otimes \mathbb{V} \arrow[r] & 0 \\
S^0_r(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"'] & \mathsf{L}^1_{r-1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\Xi"'] & V^2_{r-2}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\mathrm{vskw}"'] & V^3_{r-3}(T^{wf}) \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}
\tag{6.1}
$$

$$
\begin{tikzcd}
\mathring{S}^0_{r+1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] & \mathring{S}^1_r(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] & \mathring{\mathsf{L}}^2_{r-1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] & \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\int"] & 0 \\
\mathring{S}^0_r(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{grad}"] \arrow[ur, "-\mathrm{mskw}"'] & \mathring{\mathsf{L}}^1_{r-1}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{curl}"] \arrow[ur, "\Xi"'] & \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} \arrow[r, "\mathrm{div}"] \arrow[ur, "2\mathrm{vskw}"'] & \mathring{V}^3_{r-3}(T^{wf}) \otimes \mathbb{V} \arrow[r] & 0
\end{tikzcd}
\tag{6.2}
$$

Note that the top sequence of (6.2) is slightly different from (2.22f), as the mean-value constraint is not imposed on $\mathcal{V}_{r-2}(T^{wf}) \otimes \mathbb{V}$. This is due to the surjective property of the mapping $\mathrm{vskw} : (\mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}) \to \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ established in Lemma 6.1.2.

Theorem 6.1.3. *The following sequences are exact for any $r \geq 3$:*

$$
\begin{bmatrix} S^0_{r+1}(T^{wf}) \otimes \mathbb{V} \\ S^0_r(T^{wf} \otimes \mathbb{V}) \end{bmatrix} \xrightarrow{[\mathrm{grad}, -\mathrm{mskw}]} S^1_r(T^{wf}) \otimes \mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}} V^2_{r-2}(T^{wf}) \otimes \mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} V^3_{r-2}(T^{wf}) \otimes \mathbb{V} \\ V^3_{r-3}(T^{wf}) \otimes \mathbb{V} \end{bmatrix}. \tag{6.3}
$$

$$
\begin{bmatrix} \mathring{S}^0_{r+1}(T^{wf}) \otimes \mathbb{V} \\ \mathring{S}^0_r(T^{wf} \otimes \mathbb{V}) \end{bmatrix} \xrightarrow{[\mathrm{grad}, -\mathrm{mskw}]} \mathring{S}^1_r(T^{wf}) \otimes \mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}} \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V} \\ \mathring{V}^3_{r-3}(T^{wf}) \otimes \mathbb{V} \end{bmatrix}. \tag{6.4}
$$

Moreover, the last operator in (6.3) is surjective.

Proof. Lemma 6.1.2 tells us that $\Xi : \mathsf{L}^1_{r-1}(T^{wf}) \otimes \mathbb{V} \to \mathsf{L}^2_{r-1}(T^{wf}) \otimes \mathbb{V}$ is a bijection. With the exactness of (2.22c)-(2.22f) for $r \geq 3$ and Proposition 2.3.1, we see that these two sequences are exact. The surjectivity the last map is guaranteed by Proposition 2.3.1 and Lemma 6.1.2. ∎

6.1.2 Elasticity sequence with strong symmetry

Now we are ready to describe the local discrete elasticity sequence on Worsey-Farin splits. The discrete elasticity sequences with strong symmetry are formed by the following spaces:

$$
\begin{aligned}
&U^0_{r+1}(T^{wf}) = S^0_{r+1}(T^{wf}) \otimes \mathbb{V}, \qquad \mathring{U}^0_{r+1}(T^{wf}) = \mathring{S}^0_{r+1}(T^{wf}) \otimes \mathbb{V},\\
&U^1_r(T^{wf}) = \{\mathrm{sym}(u) : u \in S^1_r(T^{wf}) \otimes \mathbb{V}\},\\
&\mathring{U}^1_r(T^{wf}) = \{\mathrm{sym}(u) : u \in \mathring{S}^1_r(T^{wf}) \otimes \mathbb{V}\},\\
&U^2_{r-2}(T^{wf}) = \{u \in V^2_{r-2}(T^{wf}) \otimes \mathbb{V} : \mathrm{skw}\, u = 0\},\\
&\mathring{U}^2_{r-2}(T^{wf}) = \{u \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} : \mathrm{skw}\, u = 0\},\\
&U^3_{r-3}(T^{wf}) = V^3_{r-3}(T^{wf}) \otimes \mathbb{V}, \qquad \mathring{U}^3_{r-3}(T^{wf}) = \{u \in V^3_{r-3}(T^{wf}) \otimes \mathbb{V} : u \perp \mathsf{R}\},
\end{aligned}
$$

where we recall R, defined in (2.15), is the space of rigid body displacements.

Theorem 6.1.4. *The following two sequences are discrete elasticity sequences and are exact for $r \geq 3$:*

$$\mathsf{R} \to U^0_{r+1}(T^{wf}) \xrightarrow{\varepsilon} U^1_r(T^{wf}) \xrightarrow{\mathrm{inc}} U^2_{r-2}(T^{wf}) \xrightarrow{\mathrm{div}} U^3_{r-3}(T^{wf}) \to 0, \tag{6.5}$$

and

$$0 \to \mathring{U}^0_{r+1}(T^{wf}) \xrightarrow{\varepsilon} \mathring{U}^1_r(T^{wf}) \xrightarrow{\mathrm{inc}} \mathring{U}^2_{r-2}(T^{wf}) \xrightarrow{\mathrm{div}} \mathring{U}^3_{r-3}(T^{wf}) \to 0. \tag{6.6}$$

Proof. We first show that (6.5) is a sequence. In order to do this, it suffices to show the operators map the space they are acting on into the subsequent space. To this end, let $u \in U^0_{r+1}(T^{wf})$, then by (2.22e) we have $\mathrm{grad}\,(u) \in S^1_r(T^{wf}) \otimes \mathbb{V}$. Hence, $\varepsilon(u) = \mathrm{sym}\,\mathrm{grad}\,(u) \in U^1_r(T^{wf})$. Now let $u \in U^1_r(T^{wf})$ which implies that $u = \mathrm{sym}(w)$ with $w \in S^1_r(T^{wf}) \otimes \mathbb{V}$. Thus by (2.17c) we have $\mathrm{curl}\,\Xi^{-1}\mathrm{curl}\, u =$

$\operatorname{curl}\Xi^{-1}\operatorname{curl} w \in V^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ and $\operatorname{skw}(u) = 0$ due to (2.17d). Therefore, there holds $\operatorname{curl}\Xi^{-1}\operatorname{curl}(u) \in U^2_{r-2}(T^{wf})$. Finally, for any $u \in U^2_{r-2}(T^{wf}) \subset V^2_{r-2}(T^{wf}) \otimes \mathbb{V}$, $\operatorname{div} u \in V^3_{r-3}(T^{wf}) \otimes \mathbb{V}$.

Next, we prove exactness of the sequence (6.5). Let $w \in U^3_{r-3}(T^{wf})$ and consider $(0, w) \in [V^3_{r-2}(T^{wf}) \otimes \mathbb{V}] \times [V^3_{r-3}(T^{wf}) \otimes \mathbb{V}]$. Due to the exactness of (6.3) in Theorem 6.1.3, there exists $v \in V^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{div} v = w$ and $2\operatorname{vskw}(v) = 0$. Thus, $v \in U^2_{r-2}(T^{wf})$.

Now let $w \in U^2_{r-2}(T^{wf})$ with $\operatorname{div} w = 0$. Then by the exactness of (6.3), we have the existence of $v \in S^1_r(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{curl}\Xi^{-1}\operatorname{curl} v = w$. Setting $u = \operatorname{sym}(v) \in U^1_r(T^{wf})$ yields $\operatorname{inc} u = w$ by (2.17c).

Finally, let $w \in U^1_r(T^{wf})$ with $\operatorname{inc} w = 0$. Then $w = \operatorname{sym}(v)$ for some $v \in S^1_r(T^{wf}) \otimes \mathbb{V}$ and with (2.17c), $\operatorname{curl}\Xi^{-1}\operatorname{curl} v = \operatorname{curl}\Xi^{-1}\operatorname{curl} w = 0$. Due to the exactness of (6.3), we could find $(u, z) \in [S^0_{r+1}(T^{wf}) \otimes \mathbb{V}] \times [S^0_r(T^{wf}) \otimes \mathbb{V}]$ such that $v = \operatorname{grad} u - \operatorname{mskw}(z)$. Therefore, $\varepsilon(u) = \operatorname{sym}(v) = w$.

We can prove that (6.6) is a sequence and it is exact very similar to above. The main difference is the surjectivity of the last map which we prove now. Let $w \in \mathring{U}^3_{r-3}(T^{wf}) \subset \mathring{V}^3_{r-3} \otimes \mathbb{V}$. Then by the exactness of (2.22d), there exists $v \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{div} v = w$. For any $c \in \mathbb{R}^3$ we have $\operatorname{grad}(c \times x) = \operatorname{mskw} c$ and hence, using integration by parts

$$\begin{aligned}\int_T 2\operatorname{vskw} v \cdot c &= \int_T v : \operatorname{mskw} c = \int_T v : \operatorname{grad}(c \times x) \\ &= -\int_T \operatorname{div} v \cdot (c \times x) = -\int_T w \cdot (c \times x) = 0,\end{aligned}$$

where the last equality uses the fact $w \perp \mathsf{R}$. Therefore, $\operatorname{vskw} v \in \mathring{\mathcal{V}}^3_{r-2}(T^{wf}) \otimes \mathbb{V}$ and by the exactness of (2.22f), we have a $m \in \mathring{\mathsf{L}}^2_{r-1}(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{div} m = 2\operatorname{vskw} v$. Let $u = v - \operatorname{curl}(\Xi^{-1} m) \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$ and we see that $2\operatorname{vskw} u = 2\operatorname{vskw} v - 2\operatorname{vskw}\operatorname{curl}(\Xi^{-1} m) = 0$ by (2.17a). Hence, $u \in \mathring{U}^2_{r-2}(T^{wf})$ and $\operatorname{div} u = w$. ■

When $r \geq 4$, there holds $\mathsf{R} \subset U^3_{r-3}(T^{wf})$, so it is clear that

$$U^3_{r-3}(T^{wf}) = \mathsf{R} \oplus \mathring{U}^3_{r-3}(T^{wf}) \qquad \text{for } r \geq 4. \tag{6.7}$$

On the other hand, when $r = 3$, we need the following lemma for the calculation of dimensions of $\mathring{U}^3_{r-3}(T^{wf})$. Let P_U be the L^2-orthogonal projection onto $U^3_0(T^{wf})$ and let $P_U \mathsf{R} := \{P_U u : \ u \in \mathsf{R}\}$. The proof of the following lemma is provided in the appendix.

Lemma 6.1.5. *It holds,*

$$U^3_0(T^{wf}) = P_U \mathsf{R} \oplus \mathring{U}^3_0(T^{wf}), \tag{6.8}$$

and $\dim P_U \mathsf{R} = \dim \mathsf{R} = 6$.

Using the exactness of the sequences (6.5)–(6.6) along with Table 2.3, we calculate the dimensions of the spaces in the next lemma.

Lemma 6.1.6. *When $r \geq 3$, we have:*

$$\dim U^0_{r+1}(T^{wf}) = 6r^3 + 12r + 12, \qquad \dim \mathring{U}^0_{r+1}(T^{wf}) = 6r^3 - 36r^2 + 66r - 36, \tag{6.9}$$

$$\dim U^1_{r}(T^{wf}) = 12r^3 - 9r^2 + 15r + 6, \quad \dim \mathring{U}^1_{r}(T^{wf}) = 12r^3 - 63r^2 + 87r - 18, \tag{6.10}$$

$$\dim U^2_{r-2}(T^{wf}) = 12r^3 - 27r^2 + 15r, \quad \dim \mathring{U}^2_{r-2}(T^{wf}) = 12r^3 - 45r^2 + 33r + 12, \tag{6.11}$$

$$\dim U^3_{r-3}(T^{wf}) = 6r^3 - 18r^2 + 12r, \quad \dim \mathring{U}^3_{r-3}(T^{wf}) = 6r^3 - 18r^2 + 12r - 6. \tag{6.12}$$

Proof. By Lemma 6.1.2 and the rank-nullity theorem, we have

$$\begin{aligned}
\dim U^2_{r-2}(T^{wf}) &= \dim \ker(V^2_{r-2}(T^{wf}) \otimes \mathbb{V}, \mathrm{vskw}) \\
&= \dim V^2_{r-2}(T^{wf}) \otimes \mathbb{V} - \dim V^3_{r-2}(T^{wf}) \otimes \mathbb{V} \\
&= (6r^3 - 9r^2 + 3r) \times 3 - 2r(r+1)(r-1) \times 3 = 12r^3 - 27r^2 + 15r, \\
\dim \mathring{U}^2_{r-2}(T^{wf}) &= \dim \ker(\mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}, \mathrm{vskw}) \\
&= \dim \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V} - \dim \mathcal{V}^3_{r-2}(T^{wf}) \otimes \mathbb{V} \\
&= (6(r-2)^3 + 21(r-2)^2 + 9(r-2) + 2) \times 3 \\
&\quad - 2((r-2)^3 + 6(r-2)^2 + 5(r-2) + 2) \times 3 \\
&= 18r^3 - 45r^2 - 9r + 60 - (6r^3 - 42r + 48) \\
&= 12r^3 - 45r^2 + 33r + 12.
\end{aligned}$$

The dimensions of $U^0_{r+1}(T^{wf})$, $\mathring{U}^0_{r+1}(T^{wf})$ and $U^3_{r-3}(T^{wf})$ are computed similarly using the dimensions of $S^0_{r+1}(T^{wf})$, $\mathring{S}^0_{r+1}(T^{wf})$ and $V^3_{r-3}(T^{wf})$. Also, using Lemma 6.1.5 when $r = 3$ or (6.7) when $r \geq 4$, we obtain

$$\dim \mathring{U}^3_{r-3}(T^{wf}) = \dim U^3_{r-3}(T^{wf}) - 6.$$

Using the exactness of the sequences (6.5) and (6.6) in Theorem 6.1.4, with the rank-nullity theorem, we have

$$\dim U^1_r(T^{wf}) = \dim U^0_{r+1}(T^{wf}) + \dim U^2_{r-2}(T^{wf}) - \dim U^3_{r-3}(T^{wf}) - \dim \mathsf{R}$$

$$= 12r^3 - 9r^2 + 15r + 6,$$

$$\dim \mathring{U}_r^1(T^{wf}) = \dim \mathring{U}_{r+1}^0(T^{wf}) + \dim \mathring{U}_{r-2}^2(T^{wf}) - \dim \mathring{U}_{r-3}^3(T^{wf})$$

$$= 12r^3 - 63r^2 + 87r - 18.$$

■

6.1.3 An equivalent characterization of $U_r^1(T^{wf})$ and $\mathring{U}_r^1(T^{wf})$

In order to find the local degrees of freedom of $U_r^1(T^{wf})$ and $\mathring{U}_r^1(T^{wf})$, we first provide a characterization of these two spaces.

