A Dynamical System Approach to the Inverse Spectral Problem for Hankel Operators

by

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Abstract of "An Inverse Spectral Problem for Hankel Operators" by Zhehui Liang, Ph.D., Brown University, May 2021

This dissertation is devoted to the study of inverse spectral problem of Hankel operators. It is well-known that spectral characteristics of a Hankel operator does not uniquely define it: there are many unitarily equivalent Hankel operators. However, as it was noticed in breakthrough papers by P. Gerald and S. Grellier the spectral characteristics of the Hankel operator and its one column truncation completely determine the compact Hankel operator. This turns out to be true for general Hankel operators, which is one of the results of the thesis.

For self-adjoint Hankel operators it is pretty easy to understand what the spectral characteristics are. For the general case the approach is more involved and based on the theory of complex symmetric operators. In both cases we state and prove an abstract theorem that reduces the existence of a Hankel operator with prescribed spectral properties to the asymptotic stability of some contractions, hence the "dynamical system approach" in the title.

We then apply this abstract theorem to the particular case of compact operators; the asymptotic stability there is obtained almost for free, and the description of spectral characteristics can be greatly simplified. We are able to give new proofs for known results by Gerard–Grillier, as well as to prove some new results.

For compact Hankel operator with simple singular values, we will show that it can be uniquely characterized by two sequences of complex numbers whose modulus part satisfy an intertwining relation. While for Hankel operator with non-simple singular values, we will show that it will be uniquely characterized by two sequences of complex numbers with the same requirement, together with two sequences of discrete probability measures on the unit circle.

CONTENTS

• Disclaimer

This thesis is highly based on the two submitted preprints on arXiv, and the author of this thesis is one of the coauthors of these two papers.

(i) S. R. Treil, Z. Liang, *A dynamical system approach to the inverse spectral problem for Hankel operators: the case of compact operators with simple singular values*, preprints

(ii) S. R. Treil, Z. Liang, *An inverse spectral problem for Hankel operators: the case of compact operators with non-simple singular values*, preprints

Chapter 0

NOTATION

This short section contains the list of main symbols. We consider only separable Hilbert spaces, usually denoted by $\mathcal{H}$.

$:=$	Equal by definition;
ℓ^2	The Hilbert space formed by infinite complex sequence $x = (x_1, x_2, ...)$ with constraint $\sum_{i=1}^{\infty} \lvert x_i\rvert^2 < \infty$;
S	Symbol for shift operator on spaces ℓ^2 or $H^2(\mathbb{T})$;
$\mathbb{T}$	The unit circle on the complex plane, $\mathbb{T} := \{z \in \mathbb{C} : \lvert z\rvert = 1\}$;
$\mathbb{D}$	The unit disk on the complex plane, $\mathbb{D} := \{z \in \mathbb{C} : \lvert z\rvert \leq 1\}$;
$H^p(\mathbb{T})$	The Hardy space on $\mathbb{T}$ defined as $H^p(\mathbb{T}) := \{f : f \in L^p(\mathbb{T}), f = \sum_{k=0}^{\infty} a_k z^k\}$ for $1 \leq p < \infty$;
A^*	Adjoint of an operator A;
$\lvert A\rvert$	Modulus of an operator A, $\lvert A\rvert := (A^*A)^{1/2}$;
$\mathrm{Ker}\, A$	The kernel space of an operator A;
$\mathrm{Ran}\, A$	The range space of an operator A;

$P_{\mathcal{H}}x$	The orthogonal projection of a vector x onto a subspace $\mathcal{H}$.
x^*	For $x \in \mathcal{H}$, x^* is the bounded linear functional on $\mathcal{H}$ defined as $y \mapsto \langle x, y \rangle$.
$\mathcal{R}$	Symbol for a self-adjoint operator;
R	Symbol for a positive self-adjoint operator;
$\mathcal{T}$	Symbol for a contraction;
$\widetilde{\Gamma}$	The restriction of an operator Γ on its essential part, that is $\Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}$.
$\mathfrak{C}, \mathfrak{J}$	Symbols for conjugations;
$\mathcal{H}_0^\perp$	The orthogonal complement of $\mathcal{H}_0$ on the whole space;
$\sigma(A)$	The spectrum of an operator A;
$\sigma_p(A)$	The point spectrum of an operator A;
$\widetilde{\rho}$	The normalization of a finite measure ρ, i.e., the total measure of $\widetilde{\rho}$ equals 1;
$D_{\mathcal{T}}$	Defect operator of a contraction $\mathcal{T}$, defined as $D_{\mathcal{T}} := (I - \mathcal{T}^*\mathcal{T})^{1/2}$;
$\mathcal{D}_{\mathcal{T}}$	Defect space of a contraction $\mathcal{T}$, defined as $\mathcal{D}_{\mathcal{T}} := \operatorname{Clos}\operatorname{Ran} D_{\mathcal{T}}$;
$\mathfrak{D}_{\mathcal{T}}$	Defect indice of a contraction $\mathcal{T}$, defined as $\mathfrak{D}_{\mathcal{T}} := \dim \mathcal{D}_{\mathcal{T}}$;
$\rightharpoonup$	Symbol for weak convergence. We say a sequence of vector $\{x_n\} \rightharpoonup x_0$ if and only if $\lim\limits_{n\to\infty} \langle x_n, y\rangle \to \langle x, y\rangle$ holds for all $y \in \mathcal{H}$.

Chapter 1

INTRODUCTION TO HANKEL OPERATORS

In this chapter, we will first give the definition and some well-known results about Hankel operators in section 1.1. To follow up, we will introduce some substantial historical progress on the inverse spectral problem of Hankel operators in section 1.2.

1.1 Introduction to Hankel Operators

Definition 1.1.1. *A* Hankel matrix *is an infinite matrix of form* $\{\gamma_{j+k}\}_{j,k>=0}^{\infty}$, *i.e. a matrix whose entries depend only on the sum of indices.*

A Hankel operator Γ *is a bounded operator in* $\ell^2 = \ell^2(\mathbb{Z}_+)$ *whose matrix in the standard basis* $\{e_n\}_{n=0}^{\infty}$ *of* ℓ^2 *is Hankel, i.e.* $\langle \Gamma e_j, e_k \rangle = \gamma_{j+k}$ *holds for all* $j, k \in N$. *And we call* $\{\gamma_n\}_{n=0}^{\infty}$ *to be the Hankel coefficients of* Γ.

Remark 1.1.2. Let S be the *shift* operator on ℓ^2 defined as

$$S(x_0, x_1, x_2, \ldots) = (0, x_0, x_1, x_2, \ldots),$$

and its adjoint S^* given by

$$S(x_0, x_1, x_2, \ldots) = (x_1, x_2, \ldots).$$

The fact that an operator Γ is Hankel is equivalent to the following equation:

$$\Gamma S = S^*\Gamma, \tag{1.1.1}$$

so the above identity (1.1.1) can also be used as the definition for Hankel operators.

The following result is a well-known theorem proved by Nehari in [2]. We first recall the definition of *bounded mean oscillation functions* (BMO for abbreviation).

Definition 1.1.3 (Definition of BMO function)**.** *We say an integrable function f on $\mathbb{T}$ is of bounded mean oscillation (BMO) if*

$$\|f\| := \sup_I \frac{1}{|I|}\int_I |f - f_I| dm < \infty,$$

where the supremum runs over all arcs of $\mathbb{T}$ and $f_I = \frac{1}{|I|}\int_I f dm$ is the mean of f over I.

Theorem 1.1.4 (Nehari, 1957)**.** *A Hankel operator $\Gamma : \ell^2 \to \ell^2$ with coefficients $\{\gamma_n\}_{n=0}^{\infty}$ is bounded if and only if there exists a function $f \in L^\infty(\mathbb{T})$, such that $\forall n \geq 0$, $\hat{f}(n) = \gamma_n$. Or equivalently, the Fourier series $\sum_{n\geq 0} \gamma_n e^{inz}$ is BMO.*

Another famous result is proved by Hartman in [4], given an equivalent condition for compact Hankel operators on ℓ^2. We first recall the definition of *vanishing mean oscillation functions* (VMO for abbreviation).

Definition 1.1.5 (Definition of VMO functions)**.** *We say an integrable function f on $\mathbb{T}$ is of vanishing mean oscillation (VMO) if $f \in BMO(\mathbb{T})$, and additionally, the mean oscillation of f on I goes to 0 uniformly when the arc length of I goes to 0.*

Theorem 1.1.6 (Hartman, 1958)**.** *A Hankel operator $\Gamma : \ell^2 \to \ell^2$ with coefficients $\{\gamma_n\}_{n=0}^{\infty}$ is compact if and only if there exists a continuous function f on $\mathbb{T}$, such that $\hat{f}(n) = \gamma_n$ for $n \geq 0$. Or equivalently, the Fourier series $\sum_{n\geq 0} \gamma_n e^{inz}$ is a function belongs to $VMO(\mathbb{T})$.*

As for finite rank Hankel operators, the following theorem is proved by L.Kronecker in [5], stated as follow:

Theorem 1.1.7 (Kronecker 1881)**.** *A Hankel operator* Γ *with coefficients* $\{\gamma_n\}_{n=0}^{\infty}$ *is finite-rank, if and only if the sum of series*

$$R(z) = \sum_{i=0}^{\infty} \frac{\gamma_i}{z^i}$$

is a rational function of z*.*

Remark 1.1.8. There is also another Hardy space representation for Hankel operators, given in [1, Part B Chapter 1, p. 180], which we will just briefly introduce it here. (We only need Definition 1.1.1 in the main part of this thesis).

We first recall the definition of Hardy space $H^p(\mathbb{T}) := \{f : f \in L^p(\mathbb{T}), f = \sum_{k=0}^{\infty} a_k z^k\}$ for $1 \leq p < \infty$. Here H^p can be naturally identified with the spaces of analytic functions on the unit disk $\mathbb{D}$. For $p = \infty$, we define H^∞ to be the space of all bounded analytic function on the unit disk with the supremum norm.

Now we construct a map from H^2 to $H^2_- := L^2 \ominus H^2$. For a function $\varphi \in L^\infty(\mathbb{T})$, we define the operator H_φ as:

$$H_\varphi f := P_-(\varphi f),$$

here $f \in H^2$ and P_- is the natural orthogonal projection from H^2 to H^2_-.

For this $H_\varphi : H^2 \to H^2_-$, we can check that the representation matrix of H_φ, with respect to basis $\mathcal{B}_1 = \{e^{int}\}_{n\geq 0}$ on H^2, and basis $\mathcal{B}_2 = \{e^{int}\}_{n<0}$ for H^2_-, is an infinite symmetric matrix, thus H_φ has Hankel structure, and we call this H_φ as Hankel operators from H^2 to H^2_-.

Now we can use this H_φ to define an operator from ℓ^2 to ℓ^2 which is Hankel. We first define an involution operator $\mathcal{J}$ given by

$$\mathcal{J}e^{ikt} = e^{-i(k+1)t} \qquad \text{holds for all } k \in \mathbb{Z},$$

(easy to check that $\mathcal{J}H^2 = H^2_-$, $\mathcal{J}H^2_- = H^2$ and $\mathcal{J}^2 = \mathcal{I}$). Also recall the natural Fourier transform mapping $\mathcal{F}$ defined as

$$f \to (\hat{f}(0), \hat{f}(1), ...), \tag{1.1.2}$$

mapping H^2 to ℓ^2. Here $\hat{f}(n)$ is the Fourier coefficients defined as $\hat{f}(n) := \int_{\mathbb{T}} f e^{-inz} dz$. Now we define an operator $\Gamma : \ell^2 \to \ell^2$ as

$$\Gamma := \mathcal{F}\mathcal{J}H_\varphi\mathcal{F}^{-1}.$$

We can see that Γ is a Hankel operator with Hankel coefficients $\gamma_k = \hat{\varphi}(-k-1)$.

Remark 1.1.9. For this defined H_φ, we can see that it satisfies the following equation

$$H_\varphi S = P_- S H_\varphi, \tag{1.1.3}$$

here S is the shift operator on H^2 defined as $Sf := e^{it}$. The above identity can also be used as the definition of Hankel operators from H^2 to H^2_-.

Remark 1.1.10. From the definition of Hankel operators on Hardy spaces, it's easy to see that $\operatorname{Ker} H_\varphi$ is either trivial or infinite-dimensional (thus $\operatorname{Ker}\Gamma$ is also either trivial or infinite-dimensional for Hankel operator $\Gamma : \ell^2 \to \ell^2$). In fact, from (1.1.3), we can see that $\operatorname{Ker} H_\varphi$ is S-invariant, thus by Beurling theorem (see [7, Chapter 1]), it must has the form of θH^2 where θ is an inner function, thus it must be trivial or infinite-dimensional.

1.2 Historical Development on the Spectral Problem of Hankel Operators

1.2.1 Modulus of Hankel Operators

The spectral properties of Hankel operators was first studied in [6]. In this paper, Khrushchev and Peller raised the following question:

Problem 1.2.1. *Given R to be a non-negative self-adjoint operator on $\mathcal{H}$, does there exist a Hankel operator Γ such that its modulus $|\Gamma|$ is unitary equivalent to R?*

Easy to see that R needs to satisfy the following two conditions:

(i) $\dim \operatorname{Ker} R = 0$ or ∞;

(ii) R is non-invertible.

We have discussed the proof of condition (i) in Remark 1.1.10. As for condition (ii), it follows from the fact that if Γ is Hankel, then

$$\|\Gamma e_k\|^2 = \sum_{j \geq k} |\alpha_j|^2 \to 0 \qquad \text{as } k \to \infty$$

So now the question is whether (i) and (ii) are also sufficient for Problem 1.2.1.

The first progress was made in [8]. It was shown that if R satisfies (i), (ii), and additionally R has simple discrete spectrum, then R is unitary equivalent to $|\Gamma|$ for some Hankel Γ.