Theorem 6.1.7. *We have the following equivalent definitions of $U_r^1(T^{wf})$ and $\mathring{U}_r^1(T^{wf})$:*

$$U_r^1(T^{wf}) = \{u \in H^1(T;\mathbb{S}) : u \in \mathcal{P}_r(T^{wf};\mathbb{S}), (\operatorname{curl} u)' \in V_{r-1}^1(T^{wf}) \otimes \mathbb{V}\}, \tag{6.13}$$

$$\mathring{U}_r^1(T^{wf}) = \{u \in \mathring{H}^1(T;\mathbb{S}) : u \in \mathcal{P}_r(T^{wf};\mathbb{S}), (\operatorname{curl} u)' \in \mathring{V}_{r-1}^1(T^{wf}) \otimes \mathbb{V}, \tag{6.14}$$
$$\operatorname{inc}(u) \in \mathring{\mathcal{V}}_{r-2}^2(T^{wf}) \otimes \mathbb{V}\}.$$

Proof. Let the right-hand side of (6.13) and (6.14) be denoted by M_r and $\mathring{M}_r$, respectively. If $u \in U_r^1(T^{wf})$, then $u = \operatorname{sym}(z)$ for some $z \in S_r^1(T^{wf}) \otimes \mathbb{V}$, so (2.17e), (2.17b) and Definition 2.4.2 give

$$(\operatorname{curl} u)' = \Xi^{-1}\operatorname{curl} u = \Xi^{-1}\operatorname{curl} z + \operatorname{grad} \operatorname{vskw}(z), \tag{6.15}$$

from which we conclude $(\operatorname{curl} u)' \in V_{r-1}^1(T^{wf}) \otimes \mathbb{V}$. This proves the inclusion

$$U_r^1(T^{wf}) \subset M_r. \tag{6.16}$$

Similarly, if $u \in \mathring{U}_r^1(T^{wf})$, then (6.15) for $z \in \mathring{S}_r^1(T^{wf}) \otimes \mathbb{V}$, hence we have $(\operatorname{curl} u)' \in \mathring{V}_{r-1}^1(T^{wf}) \otimes \mathbb{V}$. Moreover, using (2.17c) and the exact sequence 2.22f, we obtain

$$\operatorname{inc}(u) = \operatorname{curl} \Xi^{-1} \operatorname{curl}(u) = \operatorname{curl} \Xi^{-1} \operatorname{curl}(z) \in \operatorname{curl}(\mathring{\mathsf{L}}_{r-1}^1(T^{wf}) \otimes \mathbb{V}) \subset \mathring{\mathcal{V}}_{r-2}^2 \otimes \mathbb{V}.$$

This proves

$$\mathring{U}_r^1(T^{wf}) \subset \mathring{M}_r. \tag{6.17}$$

We continue to prove the reverse inclusion of (6.13). For any $m \in M_r$, let $\sigma = \operatorname{curl}(\operatorname{curl} m)'$ which immediately implies that $\operatorname{div} \sigma = 0$. Moreover, by (2.17e) $\sigma = \operatorname{curl} \Xi^{-1} \operatorname{curl}(m)$ and by (2.17d) $\operatorname{vskw}(\sigma) = 0$. Hence, we have $\sigma \in V_{r-2}^2(T^{wf}) \otimes \mathbb{V}$, and by the exact sequence (6.3) there exists $w \in S_r^1(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{curl} \Xi^{-1} \operatorname{curl}(w) = \sigma$. Therefore, $w - m \in V_r^1(T^{wf}) \otimes \mathbb{V}$ with $\operatorname{curl} \Xi^{-1} \operatorname{curl}(w - m) = 0$ and hence, by the exact sequence (2.22a), there exists $v \in \mathsf{L}_{r+1}^0(T^{wf}) \otimes \mathbb{V}$ such that $\operatorname{grad} v = \Xi^{-1} \operatorname{curl}(w - m)$. Setting $z = m + \operatorname{vskw}(v)$ gives $\operatorname{sym}(z) = m$ and by (2.17b),

$$\operatorname{curl} z = \operatorname{curl} m + \operatorname{curl} \operatorname{mskw} v = \operatorname{curl} m - \Xi \operatorname{grad} v = \operatorname{curl} w \in \mathsf{L}_r^1(T^{wf}) \otimes \mathbb{V}.$$

We conclude

$$M_r \subset U_r^1(T^{wf}). \tag{6.18}$$

The reverse inclusion to prove (6.14) follows similar arguments, using the exact sequence (6.4) and (2.22b) in place of (6.3) and (2.22a), respectively. ∎

6.2 Local degrees of freedom for the elasticity sequence on Worsey-Farin splits

In this section we present degrees of freedom for the discrete spaces arising in the elasticity sequence. We first need to introduce some notation as follows. Recall that T^a is the set of four tetrahedra obtained by connecting the vertices of T with its incenter. For each $K \in T^a$, we denote the local Worsey-Farin splits of K as K^{wf}, i.e.,

$$K^{wf} = \{S \in T^{wf} : \bar{S} \subset \bar{K}\}.$$

Then, similar to the discrete functions spaces on T^{wf} defined in Section 2.5.3, we define spaces on K^{wf} by taking their restriction:

$$\mathsf{L}_r^0(K^{wf}) := \{u|_K : u \in \mathsf{L}_r^0(T^{wf})\}; \qquad S_r^0(K^{wf}) := \{u|_K : u \in S_r^0(T^{wf})\}.$$

Lemma 6.2.1. *Let $T \in \mathcal{T}_h$, and let $F \in \Delta_2(T)$. If $p \in \mathsf{L}_r^0(T^{wf})$ with $p = 0$ on F, then* $\operatorname{grad} p$ *is continuous on F. In particular, the normal derivative $\partial_n p$ is continuous on F. In addition, if $p \in S_r^0(T^{wf})$ with $p = 0$ on F, then* $\operatorname{grad} p|_F \in S_{r-1}^0(F^{\mathrm{ct}}) \otimes \mathbb{V}$ *and in particular, $\partial_n p|_F \in S_{r-1}^0(F^{\mathrm{ct}})$.*

Proof. Let $K \in T^a$ such that $F \in \Delta_2(K)$. Then, since p vanishes on F, we have that $p = \mu q$ on K where $q \in \mathsf{L}_{r-1}^0(K^{wf})$ and μ is the piecewise linear polynomial in Definition 6.1.1. We write $\operatorname{grad} p = \mu \operatorname{grad} q + q \operatorname{grad} \mu$, and since μ vanishes on F and $\operatorname{grad} \mu$ is constant on F, we have $\operatorname{grad} p$ is continuous on F.

Furthermore, if $p \in S_r^0(T^{wf})$, then $p = \mu q$ on K where $q \in S_{r-1}^0(K^{wf})$ because μ is a strictly positive polynomial on K. Hence by the same reasoning as the previous case, $\operatorname{grad} p|_F \in S^0_{r-1}(F^{\mathrm{ct}}) \otimes \mathbb{V}$. ∎

6.2.1 Dofs of U^0 space

Lemma 6.2.2. *A function $u \in U^0_{r+1}(T^{wf})$, with $r \geq 3$, is fully determined by the following DOFs:*

$$u(a), \qquad a \in \Delta_0(T), \qquad 12 \tag{6.19a}$$

$$\operatorname{grad} u(a), \qquad a \in \Delta_0(T), \qquad 36 \tag{6.19b}$$

$$\int_e u \cdot \kappa, \qquad \kappa \in [\mathcal{P}_{r-3}(e)]^3,\ \forall e \in \Delta_1(T), \qquad 18(r-2) \tag{6.19c}$$

$$\int_e \frac{\partial u}{\partial n_e^\pm} \cdot \kappa, \qquad \kappa \in [\mathcal{P}_{r-2}(e)]^3,\ \forall e \in \Delta_1(T), \qquad 36(r-1) \tag{6.19d}$$

$$\int_F \varepsilon_F(u_F) : \varepsilon_F(\kappa), \qquad \kappa \in [\mathring{S}^0_{r+1}(F^{\mathrm{ct}})]^2, \forall F \in \Delta_2(T) \qquad 12r^2 - 36r + 24 \tag{6.19e}$$

$$\int_F [\varepsilon(u)]_{Fn} \cdot \kappa, \qquad \kappa \in \operatorname{grad}_F \mathring{S}^0_{r+1}(F^{\mathrm{ct}}), \forall F \in \Delta_2(T) \qquad 6r^2 - 18r + 12 \tag{6.19f}$$

$$\int_F \partial_n (u \cdot n_F)\kappa, \qquad \kappa \in \mathcal{R}^0_r(F^{\mathrm{ct}}), \forall F \in \Delta_2(T) \qquad 6r^2 - 18r + 12 \tag{6.19g}$$

$$\int_F \partial_n u_F \cdot \kappa, \qquad \kappa \in [\mathcal{R}^0_r(F^{\mathrm{ct}})]^2, \forall F \in \Delta_2(T) \qquad 12r^2 - 36r + 24 \tag{6.19h}$$

$$\int_T \varepsilon(u) : \varepsilon(\kappa), \qquad \kappa \in \mathring{U}^0_{r+1}(T^{wf}), \qquad 6(r-1)(r-2)(r-3), \tag{6.19i}$$

where $\frac{\partial}{\partial n_e^\pm}$ represents two normal derivatives to edge e and $\{n_e^+, n_e^-, t_e\}$ forms an edge-based orthonormal basis of $\mathbb{R}^3$.

Proof. The dimension of $U^0_{r+1}(T^{wf})$ is $6r^3 + 12r + 12$, which is equal to the sum of the given DOFs.

Let $u \in U^0_{r+1}(T^{wf})$ such that it vanishes on the DOFs (6.19). On each edge $e \in \Delta_1(T)$, $u|_e = 0$ by (6.19a)-(6.19c). Furthermore, $\operatorname{grad} u|_e = 0$ by (6.19b) and (6.19d). Hence on any face $F \in \Delta_2(T)$, we have $u_F \in [\mathring{S}^0_{r+1}(F^{\mathrm{ct}})]^2$. Then with DOFs (6.19e), $u_F = 0$ on F. Now with Lemma 6.2.1 applied to $u_F \in S^0_{r+1}(T^{wf}) \otimes \mathbb{V}_2$, we have $\partial_n u_F \in S^0_r(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$. In addition, since $\operatorname{grad} u_F|_{\partial F} = 0$, it follows that $\partial_n u_F \in [\mathcal{R}^0_r(F^{\mathrm{ct}})]^2$ and with (6.19h), we have $\partial_n u_F = 0$.

Using the identity (2.19i), we have $2[\varepsilon(u)]_{Fn} = \partial_n u_F + \operatorname{grad}_F(u \cdot n_F) = \operatorname{grad}_F(u \cdot n_F)$. With $u \cdot n_F \in S^0_{r+1}(F^{\mathrm{ct}})$, we have in (6.19f), $[\varepsilon(u)]_{Fn} = 0$ and thus $u \cdot n_F = 0$ on F. Now similar to u_F, with Lemma 6.2.1 applied to $u \cdot n_F$, we have $\partial_n(u \cdot n_F) \in \mathcal{R}^0_r(F^{\mathrm{ct}})$ and with (6.19g), we have $\partial_n(u \cdot n_F) = 0$.

Since $u|_{\partial T} = 0$, all the tangential derivatives of u vanish. With $\partial_n(u \cdot n_F) = 0$ and $\partial_n u_F = 0$, we conclude that $\operatorname{grad} u|_{\partial T} = 0$. Thus $u \in \mathring{U}^0_{r+1}(T^{wf})$, and (6.19i) shows that u vanishes. ■

6.2.2 Dofs of U^1 space

Before giving the DOFs of the space U^1 we need preliminary results to see the continuity of the functions involved. In the following lemmas, we use the jump operator $[\![\cdot]\!]$ and the set of internal edges of a split face $\Delta^I_1(F^{\mathrm{ct}})$ given in Section 2.2.1.2. The proofs of the next four results are found in the appendix.

Lemma 6.2.3. *Let $\sigma \in V^2_r(T^{wf}) \otimes \mathbb{V}$ with $\operatorname{skw}(\sigma) = 0$. If $n'_F \sigma \ell = 0$ on ∂T for some $\ell \in \mathbb{R}^3$, then $\sigma_{F\ell} \in V^1_{\mathrm{div},r}(F^{\mathrm{ct}})$ on each $F \in \Delta_2(T)$.*

Lemma 6.2.4. *Let $w \in V^1_{r-1}(T^{wf}) \otimes \mathbb{V}$ such that $w' \in V^2_{r-1}(T^{wf}) \otimes \mathbb{V}$. If $w_{Fn} = 0$ on some $F \in \Delta_2(T)$, then we have*

$$[\![t'_e w n_f]\!]_e = 0; \quad [\![s'_e w s_e]\!]_e = 0, \qquad \forall e \in \Delta^I_1(F^{\mathrm{ct}}). \tag{6.20}$$

On the other hand, if $w_{FF} = 0$ on F, then we have

$$[\![t'_e w n_f]\!]_e = 0; \quad [\![t'_e w n_F]\!]_e = 0, \qquad \forall e \in \Delta^I_1(F^{\mathrm{ct}}). \tag{6.21}$$

Lemma 6.2.5. *Let T be a tetrahedron, and let ℓ, m be two tangent vectors to a face $F \in \Delta_2(T)$ such that $\ell \cdot m = 0$ and $\ell \times m = n_F$. Let $u \in \mathsf{L}^1_r(T^{wf}) \otimes \mathbb{V}$ for some $r \geq 0$. If $u_{FF} = 0$ on some $F \in \Delta_2(T)$, then*

$$[\![\ell'(\operatorname{curl} u) m]\!]_e = -[\![\operatorname{grad}_F(u_{Fn} \cdot \ell) \cdot \ell]\!]_e, \qquad \forall e \in \Delta^I_1(F^{\mathrm{ct}}), \tag{6.22}$$

$$[\![\ell'(\operatorname{curl} u) \ell]\!]_e = -[\![\operatorname{grad}_F(u_{Fn} \cdot \ell) \cdot m]\!]_e, \qquad \forall e \in \Delta^I_1(F^{\mathrm{ct}}). \tag{6.23}$$

On the other hand, if $u_{nF} = 0$ on F, then

$$[\![n'_F(\operatorname{curl} u) \ell]\!]_e = [\![(\operatorname{grad}_F u_{nn}) \cdot m]\!]_e, \qquad \forall e \in \Delta^I_1(F^{\mathrm{ct}}). \tag{6.24}$$

Lemma 6.2.6. *Suppose $u \in U^1_r(T^{wf})$ and $w = (\operatorname{curl} u)'$ are such that u_{FF} and w_{Fn} vanish on a face $F \in \Delta_2(T)$. Then $w_{FF} - \operatorname{grad}_F u^{\perp}_{Fn}$ is continuous on F. Furthermore, if $u = \varepsilon(v)$ for some $v \in U^0_{r+1}(T^{wf})$, then the following identity holds:*

$$w_{FF} = [(\operatorname{curl} \varepsilon(v))']_{FF} = \operatorname{grad}_F u^{\perp}_{Fn} + \operatorname{grad}_F(\partial_n v_F \times n_F). \tag{6.25}$$

The following two identities are needed to proceed with our construction and the proofs of these two lemmas are shown in [27, Lemma 5.8].

Lemma 6.2.7. *Let u be a sufficiently smooth matrix-valued function, and let ϕ be a smooth scalar-valued function. Then there holds the following integration-by-parts identity:*

$$\int_F (\mathrm{inc}_F\, u)\, \phi = \int_F u \colon \mathrm{airy}_F(\phi) + \int_{\partial F} (\mathrm{curl}_F\, u) t\, \phi ds + \int_{\partial F} ut \cdot (\mathrm{rot}_F \phi)'. \tag{6.26}$$

Consequently, if $u \in \mathring{Q}^1_{\mathrm{inc},r-1}(F^{\mathrm{ct}})$ is symmetric and $\phi \in \mathcal{P}_1(F)$, then $\int_F (\mathrm{inc}_F\, u)\, \phi = 0$.

Lemma 6.2.8. *Let u be a symmetric matrix-valued function with $[(\mathrm{curl}\, u)']_{FF} t|_{\partial F} = 0$, $u|_{\partial F} = 0$. Let $q \in \mathsf{R}_2$ be defined in (2.16). Then there holds*

$$\int_F (\mathrm{inc}\, u)_{Fn} \cdot q = 0. \tag{6.27}$$

We now define a function space needed for the following lemma. Let $Q_r^{\perp}(F^{\mathrm{ct}})$ be the subspace of $Q_r^1(F^{\mathrm{ct}})$ (cf. (5.1c)) that is $L^2(F)$-orthogonal to $\tilde{Q}_r^1(F^{\mathrm{ct}})$ (cf. (5.1d)). We then have $Q_r^1(F^{\mathrm{ct}}) = Q_r^{\perp}(F^{\mathrm{ct}}) \oplus \tilde{Q}_r^1(F^{\mathrm{ct}})$, and

$$\dim Q_r^{\perp}(F^{\mathrm{ct}}) = \dim Q_r^1(F^{\mathrm{ct}}) - \dim \tilde{Q}_r^1(F^{\mathrm{ct}}). \tag{6.28}$$

Lemma 6.2.9. *A function $u \in U_r^1(T^{wf})$, with $r \geq 3$, is fully determined by the following DOFs:*

$$
\begin{aligned}
&u(a), && a \in \Delta_0(T), && 24 && \text{(6.29a)}\\
&\int_e u \colon \kappa, && \kappa \in \mathrm{sym}[\mathcal{P}_{r-2}(e)]^{3\times 3},\ \forall e \in \Delta_1(T), && 36(r-1) && \text{(6.29b)}\\
&\int_e (\mathrm{curl}\, u)' t_e \cdot \kappa, && \kappa \in [\mathcal{P}_{r-1}(e)]^3,\ \forall e \in \Delta_1(T), && 18r && \text{(6.29c)}\\
&\int_F (\mathrm{inc}\ u)_{FF} \colon \kappa, && \kappa \in Q_{r-2}^{\perp},\ \forall F \in \Delta_2(T), && 12(r-2) && \text{(6.29d)}
\end{aligned}
$$

$$\int_F (\text{inc } u)_{nn}\kappa, \qquad \kappa \in \mathring{Q}^2_{r-2}(F^{\text{ct}}), \forall F \in \Delta_2(T) \qquad 6r^2 - 6r - 12 \tag{6.29e}$$

$$\int_F (\text{inc } u)_{Fn}\cdot\kappa, \qquad \kappa \in V^1_{\text{div},r-2}(F^{\text{ct}})/\mathsf{R}_2, \forall F \in \Delta_2(T) \qquad 12r^2 - 24r \tag{6.29f}$$

$$\int_F u_{FF} : \kappa, \qquad \kappa \in \varepsilon_F([\mathring{S}^0_{r+1}(F^{\text{ct}})]^2), \forall F \in \Delta_2(T), \qquad 12(r^2 - 3r + 2) \tag{6.29g}$$

$$\int_F ([(\operatorname{curl} u)']_{FF} - \operatorname{grad}_F(u^{\perp}_{Fn})) : \kappa,$$

$$\kappa \in \operatorname{grad}_F[(\mathcal{R}^0_r(F^{\text{ct}})]^2, \forall F \in \Delta_2(T), \quad 12(r^2 - 3r + 2) \tag{6.29h}$$

$$\int_F u_{Fn}\cdot\kappa, \qquad \kappa \in \operatorname{grad}_F([\mathring{S}_{r+1}(F^{\text{ct}})]), \forall F \in \Delta_2(T), \quad 6(r^2 - 3r + 2) \tag{6.29i}$$

$$\int_F u_{nn}\kappa, \qquad \kappa \in \mathcal{R}^0_r(F^{\text{ct}}), \forall F \in \Delta_2(T), \qquad 6(r^2 - 3r + 2) \tag{6.29j}$$

$$\int_T \text{inc}(u) : \text{inc}(\kappa), \quad \kappa \in \mathring{U}^1_r(T^{wf}), \qquad 6r^3 - 27r^2 + 21r + 18 \tag{6.29k}$$

$$\int_T \varepsilon(u) : \varepsilon(\kappa), \qquad \kappa \in \mathring{U}^0_{r+1}(T^{wf}), \qquad 6(r-1)(r-2)(r-3). \tag{6.29l}$$

Proof. The dimension of $U^1_r(T^{wf})$ is $12r^3 - 9r^2 + 15r + 6$, which is equal to the sum of the given DOFs. Suppose that all DOFs (6.29) vanish for a $u \in U^1_r(T^{wf})$.

Step 0: Using the DOFs (6.29a, 6.29b) and (6.29c), we conclude

$$u|_e = 0, \quad (\operatorname{curl} u)'t|_e = 0, \qquad \text{for } e \in \Delta_1(T). \tag{6.30}$$

Step 1: We show $\operatorname{inc} u \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$.

By (2.19b) and (6.30), we have

$$0 = n_F'(\operatorname{curl} u)' t = (\operatorname{curl}_F u_{FF}) t \quad \text{on } \partial F \text{ for each } F \in \Delta_2(T).$$

Since u is symmetric and continuous, by (2.19c), we see that $(\operatorname{inc} u)_{nn} = \operatorname{inc}_F u_{FF}$ with $u_{FF} \in \mathring{Q}^{1,s}_{\mathrm{inc},r}(F^{\mathrm{ct}}) \subset \mathring{Q}^{1}_{\mathrm{inc},r}(F^{\mathrm{ct}})$. Thus, the sequence (5.2a) in Theorem 5.1.1 and the DOFs (6.29e) yield

$$(\operatorname{inc} u)_{nn} = 0 \qquad \text{on each } F \in \Delta_2(T). \tag{6.31}$$

Next, Lemma 6.2.3 (with $\ell = n_F$ and $\sigma = \operatorname{inc} u$) shows $(\operatorname{inc} u)_{Fn} \in V^1_{\mathrm{div},r-2}(F^{\mathrm{ct}})$. Therefore using the DOFs (6.29f) and (6.27) in Lemma 6.2.8, we conclude $(\operatorname{inc} u)_{Fn} = 0$.

The identities $(\operatorname{inc} u)_{nn} = 0$ and $(\operatorname{inc} u)_{Fn} = 0$ yield $(\operatorname{inc} u) n_F = 0$. So, by Lemma 6.2.3 (with $\ell = t_1, t_2$), we see that $(\operatorname{inc} u)_{FF} \in V^1_{\mathrm{div},r-2}(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$. In particular, since $(\operatorname{inc} u)_{FF}$ is symmetric, there holds $(\operatorname{inc} u)_{FF} \in Q^1_{r-2}(F^{\mathrm{ct}})$ (cf. (5.1c)). Thus by the DOFs (6.29d) and the definition of $Q^{\perp}_{r-2}(F^{\mathrm{ct}})$ in Section 5.1, we have $(\operatorname{inc} u)_{FF} \in L^1_r(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$. Therefore, we conclude $\operatorname{inc} u \in \mathring{\mathcal{V}}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$.

Step 2: We show $(\operatorname{curl} u)' \in \mathring{V}^1_{r-1}(T^{wf}) \otimes \mathbb{V}$.

Using (6.31) and (2.19c), we have $0 = (\operatorname{inc} u)_{nn} = \operatorname{inc}_F u_{FF}$. Thus by the exact sequence (5.2a) in Theorem 5.1.1, there holds $u_{FF} = \varepsilon_F(\kappa)$ for some $\kappa \in \mathring{S}^0_{r+1}(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$. We then conclude from the DOFs (6.29g) that $u_{FF} = 0$ on each $F \in \Delta_2(F)$. Furthermore by (2.19b), $[(\operatorname{curl} u)']_{Fn} = \operatorname{curl}_F u_{FF} = 0$.

Since $(\operatorname{curl} u)' \in V^1_{r-1}(T^{wf}) \otimes \mathbb{V}$ by Theorem 6.1.7 and from (6.30)

$$[(\operatorname{curl} u)']_{FF} t_e|_e = (\operatorname{curl} u)' t_e|_e = 0, \qquad \text{for all } e \in \Delta_1(T),$$

we have $[(\operatorname{curl} u)']_{FF} \in \mathring{V}^1_{\mathrm{curl},r-1}(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$ on $F \in \Delta_2(T)$. In addition, by the identity $(\operatorname{inc} u)_{Fn} = \operatorname{curl}_F[(\operatorname{curl} u)']_{FF}$ (cf. (2.19d)) and $(\operatorname{inc} u)_{Fn} = 0$ derived in **Step 1**, there exists $\phi \in \mathring{\mathsf{L}}^0_r(F^{\mathrm{ct}}) \otimes \mathbb{V}_2$ such that $\operatorname{grad}_F \phi = [(\operatorname{curl} u)']_{FF}$. With Lemma 6.2.6, we further have $\phi - u^{\perp}_{Fn} \in [\mathcal{R}^0_r(F^{\mathrm{ct}})]^2$. Therefore, using the DOFs (6.29h) we conclude

$$[(\operatorname{curl} u)']_{FF} = \operatorname{grad}_F u^{\perp}_{Fn}. \tag{6.32}$$

Since with (2.19e), we have

$$-\operatorname{curl}_F(u_{Fn})' = \operatorname{tr}_F \operatorname{curl} u = \operatorname{tr}_F(\operatorname{curl} u)' = \operatorname{tr}_F(\operatorname{curl} u)'_{FF}.$$

With (6.32) and (2.14), we have

$$-\operatorname{curl}_F(u_{Fn})' = \operatorname{tr}_F(\operatorname{curl} u)'_{FF} = \operatorname{div}_F u^{\perp}_{Fn} = \operatorname{curl}_F(u_{Fn})',$$

and this implies that $\operatorname{curl}_F(u_{Fn})' = 0$. Since $u_{Fn} \in \mathring{\mathsf{L}}^1_r(F^{\mathrm{ct}})$, the exact sequence (2.20f) yields $u_{Fn} \in \operatorname{grad}_F([\mathring{S}_{r+1}(F^{\mathrm{ct}})])$. Therefore by (6.29i), we have $u_{Fn} = 0$. Now with (6.32) and $u_{Fn} = 0$, we have $[(\operatorname{curl} u)']_{FF} = 0$ and so $(\operatorname{curl} u)' \in \mathring{V}^1_{r-1}(T^{wf}) \otimes \mathbb{V}$.

Step 3: We show $u \in \mathring{H}^1(T; \mathbb{S})$.

From **Step 2**, we already see that $u_{FF} = 0$ and $u_{Fn} = 0$, so we only need to show $u_{nn} = 0$. Since $(\operatorname{curl} u)' \in \mathring{V}^1_{r-1}(T^{wf}) \otimes \mathbb{V}$ with $\operatorname{curl} u \in V^2_{r-1}(T^{wf}) \otimes \mathbb{V}$ and $[(\operatorname{curl} u)']_{FF} = 0$

on F, then by (6.21), we have

$$[\![t_e'(\operatorname{curl} u)' n_F]\!]_e = 0, \qquad \text{for all } e \in \Delta_1^I(F^{\mathrm{ct}}). \tag{6.33}$$

We know that $u \in \mathsf{L}_r^1(T^{wf})$, $u_{Fn} = 0$ and by (6.24) in Lemma 6.2.5 with $\ell = t_e$, $m = s_e$,

$$0 = [\![t_e'(\operatorname{curl} u)' n_F]\!]_e = [\![n_F'(\operatorname{curl} u) t_e]\!]_e = [\![(\operatorname{grad}_F u_{nn}) \cdot s_e]\!].$$

Therefore, we have $u_{nn} \in \mathcal{R}_r^0(F^{\mathrm{ct}})$ and (6.29j) implies $u_{nn} = 0$ on F. Thus $u|_{\partial T} = 0$.

Step 4:

Using the second characterization of Theorem 6.1.7, $u \in \mathring{U}_r^1(T^{wf})$. Hence (6.29k) implies $\operatorname{inc} u = 0$ on T and using the exactness of the sequence (6.6) and the DOFs of (6.29l), we see that $u = 0$ on T. ∎

6.2.3 Dofs of the U^2 and U^3 spaces

Lemma 6.2.10. *A function $u \in U_{r-2}^2(T^{wf})$, with $r \geq 3$, is fully determined by the following DOFs:*

$$\int_F u_{FF} : \kappa, \qquad \kappa \in Q_{r-2}^{\perp}, \forall F \in \Delta_2(T), \qquad 12(r-2) \tag{6.34a}$$

$$\int_F u_{nn} \kappa, \qquad \kappa \in V_{r-2}^2(F^{\mathrm{ct}}), \forall F \in \Delta_2(T), \qquad 6r^2 - 6r \tag{6.34b}$$

$$\int_F u_{nF} \cdot \kappa, \qquad \kappa \in V_{\mathrm{div},r-2}^1(F^{\mathrm{ct}}), \forall F \in \Delta_2(T), \qquad 12r^2 - 24r + 12 \tag{6.34c}$$

$$\int_T \operatorname{div} u \cdot \kappa, \qquad \kappa \in \mathring{U}_{r-3}^3(T^{wf}), \qquad 6r^3 - 18r^2 + 12r - 6 \tag{6.34d}$$

$$\int_T u : \kappa, \qquad \kappa \in \operatorname{inc} \mathring{U}_r^1(T^{wf}), \qquad 6r^3 - 27r^2 + 21r + 18. \tag{6.34e}$$

Proof. The dimension of $U^2_{r-2}(T^{wf})$ is $12r^3 - 27r^2 + 15r$, which is equal to the sum of the given DOFs.

Let $u \in U^2_{r-2}(T^{wf})$ such that u vanishes on the DOFs (6.34). By DOFs (6.34b), we have $u_{nn} = 0$ on each $F \in \Delta_2(T)$. By Lemma 6.2.3 and DOFs (6.34c), we have $u_{nF} = 0$ on each $F \in \Delta_2(T)$. Then, $u \in \mathring{V}^2_{r-2}(T^{wf}) \otimes \mathbb{V}$. With the definition of $Q^{\perp}_{r-2}$ in Section 5.1 and (6.34a), we have $u \in \mathring{\mathcal{V}}^2_r(T^{wf}) \otimes \mathbb{V}$ and thus $u \in \mathring{U}^2_{r-2}(T^{wf})$. In addition, since $\operatorname{div} u \in \operatorname{div}(\mathring{U}^2_{r-2}(T^{wf})) \subset \mathring{U}^3_{r-3}(T^{wf})$, we have $\operatorname{div} u = 0$ by DOFs (6.34d). Using the exactness of (6.6), there exist $\kappa \in \mathring{U}^1_r(T^{wf})$ such that $\operatorname{inc} \kappa = u$. With DOFs (6.34e), we have $u = 0$, which is the desired result. ∎

Lemma 6.2.11. *A function $u \in U^3_{r-3}(T^{wf})$, with $r \geq 3$, is fully determined by the following DOFs:*

$$\int_T u \cdot \kappa, \qquad \kappa \in \mathsf{R}, \qquad 6 \tag{6.35a}$$

$$\int_T u \cdot \kappa, \qquad \kappa \in \mathring{U}^3_{r-3}(T^{wf}), \qquad 6r^3 - 18r^2 + 12r - 6. \tag{6.35b}$$

Remark 6.2.12. Note that (6.35a) is equivalent to

$$\int_T u \cdot \kappa, \quad \kappa \in P_U \mathsf{R},$$

since by the definition of L^2-projection, for any $\kappa \in \mathsf{R}$,

$$\int_T u \cdot \kappa = \int_T u \cdot P_U \kappa, \quad \forall u \in U^3_{r-3}(T^{wf}).$$

6.3 Commuting projections

In this section, we show that the degrees of freedom constructed in the previous sections induce projections which satisfy commuting properties.