Then in 1990, Treil solved Problem 1.2.1, claiming that the discrete spectrum condition is unnecessary. That is,

Theorem 1.2.2. *R is a non-negative self-adjoint operator satisfying the following assumptions:*

(i) *R is non-invertible;*

(ii) $\dim \operatorname{Ker} R = 0$ *or* ∞.

Then there exists a Hankel operator Γ *, such that* $|\Gamma|$ *is unitary equivalent to* R*.*

1.2.2 Self-adjoint Hankel Operators

Now we consider the inverse spectral problem for Hankel operators.

Problem 1.2.3. *For a given self-adjoint operator* $\mathcal{R}$*, can we find out a Hankel operator* Γ*, such that* $\mathcal{R}$ *is unitary equivalent to* Γ*?*

To begin stating the equivalent conditions for this problem, we need some preparation on the definition of *Von Neumann integral* and *scalar multiplicity function.*

Given μ to be a finite positive Borel measure on $\mathbb{R}$, and $\{\mathcal{H}(t)\}_{t\in\mathbb{R}}$ be a measurable family of Hilbert spaces. Here the definition of measurable families is given as follow:

Definition 1.2.4 (Definition of measurable families)**.** *Let* $\{\mathcal{H}(t)\}_{t\in\mathbb{R}}$ *be a family of Hilbert spaces, then we say* $\{\mathcal{H}(t)\}_{t\in\mathbb{R}}$ *is a measurable family if and only if there exists at most a countable set* Ω *of functions* f *such that* $f(t) \in \mathcal{H}(t)$*,* μ*-a.e., such that*

$$\overline{\operatorname{Span}}\{f(t) : f \in \Omega\} = \mathcal{H}(t) \qquad \textit{for } \mu - \textit{almost all } t,$$

and the function

$$t \to \langle f_1(t), f_2(t)\rangle_{\mathcal{H}(t)}$$

is $\mu-$*measurable for any* $f_1, f_2 \in \Omega$*.*

If we denote $N(t) := \dim \mathcal{H}(t)$, then we can indeed embed all $\mathcal{H}(t)$ in a Hilbert space with orthogonal basis $\{e_n\}_{n=1}^{\infty}$, and set

$$\mathcal{H}(t) = \overline{\operatorname{Span}}\{e_k : 1 \leq k \leq N(t)\}$$

Definition 1.2.5. *We say a function g with values $g(t) \in \mathcal{H}(t)$ is* measurable *if the scalar function*

$$t \to \langle g(t), f(t) \rangle_{\mathcal{H}(t)}$$

is measurable for all $f \in \Omega$.

Now we are ready to define the *Von-Neumann integral space.*, and then followed by the *Von-Neumann theorem.*

Definition 1.2.6. *The* Von-Neumann integral *(or direct integral) $\int \oplus \mathcal{H}(t) d\mu(t)$ is a Hilbert space consists of measurable functions f (see definition 1.2.5), such that*

$$\|f\| = \left(\int \|f(t)\|^2_{\mathcal{H}(t)} d\mu(t) \right)^{1/2} < \infty.$$

And the inner product on the Von-Neumann integral is defined as

$$\langle f, g \rangle = \int \langle f(t), g(t) \rangle_{\mathcal{H}(t)} d\mu(t),$$

where $f, g \in \int \oplus \mathcal{H}(t) d\mu(t)$.

With this newly defined direct integral space, Von-Neumann proved that every self-adjoint operators is unitary equivalent to a multiplication by independent variable on some L^2 space. The theorem is stated as follow:

Theorem 1.2.7 (Von Neumann' theorem, see [11])**.** *Each self-adjoint operator defined on a separable Hilbert space is unitary equivalent to an operator A, which is a multiplication by independent variable defined on a direct integral $\int \oplus \mathcal{H}(t) d\mu(t)$:*

$$(Af)(t) = tf(t), \qquad f \in \int \oplus \mathcal{H}(t) d\mu(t).$$

And we say μ to be the scalar spectral measure *of A, and we say the function $N(t) = \dim \mathcal{H}(t)$ to be the* spectral multiplicity function *of A.*

It is also stated in [11] that two self-adjoint operators R_1, R_2 are unitary equivalent if and only if their scalar spectral measures are mutually absolutely continuous and their spectral multiplicity functions are equal almost everywhere.

Now back to the Problem 1.2.3. A.V. Megretskii, V.V.Peller, and S.R.Treil solved problem 1.2.3 in [12, Theorem 1, p. 245]. The equivalent condition of such existence is stated as follow:

Theorem 1.2.8. *Let $\mathcal{R}$ be a bounded self-adjoint operator on Hilbert space, denote μ be its scalar spectral measure, and $N(t)$ be its spectral multiplicity function. Then $\mathcal{R}$ is unitary equivalent to a Hankel operator Γ if and only if all the following conditions hold*

(i) $\dim \operatorname{Ker} \mathcal{R} = 0$ *or* ∞*;*

(ii) $\mathcal{R}$ *is non-invertible;*

(iii) *The spectral multiplicity function $N(t)$ satisfies: $|N(t) - N(-t)| \leq 2$, μ_a-a.e. (absolute continuous part of μ), and $|N(t) - N(-t)| \leq 1$, μ_s-a.e. (singular part of μ).*

Here (iii) *implies that if one of the $N(t), N(-t)$ is ∞, then the other one must also be ∞.*

Remark 1.2.9. Notice that condition (iii) also implies that for any self-adjoint Hankel Γ, we have

$$|\dim \operatorname{Ker}(\Gamma - \lambda I) - \dim \operatorname{Ker}(\Gamma + \lambda I)| \leq 1 \qquad \text{holds for all } \lambda \in \mathbb{R}. \tag{1.2.1}$$

Furthermore, [12] also shows that the inequality (1.2.1) is true for all bounded Hankel Γ and all $\lambda \in \mathbb{C}$, i.e.,

$$|\dim \operatorname{Ker}(\Gamma - \lambda I) - \dim \operatorname{Ker}(\Gamma + \lambda I)| \leq 1 \tag{1.2.2}$$

holds for all $\lambda \in \mathbb{C}$.

From Theorem 1.2.8, we can easily generate another version when R is compact.

Theorem 1.2.10. *Let $\mathcal{R}$ be a bounded self-adjoint compact operator on Hilbert space, then $\mathcal{R}$ is unitary equivalent to a Hankel operator Γ if and only if all the following conditions hold*

(i) $\dim \operatorname{Ker} \mathcal{R} = 0$ *or* ∞*;*

(ii) $\mathcal{R}$ *is non-invertible;*

(iii) $|\dim \operatorname{Ker}(\mathcal{R} - \lambda I) - \dim \operatorname{Ker}(\mathcal{R} + \lambda I)| \leq 1$ *holds for all* $\lambda \in \mathbb{R}$.

1.2.3 Hankel Operators not Self-adjoint

For a general Hankel Γ which is not self-adjoint, the spectrum properties of Γ is also deeply studied. In Remark 1.2.9, we have already given several spectrum properties for a general Hankel Γ:

(i) $0 \in \sigma(\Gamma)$;

(ii) $|\dim \operatorname{Ker}(\Gamma - \lambda I) - \dim \operatorname{Ker}(\Gamma + \lambda I)| \leq 1$ holds for all $\lambda \in \mathbb{C}$. In particular, if λ is a multiple eigenvalue of a Hankel Γ, then $-\lambda$ must be an eigenvalue.

There are a lot of progress concentrating on whether there exists a Hankel Γ, such that $\sigma(\Gamma) = \{0\}$. The first progress was made by S. Power in [13], proving that there exists no non-trivial nilpotent Hankel operators ($\Gamma^n = 0$ for some $n > 0$). Then in 1991, A. Metretskii constructed a non-trivial quasinilpotent Hankel operator in [14], i.e., a Hankel operator Γ such that $\|\Gamma^n\|^{1/n} \to 0$ as $n \to \infty$ (or equivalently, $\sigma(\Gamma) = \{0\}$).

Another substantial result is proved in [15], saying that there are no other constraints on the spectrum of Γ except for $\{0\}$. That is:

Theorem 1.2.11. *Let σ be any compact subset on the complex plane containing zero. Then there exists a Hankel operator Γ such that $\sigma(\Gamma) = \sigma$*

When considering other spectral properties of Hankel operators, recall that the inequality (1.2.2) gives a restriction on the geometric multiplicities of eigenvalues. However, if we consider the algebraic multiplicities of eigenvalues (recall that the *algebraic multiplicity* of an eigenvalue λ is the dimension of the space of all generalized eigenvectors, i.e. the dimension of the space $\bigcup_{n\geq 1} \mathrm{Ker}(\Gamma - \lambda I)^n$), E. Abakumov proved in [16] that there are no restrictions on the algebraic multiplicities of the eigenvalues, for the special case of finite rank Hankel operators.

Theorem 1.2.12. *Given a finite number of non-zero points $\lambda_1, \lambda_2, ..., \lambda_n$ and multiplicities $k_1, ..., k_n$, there exists a finite rank Hankel operator Γ, such that its non-zero eigenvalues are exactly $\lambda_1, ..., \lambda_n$, and the corresponding algebraic multiplicities are exactly $k_1, ..., k_n$ with 0 an eigenvalue of infinite multiplicity.*

1.2.4 The Spectral Data that Uniquely Determines a Hankel Operator

However, for all the results we stated in previous subsections (Theorem 1.2.8, Theorem 1.2.10, Theorem 1.2.11 and Theorem 1.2.12), the constructed Hankel operator is not unique. In other words, there are many unitary invariants of a Hankel operator that share the same spectral property with a given operator R.

Thus we wonder, what type of spectral data that can uniquely determine a Hankel operator?

The first pioneering progress of this problem was made by P. Gérard and S. Grellier in [17]. Their results came up as a byproduct of the study of an integrable Hamiltonian system called the cubic Szegö equation. For every $u \in H^2(\mathbb{T})$, they define the Hankel operator of symbol u as:

$$H_u(h) := P_+(u\overline{h}), \qquad h \in H^2(\mathbb{T}).$$

We need to emphasize that this H_u is conjugate-linear, which is different from the standard definition given in Definition 1.1.1 and Remark 1.1.8. In fact, we can set up a correspondence between Gérard-Grellier's definition and Definition 1.1.1.

For a given $u \in H^2(\mathbb{T})$, we define a Hankel operator Γ_u on ℓ^2 with related Hankel coefficients $\{\gamma_k\}_{k=0}^{\infty}$ satisfies $\gamma_k = \hat{u}(k)$, here $\hat{u}(k)$ is the Fourier coefficients $\hat{u}(k) = \int_{\mathbb{T}} ue^{-ikz}dz$. Then we will have the defined H_u is unitary equivalent to a conjugate-linear operator $\mathfrak{C}\Gamma$ on ℓ^2. In fact, we have $H_u = \mathcal{F}^{-1}(\mathfrak{C}\Gamma)\mathcal{F}$, where $\mathcal{F}$ is the Fourier transform given in (1.1.2).

Now for the defined H_u, we can easily set up a rank-one perturbation relation. We define the shift operator on $H^2(\mathbb{T})$ as $S_u := e^{ix}u$, and then we have $S^*H_u = H_uS = H_{S^*u}$, where S^* is the adjoint of S with the representation $S^*u = P_+(e^{-ix}u)$. We denote this S^*H_u as a new operator K_u (which is also conjugate-linear), and we call this K_u to be the *shifted Hankle operator* with symbol u. Then we have H_u and K_u is related by the following identity

$$K_u^2 = H_u^2 - u\langle , u\rangle.$$

For this H_u^2, K_u^2, we can show that they are unitary equivalent to $|\Gamma|^2, |\Gamma_1|^2$ respectively:

$$H_u^2 = \mathcal{F}^{-1}|\Gamma|^2\mathcal{F}, \qquad K_u^2 = \mathcal{F}^{-1}|\Gamma_1|^2\mathcal{F}.$$

In the Gérard-Grellier's work [17], they first considered a symbol u such that H_u has rank N, and $1 \notin \operatorname{Ran} H_u$, then they showed that the singular values of H_u^2, K_u^2, denoted as $\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N$, are simple and satisfies the following intertwining relation

$$\lambda_1^2 > \mu_1^2 > ... > \lambda_N^2 > \mu_N^2. \tag{1.2.3}$$

And conversely, given two sequences $\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N$ satisfy the intertwining relation (1.2.3), we can find a unique u such that H_u^2, K_u^2 have non-zero singullar values as $\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N$ respectively. If we further assume that $\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N$ are real numbers, then Gérard-Grellier proved in the same paper [17] that we can find a symbol u with real Fourier coefficients $\{\hat{u}(k)\}_{k=0}^{\infty}$, such that the induced Hankel operator $\Gamma_u, S^*\Gamma_u$ on ℓ^2 are self-adjoint and have simple eigenvalues as $\{\lambda_n\}_{n=1}^N, \{\mu_n\}_{n=1}^N$ respectively.

Later in a followup paper [36], Gérard-Grellier' solved the case for general self-adjoint Hankel operators with simple singular values. The result is stated as follow:

Theorem 1.2.13. *Let $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ be two sequences of real numbers such that*

$$|\lambda_1| > |\mu_1| > ... > |\lambda_n| > |\mu_n| > ... \to 0,$$

then there exists a unique self-adjoint compact Hankel operator Γ, such that

(i) *the non-zero eigenvalues of Γ are simple, and coincides with $\{\lambda_n\}_{n=1}^{\infty}$;*

(ii) *the non-zero eigenvalues of $S^*\Gamma$ are simple, and coincides with $\{\mu_n\}_{n=1}^{\infty}$.*

Then in 2014, Gérard-Grellier also solved the case for compact Hankel operators with multiple singular values. In [19], they described the spectral data by the corresponding singular values together with some Blaschke products.