Theorem 6.3.1. *Let $r \geq 3$. Let $\Pi^0_{r+1} : C^\infty(\bar{T}) \otimes \mathbb{V} \to U^0_{r+1}(T^{wf})$ be the projection defined in Lemma 6.2.2, let $\Pi^1_r : C^\infty(\bar{T}) \otimes \mathbb{V} \to U^1_r(T^{wf})$ be the projection defined in Lemma 6.2.9, let $\Pi^2_{r-2} : C^\infty(\bar{T}) \otimes \mathbb{V} \to U^2_{r-2}(T^{wf})$ be the projection defined in Lemma 6.2.10, and let $\Pi^3_{r-3} : C^\infty(\bar{T}) \otimes \mathbb{V} \to U^3_{r-3}(T^{wf})$ be the projection defined in Lemma 6.2.10. Then the following commuting properties are satisfied.*

$$\varepsilon\big(\Pi^0_{r+1}u\big) = \Pi^1_r\varepsilon(u), \qquad \forall u \in C^\infty(\bar{T}) \otimes \mathbb{V} \tag{6.36a}$$

$$\operatorname{inc}\Pi^1_r v = \Pi^2_{r-2}\operatorname{inc} v, \quad \forall v \in C^\infty(\bar{T}) \otimes \mathbb{S} \tag{6.36b}$$

$$\operatorname{div}\Pi^2_{r-2} w = \Pi^3_{r-3}\operatorname{div} w, \quad \forall w \in C^\infty(\bar{T}) \otimes \mathbb{S} \tag{6.36c}$$

Proof. (i) Proof of (6.36a): Given $u \in C^\infty(\bar{T}) \otimes \mathbb{V}$, let $\rho = \varepsilon\big(\Pi^0_{r+1}u\big) - \Pi^1_r\varepsilon(u) \in U^1_r(T^{wf})$. To prove that (6.36a) holds, it suffices to show that ρ vanishes on the DOFs (6.29) in Lemma 6.2.9. Since $\operatorname{inc} \circ \varepsilon = 0$, we have DOFs of (6.29d), (6.29e), (6.29f) and (6.29k) applied to ρ vanish. Next, with (6.19b), (6.19e), (6.19f), (6.19g), (6.19i) applied to u, and with (6.29a), (6.29g), (6.29i), (6.29j), (6.29l) applied to $\varepsilon(u)$, each term respectively imply that (6.29a), (6.29g), (6.29i), (6.29j), (6.29l) applied to ρ vanish. By the identity (6.25) in Lemma 6.2.6, for any $\kappa \in \operatorname{grad}_F[(\mathcal{R}^0_r(F^{\mathrm{ct}}))]^2, \forall F \in \Delta_2(T)$, we have:

$$\int_F ([(\operatorname{curl}\rho)']_{FF} - \operatorname{grad}_F(\rho^\perp_{Fn})) : \kappa = \int_F \operatorname{grad}_F(\partial_n(\Pi^0_{r+1}u)_F - \partial_n u_F) : \kappa = 0,$$

where the last equality holds with (6.19h) applied to u. Thus, the DOFs (6.29h) applied to ρ vanish. It only remains to prove that the DOFs of (6.29b), (6.29c) applied to ρ vanish. To show this, we need to employ the edge-based orthonormal basis $\{n_e^+, n_e^-, t_e\}$ and write $\kappa \in \mathrm{sym}[\mathcal{P}_{r-2}(e)]^{3\times 3}$ as $\kappa = \kappa_{11} n_e^+ (n_e^+)' + \kappa_{12}\big(n_e^+ (n_e^-)' + n_e^- (n_e^+)'\big) + \kappa_{13}\big(n_e^+ t_e' + t_e (n_e^+)'\big) + \kappa_{22} n_e^- (n_e^-)' + \kappa_{23}\big(n_e^+ (n_e^-)' + n_e^- (n_e^+)'\big) + \kappa_{33} t_e t_e'$ where $\kappa_{ij} \in \mathcal{P}_{r-2}(e)$. Then,

$$
\begin{aligned}
\int_e \rho : \kappa &= \int_e [\varepsilon(\Pi^0_{r+1} u) - \Pi^1_r \varepsilon(u)] : \kappa = \int_e \varepsilon(\Pi^0_{r+1} u - u) : \kappa && \text{by (6.29b)} \\
&= \int_e \operatorname{grad}(\Pi^0_{r+1} u - u) : \kappa \\
&= \int_e \operatorname{grad}(\Pi^0_{r+1} u - u) t_e \cdot (\kappa_{13} n_e^+ + \kappa_{33} t_e) && \text{by (6.19d)} \\
&= \int_e (\Pi^0_{r+1} u - u) \cdot \frac{\partial}{\partial t_e} (\kappa_{13} n_e^+ + \kappa_{33} t_e) && \text{integration by parts} \\
&= 0 && \text{by (6.19a) and (6.19c).}
\end{aligned}
$$

Thus the DOFs of (6.29b) applied to ρ vanish. Next, letting $\kappa \in [\mathcal{P}_{r-1}(e)]^3$, we note that

$$
\begin{aligned}
\int_e (\operatorname{curl} \rho)' t_e \cdot \kappa &= \int_e \big[\operatorname{curl} \varepsilon(\Pi^0_{r+1} u - u)\big]' t_e \cdot \kappa && \text{by (6.29c)} \\
&= \frac{1}{2} \int_e \big[\operatorname{grad}\, \operatorname{curl}(\Pi^0_{r+1} u - u)\big] t_e \cdot \kappa && \text{by (2.19f)} \\
&= -\frac{1}{2} \int_e \operatorname{curl}(\Pi^0_{r+1} u - u) \cdot \partial_t \kappa && \text{by (6.19a) and (6.19b)}
\end{aligned}
$$

where in the last step, we have integrated by parts, and put $\partial_t \kappa = (\operatorname{grad} \kappa) t_e$. The curl in the integrand above can be decomposed into terms involving $\partial_t(\Pi^0_{r+1} u - u)$ and those involving $\partial_{n_e^\pm}(\Pi^0_{r+1} u - u)$. The former terms can be integrated by parts yet again, which after using (6.19a), (6.19b) and (6.19c), vanish. The latter terms also vanish by (6.19d), noting that $\partial_t \kappa$ is of degree at most $r - 2$.

(ii) Proof of (6.36b): Given $v \in C^\infty(\bar{T}) \otimes \mathbb{S}$, let $\rho = \operatorname{inc} \Pi^1_r v - \Pi^2_{r-2} \operatorname{inc} v \in$

$U^2_{r-2}(T^{wf})$. To prove that (6.36b) holds, we need to show that ρ vanishes on the DOFs (6.34) in Lemma 6.2.10. By using (6.34b) on $\operatorname{inc} v$, we have

$$\int_F \rho_{nn}\kappa = \int_F [\operatorname{inc}(\Pi^1_r v - v)]_{nn}\kappa, \qquad \forall \kappa \in V^2_{r-2}(F^{\mathrm{ct}}). \tag{6.37}$$

From (6.29e), we have that the right-hand side of (6.37) vanishes for $\kappa \in V^2_{r-2}(F^{\mathrm{ct}})/\mathcal{P}_1(F)$. With (6.26) of Lemma 6.2.7, we have for any $\kappa_1 \in \mathcal{P}_1(F)$,

$$\int_F \rho_{nn}\kappa_1 = \int_{\partial F} (\operatorname{curl}_F(\Pi^1_r v - v)_{FF})t\kappa_1 + \int_{\partial F} (\Pi^1_r v - v)_{FF}t \cdot (\operatorname{rot}_F \kappa_1)'. \tag{6.38}$$

By (2.19b), $\operatorname{curl}_F(\Pi^1_r v - v)_{FF}t\,\kappa_1 = [\operatorname{curl}(\Pi^1_r v - v)']_{Fn}t\,\kappa_1 = \operatorname{curl}(\Pi^1_r v - v)' : \kappa_1 n t'$, so the first term on the right-hand side of (6.38) vanishes by (6.29c). The last term in (6.38) also vanishes because

$$\begin{aligned}
\int_{\partial F} (\Pi^1_r v - v)_{FF}t \cdot (\operatorname{rot}_F \kappa_1)' &= \int_{\partial F} Q(\Pi^1_r v - v)Qt \cdot (\operatorname{rot}_F \kappa_1)' \\
&= \int_{\partial F} (\Pi^1_r v - v)Qt \cdot Q(\operatorname{rot}_F \kappa_1)' \\
&= \int_{\partial F} (\Pi^1_r v - v) : \operatorname{sym}(Q(\operatorname{rot}_F \kappa_1)'t) = 0,
\end{aligned}$$

where we used (6.29b) in the last equality. Thus, the right-hand side of (6.38) vanishes, and therefore the right-hand side of (6.37) vanishes, i.e., the DOFs (6.34b) vanish for ρ.

Next using (6.34c) we have

$$\int_F \rho_{nF} \cdot \kappa = \int_F [\operatorname{inc}(\Pi^1_r v - v)]_{nF} \cdot \kappa, \qquad \forall \kappa \in V^1_{\mathrm{div},r-2}(F^{\mathrm{ct}}). \tag{6.39}$$

The DOFs (6.29f) imply the right-hand side of (6.39) vanishes for all $\kappa \in V^1_{\mathrm{div},r-2}(F^{\mathrm{ct}})/\mathsf{R}_2$. Considering $\kappa \in \mathsf{R}_2$ in (6.39), we may conduct a similar argument as above, but now using (6.27) of Lemma 6.2.8, to conclude the right-hand side of (6.39) vanishes. Thus, we conclude that (6.34c) vanishes for ρ.

In addition, note that (6.29d) and (6.34a) imply that the DOFs (6.34a) vanish for ρ. Finally, the remaining DOFs of (6.34d) and (6.34e) applied to ρ also vanish, thus leading to (6.36b).

(iii) Proof of (6.36c): Given $w \in C^\infty(\bar{T}) \otimes \mathbb{S}$, let $\rho = \operatorname{div} \Pi^2_{r-2} w - \Pi^3_{r-3} \operatorname{div} w \in U^3_{r-3}(T^{wf})$. To prove that (6.36c) holds, we will show that ρ vanishes on the DOFs (6.35) in Lemma 6.2.11. Using (6.34d) and (6.35b), we have for any $\kappa \in \mathring{U}^3_{r-3}(T^{wf})$,

$$\int_T \rho \cdot \kappa = \int_T (\operatorname{div} \Pi^2_{r-2} w - \operatorname{div} w) \cdot \kappa = \int_T (\operatorname{div} w - \operatorname{div} w) \cdot \kappa = 0.$$

For $\kappa \in \mathsf{R}$, we find

$$\begin{aligned}
\int_T \rho \cdot \kappa &= \int_T (\operatorname{div} \Pi^2_{r-2} w - \operatorname{div} w) \cdot \kappa && \text{by (6.35a)} \\
&= \int_{\partial T} (\Pi^2_{r-2} w - w) n_F \cdot \kappa \\
&= \sum_{F \in \Delta_2(T)} \int_{\partial F} (\Pi^2_{r-2} w - w)_{nn} (\kappa \cdot n_F) - \int_{\partial F} (\Pi^2_{r-2} w - w)_{nF} \cdot \kappa \\
&= 0 && \text{by (6.34b) and (6.34c)},
\end{aligned}$$

Thus, $\rho = 0$, and so the commuting property (6.36c) is satisfied. ■

6.4 Global sequences

In this section, we construct the discrete elasticity sequence globally by putting the local spaces together. Recall that $\Omega \subset \mathbb{R}^3$ is a contractible polyhedral domain, and

$\mathcal{T}_h^{wf}$ is the Worsey-Farin refinement of the mesh $\mathcal{T}_h$ on Ω. Global de-Rham sequences on Worsey-Farin splits are shown in Section 2.5.5 and therefore, the global analogue of Theorem 6.1.3 is now given.

Theorem 6.4.1. *The following sequence is exact for any $r \geq 3$:*

$$\begin{bmatrix} \mathcal{S}_{r+1}^0(\mathcal{T}_h^{wf}) \otimes \mathbb{V} \\ \mathcal{S}_r^0(\mathcal{T}_h^{wf} \otimes \mathbb{V}) \end{bmatrix} \xrightarrow{[\mathrm{grad}, -\mathrm{mskw}]} \mathcal{S}_r^1(\mathcal{T}_h^{wf}) \otimes \mathbb{V} \xrightarrow{\mathrm{curl}\,\Xi^{-1}\mathrm{curl}}$$

$$\mathscr{V}_{r-2}^2(\mathcal{T}_h^{wf}) \otimes \mathbb{V} \xrightarrow{\begin{bmatrix} 2\mathrm{vskw} \\ \mathrm{div} \end{bmatrix}} \begin{bmatrix} \mathscr{V}_{r-2}^3(\mathcal{T}_h^{wf}) \otimes \mathbb{V} \\ V_{r-3}^3(\mathcal{T}_h^{wf}) \otimes \mathbb{V} \end{bmatrix}. \quad (6.40)$$

Moreover, the kernel of the first operator is isomorphic to R *and the last operator are surjective.*

Proof. The result follows from the exactness of the sequences (2.26a)– (2.26b), Proposition 2.3.1, and the exact same arguments in the proof of Theorem 6.1.3. ∎

The global spaces involved in the elasticity sequence, induced by the local spaces in Section 6.1.2 are defined as follows:

$$\begin{aligned}
U_{r+1}^0(\mathcal{T}_h^{wf}) &= \mathcal{S}_{r+1}^0(\mathcal{T}_h^{wf}) \otimes \mathbb{V}, \\
U_r^1(\mathcal{T}_h^{wf}) &= \{\mathrm{sym}(u) : u \in \mathcal{S}_r^1(\mathcal{T}_h^{wf}) \otimes \mathbb{V}\}, \\
U_{r-2}^2(\mathcal{T}_h^{wf}) &= \{u \in \mathscr{V}_{r-2}^2(\mathcal{T}_h^{wf}) \otimes \mathbb{V} : \mathrm{skw}\, u = 0\}, \\
U_{r-3}^3(\mathcal{T}_h^{wf}) &= V_{r-3}^3(\mathcal{T}_h^{wf}) \otimes \mathbb{V}.
\end{aligned}$$

Theorem 6.4.2. *We have the following equivalent definition of* $U_r^1(\mathcal{T}_h^{wf})$*:*

$$U_r^1(\mathcal{T}_h^{wf}) = \{u \in H^1(\Omega;\mathbb{S}) : u|_T \in U_r^1(T^{wf}),\ \forall T \in \mathcal{T}_h,$$
$$(\operatorname{curl} u)' \in V_{r-1}^1(\mathcal{T}_h^{wf}) \otimes \mathbb{V}, \operatorname{inc}(u) \in \mathscr{V}_{r-2}^2(\mathcal{T}_h^{wf}) \otimes \mathbb{V}\}.$$

Proof. This is proved similarly as the proof of Theorem 6.1.7 using Theorem 6.4.1 in place of Theorem 6.1.3. ■

Then we have the global sequence summarized in the following theorem. Its proof is the same as Theorem 6.1.4, with Theorem 6.4.1 in place of Theorem 6.1.3.

Theorem 6.4.3. *The following sequence of global finite element spaces*

$$0 \to \mathsf{R} \xrightarrow{\subset} U_{r+1}^0(\mathcal{T}_h^{wf}) \xrightarrow{\varepsilon} U_r^1(\mathcal{T}_h^{wf}) \xrightarrow{\operatorname{inc}} U_{r-2}^2(\mathcal{T}_h^{wf}) \xrightarrow{\operatorname{div}} U_{r-3}^3(\mathcal{T}_h^{wf}) \to 0 \tag{6.41}$$

is a discrete elasticity sequence and is exact for $r \geq 3$*.*

6.5 Conclusions

This chapter constructed both local and global finite element elasticity sequences with respect to three-dimensional Worsey-Farin splits. A notable feature of the discrete spaces is the lack of extrinsic supersmoothess and accompanying DOFs at vertices in the triangulation. For example, the $H(\mathrm{div})$-conforming space does not involve vertex or edge DOFs and is therefore conducive for hybridization.

These results suggest that the last two pairs in the sequence (6.41) may be suitable to construct mixed finite element methods for three-dimensional elasticity. However, due to the assumed regularity in Theorem 6.3.1, the result does not automatically yield an inf-sup stable pair. In our future work, we plan to working on the pair of spaces $U^2_{r-2}(\mathcal{T}_h^{wf}) \times U^3_{r-3}(\mathcal{T}_h^{wf})$ and prove the inf-sup stability by modifying the commuting projections. Since the last two spaces lack vertex or edge DOFs, we plan to explore the potential of a hybridization finite element method, using linear and constant functions. We also plan to conduct numerical experiments to evaluate the performance of these methods.

Chapter Seven

Conclusion and Future Work

In this thesis, we delve into numerical methods for partial differential equations on three-dimensional Worsey-Farin splits using a combination of FEEC and spline theories with smoother sequences and macro-elements. We prove a crucial conjecture about the shape regularity of Worsey-Farin refinements, which is essential for obtaining approximation results for splines. Utilizing the shape regularity result and FEEC theories, we present a framework for the convergence theory of the three-dimensional Maxwell eigenvalue problem using Lagrange finite element spaces on Worsey-Farin splits. This framework can also be applied to other finite element spaces that satisfy certain assumptions, with bounded commuting projections constructed case-by-case. Furthermore, we use the BGG method to construct new discrete exact sequences imposed with strong symmetry, and provide the degrees of freedom of all the spaces involved. Notably, the last two spaces used in the linear elasticity problems lack vertex degrees of freedom.

Going forward, we have several interesting directions. Firstly, our current shape regularity result is not optimal based on numerical results, indicating the need for further investigations to obtain sharper estimates. Additionally, we plan to cover accuracy analysis and apply our stability analysis framework to more complex problems such as magneto-hydrodynamics. Finally, we intend to conduct numerical experiments on linear elasticity problems with the finite element spaces used in our sequences and carry out corresponding numerical analysis.

Appendix A

Additional details for Chapter 4

In this appendix, we covered some proofs of the theorems in Chapter 4.

A.1 Proof of Theorem 4.1.3

Before we prove Theorem 4.1.3 we will need a few lemmas.

Lemma A.1.1. *With Assumption 4.1.2, there exists a positive constant C such that*

$$\|\boldsymbol{A}\mathbb{P}_Q f - \boldsymbol{A}f\|_{L^2(\Omega)} + \|\boldsymbol{T}\mathbb{P}_Q f - \boldsymbol{T}f\|_{L^2(\Omega)} \leq C\omega_1(h)\|f\|_{L^2(\Omega)} \quad \forall f \in L^2(\Omega).$$

Proof. Let $f \in L^2(\Omega)$ and set $\sigma = \boldsymbol{A}f$, $u = \boldsymbol{T}f$,$\psi = \boldsymbol{A}\mathbb{P}_Q f$ and $w = \boldsymbol{T}\mathbb{P}_Q f$. With f and $\mathbb{P}_Q f$ in (4.3), we see that

$$\begin{aligned}
(\sigma - \psi, \tau) + (u - w, \operatorname{curl}\tau) &= 0 \qquad && \forall \tau \in \mathring{H}(\operatorname{curl}, \Omega), \\
(\operatorname{curl}(\sigma - \psi), q) &= (f - \mathbb{P}_Q f, q) \qquad && \forall q \in \mathring{H}(\operatorname{div}^0, \Omega).
\end{aligned} \tag{A.1}$$

Setting $q = w - u$ and $\tau = \sigma - \psi$ in the above equations gives $\|\sigma - \psi\|^2_{L^2(\Omega)} = (f - \mathbb{P}_Q f, w - u)$, and the above equations also tell us that $\operatorname{curl}(u - w) = \sigma - \psi$. Furthermore, $u - w \in H(\operatorname{curl}, \Omega) \cap \mathring{H}(\operatorname{div}^0, \Omega)$ and there holds

$$\|\sigma - \psi\|_{L^2(\Omega)} \leq \sup_{\phi \in H(\operatorname{curl},\Omega)\cap\mathring{H}(\operatorname{div}^0,\Omega)} \frac{(f - \mathbb{P}_Q f, \phi)}{\|\operatorname{curl}\phi\|_{L^2(\Omega)}}.$$

Moreover, Assumption 4.1.2 gives us

$$\begin{aligned}
\sup_{\phi \in H(\operatorname{curl},\Omega)\cap\mathring{H}(\operatorname{div}^0,\Omega)} \frac{(f - \mathbb{P}_Q f, \phi)}{\|\operatorname{curl}\phi\|_{L^2(\Omega)}} &= \sup_{\phi \in H(\operatorname{curl},\Omega)\cap\mathring{H}(\operatorname{div}^0,\Omega)} \frac{(f, \phi - \mathbb{P}_Q \phi)}{\|\operatorname{curl}\phi\|_{L^2(\Omega)}} \\
&\leq \omega_1(h)\|f\|_{L^2(\Omega)}.
\end{aligned}$$

Thus, we have shown

$$\|\boldsymbol{A}\mathbb{P}_Q f - \boldsymbol{A} f\|_{L^2(\Omega)} \leq \omega_1(h)\|f\|_{L^2(\Omega)}$$

On the other hand, since $\boldsymbol{T}(\mathbb{P}_Q f - f) \in H(\mathrm{curl}, \Omega) \cap \mathring{H}(\mathrm{div}^0, \Omega)$, we have by Lemma 2.1.5,

$$\begin{aligned}\|\boldsymbol{T}\mathbb{P}_Q f - \boldsymbol{T} f\|_{L^2(\Omega)} &\leq C\|\mathrm{curl}\,(\boldsymbol{T}\mathbb{P}_Q f - \boldsymbol{T} f)\|_{L^2(\Omega)} \\ &= C\|\boldsymbol{A}\mathbb{P}_Q f - \boldsymbol{A} f\|_{L^2(\Omega)} \leq C\omega_1(h)\|f\|_{L^2(\Omega)}.\end{aligned}$$

∎

Since the domain Ω is contractible, then the set of harmonic forms is trivial and we have (cf. [32, Theorem 4.9]):

Lemma A.1.2. *Let Ω be a bounded, contractible, Lipschitz domain in $\mathbb{R}^3$. Then for all $u \in \mathring{H}(\mathrm{div}^0, \Omega)$, there exists $v \in H_0^1(\Omega)$ such that $u = \mathrm{curl}\, v$ and $\|v\|_{H^1(\Omega)} \leq C\|u\|_{L^2(\Omega)}$.*

Lemma A.1.3. *With Assumption 4.1.2, there exists a positive constant C such that for every $p_h \in \mathscr{Q}_h$, there exists $\tau_h \in \mathscr{V}_h$ such that $\mathrm{curl}\,\tau_h = p_h$ and $\|\tau_h\|_{L^2(\Omega)} \leq C\|p_h\|_{L^2(\Omega)}$.*

Proof. By Lemma A.1.2, for every $p_h \in \mathscr{Q}_h \subset \mathring{H}(\mathrm{div}^0, \Omega)$, there exists $\tau \in H_0^1(\Omega)$ such that $\mathrm{curl}\,\tau = p_h$ and $\|\tau\|_{H^1(\Omega)} \leq C\|p_h\|_{L^2(\Omega)}$. Note that since $\tau \in \mathscr{V}^t(\mathscr{Q}_h)$, we can set $\tau_h = \mathbf{\Pi}_{\mathscr{V}}\tau$. By Assumption 4.1.2, we have $\mathrm{curl}\,\tau_h = \mathrm{curl}\,\tau = p_h$. Additionally,

$$\|\tau_h\|_{L^2(\Omega)} \leq C(\|\tau\|_{H^{1/2+\delta}(\Omega)} + \|\mathrm{curl}\,\tau\|_{L^2(\Omega)}) \leq C\|\tau\|_{H^1(\Omega)} \leq C\|p_h\|_{L^2(\Omega)}.$$

∎

We are now in position to prove Theorem 4.1.3.

Proof of Theorem 4.1.3. Let $f \in L^2(\Omega)$ and set $\sigma = \boldsymbol{A}f$, $u = \boldsymbol{T}f$,$\sigma_h = \boldsymbol{A}_h f$, $u_h = \boldsymbol{T}_h f$, $\psi = \boldsymbol{A}\mathbb{P}_Q f$ and $w = \boldsymbol{T}\mathbb{P}_Q f$. The first step is to estimate $\psi - \sigma_h$. We note that $\operatorname{curl}\psi = \mathbb{P}_Q f$ and $\operatorname{curl} w = \psi$. From this we have that $\psi \in \mathscr{V}^t(\mathscr{Q}_h)$, $\operatorname{div}\psi = 0$ and $w \in H(\operatorname{curl}, \Omega) \cap \mathring{H}(\operatorname{div}^0, \Omega)$. Hence, by Assumption 1 we get $\operatorname{curl}(\boldsymbol{\Pi}_{\mathscr{V}}\psi) = \operatorname{curl}\psi = \mathbb{P}_Q f$. We can write the error equations

$$(\boldsymbol{\Pi}_{\mathscr{V}}\psi - \sigma_h, \tau_h) + (\mathbb{P}_Q w - u_h, \operatorname{curl}\tau_h) = (\boldsymbol{\Pi}_{\mathscr{V}}\psi - \psi, \tau_h) \qquad \forall \tau_h \in \mathscr{V}_h, \tag{A.2a}$$

$$(\operatorname{curl}(\boldsymbol{\Pi}_{\mathscr{V}}\psi - \sigma_h), q_h) = 0 \qquad \forall q_h \in \mathscr{Q}_h. \tag{A.2b}$$

Setting $\tau_h = \boldsymbol{\Pi}_{\mathscr{V}}\psi - \sigma_h$ and using the Cauchy-Schwarz inequality provides:

$$\|\boldsymbol{\Pi}_{\mathscr{V}}\psi - \sigma_h\|_{L^2(\Omega)} \le \|\boldsymbol{\Pi}_{\mathscr{V}}\psi - \psi\|_{L^2(\Omega)} \le \omega_0(h)(\|\psi\|_{H^{1/2+\delta}(\Omega)} + \|\operatorname{curl}\psi\|_{L^2(\Omega)}). \tag{A.3}$$

Next, with the embedding result in Proposition 2.1.5 and noting that $\operatorname{div}\psi = 0$, we obtain

$$\begin{aligned} \|\psi\|_{H^{1/2+\delta}(\Omega)} &\le C(\|\operatorname{curl}\psi\|_{L^2(\Omega)} + \|\operatorname{div}\psi\|_{L^2(\Omega)}) \\ &\le C\|\operatorname{curl}\psi\|_{L^2(\Omega)} = C\|\mathbb{P}_Q f\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)}. \end{aligned} \tag{A.4}$$

Thus, we have $\|\boldsymbol{\Pi}_{\mathscr{V}}\psi - \sigma_h\|_{L^2(\Omega)} \le C\omega_0(h)\|f\|_{L^2(\Omega)}$. Lemma A.1.1 shows that

$$\|\sigma - \psi\|_{L^2(\Omega)} + \|w - u\|_{L^2(\Omega)} \le C\omega_1(h)\|f\|_{L^2(\Omega)}, \tag{A.5}$$

and therefore, combining (A.3), (A.4), (A.5) we obtain

$$\begin{aligned}\|(\boldsymbol{A}-\boldsymbol{A}_h)f\|_{L^2(\Omega)} &= \|\sigma-\sigma_h\|_{L^2(\Omega)} \\ &\le \|\sigma-\psi\|_{L^2(\Omega)} + \|\sigma_h - \boldsymbol{\Pi}_{\mathscr{V}}\psi\|_{L^2(\Omega)} + \|\boldsymbol{\Pi}_{\mathscr{V}}\psi-\psi\|_{L^2(\Omega)} \\ &\le C(\omega_0(h)+\omega_1(h))\|f\|_{L^2(\Omega)}.\end{aligned}$$

By Lemma A.1.3, we know that there exists $\tau_h \in \mathscr{V}_h$ such that $\operatorname{curl}\tau_h = \mathbb{P}_Q w - u_h$ and $\|\tau_h\|_{L^2(\Omega)} \le C\|\mathbb{P}_Q w - u_h\|_{L^2(\Omega)}$. Then using (A.2a) and the Cauchy-Schwarz inequality, we have

$$\begin{aligned}\|\mathbb{P}_Q w - u_h\|^2_{L^2(\Omega)} = (\mathbb{P}_Q w - u_h, \operatorname{curl}\tau_h) &= -(\psi-\sigma_h,\tau_h) \\ &\le C\|\mathbb{P}_Q w - u_h\|_{L^2(\Omega)}\|\psi-\sigma_h\|_{L^2(\Omega)}.\end{aligned}$$

Furthermore, using the triangle inequality, and Assumption 1, we have

$$\begin{aligned}\|w-u_h\|_{L^2(\Omega)} &\le \|\mathbb{P}_Q w - u_h\|_{L^2(\Omega)} + \|w-\mathbb{P}_Q w\|_{L^2(\Omega)} \\ &\le C\|\psi-\sigma_h\|_{L^2(\Omega)} + \omega_1(h)\|\operatorname{curl} w\|_{L^2(\Omega)} \\ &\le C(\omega_0(h)+\omega_1(h))\|f\|_{L^2(\Omega)} + \omega_1(h)\|\operatorname{curl} w\|_{L^2(\Omega)}. \qquad \text{(A.6)}\end{aligned}$$

We also have using Lemma 2.1.6,

$$\|\operatorname{curl} w\|_{L^2(\Omega)} = \|\psi\|_{L^2(\Omega)} \le C\|\operatorname{curl}\psi\|_{L^2(\Omega)} = C\|\mathbb{P}_Q f\|_{L^2(\Omega)} \le C\|f\|_{L^2(\Omega)},$$

and therefore using this inequality with (A.6) and (A.5) we arrive at

$$\begin{aligned}\|(\boldsymbol{T}-\boldsymbol{T}_h)f\|_{L^2(\Omega)} = \|u-u_h\|_{L^2(\Omega)} &\le \|u-w\|_{L^2(\Omega)} + \|w-u_h\|_{L^2(\Omega)} \\ &\le C(\omega_0(h)+\omega_1(h))\|f\|_{L^2(\Omega)}.\end{aligned}$$

■

A.2 Proof of Proposition 4.4.1 (Exact sequences)

Proof. In order to prove the sequence (2.26a) is an exact sequence, we only need to prove $\mathbf{grad}\ \mathfrak{S}_r = \mathbf{Ker}\ (\mathfrak{L}_{r-1}, \mathrm{curl}\,)$, curl $\mathfrak{L}_{r-1} = \mathbf{Ker}\ (\mathfrak{N}_{r-1}, \mathrm{div}\,)$ and $\mathrm{div}\,\mathfrak{N}_{r-2} = \mathfrak{W}_{r-3}$.

(i) $\forall v \in \mathfrak{S}_r$, $\mathbf{grad}\, v \in C[\Omega]^3$, local exact sequence of (3.2b) in [44] gives that $\mathbf{grad}\ \mathfrak{S}_r \subset \mathbf{Ker}\ (\mathfrak{L}_{r-1}, \mathrm{curl}\,)$. $\forall v \in \mathfrak{L}_{r-1}$, local exact sequence of (3.2b) in [44] also gives that there exists $\phi_T \in S_r(T^{wf})$ such that $\mathbf{grad}\ \phi_T = v|_T$, so we can find $\phi \in C^1(\Omega)$ such that $\phi|_T = \phi_T$ and then $\phi \in \mathfrak{S}_r$. Therefore, $\mathbf{grad}\ \mathfrak{S}_r = \mathbf{Ker}\ (\mathfrak{L}_{r-1}, \mathrm{curl}\,)$.

(ii) Since $\mathrm{div}\,\mathfrak{N}_{r-2} \subset \mathrm{div}\, H(\mathrm{div}\,, \Omega) = L^2(\Omega)$, we have $\mathrm{div}\,\mathfrak{N}_{r-2} \subset \mathfrak{W}_{r-3}$. On the other hand, $\forall q \in \mathfrak{W}_{r-3} \subset L^2(\Omega) = \mathrm{div}\, H(\mathrm{div}\,, \Omega)$, by [32, Theorem 4.9], there exists $v \in H^1(\Omega)$ such that $\mathrm{div}\, v = q$ and then we have $q = \Pi_W q = \Pi_W \mathrm{div}\, v = \mathrm{div}\,\mathbf{\Pi}_N v$. Therefore $\mathrm{div}\,\mathfrak{N}_{r-2} = \mathfrak{W}_{r-3}$.

(iii) We already have curl $\mathfrak{L}_{r-1} \subset \mathbf{Ker}\ (\mathfrak{N}_{r-1}, \mathrm{div}\,)$. On the other hand, $\forall \tau \in \mathbf{Ker}(\mathfrak{N}_{r-1}, \mathrm{div}\,) \subset H(\mathrm{div}^{\,0}, \Omega)$, by [32, Theorem 4.9], there exists $v \in H^1(\Omega)$ such that $\mathrm{curl}\, v = \tau$ and then we have $\tau = \mathbf{\Pi}_N \tau = \mathbf{\Pi}_N \mathrm{curl}\, v = \mathrm{curl}\,\mathbf{\Pi}_L v$. Therefore, curl $\mathfrak{L}_{r-1} = \mathbf{Ker}\ (\mathfrak{N}_{r-1}, \mathrm{div}\,)$

Next, we are ready to prove the exactness of (4.9). By direct checking the boundary conditions, we can get the result. ■

A.3 Proof of Lemma 4.2.5

Before proving Lemma 4.2.5 let us develop some notations. Let $\{E_1, \ldots, E_M\}$ and $\{v_1, \ldots, v_m\}$ be the sets of edges and (corner) vertices, respectively, of the polyhedral domain Ω. For each E_j there exists two faces of $\partial\Omega$, f_j^1 and f_j^2 that share E_j, and we denote their unit-normal vectors by n_j^i for $i = 1, 2$. We let $\tilde{\boldsymbol{T}}_j^3$ be tangent to E_j and set $\tilde{\boldsymbol{T}}_j^i = n_j^i \times \tilde{\boldsymbol{T}}_j^3$ for $i = 1, 2$. Note that $\tilde{\boldsymbol{T}}_j^i$ for $i = 1, 2, 3$ are linearly independent.

Let $\mathcal{V}_h^j$ denote all the vertices of $\mathcal{T}_h^{wf}$ that are in the interior of E_j. Recall that $\mathcal{V}_h$ is the set of vertices of $\mathcal{T}_h^{wf}$. We decompose them in the following form

$$\mathcal{V}_h = \mathcal{V}_h^C \cup \mathcal{V}_h^E \cup \mathcal{V}_h^0,$$

where $\mathcal{V}_h^C = \{v_1, \ldots, v_m\}$ are the corner points and $\mathcal{V}_h^E = \cup_{1 \le j \le M} \mathcal{V}_h^j$ are the vertices lying on edges of $\partial\Omega$. Finally, $\mathcal{V}_h^0 = \mathcal{V}_h \backslash (\mathcal{V}_h^C \cup \mathcal{V}_h^E)$.