In this thesis, we will present a dynamical system approach for the inverse spectral problems of Hankel operators by reducing the problem to the asymptotic stability of a certain contraction. Different from the work by Gérard-Grellier, we solve the problem by means of:

(i) Instead of using the language of conjugate-linear operators in [17], [36] and [19]; we treat Hankel operatoars as complex symmetric operators in the non self-adjoint case;

(ii) To reconstruct the compact Hankel operators according to their singular values, we will present a theorem so-called Abstract Borg's theorem (see Theorem 8.0.1), and then we construct the scalar spectral measure of the Hankel operator from this theorem;

(iii) For the multiple singular values case, we translate the spectral data by their singular values and some probability measures, where those probability measures are exactly the Clark measures of the Blaschke products given in [19].

Chapter 2

STRUCTURE OF THIS THESIS

In this chapter, we will briefly introduce the structure of this thesis.

Throughout the historical studies of inverse spectral problems of Hankel operators (see section 1.2), we can see that

(i) A Hankel operator cannot be uniquely characterized by the spectral data of its own. There are many unitary invariants of it on ℓ^2 which are also Hankel, and share the same spectral properties;

(ii) From the studies of P. Gérard and S. Grellier in [17] and [18], we can see that the spectral information of Γ and $S^*\Gamma$ can completely characterize a Hankel operator.

In this thesis, we will further discuss the topic that what spectral data can uniquely determine a Hankel operator.

In chapter 3, we mainly discuss the inverse spectral problem for self-adjoint Hankel operators. A self-adjoint Hankel operator Γ satisfies $\Gamma^2 - \Gamma_1^2 = uu^*$ where $\Gamma_1 := \Gamma S = S^*\Gamma$ and $u = \Gamma e_0$, thus we construct a tuple $(\mathcal{R}, \mathcal{R}_1, p)$ with relation $\mathcal{R}^2 - \mathcal{R}_1^2 = pp^*$ where $\mathcal{R}, \mathcal{R}_1$ are self-adjoint and $\operatorname{Ker}\mathcal{R} = \{0\}$. We want to find out whether there exists a self-adjoint Γ, such that the tuple restricted on the essential part of Γ, i.e., $\big(\widetilde{\Gamma} := \Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}, \widetilde{\Gamma}_1 := \Gamma_1\big|_{(\operatorname{Ker}\Gamma)^\perp}, u\big)$ is unitary equivalent to $\big(R, R_1, p\big)$. If such Γ exists, since we have

$$\widetilde{\Gamma}_1 = S^*\big|_{(\operatorname{Ker}\Gamma)^\perp}\widetilde{\Gamma},$$

we require the contraction $\mathcal{T}$ defined by $\mathcal{T} := \mathcal{R}_1\mathcal{R}^{-1}$ to be unitary equivalent to $S^*\big|_{(\operatorname{Ker}\Gamma)^\perp}$, thus $\mathcal{T}$ is asymptotically stable. In fact, we show in Proposition 3.1.3 tells us that there exists a unique Hankel Γ such that $\big(\widetilde{\Gamma}, \widetilde{\Gamma}_1, u\big)$ is unitary equivalent to $\big(\mathcal{R}, \mathcal{R}_1, p\big)$ as long as we guarantee the asymptotic stability of $\mathcal{T}$. Thus a self-adjoint Hankel can be uniquely determined by two operators $\mathcal{R}, \mathcal{R}_1$.

Then in chapter 5, we discuss the inverse spectral problem for general Hankel operators Γ. Under this case we take Γ as a special type of operators called $\mathfrak{C}$-symmetric operators (See Definition 4.1.6), and write the essential part $\widetilde{\Gamma}, \widetilde{\Gamma}$ as (5.1.5):

$$\widetilde{\Gamma} = \mathfrak{C}\widetilde{\phi}|\widetilde{\Gamma}|\widetilde{\mathfrak{J}}_u, \qquad \widetilde{\Gamma}_1 = \mathfrak{C}\widetilde{\phi}_1|\widetilde{\Gamma}_1|\widetilde{\mathfrak{J}}_u,$$

where here $\mathfrak{J}_u$ is a conjugation commutes with $|\Gamma|, |\Gamma_1|$, $\widetilde{\phi}$ is unitary and $\widetilde{\phi}_1$ is partial isometry which is unitary on $\ell^2 \ominus \operatorname{Ker}\Gamma_1$.

Since a general Hankel operator Γ satisfies $|\widetilde{\Gamma}|^2 - |\widetilde{\Gamma}_1|^2 = uu^*$ where $u := \Gamma^* e_0$, we construct a tuple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ that satisfies

(i) R, R_1 are two positive, self-adjoint compact operators defined on a Hilbert space $\mathcal{H}$. In addition we have $\operatorname{Ker} R = \{0\}$;

(ii) $R^2 - R_1^2 = pp^*$ for a vector p with $\|R^{-1}p\| \leq 1$;

(iii) $\mathfrak{J}_p$ is a conjugation commutes with R, R_1 and preserves p;

(iv) φ is a $\mathfrak{J}_p$-symmetric unitary operator, which commutes with R;

(v) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry with $\operatorname{Ker}\varphi_1 = \operatorname{Ker} R_1$, which commutes with R_1. In addition, we have $\varphi_1\big|_{(\operatorname{Ker} R_1)^\perp}$ is unitary.

We want to find out whether there exists a Hankel operator Γ, such that there exists a conjugation $\mathfrak{J}_u$, commuting with $|\Gamma|$ and $|\Gamma_1|$ and preserving u, such that induced tuple $\big(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \widetilde{\mathfrak{J}}_u\big)$ is unitary equivalent to $\big(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p\big)$. In fact, we show in Proposition (5.1.6) that with the asymptotic stability of $\mathcal{T} := R_1\varphi_1\varphi^* R^{-1}$, there exists a unique Hankel operator Γ such that $\big(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \widetilde{\mathfrak{J}}_u\big)$ is unitary equivalent to $\big(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p\big)$. Thus a general Hankel operator can be uniquely determined by two operators $\mathcal{R}_1 := R_1\varphi_1, \mathcal{R} := R\varphi$ (which also means that a Hankel operator can be uniquely determined by its essential part).

For the following chapters, we will further consider the case of compact Hankel operators, and describe the spectral data of these two operators in a different way from the results in [36].

We first need to consider the assumptions needed for $R\varphi, R_1\varphi_1$ to guarantee the asymptotic stability of $\mathcal{T}$. Finding out the equivalent requirement for the asymptotic stability of $\mathcal{T}$ is usually a hard issue, but things become easier if we have R, R_1 are compact. In chapter 7, we discuss the requirements we need to guarantee the asymptotic stability of $\mathcal{T}$.

(i) If p is cyclic with respect to R in $\mathcal{H}$, then we will show that $\mathcal{T}$ will be asymptotic stable for free.

(ii) If p is not cyclic with respect to R in $\mathcal{H}$, then Proposition 7.5.1 gives a criterion for the asymptotic stability of $\mathcal{T}$. In addition, if we set a canonical choice for φ, φ_1 given in Lemma 6.3.2, then Proposition 7.6.3 gives an equivalent condition for φ and φ_1 for the asymptotic stability of $\mathcal{T}$.

Secondly, to translates the spectral data of $\mathcal{R}_1, \mathcal{R}$ in compact case, we first need to analyze the eigenspace structure R_1, R, which will be the work in chapter 6. Denoting $\mathcal{H}_0 = \overline{\operatorname{Span}}\{R^n p \big| n \geq 0\}$, we have $R|_{\mathcal{H}_0}, R_1|_{\mathcal{H}_0}$ both have simple eigenvalues, and $R = R_1$ on $\mathcal{H}_0^\perp$. The complete description of structure of tuple (R, R_1, p) is stated in Proposition 6.2.10.

With these preparations, now we are ready to translate the spectral data of $\mathcal{R}_1, \mathcal{R}_1$. We classify the compact $\mathcal{R}, \mathcal{R}_1$ into the following situations:

(a) $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{R}_1, \mathcal{R}$ are self-adjoint. Under this case, we have $\sigma(\varphi) = \sigma_p(\varphi) \subseteq \{\pm 1\}$, $\sigma(\varphi_1) = \sigma_p(\varphi_1) \subseteq \{\pm 1, 0\}$ with $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, thus we can write φ, φ_1 as $f(R), f_1(R_1)$ where f, f_1 are some measurable unimodular functions that take values in $\{\pm 1\}$. With these assumptions, we show in Theorem 9.1.1 that a self-adjoint compact Hankel operator with simple singular values can be uniquely characterized by two sets of real sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ satisfying the following intertwining relation:

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0.$$

(b) $\mathcal{H}_0 = \mathcal{H}$ and $\mathcal{R}_1, \mathcal{R}$ are not self-adjoint. Under this case, φ, φ_1 acts unitarily on each one-dimensional eigenspace of R, R_1 respectively. Under this condition, we show in Theorem 9.2.1 that a a compact Hankel operator with simple singular values can be uniquely characterized by two sets of complex sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$, whose modulus part satisfy an intertwining relation:

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0$$

(c) $\mathcal{H}_0 \subsetneq \mathcal{H}$. Under this case, if we consider the canonical choice of φ, φ_1, then Lemma 6.3.2 and Proposition 7.6.3 give the behavior of φ, φ_1 on each eigenspaces of R, R_1. Denoting $\{\rho_k\}_{k=1}^{\infty}$, $\{\rho_k^1\}_{k=1}^{\infty}$ as the scalar spectral measure of φ, φ_1 restricted to the corresponding eigenspaces with respect to their *-cyclic vector, then we show in Proposition 10.1.5 that a compact Hankel operator Γ can be uniquely determined by two complex sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$, which serves as the singular values of $|\Gamma|$ and $|\Gamma_1|$, together with two normalized discrete probability measure $\{\widetilde{\rho}_k\}_{k=1}^{\infty}$, $\{\widetilde{\rho}_k^1\}_{k=1}^{\infty}$.

Hence the four sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$, $\{\widetilde{\rho}_k\}_{k=1}^{\infty}$, $\{\widetilde{\rho}_k^1\}_{k=1}^{\infty}$ as a whole can be regarded as the spectral data of a general compact Hankel operator.

Chapter 3

A RESULT FOR SELF-ADJOINT HANKEL OPERATORS

In this chapter, we discuss the inverse spectral problem for self-adjoint Hankel operators.

First in section 3.1, we introduce the setting and some preparation work for the work. For a self-adjoint Hankel operator Γ, we can show that Γ and $\Gamma_1 := \Gamma S$ satisfy a rank-one perturbation relation (3.1.1), thus restricted on $(\operatorname{Ker}\Gamma)^\perp$ we get a triple $\big(\widetilde{\Gamma} := \Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}, \widetilde{\Gamma}_1 := \Gamma_1\big|_{(\operatorname{Ker}\Gamma)^\perp}, u := \Gamma e_0\big)$.

Next in section 3.2, we prove the main result (Proposition 3.1.3) in this chapter. That is, for a given rank-one perturbation $(\mathcal{R}, \mathcal{R}_1, p)$ satisfying the following conditions:

(i) $\mathcal{R}, \mathcal{R}_1$ are self-adjoint operators, $\operatorname{Ker}\mathcal{R} = \{0\}$;

(ii) $\mathcal{R}^2 - \mathcal{R}_1^2 = pp^*$;

(iii) The contraction $\mathcal{T} := \mathcal{R}_1\mathcal{R}^{-1}$ is asymptotically stable,

then there exists a unique self-adjoint Hankel operator Γ, such that the triple restricted on the essential part of Γ: $\big(\widetilde{\Gamma}, \widetilde{\Gamma}_1, u\big)$ is unitary equivalent to $(\mathcal{R}, \mathcal{R}_1, p)$. Thus from this proposition, we can see that a self-adjoint Hankel operator can be uniquely determined by the spectral data of two operators $\mathcal{R}, \mathcal{R}_1$ which satisfy a rank-one perturbation relation.

In addition, we can further translate the spectral data of $\mathcal{R}, \mathcal{R}_1$ under the self-adjoint compact case. If we further assume that p is cyclic with respect to $|\mathcal{R}|$ in $\mathcal{H}$ (under this case we can see that $\mathcal{R}$ has simple singular values), then we can show that the spectral data of $\mathcal{R}, \mathcal{R}_1$ can be characterized by two sets of real sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ satisfying an intertwining relation

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0. \qquad (3.0.1)$$

In other words, given two real sequences $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ with the intertwining relation (3.0.1), there exists a unique self-adjoint compact Hankel operator Γ such that the non-zero eigenvalues of Γ, Γ_1 are simple, and coincide with $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ respectively. The detail of work in this paragraph can be found in chapter 8.

3.1 Setup of the Inverse Spectral Problem for Self-adjoint Hankel Operators

Let Γ be a self-adjoint Hankel operator. Define $\Gamma_1 := \Gamma S = S^*\Gamma$, which is also a self-adjoint Hankel operator, then we can write

$$\Gamma_1^2 = \Gamma S S^* \Gamma = \Gamma(I - e_0 e_0^*)\Gamma = \Gamma^2 - uu^*, \qquad (3.1.1)$$

where $u := \Gamma e_0$.

Since Γ is self-adjoint, we have $\operatorname{Ran}\Gamma \oplus \operatorname{Ker}\Gamma = \ell^2$, and $(\operatorname{Ker}\Gamma)^{\perp}$ is a reducing subspace for both Γ and Γ_1. Hence we restrict everything on $(\operatorname{Ker}\Gamma)^{\perp}$, and denote

$$\widetilde{\Gamma} := \Gamma\big|_{(\operatorname{Ker}\Gamma)^{\perp}}, \qquad \widetilde{\Gamma}_1 := \Gamma_1\big|_{(\operatorname{Ker}\Gamma)^{\perp}},$$

then (3.1.1) can be transformed to

$$\widetilde{\Gamma}^2 - \widetilde{\Gamma}_1^2 = uu^* \tag{3.1.2}$$

Now let us try to go from the opposite direction. Suppose that we are given two self-adjoint operators $\mathcal{R}$, $\mathcal{R}_1$ on a Hilbert space $\mathcal{H}$, and R, R_1 are their modulus operator respectively

$$\mathcal{R} = R\varphi, \qquad \mathcal{R}_1 = R_1\varphi_1,$$

where here φ, φ_1 are partial isometries commuting with R, R_1 respectively.

Our goal here to is find a Hankel operator Γ, such that the essential part $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_1$ are simultaneously unitary equivalent to $\mathcal{R}$ and $\mathcal{R}_1$ respectively, meaning that we have $\widetilde{\Gamma} = \mathcal{V}\mathcal{R}\mathcal{V}^*$, $\widetilde{\Gamma}_1 = \mathcal{V}\mathcal{R}_1\mathcal{V}^*$ for some unitary operator $\mathcal{V}$. We can see from (3.1.1) that $\mathcal{R}$ and $\mathcal{R}_1$ should satisfy the relation

$$\mathcal{R}^2 - \mathcal{R}_1^2 = pp^* \tag{3.1.3}$$

for a vector $p \in \operatorname{Ran} \mathcal{R}$.

To begin with solving the inverse spectral problem, we first need to find a contraction $\mathcal{T}$, (hopefully unitary equivalent to the backward shift S^*) such that $\mathcal{R}_1 = \mathcal{T}\mathcal{R}$. The tool to find $\mathcal{T}$ is by using the following simple lemma.

Lemma 3.1.1 (Douglas Lemma)**.** *Let A and B be two bounded operators on a Hilbert space $\mathcal{H}$ such that*

$$\|Bh\| \le \|Ah\| \qquad \forall h \in \mathcal{H},$$

or, equivalently, $B^*B \le A^*A$.

Then there exists a contraction T (i.e. $\|T\| \le 1$) such that $B = TA$.

In addition, if $\operatorname{Ker} A = \operatorname{Ker} A^* = \{0\}$, then the operator $\widetilde{T} := BA^{-1}$ (initially *defined on* $\operatorname{Clos}\operatorname{Ran} A$*) can be extended to a contraction* T *on the whole space* $\mathcal{H}$*, and its adjoint is given by* $T^* = (A^*)^{-1}B^*$*; notice that the condition* $\|T\| \leq 1$ *implies that* $\operatorname{Ran} B^* \in \operatorname{Dom}(A^*)^{-1}$*, so the operator* T^* *is well defined on all* $x \in \mathcal{H}$

For the specific condition, if A *has dense range, the operator* T *is unique.*

The specific proof of Lemma 3.1.1 can be found in [20].

When finding a contraction $\mathcal{T}$ satisfying $\mathcal{R}_1 = \mathcal{T}\mathcal{R}$, we usually assume $\operatorname{Ker}\mathcal{R} = \{0\}$ to avoid the non-uniqueness of such $\mathcal{T}$, meaning that we want R to be unitarily equivalent to the *esssential part of* Γ (instead of Γ itself), i.e., to the operator $\widetilde{\Gamma} := \Gamma|_{(\operatorname{Ker}\Gamma)^\perp}$. In this case the operator $\mathcal{T}$ should be unitarily equivalent to S^* restricted on $(\operatorname{Ker}\Gamma)^\perp$, meaning that

$$\mathcal{T} = \widetilde{V}^* S^*|_{(\operatorname{Ker}\Gamma)^\perp}\widetilde{V} \tag{3.1.4}$$

for an unitary $\widetilde{V}$.

In this situation the vector p from (3.1.3) should be a vector satisfying $\|\mathcal{R}^{-1}p\| \leq 1$, where here $q := \mathcal{R}^{-1}p$ can be thinking as

$$q = \widetilde{V}^* P_{(\operatorname{Ker}\Gamma)^\perp} e_0.$$

So now we arrive at the following setup. Given a self-adjoint $\mathcal{R}$ with trivial kernel on $\mathcal{H}$, and a vector p with property $q := \mathcal{R}^{-1}p$ with norm $\|q\| \leq 1$, then $\mathcal{R}^2 - pp^* \geq 0$, and we take a self-adjoint $\mathcal{R}_1$ such that

$$\mathcal{R}^2 - \mathcal{R}_1^2 = pp^*. \tag{3.1.5}$$

Now applying the Douglas Lemma 3.1.1, there exists a unique contraction $\mathcal{T}$ such that

$$\mathcal{R}_1 = \mathcal{T}\mathcal{R} = \mathcal{R}\mathcal{T}^*.$$

Thus we can rewrite (3.1.5) as

$$\mathcal{R}(I - \mathcal{T}^*\mathcal{T})\mathcal{R} = \mathcal{R}qq^*\mathcal{R},$$

which implies that

$$I - \mathcal{T}^*\mathcal{T} = qq^* = \langle \cdot, q \rangle q. \tag{3.1.6}$$

Before stating the main result in this chapter, we need to define the *asymptotic stability* of a contraction.

Definition 3.1.2. *We say an operator $\mathcal{T}$ is* asymptotically stable *iff $\mathcal{T}^n \to 0$ in the strong operator topology as $n \to \infty$, i.e. iff for $\forall x \in \mathcal{H}$*

$$\lim_{n \mapsto \infty} \|\mathcal{T}^n x\| \to 0$$

Now go back to identity (3.1.4). In order to find a Hankel operator Γ such that $(\widetilde{\Gamma}, \widetilde{\Gamma}_1, u)$ is unitary equivalent to $(\mathcal{R}, \mathcal{R}_1, p)$, we need $\mathcal{T}$ to be unitary equivalent to $\widetilde{S}^* := S^*\big|_{(\operatorname{Ker}\Gamma)^\perp}$, thus we need $\mathcal{T}$ to be asymptotically stable. In addition, the following proposition shows that this requirement is also sufficient.

Proposition 3.1.3. *If $\mathcal{T}$ is asymptotically stable, then there exist a unique self-adjoint Hankel operator Γ and an unitary operator $\widetilde{\mathcal{V}} : \mathcal{H} \to (\operatorname{Ker}\Gamma)^\perp$ such that*

$$\widetilde{\Gamma} := \Gamma|_{(\operatorname{Ker}\Gamma)^\perp} = \widetilde{\mathcal{V}}\mathcal{R}\widetilde{\mathcal{V}}^*, \tag{3.1.7}$$

$$\widetilde{\Gamma}_1 := \Gamma_1|_{(\operatorname{Ker}\Gamma)^\perp} = \widetilde{\mathcal{V}}\mathcal{R}_1\widetilde{\mathcal{V}}^*, \tag{3.1.8}$$

$$\Gamma e_0 = \widetilde{\mathcal{V}}p; \tag{3.1.9}$$

note that (3.1.9) implies that

$$P_{(\operatorname{Ker}\Gamma)^\perp} e_0 = \widetilde{\mathcal{V}}q, \tag{3.1.10}$$

And the Hankel coefficients $\{\gamma_k\}_{k=0}^{\infty}$ *can be expressed by*

$$\gamma_k = \langle \mathcal{T}^k p, q \rangle.$$

Moreover, $\operatorname{Ker}\Gamma = \{0\}$ *if and only if* $\|q\| = 1$ *and* $q \notin \operatorname{Ran}\{\mathcal{R}\}$.

3.2 Proof of Proposition 3.1.3

3.2.1 Existence of Hankel Γ

Proof. Treating (3.1.6) as an identity for quadratic forms and substituting $x \in \mathcal{H}$ into it we get

$$\|x\|^2 - \|\mathcal{T}x\|^2 = |\langle x, q \rangle|^2.$$

Then we substitute x by $\mathcal{T}^k x$ for an arbitrary $k \in \mathbb{N}$, we get

$$\|\mathcal{T}^k x\|^2 - \|\mathcal{T}^{k+1}x\|^2 = |\langle \mathcal{T}^k x, q \rangle|^2,$$

Now taking the sum from $k = 0$ to $k = n$ we get

$$\|x\|^2 - \|\mathcal{T}^{n+1}x\|^2 = \sum_{k=0}^{n} |\langle \mathcal{T}^k x, q \rangle|^2.$$

Let $n \to \infty$ and using the asymptotic stability of $\mathcal{T}$ we see that

$$\|x\|^2 = \sum_{k=0}^{\infty} |\langle \mathcal{T}^k x, q \rangle|^2,$$

which means that the operator $V : \mathcal{H} \to \ell^2$

$$\mathcal{V}x = \big(\langle x, q \rangle, \langle \mathcal{T}x, q \rangle, \langle \mathcal{T}^2 x, q \rangle, \ldots\big) = \big(\langle \mathcal{T}^k x, q \rangle\big)_{k=0}^{\infty} \tag{3.2.1}$$

is an isometry.

We can see that

$$\mathcal{V}\mathcal{T}x = \big(\langle \mathcal{T}x, q\rangle, \langle \mathcal{T}^2x, q\rangle, \langle \mathcal{T}^3x, q\rangle, \ldots\big) = S^*\mathcal{V}x,$$

so $\mathcal{T}$ is unitarily equivalent to either S^* (if $\operatorname{Ran}\mathcal{V} = \ell^2$) or to the restriction of S^* to $\operatorname{Ran}\mathcal{V}$, which is a S^*-invariant subspace (if $\operatorname{Ran}\mathcal{V} \neq \ell^2$). (Here we use a simple fact that an onto isometry is unitary.)

Let $\widetilde{\mathcal{V}}$ be the operator $\mathcal{V}$ with the target space restricted to $\operatorname{Ran}\mathcal{V}$, so $\widetilde{\mathcal{V}} : \mathcal{H} \to \operatorname{Ran}\mathcal{V}$ is an unitary operator. Denoting by $\widetilde{S}^* := S^*|_{\operatorname{Ran}\mathcal{V}}$ (so $\widetilde{S}^* = S^*$ if $\operatorname{Ran}\mathcal{V} = \ell^2$), we see that

$$\widetilde{\mathcal{V}}\mathcal{T}\widetilde{\mathcal{V}}^* = \widetilde{S}^*, \qquad \widetilde{\mathcal{V}}\mathcal{T}^*\widetilde{\mathcal{V}}^* = (\widetilde{S}^*)^* = P_{\operatorname{Ran}\mathcal{V}} S|_{\operatorname{Ran}\mathcal{V}} =: \widetilde{S}.$$

Define

$$\widetilde{\Gamma} := \widetilde{\mathcal{V}}\mathcal{R}\widetilde{\mathcal{V}}^*, \qquad \widetilde{\Gamma}_1 := \widetilde{\mathcal{V}}\mathcal{R}_1\widetilde{\mathcal{V}}^*. \tag{3.2.2}$$

Then the relation $\mathcal{T}\mathcal{R} = \mathcal{R}_1 = \mathcal{R}_1^* = \mathcal{R}\mathcal{T}^*$ (remember that $\mathcal{R}$ and $\mathcal{R}_1$ are self-adjoint) translates to

$$\widetilde{\Gamma}\widetilde{S} = \widetilde{\Gamma}_1 = \widetilde{S}^*\widetilde{\Gamma}.$$

Extending $\widetilde{\Gamma}$ and $\widetilde{\Gamma}_1$ to operators Γ and Γ_1 on the whole space ℓ^2 by setting them to be 0 on $(\operatorname{Ran}\mathcal{V})^\perp$, we can see that $\operatorname{Ker}\Gamma = (\operatorname{Ran} V)^\perp$ and that

$$\Gamma = \mathcal{V}\mathcal{R}\mathcal{V}^*, \qquad \Gamma_1 = \mathcal{V}\mathcal{R}_1\mathcal{V}^*.$$

Let us now show that Γ satisfies the identity $\Gamma S = S^*\Gamma$, i.e. that Γ is indeed a

Hankel operator. For $\forall f \in \ell^2$ we decompose

$$f = P_{\operatorname{Ran}\mathcal{V}} f + P_{(\operatorname{Ran}\mathcal{V})^\perp} f =: f_1 + f_2.$$

We know that $S^* \operatorname{Ran}\mathcal{V} \subset \operatorname{Ran}\mathcal{V}$, so $S(\operatorname{Ran}\mathcal{V})^\perp \subset (\operatorname{Ran}\mathcal{V})^\perp$. Therefore $Sf_2 \perp \operatorname{Ran} V$, so $\Gamma S f_2 = 0 = S^*\Gamma f_2$. As for f_1, since $\operatorname{Ker}\Gamma = (\operatorname{Ran}\mathcal{V})^\perp$, we have

$$\Gamma S f_1 = \Gamma P_{\operatorname{Ran}\mathcal{V}} S f_1 = \Gamma \widetilde{S} f_1 = \widetilde{\Gamma}\widetilde{S} f_1 = \widetilde{S}^*\widetilde{\Gamma} f_1 = S^*\Gamma f_1,$$

so Γ is indeed a Hankel operator.

Now it remains to show the identities (3.1.7), (3.1.8) and (3.1.9). The first two identities are exactly from (3.2.2).To show (3.1.9), we first derive a representation for $\mathcal{V}^*$.

For $\forall x, y \in \mathcal{H}$, we have

$$\begin{aligned} \langle \mathcal{V}x, y\rangle &= \sum_k \bar{y_k} \langle \mathcal{T}^k x, q\rangle \\ &= \langle x, \sum_k y_k (\mathcal{T}^*)^k q\rangle, \end{aligned}$$

thus $\mathcal{V}^* y = \sum_k y_k (\mathcal{T}^*)^k q$. In specific, we have $\mathcal{V}^* e_0 = q$. In conclusion, we have

$$\Gamma e_0 = \mathcal{V}\mathcal{R}\mathcal{V}^*\mathcal{V} q = \mathcal{V}\mathcal{R} q = \mathcal{V} p = \widetilde{\mathcal{V}} p,$$

and we finish the proof of existence part. □

3.2.2 Uniqueness of Hankel Γ

Proof. Suppose that the identities (3.1.7), (3.1.8), (3.1.9) hold for some unitary operator $\widetilde{\mathcal{V}} : \mathcal{H} \to \operatorname{Clos}\operatorname{Ran}\Gamma = (\operatorname{Ker}\Gamma)^\perp \subset \ell^2$.