For any $z \in \mathcal{V}_h^j$ we choose $F_z^i \in \Delta_2(\mathcal{T}_h^{wf})$ such that z is a vertex of F_z^i and $F_z^i \subset f_j^i$ for $i = 1, 2$. We then set $F_z^3 = F_z^2$. We also set $\boldsymbol{T}_z^i = \tilde{\boldsymbol{T}}_j^i$ for $i = 1, 2, 3$. If $z \in \mathcal{V}_h^C$ then z is an end point of some E_j and we define F_z^i and $\boldsymbol{T}_z^i$ for $i = 1, 2, 3$ in the same way.

Proof of Lemma 4.2.5. First, we summarize the construction of the Scott-Zhang interpolant in [60]. Let $\mathcal{V}_h$ be the set of all the vertices of $\mathcal{T}_h$. For each $z \in \mathcal{V}_h$, let ϕ_z be corresponding nodal basis of $\mathcal{P}_1^c(\mathcal{T}_h)$, i.e., $\phi_z \in \mathcal{P}_1^c(\mathcal{T}_h)$ satisfies $\phi_z(y) = \delta_{yz}$ for all $y \in \mathcal{V}_h$. For every $z \in \mathcal{V}_h$, we identify an arbitrary face F_z of the mesh that contains z with the only constraint that F_z is a boundary face if z is a boundary vertex. Then there exists function $\psi_z \in L^\infty(F_z)$ such that

$$\int_{F_z} \psi_z \phi_y = \delta_{yz}, \qquad \forall y \in \mathcal{V}_h. \tag{A.7}$$

Furthermore, the function ψ satisfies the estimate:

$$\|\psi_z\|_{L^\infty(F_z)} \leq \frac{C}{|F_z|}. \tag{A.8}$$

The Scott-Zhang interpolant $\tilde{\boldsymbol{I}}_h$ is given by:

$$\tilde{\boldsymbol{I}}_h \tau(x) = \sum_{z \in \mathcal{V}_h} (\int_{F_z} \psi_z \tau) \phi_z(x). \tag{A.9}$$

Construction of $\boldsymbol{I}_h^{curl}$: Similar to the construction in [20] for the two-dimensional case, we modify the Scott-Zhang interpolant $\tilde{\boldsymbol{I}}_h$ on edges and corner vertices of Ω to preserve the vanishing tangential trace. We also let ψ_z^i (for $i = 1, 2, 3$) satsify:

$$\int_{F_z^i} \psi_z^i \phi_y = \delta_{yz}, \qquad \forall y \in \mathcal{V}_h, \;\; i = 1, 2, 3 \tag{A.10}$$

$$\|\psi_z^i\|_{L^\infty(F_z^i)} \leq \frac{C}{|F_z^i|}. \tag{A.11}$$

The modified Scott-Zhang interpolant $\boldsymbol{I}_h^{curl}$ is given as

$$\boldsymbol{I}_h^{curl} \tau(x) = \sum_{z \in \mathcal{V}_h^0} (\int_{F_z} \psi_z \tau) \phi_z(x) + \sum_{z \in \mathcal{V}_h^C \cup \mathcal{V}_h^E} \boldsymbol{\beta}_z^t(\tau) \phi_z(x) \tag{A.12}$$

where

$$\boldsymbol{\beta}_z^t(\tau) := \sum_{i=1}^{3} \frac{\boldsymbol{C}_i}{(\boldsymbol{T}_z^1 \times \boldsymbol{T}_z^2) \cdot \boldsymbol{T}_z^3} \int_{F_z^i} (\tau \cdot \boldsymbol{T}_z^i) \psi_z^i,$$

with $\boldsymbol{C}_1 = \boldsymbol{T}_z^2 \times \boldsymbol{T}_z^3$,$\boldsymbol{C}_2 = -\boldsymbol{T}_z^1 \times \boldsymbol{T}_z^3$ and $\boldsymbol{C}_3 = \boldsymbol{T}_z^1 \times \boldsymbol{T}_z^2$. Note that $\boldsymbol{C}_i \cdot \boldsymbol{T}_z^\ell = \delta_{i\ell}$.

Construction of $\boldsymbol{I}_h^{div}$: If $z \in \mathcal{V}_h^E$ we define F_z^i for $i = 1,2,3$ as above. Moreover, we set $n_z^i = n_j^i$ for $i = 1,2$ and $n_z^3 = \boldsymbol{T}_z^3$. On the other hand, if $z \in \mathcal{V}_h^C$ we let $F_z^i \in \Delta_2(\mathcal{T}_h^{wf})$ for $i = 1,2,3$ be such that $F_z^i \subset \partial\Omega$ with each z being a vertex of F_z^i, and such that each lies on a distinct plane. We then let n_z^i be unit normal vectors to F_z^i. We then define

$$\boldsymbol{I}_h^{div}\tau(x) = \sum_{z\in\mathcal{V}_h^0}(\int_{F_z}\psi_z\tau)\phi_z(x) + \sum_{z\in\mathcal{V}_h^E\bigcup\mathcal{V}_h^C}\boldsymbol{\beta}_z^n(\tau)\phi_z(x), \tag{A.13}$$

where

$$\boldsymbol{\beta}_z^n(\tau) := \sum_{i=1}^3 \frac{\boldsymbol{D}_i}{(n_z^1\times n_z^2)\cdot n_z^3}\int_{F_z^i}(\tau\cdot n_z^i)\psi_z^i,$$

with $\boldsymbol{D}_1 = n_z^2\times n_z^3$, $\boldsymbol{D}_2 = -n_z^1\times n_z^3$ and $\boldsymbol{D}_3 = n_z^1\times n_z^2$. Note that $\boldsymbol{D}_i\cdot n_z^\ell = \delta_{i\ell}$.

Proof of estimates (4.12)–(4.13)**:** We prove the estimates in four steps:

(ia) $\boldsymbol{I}_h^{curl} : H^{1/2+\delta}(\Omega)\cap\mathring{H}(\mathrm{curl},\Omega) \to \boldsymbol{\mathcal{P}}_1^c(\mathcal{T}_h)\cap\mathring{H}(\mathrm{curl},\Omega)$: if $\tau\in H^{1/2+\delta}(\Omega)\cap\mathring{H}(\mathrm{curl},\Omega)$, then $\tau\times n|_{\partial\Omega} = 0$, and so $(\tau\cdot\boldsymbol{T})(z) = 0$ for any tangential vector $\boldsymbol{T}$ at $z\in\partial\Omega$. Therefore, for every $z\in\mathcal{V}_h^E\bigcup\mathcal{V}_h^C$, $\boldsymbol{I}_h^{curl}\tau(z) = \boldsymbol{\beta}_z^t(\tau) = 0$. On the other hand, for every $z\in\mathcal{V}_h^0\cap\partial\Omega$, we have $\boldsymbol{I}_h^{curl}\tau(z)\times n_{F_z} = \int_{F_z}\psi_z(\tau\times n_{F_z}) = 0$, where n_{F_z} is the outward normal vector of $F_z\subset\partial\Omega$. These two identities yield $(\boldsymbol{I}_h^{curl}\tau\times n)|_{\partial\Omega} = 0$.

(ib) $\boldsymbol{I}_h^{div} : H^{1/2+\delta}(\Omega)\cap\mathring{H}(\mathrm{div},\Omega) \to \boldsymbol{\mathcal{P}}_1^c(\mathcal{T}_h)\cap\mathring{H}(\mathrm{div},\Omega)$: if $\tau\in H^{1/2+\delta}(\Omega)\cap\mathring{H}(\mathrm{div}\,,\Omega)$, then $\tau\cdot n|_{\partial\Omega} = 0$. Suppose $z\in\mathcal{V}_h^E$, then the definition of $\boldsymbol{I}_h^{div}$ shows $\boldsymbol{I}_h^{div}\tau(z)\cdot n_z^i = \boldsymbol{\beta}_z^n(\tau)\cdot n_z^i = 0$ for $i = 1,2$. On the other hand, if $z\in\mathcal{V}_h^C$ then $\boldsymbol{I}_h^{div}\tau(z)\cdot n_z^i = \boldsymbol{\beta}_z^n(\tau)\cdot n_z^i = 0$, $i = 1,2,3$, and so in this case $\boldsymbol{I}_h^{div}\tau(z) = 0$. Finally, if $z\in\mathcal{V}_h^0\cap\partial\Omega$, we have $\boldsymbol{I}_h^{div}\tau(z)\cdot n_{F_z} = \int_{F_z}\psi_z(\tau\cdot n_{F_z}) = 0$, where n_{F_z} is the outward normal vector of $F_z\subset\partial\Omega$. We conclude that $(\boldsymbol{I}_h^{curl}\tau)\cdot n|_{\partial\Omega} = 0$.

(ii) $\boldsymbol{I}_h^{curl}$ and $\boldsymbol{I}_h^{div}$ *are projections.* We show that if $\tau_1 \in \boldsymbol{\mathcal{P}}_1(\mathcal{T}_h) \cap H(\mathrm{curl}, \Omega)$ and $\tau_2 \in \boldsymbol{\mathcal{P}}_1(\mathcal{T}_h) \cap H(\mathrm{div}, \Omega)$, then $\boldsymbol{I}_h^{curl}\tau_1 = \tau_2$ and $\boldsymbol{I}_h^{div}\tau_2 = \tau_2$. Since $\tau_i \in \boldsymbol{\mathcal{P}}_1^c(\mathcal{T}_h)$ $(i = 1, 2)$, we can write

$$\tau_i(x) = \sum_{z \in \mathcal{V}_h} \tau_i(y)\phi_y(x).$$

If $z \in \mathcal{V}_h^0$, $\boldsymbol{I}_h^{curl}\tau_1(z) = \int_{F_z} \psi_z \tau_1 = \sum_{y \in \mathcal{V}_h} \tau_1(y) \int_{F_z} \psi_z \phi_y = \tau_1(z)$ by (A.7). Similarly, $\boldsymbol{I}_h^{div}\tau_2(z) = \tau_2(z)$. However, if $z \in \mathcal{V}_h^E \bigcup \mathcal{V}_h^C$ then $\boldsymbol{I}_h^{curl}\tau_1(z) = \boldsymbol{\beta}_z^t(\tau_1)$ and $\boldsymbol{I}_h^{div}\tau_2(z) = \boldsymbol{\beta}_z^n(\tau_2)$. We have $\boldsymbol{\beta}_z^t(\tau_1) \cdot \boldsymbol{T}_z^i = \int_{F_z^i} (\tau_1 \cdot \boldsymbol{T}_z^i)\psi_z^i = \tau_1(z) \cdot \boldsymbol{T}_z^i$ for $i = 1, 2, 3$. Recalling these three tangential vectors are linearly independent, we conclude $\boldsymbol{I}_h^{curl}\tau_1(z) = \tau_1(z)$.

Similarly, since $\boldsymbol{\beta}_z^n(\tau_2) \cdot n_z^i = \int_{F_z^i} (\tau_2 \cdot n_z^i)\psi_z^i = \tau_2(z) \cdot n_z^i$, $i = 1, 2, 3$, we have $\boldsymbol{I}_h^{div}\tau_2(z) = \tau_2(z)$.

(iii) *Stability estimate.* By an inverse estimate (2.5a) we have

$$|\boldsymbol{I}_h^{curl}\tau|_{H^{1/2+\delta}(T)} \leq C h_T^{-1/2-\delta} \|\boldsymbol{I}_h^{curl}\tau\|_{L^2(T)}.$$

We will use the following trace inequality (see [30, Proposition 3.1]; also follows from (2.3), (2.5) and a scaling argument),

$$\|\tau\|_{L^1(F_z)} \leq C(h_T^{\frac{1}{2}}\|\tau\|_{L^2(T)} + h_T^{1+\delta}|\tau|_{H^{\frac{1}{2}+\delta}(T)}), \quad T \in \mathcal{T}_h, F_z \in \Delta_2(T). \tag{A.14}$$

Since the number of edges and vertices of $\partial\Omega$ is finite we have $M_t := \max_{z \in \mathcal{V}_h^E \bigcup \mathcal{V}_h^C} \frac{1}{(\boldsymbol{T}_z^1 \times \boldsymbol{T}_z^2) \cdot \boldsymbol{T}_z^3}$ is finite. Using the L^∞ estimates of ψ_z, (A.8), (A.11), (A.14) and the estimate $\|\phi_z\|_{L^2(T)} \leq C h_T^{3/2}$ for $z \in \bar{T}$, we have

$$\begin{aligned}
\|\boldsymbol{I}_h^{curl}\tau\|_{L^2(T)} \leq & \sum_{\substack{z\in\mathcal{V}_h^0 \\ z\in\bar{T}}} \|\phi_z\|_{L^2(T)}\|\psi_z\|_{L^\infty(F_z)}\|\tau\|_{L^1(F_z)} \\
& + M_t \sum_{\substack{z\in\mathcal{V}_h^E\bigcup\mathcal{V}_h^C \\ z\in\bar{T}}} \|\phi_z\|_{L^2(T)} \sum_{i=1}^{3}(\|\psi_z^i\|_{L^\infty(F_z^i)}\|\tau\|_{L^1(F_z^i)}) \\
& \leq C(1+M_t)\big(\|\tau\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\tau|_{H^{\frac{1}{2}+\delta}(\omega(T))}\big).
\end{aligned}$$

Therefore, we conclude

$$\begin{aligned}
h_T^{1/2+\delta}|\boldsymbol{I}_h^{curl}\tau|_{H^{1/2+\delta}(T)}+&\|\boldsymbol{I}_h^{curl}\tau\|_{L^2(T)} \\
&\leq C(1+M_t)(\|\tau\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\omega(T))}).
\end{aligned} \tag{A.15}$$

On the other hand, by following the same process, we obtain

$$h_T^{1/2+\delta}|\boldsymbol{I}_h^{div}\tau|_{H^{1/2+\delta}(T)}+\|\boldsymbol{I}_h^{div}\tau\|_{L^2(T)} \tag{A.16}$$

$$\leq C(1+M_n)(\|\tau\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\omega(T))}), \tag{A.17}$$

where $M_n := \max_{z\in\mathcal{V}_h^E\bigcup\mathcal{V}_h^C} \frac{1}{(n_z^1\times n_z^2)\cdot n_z^3}$.

(iv) *Estimate (4.12)*: Let $w = \frac{1}{|\omega(T)|}\int_{\omega(T)} \tau\, dx$, so that by the Poincare inequality (cf. [60, Section 4] and [33, Proposition 2.1])

$$\|\tau - w\|_{L^2(\omega(T))} \leq C h_T^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\omega(T))}. \tag{A.18}$$

Since w is a constant, there holds $\boldsymbol{I}_h^{curl} w|_T = w$. Using the estimate (A.18) and the stability result (A.15), we obtain

$$
\begin{aligned}
\|\boldsymbol{I}_h^{curl}\tau - \tau\|_{L^2(T)} &= \|\boldsymbol{I}_h^{curl}(\tau - w) - (\tau - w)\|_{L^2(T)} \\
&\leq C(1 + M_t)(\|\tau - w\|_{L^2(\omega(T))} + h_T^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\omega(T))}) \\
&\leq C(1 + M_t)h_T^{1/2+\delta}|\tau|_{H^{1/2+\delta}(\omega(T))}.
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|\boldsymbol{I}_h^{curl}\tau|_{H^{1/2+\delta}(T)} &= |\boldsymbol{I}_h^{curl}(\tau - w)|_{H^{1/2+\delta}(T)} \\
&\leq C(1 + M_t)(h_T^{-1/2-\delta}\|\tau - w\|_{L^2(\omega(T))} + |\tau|_{H^{1/2+\delta}(\omega(T))}) \\
&\leq C(1 + M_t)|\tau|_{H^{1/2+\delta}(\omega(T))}.
\end{aligned}
$$

These last two estimates complete the proof of (4.12). By the same process, (but replacing the stability estimate (A.15) with (A.16)) we obtain the estimate for $\boldsymbol{I}_h^{div}$ (4.13).