Since for a Hankel operator $\operatorname{Ker}\Gamma$ is always S-invariant, the subspace $(\operatorname{Ker}\Gamma)^\perp$ is

S^*-invariant, so the restriction $S^*|_{(\operatorname{Ker}\Gamma)^\perp}$ is well defined. The identities (3.1.7), (3.1.8) and the definition of $\mathcal{T}$ imply that

$$S^*|_{(\operatorname{Ker}\Gamma)^\perp} = \widetilde{\mathcal{V}}\mathcal{T}\widetilde{\mathcal{V}}^*. \tag{3.2.3}$$

Now denote $\widetilde{\Gamma} := \Gamma|_{(\operatorname{Ker}\Gamma)^\perp}$ and $\widetilde{\Gamma}_1 = \Gamma_1|_{(\operatorname{Ker}\Gamma)^\perp}$, we restrict both sides of identity (3.1.1) on $(\operatorname{Ker}\Gamma)^\perp$ and write

$$\widetilde{\Gamma}_1^2 = \widetilde{\Gamma}^2 - uu^*,$$

where

$$u := \Gamma e_0 = \widetilde{\Gamma} P_{(\operatorname{Ker}\Gamma)^\perp} e_0,$$

Then identities (3.1.7), (3.1.8) will be unitarily translated to

$$\mathcal{R}_1^2 = \mathcal{R}^2 - \tilde{p}\tilde{p}^*, \qquad \text{where} \quad \tilde{p} = \mathcal{R}\tilde{q},\ \ \tilde{q} = \widetilde{\mathcal{V}}^* P_{(\operatorname{Ker}\Gamma)^\perp} e_0.$$

Comparing this with $\mathcal{R}_1^2 = \mathcal{R}^2 - pp^*$ we conclude that $p = \alpha\tilde{p}$, $q = \alpha\tilde{q}$, for an $|\alpha| = 1$.

To compute the Hankel coefficients $\{\gamma_k\}_{k=1}^\infty$ of Γ, we have

$$\begin{aligned}\gamma_k &= (\Gamma e_0, S^k e_0) = \langle (S^*)^k \Gamma e_0, e_0\rangle = \langle (S^*)^k \Gamma P_{(\operatorname{Ker}\Gamma)^\perp} e_0, P_{(\operatorname{Ker}\Gamma)^\perp} e_0\rangle \\ &= \langle \mathcal{T}^k \mathcal{R}\tilde{q}, \tilde{q}\rangle = \langle \mathcal{T}^k \tilde{p}, \tilde{q}\rangle = \langle \mathcal{T}^k p, q\rangle,\end{aligned}$$

meaning that the coefficients $\{\gamma_k\}_{k=0}^\infty$ does not depend on $\widetilde{\mathcal{V}}$. So the uniqueness is proved.

□

3.2.3 Trivial Kernel Condition

Proof. As we discussed in the proof of existence part, $\operatorname{Ker}\Gamma$ is trivial if and only if the operator $\mathcal{V}$ defined by (3.2.1) satisfies $\operatorname{Ran} V = \ell^2$. The latter condition is equivalent to $\widetilde{S}^* = S^*$, which happens if and only if $\mathcal{T}$ is unitarily equivalent to S^*, or, equivalently, $\mathcal{T}^*$ is unitarily equivalent to S.

For the first direction, suppose $\operatorname{Ker}\Gamma = \{0\}$, so T is unitarily equivalent to S^*. We know that

$$I - SS^* = e_0 e_0^*, \qquad I - \mathcal{T}^*\mathcal{T} = qq^*,$$

so by the unitary equivalence $\|q\| = \|e_0\| = 1$. If $q \in \operatorname{Ran}\mathcal{R}$, i.e. $q = \mathcal{R}f$, then

$$\mathcal{R}\mathcal{T}^* f = \mathcal{T}\mathcal{R}f = \mathcal{T}q = 0,$$

and since $\operatorname{Ker}\mathcal{R} = \{0\}$, we conclude that $\mathcal{T}^* f = 0$. But if $\operatorname{Ker}\Gamma = \{0\}$. then $\mathcal{T}$ is an isometry, which contradicts $\mathcal{T}^* f = 0$. So $q \notin \operatorname{Ran}\mathcal{R}$.

For the sufficiency part, suppose $\|q\| = 1$ and $q \notin \operatorname{Ran}\mathcal{R}$. We know that

$$\mathcal{T}\mathcal{R}^2\mathcal{T}^* = \mathcal{R}_1^2 = \mathcal{R}\mathcal{T}^*\mathcal{T}\mathcal{R} = \mathcal{R}(I - qq^*)\mathcal{R},$$

and that $\operatorname{Ker}(I - qq^*) = \operatorname{Span}\{q\}$. Since $q \notin \operatorname{Ran}\mathcal{R}$, we see that $\operatorname{Ker}\mathcal{R}(I - qq^*)\mathcal{R} = \{0\}$, so $\operatorname{Ker}\mathcal{T}^* = \{0\}$.

Now applying (3.1.6) to vector q we get that

$$q - \mathcal{T}^*\mathcal{T}q = q,$$

and since $\operatorname{Ker}\mathcal{T}^* = \{0\}$ we see that $\mathcal{T}q = 0$.

Then left and right multiplying (3.1.6) by $\mathcal{T}$ and $\mathcal{T}^*$ respectively, we get

$$\mathcal{T}\mathcal{T}^* - \mathcal{T}\mathcal{T}^*\mathcal{T}\mathcal{T}^* = \mathcal{T}qq^*\mathcal{T}^*,$$

and since $\mathcal{T}q = 0$, we have $\mathcal{T}\mathcal{T}^* = (\mathcal{T}\mathcal{T}^*)^2$, which implies that $\mathcal{T}\mathcal{T}^*$ is an orthogonal projection. Since $\operatorname{Ker}\mathcal{T}^* = \{0\}$, we conclude that $\mathcal{T}\mathcal{T}^* = I$, i.e. $\mathcal{T}$ is an isometry.

In addition, we have that $\mathcal{T}$ is asymptotically stable, thus there is no reduced subspace on which $\mathcal{T}$ (and so $\mathcal{T}^*$) is unitary. The identity (3.1.6) implies that $\operatorname{rank}(I - \mathcal{T}^*\mathcal{T}) = 1$, i.e. the defect indices of $\mathfrak{D}_{\mathcal{T}} = 1$.so $\mathcal{T}^*$ is unitarily equivalent to the shift operator S, i.e. that

$$S = \mathcal{V}\mathcal{T}^*\mathcal{V}^*$$

for some unitary operator $\mathcal{V} : \mathcal{H} \to \ell^2$. Defining $\Gamma = \widetilde{\mathcal{V}}\mathcal{R}\widetilde{\mathcal{V}}^*$, $\Gamma_1 = \widetilde{\mathcal{V}}\mathcal{R}_1\widetilde{\mathcal{V}}^*$, we can see that

$$\Gamma_1 = \Gamma S = S^*\Gamma,$$

thus Γ is indeed the Hankel operator with trivial kernel, which satisfies (3.1.7), (3.1.8) and (3.1.9). Since such Hankel Γ that satisfies (3.1.7), (3.1.8) and (3.1.9) is unique, we have finished the proof of trivial kernel condition. $\square$

Chapter 4

INTRODUCTION TO CONJUGATION AND $\mathfrak{C}$–SYMMETRIC OPERATORS

In last chapter, we talk about the inverse spectral problem for self-adjoint Hankel operators. In chapter 4 and chapter 6, we discuss the same topic for a non self-adjoint Hankel operator. In this chapter, we discuss some background definition and properties before stating the main result for a non self-adjoint Hankel operator in chapter 5.

In section 4.1, we introduce the definition of conjugation and a special type of operators called complex symmetric operators, and we show that all Hankel operators are $\mathfrak{C}$-symmetric operators where $\mathfrak{C}$ is the canonical conjugation defined as (4.1.2).

Next in section 4.2 and section 4.3 we will show two propositions that we will frequently use in the following chapters. In section 4.2, we give the description of the generalized polar decomposition (see Theorem 4.2.2) for $\mathfrak{C}$-symmetric operators. In section 4.3, we prove a lemma related to rank-one perturbation, stated as below:

Lemma 4.0.1. *Let a self-adjoint operator $B = B^*$ be a rank one perturbation of a self-adjoint operator $A = A^*$,*

$$B = A - \alpha pp^*, \qquad \alpha \in \mathbb{R},\ \alpha \neq 0.$$

Then there exists a conjugation $\mathfrak{J}_p$ commuting with both A and B and preserving p,

$\mathfrak{J}_p p = p$.

Moreover, $\mathfrak{J}_p|_{\mathcal{H}_0}$ *is uniquely determined where* $\mathcal{H}_0 := \overline{\text{Span}}\{A^n p : n \geq 0\}$. *Indeed, we have* $\mathfrak{J}_p A^n p = A^n p$ *holds for all* $n \in \mathbb{N}$.

4.1 Definition of Conjugation and Complex Symmetric Operators

To begin with, we need some definitions and preparation work, borrowed from [22].

Definition 4.1.1. *An operator* $\mathfrak{C}$ *in a complex Hilbert space* $\mathcal{H}$ *is called a* conjugation*, if and only if it is*

(i) conjugate-linear*:* $\mathfrak{C}(\alpha x + \beta y) = \bar{\alpha}\mathfrak{C}x + \bar{\beta}\mathfrak{C}y$ *for all* $x, y \in \mathcal{H}$ *;*

(ii) involutive*:* $\mathfrak{C}^2 = I$ *;*

(iii) isometric*:* $\|\mathfrak{C}x\| = \|x\|$ *for all* $x \in \mathcal{H}$.

The followings are some typical examples of conjugations.

Example 4.1.2. *Let* (X, μ) *be a measurable space, then the canonical conjugation* $\mathfrak{C}$ *on* $L^2(X, \mu)$ *is just the pointwise complex conjugation:*

$$(\mathfrak{C}f)x = \overline{f(x)}.$$

Particularly, the canonical conjugation defined on $\mathbb{C}^n$ *is given by*

$$\mathfrak{C}(z_1, z_2, ..., z_n) = (\overline{z_1}, \overline{z_2}..., \overline{z_n})$$

Example 4.1.3. *The* Toeplitz conjugation *on* $\mathbb{C}^n$ *is defined as*

$$\mathfrak{C}(z_1, z_2, ..., z_n) = (\overline{z_n}, \overline{z_{n-1}}, ...\overline{z_1})$$

The following is a well-known result, giving the existence of a set of $\mathfrak{C}$-*real orthogonal basis* $\{e_n\}_{n=1}^{\infty}$, stated in [22].

Definition 4.1.4. *We call a vector $v \in \mathcal{H}$ a $\mathfrak{C}$-real vector if $\mathfrak{C}v = v$.*

Lemma 4.1.5. *If $\mathfrak{C}$ is a conjugation on $\mathcal{H}$, then there exists an orthogonal basis $\{e_n\}_{n=1}^{\infty}$ such that $\mathfrak{C}e_n = e_n$ holds for all n. In particular,*

$$\mathfrak{C}(\sum_n \alpha_n e_n) = \sum_n \overline{\alpha_n} e_n$$

holds for all $\{\alpha_n\}_{n\geq 0} \in \ell^2$. And we call such basis $\{e_n\}_{n=1}^{\infty}$ $\mathfrak{C}$-real orthogonal basis*.*

From the definition of $\mathfrak{C}$-real orthogonal basis, it's easy to see that for any two vectors $x, y \in \mathcal{H}$ and any conjugation $\mathfrak{C}$, we have

$$\langle \mathfrak{C}x, y\rangle = \langle \mathfrak{C}y, x\rangle. \tag{4.1.1}$$

Now we introduce the definition of $\mathfrak{C}$-symmetric operators.

Definition 4.1.6. *Let $\mathfrak{C}$ be a conjugation on Hilbert space $\mathcal{H}$. A bounded linear operator T on $\mathcal{H}$ is called* $\mathfrak{C}$-symmetric *iff $T^* = \mathfrak{C}T\mathfrak{C}$.*

If we have an operator T is $\mathfrak{C}-$symmetric, and we take a $\mathfrak{C}$-real orthogonal basis $\{e_n\}_{n\geq 0}$, then we can show that the representation matrix of T with respect to this basis $\{e_n\}_{n=1}^{\infty}$ is a complex symmetric matrix. In fact,

$$\begin{aligned}[T]_{ij} &= \langle Te_j, e_i\rangle = \langle \mathfrak{C}T^*\mathfrak{C}e_j, e_i\rangle = \langle \mathfrak{C}e_i, T^*\mathfrak{C}e_j\rangle \\ &= \langle e_i, T^*e_j\rangle = \langle Te_i, e_j\rangle = [T]_{ji}\end{aligned}$$

We record here some well-know examples of complex symmetric operators, and their corresponding $\mathfrak{C}$-real orthogonal basis.

Example 4.1.7. *For any Hankel operator Γ, we have $\Gamma = \mathfrak{C}\Gamma^*\mathfrak{C}$ where here $\mathfrak{C}$ is the canonical conjugation on ℓ^2:*

$$\mathfrak{C}(z_1, z_2, z_3...) = (\overline{z_1}, \overline{z_2}, \overline{z_3}...). \tag{4.1.2}$$

Hence Γ is a $\mathfrak{C}$-symmetric operator. Here the $\mathfrak{C}$-real orthogonal basis can be simply taken to be the original orthogonal basis $\{e_n\}_{n=0}^{\infty}$.

4.2 Generalization of Polar Decomposition for $\mathfrak{C}$-symmetric operators

In this section we will introduce a generalized polar decomposition for $\mathfrak{C}$-symmetric operators. We begin with the definition of partial conjugation.