■

A.4 Scaling Properties

In this section, we need to prove Proposition 4.3.7, Lemma 4.3.8, and Lemma 4.3.9.

A.4.1 Proof of Proposition 4.3.7

Proof. The proof of (i) follows from the definition of $\hat{T}$ and Definition 2.2.4. The identities given in (ii) follow from the chain rule, and the scaling result in (iii) is a direct application of [46, Lemma 2.9]. ■

To prove those two lemmas, we transform $\hat{T}$ to the standard reference tetrahedron $\tilde{T}$ with unit size.

Definition A.4.1. Let $\tilde{T}$ be the tetrahedron with vertices (0,0,0), (1,0,0), (0,1,0), (0,0,1). For any tetrahedra $\hat{T}$ set $\phi_{\hat{T}} : \tilde{T} \to \hat{T}$ to be an affine diffeomorphism with

$$\phi_{\hat{T}}(\tilde{x}) = B_{\hat{T}}\tilde{x} + b_{\hat{T}} \qquad \tilde{x} \in \tilde{T}$$

for some $B_{\hat{T}} \in \mathbb{R}^{3\times 3}$ and $b_{\hat{T}} \in \mathbb{R}^3$.

Remark A.4.2. There holds (cf. [57, Page 80 and Lemma 5.10])

$$|\det(B_{\hat{T}})| = \frac{|\hat{T}|}{|\tilde{T}|} = 6|\hat{T}|, \quad \|B_{\hat{T}}\| \leq h_{\hat{T}} = 1, \quad \|B_{\hat{T}}^{-1}\| \leq \rho_{\hat{T}}^{-1} \leq c_0.$$

A.4.2 Proof of Lemma 4.3.8

We require an intermediate result to prove Lemma 4.3.8.

Proposition A.4.3. *(Equivalence norms of Sobolev space)*

(i) $\forall \hat{v} \in H^s(\hat{T})$ *with* $0 < s < 1$*, we have*

$$C_1|\hat{v}|_{H^s(\hat{T})} \leq |\hat{v} \circ \phi_{\hat{T}}|_{H^s(\tilde{T})} \leq C_2|\hat{v}|_{H^s(\hat{T})}, \tag{A.19}$$

where C_1 *and* C_2 *depends on* c_0 *(the shape regularity of* $\mathcal{T}_h$*) and* s*.*

(ii) For all $\hat{\kappa} \in W^{s,q}(\hat{e})$ with $sq < 1$ and $0 < s < 1/2$, we have

$$C_3 \|\hat{\kappa} \circ \phi_{\hat{T}}\|_{W^{s,q}(\tilde{e})} \leq \|\hat{\kappa}\|_{W^{s,q}(\hat{e})},$$

where $\tilde{e} = \phi_{\hat{T}}^{-1}(\hat{e})$, and C_3 depends on c_0, s and q.

(iii) For all $\hat{\kappa} \in W^{1,p}(\hat{T})$, we have

$$\|\hat{\kappa}\|_{W^{1,p}(\hat{T})} \leq C_4 \|\kappa \circ \phi_{\hat{T}}\|_{W^{1,p}(\tilde{T})}. \tag{A.20}$$

Moreover, for any $\hat{F} \in \Delta_2(\hat{T})$,

$$\|\hat{\kappa}\|_{W^{1,p}(\hat{F})} \leq C_5 \|\hat{\kappa} \circ \phi_{\hat{T}}\|_{W^{1,p}(\tilde{F})} \quad \forall \hat{\kappa} \in W^{1,p}(\hat{F}), \tag{A.21}$$

where $\tilde{F} = \phi_{\hat{T}}^{-1}(\hat{F})$, and C_4 and C_5 depends on c_0 and p.

Proof. (i) This estimate follows from [46, Lemma 2.9] and Remark A.4.2.

(ii) Let $\hat{\kappa} \in W^{s,q}(\hat{e})$ and to ease notation, set $\tilde{\kappa} = \hat{\kappa} \circ \phi_{\hat{T}}$. Recalling the definition of $W^{s,q}(\hat{e})$ and applying a change of variables, we have

$$\begin{aligned} |\hat{\kappa}|^q_{W^{s,q}(\hat{e})} &= \int_{\hat{e}} \int_{\hat{e}} \frac{|\hat{\kappa}(\hat{x}) - \hat{\kappa}(\hat{y})|^q}{|\hat{x} - \hat{y}|^{1+sq}} \, d\hat{x} \, d\hat{y} = \frac{|\hat{e}|^2}{|\tilde{e}|^2} \int_{\tilde{e}} \int_{\tilde{e}} \frac{|\tilde{\kappa}(\tilde{x}) - \tilde{\kappa}(\tilde{y})|^q}{|B_{\hat{T}}(\tilde{x} - \tilde{y})|^{1+sq}} \, d\tilde{x} \, d\tilde{y} \\ &\geq C |\tilde{\kappa}|^q_{W^{s,k}(\tilde{e})}. \end{aligned}$$

Because

$$\|\hat{\kappa}\|^q_{L^q(\hat{e})} = \frac{|\hat{e}|}{|\tilde{e}|} \|\tilde{\kappa}\|^q_{L^q(\tilde{e})} \geq C \|\tilde{\kappa}\|^q_{L^q(\tilde{e})},$$

we conclude $\|\hat{\kappa}\|_{W^{s,q}(\hat{e})} \geq C_3 \|\tilde{\kappa}\|_{W^{s,q}(\tilde{e})}$.

(iii) The proof of (A.20) and (A.21) follows the same arguments as (ii); we omit the details.

■

Now we are ready to prove Lemma 4.3.8.

Proof of Lemma 4.3.8. Let $\hat{v} \in H^{1/2+\delta}(\hat{T})$ and set $\tilde{v} = \hat{v} \circ \phi_{\hat{T}}$. Then $\tilde{v} \in H^{1/2+\delta}(\tilde{T})$ by Proposition A.4.3. Setting $s = \delta - 1 + 3/p$ then we see that since $p < \frac{2}{1-\delta}$ we have $s > 1/p$. Hence by the trace inequality (2.3) we have

$$\|\tilde{v}\|_{L^p(\partial\tilde{T})} \leq \|\tilde{v}\|_{W^{s-1/p,p}(\partial\tilde{T})} \leq C\|\tilde{v}\|_{W^{s,p}(\tilde{T})}.$$

By our choice we have $s - 3/p = 1/2 + \delta - 3/2$ and hence by the Sobolev inequality (2.2)

$$\|\tilde{v}\|_{W^{s,p}(\tilde{T})} \leq C\|\tilde{v}\|_{H^{1/2+\delta}(\tilde{T})}.$$

A change of variables along with (A.19) then shows

$$\|\hat{v}\|_{L^p(\partial\hat{T})} \leq C\|\tilde{v}\|_{L^p(\partial\tilde{T})} \leq C\|\tilde{v}\|_{H^{1/2+\delta}(\tilde{T})} \leq C\|\hat{v}\|_{H^{1/2+\delta}(\hat{T})}.$$

■

A.4.3 Proof of Lemma 4.3.9

We start with the same result for the reference element $\tilde{T}$.

Lemma A.4.4. *Let $\tilde{e} \in \Delta_1(\tilde{T})$ and $\tilde{F} \in \Delta_2(\tilde{T})$ be an edge and face of the reference tetrahedron, respectively, such that $\tilde{e} \in \Delta_1(\tilde{F})$. Then for $1 < q < 2$, there exists*

$\tilde{E} : \mathcal{P}_{r-3}(\tilde{e}) \to W^{1,q}(\tilde{T})$ such that $(\tilde{E}\tilde{\kappa})|_{\tilde{e}} = \tilde{\kappa}$, $(\tilde{E}\tilde{\kappa})|_{\partial\tilde{F}\setminus\tilde{e}} = 0$, $(\tilde{E}\tilde{\kappa})|_{\partial\tilde{T}\setminus\tilde{F}} = 0$ for all $\tilde{\kappa} \in \mathcal{P}_{r-3}(\tilde{e})$, and the following estimates hold:

$$\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{F})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})},$$
$$\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{T})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})}.$$

Proof. We first extend $\tilde{\kappa} \in \mathcal{P}_{r-3}(\tilde{e}) \subset W^{1-1/q,q}(\tilde{e})$ by zero to $\partial\tilde{F}$, denoted by $\tilde{\kappa}_1$. With $q < 2$ and the definition of $W^{1-1/q,q}(\tilde{e})$, we have

$$\begin{aligned}\|\tilde{\kappa}_1\|^q_{W^{1-1/q,q}(\partial\tilde{F})} &= \|\tilde{\kappa}\|^q_{W^{1-1/q,q}(\tilde{e})} + 2\int_e\int_{\partial\tilde{F}\setminus\tilde{e}} \frac{|\tilde{\kappa}(\tilde{x})|^q}{|\tilde{x}-\tilde{y}|^q}\,\tilde{y}\,d\tilde{x}\\
&\leq \|\tilde{\kappa}\|^q_{W^{1-1/q,q}(\tilde{e})} + 2\|\tilde{\kappa}\|^q_{L^\infty(\tilde{e})}\int_{\tilde{e}}\int_{\partial\tilde{F}\setminus\tilde{e}} \frac{1}{|\tilde{x}-\tilde{y}|^q}\,d\tilde{y}\,d\tilde{x}\\
&\leq \|\tilde{\kappa}\|^q_{W^{1-1/q,q}(\tilde{e})} + C\|\tilde{\kappa}\|^q_{L^\infty(\tilde{e})} \leq C\|\tilde{\kappa}\|^q_{W^{1-1/q,q}(\tilde{e})},\end{aligned}$$

where we used that $\int_{\tilde{e}}\int_{\partial\tilde{F}\setminus\tilde{e}} \frac{1}{|\tilde{x}-\tilde{y}|^q}\,d\tilde{y}\,d\tilde{x}$ is finite for $q < 2$ and (2.5a).

We extend $\tilde{\kappa}_1$ to $\tilde{F}$ using (2.4) and denote the extension by $\tilde{\kappa}_2 \in W^{1,q}(\tilde{F})$ with the estimate:

$$\|\tilde{\kappa}_2\|_{W^{1,q}(\tilde{F})} \leq C\|\tilde{\kappa}_1\|_{W^{1-1/q,q}(\partial\tilde{F})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})}. \tag{A.22}$$

Similarly, we extend $\tilde{\kappa}_2 \in W^{1,q}(\tilde{F})$ by zero to $\partial\tilde{T}$, which we denote by $\tilde{\kappa}_3$. Set $s = 2/(2-q)$ then we see that using a Sobolev inequality $\|\tilde{\kappa}_2\|_{L^{qs}(\tilde{F})} \leq C\|\tilde{\kappa}_2\|_{W^{1,q}(\tilde{F})}$.

Using definition of $W^{1-1/q,q}(\partial\tilde{T})$ and Hölder's inequality, we have:

$$\begin{aligned}\|\tilde{\kappa}_3\|^q_{W^{1-1/q,q}(\partial\tilde{T})} &= \|\tilde{\kappa}_2\|^q_{W^{1-1/q,q}(\tilde{F})} + \int_{\tilde{F}}\int_{\partial\tilde{T}\setminus\tilde{F}} \frac{|\tilde{\kappa}_2(\tilde{x})|^q}{|\tilde{x}-\tilde{y}|^{q+1}}\, dA(y)dA(x) \\ &\leq \|\tilde{\kappa}_2\|^q_{W^{1-1/q,q}(\tilde{F})} \\ &\qquad + \|\tilde{\kappa}_2\|^q_{L^{qs}(\tilde{F})}\Big(\int_{\tilde{F}}\int_{\partial\tilde{T}\setminus\tilde{F}} \frac{1}{|x-y|^{(q+1)s'}}\, dA(y)dA(x)\Big)^{1/s'} \\ &\leq \|\tilde{\kappa}_2\|^q_{W^{1-1/q,q}(\tilde{F})} + C\|\tilde{\kappa}_2\|^q_{L^{qs}(\tilde{F})} \leq C\|\tilde{\kappa}_2\|^q_{W^{1,q}(\tilde{F})},\end{aligned}$$

where we used $s' = 2/q$ which implies $(q+1)s' = (q+1)\frac{2}{q} < 3 < 4$ and hence the double integral is finite.

Again we use (2.4) to lift $\tilde{\kappa}_3$ to $\tilde{T}$ where we denote the lifting by $\tilde{E}\tilde{\kappa} \in W^{1,q}(\tilde{T})$ and it has the estimate

$$\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{T})} \leq C\|\tilde{\kappa}_3\|_{W^{1-1/q,q}(\partial\tilde{T})} \leq C\|\tilde{\kappa}_2\|_{W^{1,q}(\tilde{F})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})}.$$

Furthermore, we have

$$\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{F})} = \|\tilde{\kappa}_2\|_{W^{1,q}(\tilde{F})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})}.$$

■

Using Lemma A.4.4 and Proposition A.4.3, we now prove Lemma 4.3.9.

Proof of Lemma 4.3.9. Let $\hat{\kappa} \in \mathcal{P}_{r-3}(\hat{e})$, and let $\tilde{\kappa} \in \mathcal{P}_{r-3}(\tilde{e})$ (with $\tilde{e} = \phi^{-1}_{\hat{T}}(\hat{e})$) be given as $\tilde{\kappa} = \hat{\kappa} \circ \phi_{\hat{T}}$. By (ii) of Proposition A.4.3 there holds

$$\|\hat{\kappa}\|_{W^{1-1/q,q}(e)} \geq C_3\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})}.$$

By Lemma A.4.4, there exists $\tilde{E} : \mathcal{P}_{r-3}(\tilde{e}) \to W^{1,q}(\tilde{T})$ such that $(\tilde{E}\tilde{\kappa})|_{\tilde{e}} = \tilde{\kappa}$ and $(\tilde{E}\tilde{\kappa})|_{\partial\tilde{F}\setminus\tilde{e}} = 0$, $(\tilde{E}\tilde{\kappa})|_{\partial\tilde{T}\setminus\tilde{F}} = 0$. Let $E : \mathcal{P}_{r-3}(e) \to W^{1,q}(\hat{T})$ be defined by $(E\hat{\kappa}) = (\tilde{E}\tilde{\kappa})\circ\phi_{\hat{T}}$. Then $E\hat{\kappa}|_{\hat{e}} = \hat{\kappa}$ $(E\hat{\kappa})|_{\partial\hat{F}\setminus\hat{e}} = 0$, $(E\hat{\kappa})|_{\partial\hat{T}\setminus\hat{F}} = 0$ and with (iii) of Proposition A.4.3, we have

$$\|E\hat{\kappa}\|_{W^{1,q}(\hat{F})} \leq C\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{F})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})} \leq C\|\hat{\kappa}\|_{W^{1-1/q,q}(\hat{e})},$$

and

$$\|E\hat{\kappa}\|_{W^{1,q}(\hat{T})} \leq C\|\tilde{E}\tilde{\kappa}\|_{W^{1,q}(\tilde{T})} \leq C\|\tilde{\kappa}\|_{W^{1-1/q,q}(\tilde{e})} \leq C\|\hat{\kappa}\|_{W^{1-1/q,q}(\hat{e})}.$$

■

Appendix B

Additional details for Chapter 6

In this appendix, we covered some proofs of the theorems in Chapter 6.

B.1 Proof of Proposition 2.3.1

Proof. The range of $[r_0\ s_0]$ is contained in the kernel of $t_1 \circ s_1^{-1} \circ r_1$ because for any $(u, v) \in A_0 \times B_0$

$$t_1 s_1^{-1} r_1(r_0 u + s_0 v) = t_1 s_1^{-1} r_1 s_0 v = t_1 s_1^{-1} s_1 t_0 v = t_1 t_0 v = 0,$$

where we have used the given assumptions that the top and bottom sequences in (2.10) are sequences and that the commutativity property, $r_1 s_0 = s_1 t_0$, holds. For the reverse inclusion, if $u \in A_1$ is in the kernel of $t_1 \circ s_1^{-1} \circ r_1$, then $s_1^{-1} r_1 u$ is in $\ker t_1 = \operatorname{range} t_0$, so there is a $v \in B_0$ such that $s_1^{-1} r_1 u = t_0 v$, i.e., $0 = r_1 u - s_1 t_0 v = r_1 u - r_1 s_0 v$ using the commutativity property again. Hence, $u - s_0 v$ is in $\ker r_1 = \operatorname{range} r_0$, i.e., $u = s_0 v + r_0 u_0$ for some $u_0 \in A_0$, thus showing that u is in the range of $[r_0\ s_0]$ and completing the proof of range $[r_0\ s_0] = \ker(t_1 \circ s_1^{-1} \circ r_1)$.