Definition 4.2.1 (See [21]). *An anti-linear operator $\mathfrak{J}$ in Hilbert space is called a* partial conjugation *if $(\operatorname{Ker}\mathfrak{J})^{\perp}$ is invariant for $\mathfrak{J}$ and $\mathfrak{J}|_{(\operatorname{Ker}\mathfrak{J})^{\perp}}$ is a conjugation. Denoting $\mathfrak{K} := (\operatorname{Ker}\mathfrak{J})^{\perp}$, then we often say that $\mathfrak{J}$ is a partial conjugation on $\mathfrak{K}$.*

Let us recall that an operator U is called a *partial isometry* if its restriction to $(\operatorname{Ker} U)^{\perp}$ is an isometry; note that $(\operatorname{Ker} U)^{\perp}$ does not need to be U-invariant. If denoting $\mathfrak{K} := (\operatorname{Ker} U)^{\perp}$, we sometimes say that U is a partial isometry on $\mathfrak{K}$.

Recall also that any bounded operator T in a Hilbert space admits a unique *polar decomposition* $T = U|R|$, where $|R| := (R^*R)^{1/2}$, and U is a partial isometry with $\operatorname{Ker} U = \operatorname{Ker} T$.

Then we have a generalized polar decomposition for complex symmetric operators, given in [21]:

Theorem 4.2.2. *If $T = U|T|$ is the polar decomposition of a $\mathfrak{C}-$symmetric operator T, then $T = \mathfrak{C}\mathfrak{J}|T|$, where $\mathfrak{J}$ is a partial conjugation satisfy $\operatorname{Ker}\mathfrak{J} = \operatorname{Ker} T$, which commutes with $|T| := \sqrt{T^*T}$. In particular, the partial isometry U is $\mathfrak{C}-$symmetric and factors as $U = \mathfrak{C}\mathfrak{J}$.*

Remark 4.2.3. Note that we can always find a conjugation $\widetilde{\mathfrak{J}}$, such that $\widetilde{\mathfrak{J}} = \mathfrak{J}$ on $(\operatorname{Ker}\mathfrak{J})^{\perp}$. In fact, we can take $\mathfrak{J}'$ is an arbitrary partial conjugation with support $\operatorname{Ker}\mathfrak{J}$, and then define $\widetilde{\mathfrak{J}} := \mathfrak{J} + \mathfrak{J}'$. With this conjugation $\widetilde{\mathfrak{J}}$, we can reach several corollaries from Theorem 4.2.2.

Corollary 4.2.4. *If T is a bounded $\mathfrak{C}$-symmetric operator on $\mathcal{H}$, then $T = \mathfrak{C}\mathfrak{J}|T|$, where $\mathfrak{J}$ is a conjugation that commutes with $|T|$. In addition, the choice of $\mathfrak{J}$ is unique if and only if* $\operatorname{Ker} T = \{0\}$.

Corollary 4.2.5. *If T is a bounded $\mathfrak{C}$-symmetric operator, then $T = W|T|$ where W is a $\mathfrak{C}$-symmetric unitary operator.*

4.3 A Property Related to Rank-one Perturbation

In last chapter, we have shown that a self-adjoint Hankel operator Γ satisfies a rank-one perturbation relation: $\Gamma^2 - \Gamma_1^2 = uu^*$, where $\Gamma_1 := \Gamma S$ and $u = \Gamma e_0$. For a general Hankel operator Γ, we can derive a similar identity. In fact, since $\Gamma_1 = \Gamma S = S^*\Gamma$, we have

$$|\Gamma|^2 - |\Gamma_1|^2 = \Gamma^*\Gamma - \Gamma^* S S^* \Gamma = \Gamma^*(I - SS^*)\Gamma = \Gamma^*(e_0 e_0^*)\Gamma,$$

so denoting $u := \Gamma^* e_0$, we have

$$|\Gamma|^2 - |\Gamma_1|^2 = uu^*, \tag{4.3.1}$$

In this section, we will prove the result in lemma 4.0.1, which will be frequently used in later discussion. Applying this lemma to (4.3.1), we can find a conjugation $\mathfrak{J}_u$ commuting with $|\Gamma|, |\Gamma_1|$ and preserves u. In addition, this conjugation is unique if u is cyclic with respect to $|\Gamma|$ in $\mathcal{H}$.

Proof of Lemma 4.0.1. To begin with, we need the following lemma.

Lemma 4.3.1. *(Modified from [24, Theorem 2.2]) Let $\mathfrak{J}$ be a conjugation on $L^2(\rho)$, where ρ is a Borel measure on $\mathbb{R}$, which is compact supported. Then the following two statements are equivalent:*

(i) $\mathcal{M}_x \mathfrak{J} = \mathfrak{J}\mathcal{M}_x$*, where* $\mathcal{M}_x$ *is the operator of multiplication by independent variable;*

(ii) *There exists a function* $\phi(x) \in L^\infty(\rho)$*,* $|\phi(x)| = 1$ $\rho-$ *a.e., such that:* $\mathfrak{J}f = \mathcal{M}_{\phi(x)}\mathfrak{J}_1 f$ *holds* $\rho-$*a.e . Here* $\mathfrak{J}_1$ *is the canonical conjugation on* $L^2(\rho)$ *defined by*

$$\mathfrak{J}_1 : f(x) \to \overline{f(x)}$$

Proof. We only need to show (i) $\Longrightarrow$ (ii), while the other direction is easy to check. We know that $\mathcal{M}_x \mathfrak{J}\mathfrak{J}_1 = \mathfrak{J}\mathcal{M}_x\mathfrak{J}_1 = \mathfrak{J}\mathfrak{J}_1\mathcal{M}_x$, hence $\mathcal{M}_x$ commutes with an unitary operator $\mathfrak{J}\mathfrak{J}_1$ on $L^2(\rho)$. By [Theorem 3.2, [25]], $\mathfrak{J}\mathfrak{J}_1 = \mathcal{M}_\phi$ for a $\phi(x) \in L^2(\rho)$, $J = \mathcal{M}_{\phi(x)}\mathfrak{J}_1$. Since $\mathfrak{J}\mathfrak{J}_1 = \mathcal{M}_\phi$ is unitary, we conclude that $|\phi| = 1$ ρ-a.e. □

Now get back to the proof of Lemma 4.0.1

By Von Neumann's theorem (See Theorem 1.2.7), up to unitary equivalence we can assume that A is the multiplication by independent variable $\mathcal{M}_x$ on $L^2(\rho)$, and p is a function $f(x) \in L^2(\rho)$. By Lemma 4.3.1, since $\mathcal{M}_x\mathfrak{J} = \mathfrak{J}\mathcal{M}_x$, we have $\mathfrak{J} = \mathcal{M}_{\phi(x)}\mathfrak{J}_1$ for a certain function $\phi(x) \in L^\infty(\rho)$, $|\phi(x)| = 1$ $\rho-$ a.e. Then from $\mathfrak{J}f = f$, we have

$$\phi(x)\overline{f(x)} = f(x)$$

Thus we can set $\phi(x) = \frac{f(x)}{\overline{f(x)}}$ where $f(x) \neq 0$, and set $\phi(x)$ to be any unit value where $f(x) = 0$. This gives the existence of such conjugation $\mathfrak{J}_p$.

To see that $\mathfrak{J}_p$ is unique on $\mathcal{H}_0$, we can show by induction that $\mathfrak{J}_p A^n p = A^n p$ for all $n \in \mathbb{N}$.

In fact, the equation holds trivially when $n = 0$. Assume that the equation holds for $n = k$, then for $n = k + 1$,

$$\mathfrak{J}_p A^{k+1} p = A\mathfrak{J}_p A^k p = AA^k p = A^{k+1} p,$$

thus $\mathfrak{J}_p$ is uniquely defined on $\mathcal{H}_0 = \overline{\text{Span}}\{A^n p : n \geq 0\}$. □

The following lemma gives all such conjugations that satisfy the property in Lemma 4.0.1. It's easy to see that such conjugation $\mathfrak{J}_p$ is unique if and only if p is a cyclic vector for A.

Lemma 4.3.2. *If $\mathfrak{J}_p$ is a conjugation from Lemma 4.0.1, then any other such conjugation $\mathfrak{J}'_p$ that also satisfies the requirement in Lemma 4.0.1 will be given by $\mathfrak{J}'_p = \psi\mathfrak{J}_p$, where ψ is unitary $\mathfrak{J}_p$-symmetric operators commuting with A and preserves p, $\psi p = p$.*

Proof. If $\mathfrak{J}'_p$ is another conjugation, commuting with A and preserving p, then $\mathfrak{J}'_p = \psi\mathfrak{J}_p$, where $\psi := \mathfrak{J}'_p\mathfrak{J}_p$. It is east to see that ψ is a unitary operator. It is also easy to see that ψ is $\mathfrak{J}_p$-symmetric operator commuting with A and that $\psi p = p$.

On the other hand, if ψ is a $\mathfrak{J}_p$-symmetric unitary operator, commuting with A and such that $\psi p = p$, then the (conjugate-linear) operator $\mathfrak{J}'_p := \psi\mathfrak{J}_p$ is a conjugation commuting with A and preserving p.

Indeed, the operator $\mathfrak{J}'_p$ is trivially conjugate-linear, isometric, preserves p and commutes with A (and so with B). To show that $\mathfrak{J}'_p$ is an involution we use the $\mathfrak{J}_p$-symmetry of ψ:

$$(\mathfrak{J}'_p)^2 = \psi\mathfrak{J}_p\psi\mathfrak{J}_p = \mathfrak{J}_p\psi^*\psi\mathfrak{J}_p = \mathfrak{J}_p^2 = I;$$

here in the second equality we use the fact that ψ is $\mathfrak{J}_p$-symmetric. □

Remark 4.3.3. Since ψ commutes with A and preserves p, it is easy to see that $\psi\big|_{\mathcal{H}_0} = I\big|_{\mathcal{H}_0}$.

Chapter 5

GENERAL HANKEL OPERATORS AS COMPLEX SYMMETRIC OPERATORS

In chapter 3, we have discussed the inverse spectral problem for a self-adjoint Hankel operator. In this chapter, we will discuss the same topic for a general Hankel operator taken as a $\mathfrak{C}$-symmetric operator. We will also show that a Hankel operator can be uniquely determined by the spectral data of two operators satisfying a rank-one perturbation relation.

In section 5.1, we introduce the setting and some preparation work for the game. Briefly speaking, we can find a conjugation $\mathfrak{J}_u$, such that this $\mathfrak{J}_u$ commutes with $|\Gamma|$ and $|\Gamma_1|$, and also preserves $u := \Gamma^* e_0$. Then we apply the generalized polar decomposition Theorem 4.2.2 to Γ, Γ_1, and restrict Γ, Γ_1 on the essential part of Γ, thus we can write them as the form in (5.1.5):

$$\mathfrak{C}\widetilde{\Gamma} = \widetilde{\phi}|\widetilde{\Gamma}|\widetilde{\mathfrak{J}}_u, \qquad \mathfrak{C}\widetilde{\Gamma}_1 = \widetilde{\phi}_1|\widetilde{\Gamma}_1|\widetilde{\mathfrak{J}}_u. \tag{5.0.1}$$

We also need to emphasize that the choice of $\mathfrak{J}_u$ is unique if and only if u is cyclic with respect to $|\Gamma|$ on $\mathcal{H}$. Under this case, the choice of unitary $\widetilde{\phi}$ and partial isometry $\widetilde{\phi}_1$ is also unique.

Next in section 5.2 and section 5.3, we state and prove the main result (Proposition

5.2.3 and Proposition 5.2.4) in this chapter. That is, given a triple $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$ satisfying the following properties

(i) $R, R_1 \geq 0$, $\operatorname{Ker} R = \{0\}$, are self-adjoint operators in $\mathcal{H}$, and $p \in \mathcal{H}$ and such that

$$R^2 - R_1^2 = pp^*.$$

(ii) $\mathfrak{J}_p$ is a conjugation commuting with R and R_1 and preserving p, $\mathfrak{J}_p p = p$.

(iii) φ is a $\mathfrak{J}_p$-symmetric unitary operator commuting with R.

(iv) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry, $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, commuting with R_1; note that if $\operatorname{Ker} R_1 = \{0\}$ then φ_1 is unitary.

(v) The contraction $\mathcal{T} := R_1 \varphi_1 \varphi^* R^{-1}$ is asymptotically stable;

then there exists a unique Hankel operator Γ, such that we can find a conjugation $\mathfrak{J}_u$ commuting with $|\Gamma|, |\Gamma_1|$ and preserves u, satisfying that the induced triple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \mathfrak{J}_u)$ is unitary equivalent to $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$.

Thus we can see that a general Hankel operator can be uniquely determined by the spectral data of two operators $R\varphi, R_1\varphi_1$, which satisfy a rank-one perturbation relation.

In addition, we can also further translate the spectral data of $R\varphi, R_1\varphi_1$ under the compact case.

Denoting $\mathfrak{H}_o := \overline{\operatorname{Span}}\{|\widetilde{\Gamma}|^n u \big| n \geq 0\}$. Then Proposition 6.2.2 implies that there exists two positive real sequences $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$, such that the non-zero eigenvalues of $|\widetilde{\Gamma}|\big|_{\mathfrak{H}_o}$, $|\widetilde{\Gamma}_1|\big|_{\mathfrak{H}_o}$ are simple, and coincide with $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ respectively.

We can further show that $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ has no non-zero eigenvalues other than $\{\lambda_k\}_{k=1}^{\infty}$,

$\{\mu_k\}_{k=1}^{\infty}$. Thus we can write all eigenspaces of $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ as the following:

$$\mathfrak{E}^1_{\lambda_k} := \mathrm{Ker}(|\widetilde{\Gamma}_1| - \lambda_k I), \qquad \mathfrak{E}_{\lambda_k} := \mathrm{Ker}(|\widetilde{\Gamma}| - \lambda_k I);$$
$$\mathfrak{E}^1_{\mu_k} := \mathrm{Ker}(|\widetilde{\Gamma}_1| - \mu_k I), \qquad \mathfrak{E}_{\mu_k} := \mathrm{Ker}(|\widetilde{\Gamma}| - \mu I),$$

For the non self-adjoint compact Γ, we can also take a canonical choice of tuple $(\mathfrak{J}_u, \widetilde{\phi}, \widetilde{\phi}_1)$, which is given by Lemma 6.3.2, together with the equivalent condition of asymptotic stability given in Lemma 7.6.3, we have the structure of φ on $\mathfrak{E}_{\lambda_k}, \mathfrak{E}_{\mu_k}$, and the structure of φ_1 on $\mathfrak{E}^1_{\lambda_k}, \mathfrak{E}^1_{\mu_k}$, which is given as follow:

(i) $\widetilde{\phi}|_{\mathfrak{E}_{\mu_k}} = I, \qquad \widetilde{\phi}_1|_{\mathfrak{E}^1_{\lambda_k}} = I;$

(ii) u_{λ_k} is a *-cyclic vector for $\widetilde{\phi}|_{\mathfrak{E}_{\lambda_k}}$, where $u_{\lambda_k} := P_{\mathfrak{E}_{\lambda_k}} u$;

(iii) u_{μ_k} is a *-cyclic vector for $\widetilde{\phi}_1|_{\mathfrak{E}^1_{\mu_k}}$, where $u_{\mu_k} := P_{\mathfrak{E}^1_{\mu_k}} u$.

Now with the preparations above, we consider two different situations under the compact case:

(i) $\mathfrak{H}_{\mathfrak{o}} = \ell^2$. Then we have $\mathfrak{E}_{\mu_k}, \mathfrak{E}^1_{\lambda_k}$ are trivial, and $\mathfrak{E}_{\lambda_k}, \mathfrak{E}^1_{\mu_k}$ are of dimension 1. Under this situation, we have the choice of $\mathfrak{J}_u$ is unique, and we conclude that Γ will be uniquely determined by the eigenvalues of two operators $|\widetilde{\Gamma}|\widetilde{\phi}, |\widetilde{\Gamma}_1|\widetilde{\phi}_1$. In other words, given two sequences of complex numbers $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$, whose modulus satisfy an intertwining relation:

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0,$$

we can find a compact Hankel operator Γ with simple singular values, with the uniquely determined tuple $(\mathfrak{J}_u, \phi, \phi_1)$, satisfying that the non-zero eigenvalues of $|\Gamma|\phi, |\Gamma_1|\phi_1$ are simple, and coincide with $\{\lambda_n\}_{n=1}^{\infty}, \{\mu_n\}_{n=1}^{\infty}$ respectively.

The detail of work under this situation can be found in section 9.2 in chapter 8.

(ii) $\mathfrak{H}_{\mathfrak{o}} \subsetneq \ell^2$, then we can define a measure $\widetilde{\rho}_k(s)$ to be the scalar spectral measure of $\widetilde{u}_{\lambda_k} := \frac{u_{\lambda_k}}{\|u_{\lambda_k}\|}$ with respect to $\widetilde{\phi}|_{\mathfrak{E}_{\lambda_k}}$. And we also similarly define $\widetilde{\rho}_k^1(s)$ to be the scalar spectral measure of $\widetilde{\mu}_k := \frac{u_{\mu_k}}{\|u_{\mu_k}\|}$ with respect to $\widetilde{\phi}_1|_{\mathfrak{E}^1_{\mu_k}}$:

$$\langle(\widetilde{\phi} - zI)^{-1}\widetilde{u}_{\lambda_k}, \widetilde{u}_{\lambda_k}\rangle = \int_{\mathbb{T}} \frac{d\widetilde{\rho}_k(s)}{s-z}, \qquad \langle(\widetilde{\phi}_1 - zI)^{-1}\widetilde{u}_{\mu_k}, \widetilde{u}_{\mu_k}\rangle = \int_{\mathbb{T}} \frac{d\widetilde{\rho}_k^1(s)}{s-z}.$$

We conclude that $\widetilde{\rho}_k(s), \widetilde{\rho}_k^1(s)$ are both positive discrete probability measures (see Proposition 10.1.2), and we conclude that the four sequences $\{\lambda_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$, $\{\widetilde{\rho}_k\}_{k=1}^{\infty}$, $\{\widetilde{\rho}_k^1\}_{k=1}^{\infty}$ will uniquely determine a Hankel operator (see Theorem 10.1.5).

The specific description of the theorem can be found in chaper 9.

5.1 Setup

Given a bounded Hankel operator Γ, we have mentioned in Example 4.1.7 that Γ and $\Gamma_1 := S^*\Gamma$ are $\mathfrak{C}$-symmetric, thus applying Theorem 4.2.2, there exists partial conjugations $\mathfrak{J}$ and $\mathfrak{J}_1$ commuting with $|\Gamma|$ and $|\Gamma_1|$ respectively, such that $\operatorname{Ker}\mathfrak{J} = \operatorname{Ker}\Gamma$, $\operatorname{Ker}\mathfrak{J}_1 = \operatorname{Ker}\Gamma_1$ and

$$\Gamma = \mathfrak{C}\mathfrak{J}|\Gamma|, \qquad \Gamma_1 = \mathfrak{C}\mathfrak{J}_1|\Gamma_1|, \tag{5.1.1}$$

where here $\mathfrak{J}, \mathfrak{J}_1$ commutes with $|\Gamma|, |\Gamma_1|$ respectively.

Now recall that we have derived in (4.3.1) that

$$|\Gamma|^2 - |\Gamma_1|^2 = uu^*, \tag{5.1.2}$$

where $u = \Gamma^* e_0$. We apply Lemma 4.0.1 to (4.3.1) with $A = |\Gamma|^2, B = |\Gamma_1|^2, \alpha = 1$, then we have that there exists a conjugation $\mathfrak{J}_u$ commuting with $|\Gamma|^2$ and $|\Gamma_1|^2$ and such that $\mathfrak{J}_u u = u$.

We want to show that $\mathfrak{J}_u$ also commutes with $|\Gamma|$ and $|\Gamma_1|$. We need the following short lemma.

Lemma 5.1.1. *Let R be a bounded, self-adjoint positive operator on Hilbert space $\mathcal{H}$, and $\mathfrak{J}$ is a conjugation on $\mathcal{H}$. Then the following two statements are equivalent:*

(i) *R commutes with $\mathfrak{J}$;*

(ii) *R^2 commutes with $\mathfrak{J}$.*

Proof. (i) $\Longrightarrow$ (ii) is trivial, we only need to prove the other direction.

Since R^2 commutes with $\mathfrak{J}$, for any polynomial $\mathfrak{p}$ with real coefficients, $\mathfrak{p}(R^2)$ also commutes with $\mathfrak{J}$: $\mathfrak{p}(R^2)\mathfrak{J} = \mathfrak{J}\mathfrak{p}(R^2)$. Since the scalar spectrum measure $\sigma(R^2)$ is compact supported, we can take a polynomial sequence $\{\mathfrak{p}_n(x)\}_{n=1}^{\infty}$ which converges uniformly to $\phi(x) = \sqrt{x}$ on $\sigma(R^2)$. Then by $\phi(R^2)\mathfrak{J} = \mathfrak{J}\phi(R^2)$ we have

$$R\mathfrak{J} = \mathfrak{J}R$$

□

Now back to identity (5.1.2), we know that $\mathfrak{J}_u$ commutes with $|\Gamma|, |\Gamma_1|$. Thus we can rewrite the polar decomposition form in (5.1.1) as

$$\Gamma = \mathfrak{C}\phi\mathfrak{J}_u|\Gamma| = \mathfrak{C}\phi|\Gamma|\mathfrak{J}_u, \qquad \Gamma_1 = \mathfrak{C}\phi_1\mathfrak{J}_u|\Gamma_1| = \mathfrak{C}\phi_1|\Gamma_1|\mathfrak{J}_u, \tag{5.1.3}$$

where $\phi := \mathfrak{J}\mathfrak{J}_u$, $\phi_1 := \mathfrak{J}_1\mathfrak{J}_u$ are partial isometries, commuting with $|\Gamma|$ and $|\Gamma_1|$ respectively. Since $\mathfrak{J}_u$ commutes with $|\Gamma|$ and with $|\Gamma_1|$, both $\operatorname{Ker}\Gamma = \operatorname{Ker}|\Gamma|$ and $\operatorname{Ker}\Gamma_1 = \operatorname{Ker}|\Gamma_1|$ are invariant for $\mathfrak{J}_u$, so $\operatorname{Ker}\phi = \operatorname{Ker}\Gamma$, $\operatorname{Ker}\phi_1 = \operatorname{Ker}\Gamma_1$.

We can rewrite those above identities (5.1.3) as

$$\mathfrak{C}\Gamma = \phi|\Gamma|\mathfrak{J}_u, \qquad \mathfrak{C}\Gamma_1 = \phi_1|\Gamma_1|\mathfrak{J}_u. \tag{5.1.4}$$

Since $\mathfrak{J}_u$ commute with $|\Gamma|$, both $\operatorname{Ker}\Gamma = \operatorname{Ker}|\Gamma|$ and $(\operatorname{Ker}\Gamma)^{\perp}$ are invariant for $\mathfrak{J}_u$, and therefore $\mathfrak{J}_u(\operatorname{Ker}\Gamma)^{\perp} = (\operatorname{Ker}\Gamma)^{\perp}$.

Now restricting everything to $(\operatorname{Ker}\Gamma)^\perp$ and denoting $\widetilde{\Gamma} := \Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}$, $\widetilde{\Gamma}_1 := \Gamma_1\big|_{(\operatorname{Ker}\Gamma)^\perp}$, $\widetilde{\mathfrak{J}}_u := \mathfrak{J}_u\big|_{(\operatorname{Ker}\Gamma)^\perp}$, $\widetilde{\phi} := \phi\big|_{(\operatorname{Ker}\Gamma)^\perp}$, $\widetilde{\phi}_1 = \phi_1\big|_{(\operatorname{Ker}\Gamma)^\perp}$ we can see that

$$\mathfrak{C}\widetilde{\Gamma} = \widetilde{\phi}|\widetilde{\Gamma}|\widetilde{\mathfrak{J}}_u, \qquad \mathfrak{C}\widetilde{\Gamma}_1 = \widetilde{\phi}_1|\widetilde{\Gamma}_1|\widetilde{\mathfrak{J}}_u. \tag{5.1.5}$$

Remark 5.1.2. We can easily see that (5.1.5) is well-defined. In fact, though $(\operatorname{Ker}\Gamma)^\perp$ is not necessarily an invariant subspace for Γ, it is a reducing subspace for $|\Gamma|$, and since $u \in \operatorname{Ran}\Gamma^* \perp \operatorname{Ker}\Gamma$, it is also reducing for $|\Gamma_1|^2$, and so for $|\Gamma_1|$. It is also invariant for ϕ_1 and for $\mathfrak{J}_u$, so everything in the right hand sides of (5.1.5) is well defined, which means that $(\operatorname{Ker}\Gamma)^\perp$ is invariant for $\mathfrak{C}\Gamma$ and for $\mathfrak{C}\Gamma_1$.

Now back to (5.1.5). Since $\mathfrak{J}_u(\operatorname{Ker}\Gamma)^\perp = (\operatorname{Ker}\Gamma)^\perp$, $\widetilde{\mathfrak{J}}_u$ is a conjugation on $(\operatorname{Ker}\Gamma)^\perp$, commuting with $|\widetilde{\Gamma}|$ and preserving u (note that $u = \Gamma^* e_0 \in \operatorname{Ran}|\Gamma|$). We can see that $\widetilde{\phi}$ is unitary since $\operatorname{Ker}\phi = \operatorname{Ker}\Gamma$. Note also that the operator $\widetilde{\Gamma}_1$ can have a non-trivial kernel (at most one-dimensional), and the action of $\widetilde{\phi}_1$ on this kernel does not matter in (5.1.5). So, let us assume for definiteness that $\widetilde{\phi}_1\big|_{\operatorname{Ker}\widetilde{\Gamma}_1} = 0$, i.e. that $\widetilde{\phi}_1$ is a partial isometry with $\operatorname{Ker}\widetilde{\phi}_1 = \operatorname{Ker}\widetilde{\Gamma}_1$.

Remark 5.1.3. As it was shown above, see Lemma 4.3.2, the conjugation $\widetilde{\mathfrak{J}}_u$ and the operators $\widetilde{\phi}$, $\widetilde{\phi}_1$ in (5.1.5) are not unique, and are defined up to equivalence. Namely, the two triples $(\widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1)$ and $(\widetilde{\mathfrak{J}}'_u, \widetilde{\phi}', \widetilde{\phi}'_1)$ are in the same equivalent class if there exists an $\widetilde{\mathfrak{J}}_u$-symmetric unitary operator ψ commuting with $|\widetilde{\Gamma}|$ and preserving u, $\psi u = u$, such that

$$\widetilde{\mathfrak{J}}'_u = \psi\widetilde{\mathfrak{J}}_u, \qquad \widetilde{\phi}' = \widetilde{\phi}\psi^*, \qquad \widetilde{\phi}'_1 = \widetilde{\phi}_1\psi^*.$$

We can easily check that (5.1.5) is still true if we substitute $(\widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1)$ by $(\widetilde{\mathfrak{J}}'_u, \widetilde{\phi}', \widetilde{\phi}'_1)$:

$$\mathfrak{C}\widetilde{\Gamma} = \widetilde{\phi}'|\widetilde{\Gamma}|\widetilde{\mathfrak{J}}'_u, \qquad \mathfrak{C}\widetilde{\Gamma}_1 = \widetilde{\phi}'_1|\widetilde{\Gamma}_1|\widetilde{\mathfrak{J}}'_u$$

Now we give an expression of the Hankel coefficients of Γ, i.e. $\{\gamma_k\}_{k=0}^\infty$, implying that Γ could be uniquely determined by $|\widetilde{\Gamma}|, u, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1$. In other words, the uniqueness of Γ only depends on the essential part of Γ, i.e. $\Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}$.