We use the other commutativity property, $r_2 s_1 = s_2 t_1$, to prove range $(t_1 \circ s_1^{-1} \circ r_1) = \ker \left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$. Consider a v_2 in the latter kernel, i.e., $s_2 v_2 = 0$ and $t_2 v_2 = 0$. Since $\ker(t_2) = \operatorname{range}(t_1)$, there is a $v_1 \in B_1$ such that $v_2 = t_1(v_1)$, so $0 = s_2 v_2 = s_2 t_1 v_1 = r_2 s_1 v_1$. Since $s_1 v_1 \in \ker(r_2) = \operatorname{range}(r_1)$, there is a $u_1 \in A_1$ such that $s_1 v_1 = r_1 u_1$, i.e., $v_1 = s_1^{-1} r_1 u_1$. Thus $v_2 = t_1 v_1 = t_1 s_1^{-1} r_1 u_1$, i.e., range $(t_1 \circ s_1^{-1} \circ r_1) \supseteq \ker \left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right]$. The reverse inclusion is easy.

To prove the final statement, consider a $u_3 \in A_3$ and $v_3 \in B_3$. If t_2 is surjective, then there is a $\tilde{v}_2 \in B_2$ such that $t_2 \tilde{v}_2 = v_3$. If r_2 is also surjective, then we can find a u_2 and $\tilde{u}_2$ in A_2 such that $r_2 u_2 = u_3$ and $r_2 \tilde{u}_2 = s_2 \tilde{v}_2$. It may now be easily verified that $v_2 = t_1 s_1^{-1} u_2 + \tilde{v}_2 - t_1 s_1^{-1} \tilde{u}_2$ in B_2 satisfies $\left[\begin{smallmatrix} s_2 \\ t_2 \end{smallmatrix}\right] v_2 = \left[\begin{smallmatrix} u_3 \\ v_3 \end{smallmatrix}\right]$. ■

B.2 Proof of Lemma 6.1.5

Proof. We first show that $\dim P_U \mathsf{R} = \dim \mathsf{R} = 6$. This follows if we show that the kernel of P_U is empty. Let $v \in \mathsf{R}$ and assume that $P_U v = 0$. Then, by the definition of P_U and the fact that v is a linear function, we must have that v vanishes on the barycenter of each $K \in T^{wf}$. This implies that $v \equiv 0$ if there are three such barycenters that are not collinear. To see that there are such barycenters, recall that the barycenter of $K \in T^{wf}$ is the average of the four vertices of K. Hence the line connecting barycenters of two adjacent $K_\pm \in T^{wf}$ is parallel to the line connecting the two vertices opposite to the common face $F = \partial K_+ \cap \partial K_-$. Thus taking, for example, three subtetrahedra in T^{wf} with a face contained in a common $F \in \Delta_2(\mathcal{T}_h)$, we see that their barycenters cannot be collinear, since no three of their vertices are collinear.

We now prove (6.8). Since $\dim \mathsf{R} = 6$ and by the definition of $\mathring{U}_0^3(T^{wf})$, we have

$$\dim \mathring{U}_0^3(T^{wf}) \geq \dim U_0^3(T^{wf}) - \dim \mathsf{R} = 36 - 6 = 30.$$

We use that

$$U_0^3(T^{wf}) = \mathring{U}_0^3(T^{wf}) \oplus [\mathring{U}_0^3(T^{wf})]^\perp,$$

and obtain $\dim[\mathring{U}_0^3(T^{wf})]^\perp \leq 6$. However, one can easily show that $P_U \mathsf{R} \subset [\mathring{U}_0^3(T^{wf})]^\perp$ which implies $\dim[\mathring{U}_0^3(T^{wf})]^\perp = 6$ and $P_U \mathsf{R} = [\mathring{U}_0^3(T^{wf})]^\perp$.

■

B.3 Proof of Lemma 6.2.3

Proof. Fix $F \in \Delta_2(T)$, and let $e \in \Delta_1^I(F^{\text{ct}})$ be an internal edge in the induced Clough-Tocher split of F. Let f be the corresponding internal face of T^{wf} with e as an edge, and let n_f is a unit-normal to f. We further set t_e to be a unit tangent vector to e and $s_e = n_F \times t_e$ to be a unit tangent vector of F orthogonal to t_e.

Since $n_f \cdot t_e = 0$, we have $n_f = (n_f \cdot n_F)n_F + (n_f \cdot s_e)s_e$. Since $\sigma \in V_r^2(T^{wf}) \otimes \mathbb{V}$, we have σn_f is single-valued on e and hence, by symmetry of σ, $(\sigma\ell) \cdot n_f$ is single-valued on e. Therefore, on e, with $(\sigma\ell) \cdot n_F = n_F'\sigma\ell = 0$, we have $(\sigma\ell) \cdot n_f = (n_f \cdot s_e)(\sigma\ell) \cdot s_e$ and so $[\![\sigma_{F\ell} \cdot s_e]\!]_e = [\![(\sigma\ell) \cdot s_e]\!]_e = 0$ for any $e \in \Delta_1^I(F^{\text{ct}})$. Therefore, $\sigma_{F\ell} \in V_{\text{div},r}^1(T^{\text{ct}})$ on each $F \in \Delta_2(T)$. ■

B.4 Proof of Lemma 6.2.4

Proof. Since $w \in V_{r-1}^1(T^{wf}) \otimes \mathbb{V}$ and $w' \in V_{r-1}^2(T^{wf}) \otimes \mathbb{V}$, then $n_f'w$, wt_e and wt_s are continuous cross e on F:

$$[\![n_f'w]\!]_e = 0, \quad [\![wt_e]\!]_e = 0, \quad [\![wt_s]\!]_e = 0. \tag{B.1}$$

Let $s_e = \alpha_1 n_f + \beta_1 t_s$, $n_F = \alpha_2 n_f + \beta_2 t_s$, and note $\alpha_1 \neq 0$ and $\beta_2 \neq 0$.

Since $n_F'wQ|_{\partial T} = 0$, for any $e \in \Delta_1^I(F^{\text{ct}})$,

$$\begin{aligned} 0 = [\![n_F'ws_e]\!]_e &= [\![(\alpha_2 n_f' + \beta_2 t_s')w(\alpha_1 n_f + \beta_1 t_s)]\!]_e \\ &= \alpha_1\alpha_2[\![n_f'wn_f]\!]_e + \alpha_2\beta_1[\![n_f'wt_s]\!]_e + \alpha_1\beta_2[\![t_s'wn_f]\!]_e + \beta_2\beta_1[\![t_s'wt_s]\!]_e \end{aligned}$$

$$= \alpha_1\beta_2[\![t_s' w n_f]\!]_e.$$

Thus, we have

$$[\![t_s' w n_f]\!]_e = 0,$$

and therefore

$$[\![s_e' w s_e]\!]_e = \alpha_1^2[\![n_f' w n_f]\!]_e + \alpha_1\beta_1[\![n_f' w t_s]\!]_e + \alpha_1\beta_1[\![t_s' w n_f]\!]_e + \beta_1^2[\![t_s' w t_s]\!]_e = 0.$$

We have $[\![t_e' w n_f]\!]_e = 0$ since $w_{FF} = 0$ and

$$0 = [\![t_e' w s_e]\!]_e = [\![t_e' w(\alpha_1 n_f + \beta_1 t_s)]\!]_e = \alpha_1[\![t_e' w n_f]\!]_e + \beta_1[\![t_e' w t_s]\!]_e = \alpha_1[\![t_e' w n_f]\!]_e,$$

where we use (B.1). This implies that

$$[\![t_e' w n_F]\!]_e = 0$$

since

$$[\![t_e' w n_F]\!]_e = [\![t_e' w(\alpha_2 n_f + \beta_2 t_s)]\!]_e = 0.$$

■

B.5 Proof of Lemma 6.2.5

Proof. Write $\ell = a_1 t_1 + a_2 t_2$, $m = a_1 t_2 - a_2 t_1$, where t_1, t_2 are tangential basis defined in Section 2.4. We also set $t_3 = n_F$, and write $u = \sum_{i,j=1}^{3} u_{ij} t_i t_j'$. We then have the following identities for the components of $\operatorname{curl} u$ ($s \in \{1, 2, 3\}$):

$$
\begin{aligned}
t_s'(\operatorname{curl} u)t_1 &= \partial_{t_2} u_{s3} - \partial_{t_3} u_{s2},\\
t_s'(\operatorname{curl} u)t_2 &= \partial_{t_3} u_{s1} - \partial_{t_1} u_{s3},\\
t_s'(\operatorname{curl} u)t_3 &= \partial_{t_1} u_{s2} - \partial_{t_2} u_{s1}.
\end{aligned}
\tag{B.2}
$$

We then compute

$$
\begin{aligned}
\ell'(\operatorname{curl} u)m &= (a_1 t_1 + a_2 t_2)'(\operatorname{curl} u)(a_1 t_2 - a_2 t_1)\\
&= (a_1)^2(\partial_{t_3} u_{11} - \partial_{t_1} u_{13}) - (a_2)^2(\partial_{t_2} u_{23} - \partial_{t_3} u_{22})\\
&\qquad + a_1 a_2(\partial_{t_3} u_{21} - \partial_{t_1} u_{23} - \partial_{t_2} u_{13} + \partial_{t_3} u_{12})\\
&= \partial_{t_3}((a_1)^2 u_{11} + (a_2)^2 u_{22} + a_1 a_2(u_{21} + u_{12}))\\
&\qquad - a_1(a_2 \partial_{t_2} u_{13} + a_1 \partial_{t_1} u_{13}) - a_2(a_1 \partial_{t_1} u_{23} + a_2 \partial_{t_2} u_{23})\\
&= \partial_{t_3}(\ell' u_{FF} \ell) - a_1 \partial_\ell u_{13} - a_2 \partial_\ell u_{23}\\
&= \partial_n(\ell' u_{FF} \ell) - \partial_l(u_{Fn} \cdot \ell) = \partial_n(\ell' u_{FF} \ell) - \operatorname{grad}_F(u_{Fn} \cdot \ell) \cdot \ell.
\end{aligned}
\tag{B.3}
$$

Similarly, by using (B.2), we have

$$
\begin{aligned}
\ell'(\operatorname{curl} u)\ell &= (a_1 t_1 + a_2 t_2)'(\operatorname{curl} u)(a_1 t_1 + a_2 t_2)\\
&= (a_1)^2(\partial_{t_2} u_{13} - \partial_{t_3} u_{12}) + (a_2)^2(\partial_{t_3} u_{21} - \partial_{t_1} u_{23})\\
&\qquad + a_1 a_2(\partial_{t_2} u_{23} - \partial_{t_3} u_{22} + \partial_{t_3} u_{11} - \partial_{t_1} u_{13})\\
&= \partial_{t_3}(-(a_1)^2 u_{12} + (a_2)^2 u_{21} + a_1 a_2(u_{11} - u_{22}))\\
&\qquad - a_1(-a_1 \partial_{t_2} u_{13} + a_2 \partial_{t_1} u_{13}) + a_2(a_1 \partial_{t_2} u_{23} - a_2 \partial_{t_1} u_{23})\\
&= -\partial_{t_3}(m' u_{FF} \ell) + a_1 \partial_m u_{13} + a_2 \partial_m u_{23}\\
&= -\partial_n(m' u_{FF} \ell) + \partial_m(u_{Fn} \cdot \ell) = -\partial_n(m' u_{FF} \ell) + \operatorname{grad}_F(u_{Fn} \cdot \ell) \cdot m.
\end{aligned}
\tag{B.4}
$$

Finally, again by using (B.2), we have

$$\begin{aligned} n_F'(\operatorname{curl} u)\ell &= t_3'(\operatorname{curl} u)(a_1t_1 + a_2t_2) \\ &= (a_1\partial_{t_2}u_{33} - a_2\partial_{t_1}u_{33}) + \partial_{t_3}(a_2u_{31} - a_1a_{32}) \\ &= \partial_m u_{33} - \partial_n(u_{nF}\cdot m) = (\operatorname{grad}_F u_{33})\cdot m - \partial_n(u_{nF}\cdot m). \end{aligned} \tag{B.5}$$

Lemma 6.2.5 now follows from (B.3)–(B.5) and the first case in Lemma 6.2.1. ∎

B.6 Proof of Lemma 6.2.6

Proof. (i) **Continuity:** we show the continuity of $w_{FF} - \operatorname{grad}_F u_{Fn}^\perp$. Recall the notation from Section 2.2.1.2. Since $w \in V_{r-1}^1(T^{wf})\otimes\mathbb{V}$ (by Theorem 6.1.7), for any $e\in\Delta_1^I(F^{\mathrm{ct}})$ we have $[\![w_{FF}t_e]\!]_e = 0$ due to $[\![wt_e]\!]_e = 0$. Consequently, because u is continuous, we have

$$[\![(w_{FF} - \operatorname{grad}_F u_{Fn}^\perp)t_e]\!]_e = 0. \tag{B.6}$$

Now to prove the continuity of $w_{FF} - \operatorname{grad}_F u_{Fn}^\perp$ on F, it suffices to prove $[\![(w_{FF} - \operatorname{grad}_F u_{Fn}^\perp)s_e]\!]_e = 0$ for all $e\in\Delta_1^I(F^{\mathrm{ct}})$. Using $w'\in V_{r-1}^2(T^{wf})\otimes\mathbb{V}$ and $w_{Fn}=0$, by Lemma 6.2.4 we have

$$[\![s_e' w_{FF} s_e]\!]_e = [\![s_e' w s_e]\!]_e = 0. \tag{B.7}$$

Next we show that $[\![s_e'\operatorname{grad}_F(u_{Fn}^\perp)s_e]\!]_e = 0$ and $[\![t_e'(w_{FF} - u_{Fn}^\perp)s_e]\!]_e = 0$. Since $u\in\mathsf{L}_r^1(T^{wf})\otimes\mathbb{V}$ and $u_{FF}=0$ on F, we have

$$\begin{aligned} [\![s_e'\operatorname{grad}_F(u_{Fn}^\perp)s_e]\!]_e &= [\![\operatorname{grad}_F(u_{Fn}^\perp\cdot s_e)\cdot s_e]\!]_e \\ &= [\![\operatorname{grad}_F(u_{Fn}\cdot t_e)\cdot s_e]\!]_e = -[\![t_e'(\operatorname{curl} u)' t_e]\!]_e = 0, \end{aligned} \tag{B.8}$$

where the second equality comes from (2.14) and the third equality uses (6.23) in Lemma 6.2.5 with $\ell = t_e$ and $m = s_e$. Similarly by (6.22) in Lemma 6.2.5 with $\ell = s_e$, $m = -t_e$ and (2.14), we have $[\![t_e' \operatorname{grad}_F(u_{Fn}^\perp)s_e]\!]_e = [\![t_e'(\operatorname{curl} u)'s_e]\!]_e$. Therefore, we have

$$[\![t_e'(w_{FF} - \operatorname{grad}_F u_{Fn}^\perp)s_e]\!]_e = [\![t_e'[(\operatorname{curl} u)']_{FF}s_e]\!]_e - [\![t_e'(\operatorname{curl} u)'s_e]\!]_e = 0. \tag{B.9}$$

Combining (B.6), (B.7), (B.8) and (B.9), we conclude that $w_{FF} - \operatorname{grad}_F u_{Fn}^\perp$ is continuous on F.

(ii) **Proof of** (6.25)**:** With (2.19g), (2.19h) and (2.13), we have

$$2w_{FF} = \operatorname{grad}_F(\operatorname{grad}_F(v \cdot n_F) \times n_F - (\partial_n v_F) \times n_F).$$

Then with (2.19i), (2.19j) and (2.14), we obtain

$$2\operatorname{grad}_F u_{nF}^\perp = 2\operatorname{grad}_F[(\varepsilon(v))_{nF}]^\perp = \operatorname{grad}_F(\operatorname{grad}_F(v \cdot n_F) \times n_F + (\partial_n v_F) \times n_F).$$

Therefore, by computing the difference of the above two equations, we conclude

$$w_{FF} - \operatorname{grad}_F u_{nF}^\perp = \operatorname{grad}_F(\partial_n v_F \times n_F).$$

$\blacksquare$

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