Proposition 5.1.4. *Given a Hankel operator Γ, with the tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \mathfrak{J}_u)$ defined in (5.1.5). Then the Hankel coefficients $\{\gamma_k\}_{k=0}^\infty$ can be expressed as*

$$\gamma_k = \langle (\mathfrak{S}^*)^k u, v \rangle, \tag{5.1.6}$$

where $\mathfrak{S}^* := |\widetilde{\Gamma}_1|\widetilde{\phi}_1^*\widetilde{\phi}|\widetilde{\Gamma}|^{-1}, v := \mathfrak{J}_u e_0$, and here $\widetilde{S}^*$ is the restriction of backward shift on $(\operatorname{Ker}\Gamma)^\perp$, i.e.

$$\widetilde{S}^* := S^*|_{(\operatorname{Ker}\Gamma)^\perp}$$

Proof. From $\Gamma_1 = S^*\Gamma$ together with (5.1.4), we have

$$\mathfrak{C}\phi_1|\Gamma_1|\mathfrak{J}_u = S^*\mathfrak{C}\phi|\Gamma|\mathfrak{J}_u,$$

Thus

$$\phi_1|\Gamma_1| = S^*\phi|\Gamma|. \tag{5.1.7}$$

Restricted on $(\operatorname{Ker}\Gamma)^\perp$ and denote $\widetilde{S}^* := S^*|_{(\operatorname{Ker}\Gamma)^\perp}$, we can rewrite it as

$$|\widetilde{\Gamma}_1|\widetilde{\phi}_1 = \widetilde{S}^*|\widetilde{\Gamma}|\widetilde{\phi}, \qquad \widetilde{S}^* = |\widetilde{\Gamma}_1|\widetilde{\phi}_1\widetilde{\phi}^*|\widetilde{\Gamma}|^{-1}.$$

Now using the definition of $\mathfrak{S}^*$, since we have $\widetilde{\phi}, \widetilde{\phi}_1$ are $\widetilde{\mathfrak{J}}_u$-symmetric, we have

$$\mathfrak{S}^*\widetilde{\mathfrak{J}}_u = \widetilde{\mathfrak{J}}_u|\widetilde{\Gamma}_1|\widetilde{\phi}_1\widetilde{\phi}^*|\widetilde{\Gamma}|^{-1} = \widetilde{\mathfrak{J}}_u\widetilde{S}^*$$

thus $\widetilde{\mathfrak{S}}^* = \widetilde{\mathfrak{J}}_u \widetilde{S}^* \widetilde{\mathfrak{J}}_u$, and $\mathfrak{S} = (\mathfrak{S}^*)^*$ satisfies

$$|\widetilde{\Gamma}_1|\widetilde{\phi}_1 = \widetilde{S}^*|\widetilde{\Gamma}|\widetilde{\phi} = |\widetilde{\Gamma}|\widetilde{\phi}\mathfrak{S}$$

Now using $v = \mathfrak{J}_u e_0 = \mathfrak{J}_u \mathfrak{C} e_0 = \mathfrak{J}_u \mathfrak{C} \mathfrak{J}_u v$, and since $(\operatorname{Ker}\Gamma)^\perp$ is an invariant subspace for $|\Gamma|, \phi$, we can write

$$\begin{aligned} u = \Gamma^* e_0 &= \Gamma^* \mathfrak{J}_u v \\ &= (|\Gamma|\mathfrak{J}\mathfrak{C})\mathfrak{J}_u(\mathfrak{J}_u \mathfrak{C} \mathfrak{J}_u v) \\ &= |\Gamma|\mathfrak{J}\mathfrak{J}_u v = |\Gamma|\phi v = |\widetilde{\Gamma}|\widetilde{\phi} v. \end{aligned} \tag{5.1.8}$$

Now using (4.1.1), we can express the Hankel symbol γ_k as

$$\begin{aligned} \gamma_k = \langle \Gamma e_k, e_0 \rangle &= \langle S^k e_0, u \rangle = \langle \mathfrak{J}_u^2 e_0, (S^*)^k u \rangle \\ &= \langle \mathfrak{J}_u (S^*)^k u, v \rangle = \langle \widetilde{\mathfrak{J}}_u (\widetilde{S}^*)^k u, v \rangle \\ &= \langle (\mathfrak{S}^*)^k u, v \rangle. \end{aligned} \tag{5.1.9}$$

□

We will use this equation (5.1.6) in a later proof.

5.2 Plan of the Game and Main Result

Given a Hankel operator Γ, in section 5.1 we have setup a tuple $(|\Gamma|, |\Gamma_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$ with the following properties:

(i) $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ are self-adjoint positive operators. In addition, $|\widetilde{\Gamma}|$ has trivial kernel;

(ii) $|\widetilde{\Gamma}|^2 - |\widetilde{\Gamma}_1|^2 = uu^*$;

(iii) $\widetilde{\mathfrak{J}}_u$ is a conjugation which commutes with $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ and preserves u, i.e. $\widetilde{\mathfrak{J}}_u u = u$;

(iv) $\widetilde{\phi}$ is a $\widetilde{\mathfrak{J}}_u$-symmetric unitary operator which commutes with $|\widetilde{\Gamma}|$;

(v) $\widetilde{\phi}_1$ is a $\widetilde{\mathfrak{J}}_u$-symmetric partial unitary operator which commutes with $|\widetilde{\Gamma}_1|$.

Now we consider the inverse direction of the problem. Assume that we are given a tuple $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$, which shares similar properties as tuple $(|\Gamma|, |\Gamma_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$:

(i) $R, R_1 \geq 0$, $\operatorname{Ker} R = \{0\}$, are self-adjoint operators in $\mathcal{H}$, and $p \in \mathcal{H}$ and such that

$$R^2 - R_1^2 = pp^*.$$

Note that the above identity implies that $p \in \operatorname{Ran} R$ and $\|R^{-1}p\| \leq 1$.

(ii) $\mathfrak{J}_p$ is a conjugation commuting with R and R_1 and preserving p, $\mathfrak{J}_p p = p$.

(iii) φ is a $\mathfrak{J}_p$-symmetric unitary operator commuting with R.

(iv) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry, $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, commuting with R_1; note that if $\operatorname{Ker} R_1 = \{0\}$ then φ_1 is unitary.

Remark 5.2.1. The first thing to notice is that the dimension of $\operatorname{Ker} R_1$ is at most 1. This can been easily seen from a later Lemma 6.2.1.

The second thing is that $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is unitary, implied by (iv). In fact, since we already have $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is isometry, it suffices to show that $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is onto. From equation $\varphi_1 \mathfrak{J}_p = \mathfrak{J}_p \varphi_1^*$, since $\operatorname{Ran} \varphi_1^* = (\operatorname{Ker} R_1)^\perp$ and $(\operatorname{Ker} R_1)^\perp$ is a reducing subspace for $\mathfrak{J}_p$, we have $(\operatorname{Ker} R_1)^\perp \subseteq \operatorname{Ran}(\mathfrak{J}_p \varphi_1^*)$, hence $(\operatorname{Ker} R_1)^\perp \subseteq \operatorname{Ran} \varphi_1$.

We want to know whether we can find a Hankel operator Γ and choose an appropriate a partial conjugation $\mathfrak{J}_u$, which commutes with $|\Gamma|$ and preserving $u = \Gamma^* e_0$ such that the tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$ defined in equation (5.1.5) is unitary equivalent to $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$, i.e., the following identities hold

$$|\widetilde{\Gamma}| = \widetilde{\mathcal{V}} R \widetilde{\mathcal{V}}^*, \qquad |\widetilde{\Gamma}_1| = \widetilde{\mathcal{V}} R_1 \widetilde{\mathcal{V}}^*; \tag{5.2.1}$$

$$\widetilde{\phi} = \widetilde{\mathcal{V}} \varphi \widetilde{V}^*, \qquad \widetilde{\phi}_1 = \widetilde{\mathcal{V}} \varphi_1 \widetilde{\mathcal{V}}^*, \qquad \widetilde{\mathfrak{J}}_u = \widetilde{\mathcal{V}} \mathfrak{J}_p \widetilde{\mathcal{V}}^*, \qquad u = \widetilde{\mathcal{V}} p, \tag{5.2.2}$$

for some unitary operator $\widetilde{\mathcal{V}} : \mathcal{H} \to (\operatorname{Ker}\Gamma)^\perp$.

Remark 5.2.2. For the case when p is a cyclic vector with respect to R in $\mathcal{H}$, then by Lemma 4.3.2, we know that $\mathfrak{J}_p$ is uniquely defined. (Thus with the unitary equivalence relation in (5.2.1) and (5.2.2), we have u is a cyclic vector with respect to $|\widetilde{\Gamma}|$ on $(\operatorname{Ker}\Gamma)^\perp$). In this case, we can write φ, φ_1 as $f(R), f_1(R_1)$ for some Borel measurable, unimodular functions f, f_1.

In this chapter, we will prove the following two propositions as our main results.

Proposition 5.2.3. *If contraction $\mathcal{T} := \varphi_1 R_1 R^{-1}\varphi^*$ (see Douglas Lemma 3.1.1) is asymptotically stable (see definition 3.1.2), then there exists a unique Hankel operator Γ such that*

$$\mathfrak{C}\widetilde{\Gamma} = \widetilde{\mathcal{V}} R\varphi\mathfrak{J}_p\widetilde{\mathcal{V}}^*, \tag{5.2.3}$$

$$\mathfrak{C}\widetilde{\Gamma}_1 = \widetilde{\mathcal{V}} R_1\varphi_1\mathfrak{J}_p\widetilde{\mathcal{V}}^*, \tag{5.2.4}$$

$$\Gamma^* e_0 = \widetilde{\mathcal{V}} p \tag{5.2.5}$$

for some unitary operator $\widetilde{\mathcal{V}} : \mathcal{H} \to (\operatorname{Ker}\Gamma)^\perp$; here, recall $\mathfrak{C}$ is the standard conjugation on ℓ^2 defined by (4.1.2).

The coefficients $\{\gamma_k\}_{k=1}^\infty$ of the Hankel operator Γ can be calculated as

$$\gamma_k = \langle \mathfrak{T}^k p, q\rangle. \tag{5.2.6}$$

where $\mathfrak{T} = \varphi_1^* R_1 R^{-1}\varphi = \mathfrak{J}_p\mathcal{T}\mathfrak{J}_p$ and $q := R^{-1}\varphi^* p = \varphi^* R^{-1} p$.

Furthermore, $\operatorname{Ker}\Gamma = \{0\}$ if and only if $\|q\| = 1$ and $q \notin \operatorname{Ran} R$ (recall that $\|R^{-1}p\| \le 1$, so $\|q\| \le 1$).

Proposition 5.2.4. *The identities* (5.2.3), (5.2.4), *and* (5.2.5) *are equivalent to the unitary equivalence of the tuples $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$ and $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$, i.e. to the identities* (5.2.1), (5.2.2). *(In other words, there exists a conjugation $\mathfrak{J}_u$ defining on*

$\mathcal{H}$, such that we can construct a tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$ which is unitary equivalent to $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$, and also satisfying (5.1.5)).

We first show that the two propositions aboveare equivalent, then we will present the proof of Proposition 5.2.3 in the next subsection.

Proof. Substituting (5.2.1) and (5.2.2) into (5.1.5) we immediately get (5.2.3), (5.2.4), and (5.2.5), so we only need to prove the other direction.

Now Assuming (5.2.3), (5.2.4) and (5.2.5), let us show that

$$|\widetilde{\Gamma}| = \widetilde{\mathcal{V}} R \widetilde{\mathcal{V}}^*, \qquad |\widetilde{\Gamma}_1| = \widetilde{\mathcal{V}} R_1 \widetilde{\mathcal{V}}^*.$$

We first show that $\widetilde{\Gamma}^* = \widetilde{\mathcal{V}} \mathfrak{J}_p \varphi^* R (\widetilde{\mathcal{V}})^* \mathfrak{C}$. Indeed,

$$\begin{aligned}
\langle \widetilde{\Gamma} x, y \rangle &= \langle \mathfrak{C} \widetilde{\mathcal{V}} R \varphi \mathfrak{J}_p \widetilde{\mathcal{V}}^* x, y \rangle = \langle \mathfrak{C} y, \widetilde{\mathcal{V}} R \varphi \mathfrak{J}_p \widetilde{\mathcal{V}}^* x \rangle \\
&= \langle \varphi^* R \widetilde{\mathcal{V}}^* \mathfrak{C} y, \mathfrak{J}_p \widetilde{\mathcal{V}}^* x \rangle = \langle \widetilde{\mathcal{V}}^* x, \mathfrak{J}_p \varphi^* R \widetilde{\mathcal{V}}^* \mathfrak{C} y \rangle \\
&= \langle x, \widetilde{\mathcal{V}}^* \mathfrak{J}_p \varphi^* R \widetilde{\mathcal{V}}^* \mathfrak{C} y \rangle
\end{aligned}$$

Thus $\widetilde{\Gamma}^* = \widetilde{\mathcal{V}}^* \mathfrak{J}_p \varphi^* R \widetilde{\mathcal{V}}^* \mathfrak{C}$, and we have

$$\begin{aligned}
|\widetilde{\Gamma}| &= \left((\widetilde{\Gamma})^* \widetilde{\Gamma} \right)^{1/2} \\
&= \left(\widetilde{\mathcal{V}} \mathfrak{J}_p \varphi^* R^2 \varphi \mathfrak{J}_p (\widetilde{\mathcal{V}})^* \right)^{1/2} \\
&= \left(\widetilde{\mathcal{V}} R^2 \widetilde{\mathcal{V}}^* \right)^{1/2} = \widetilde{\mathcal{V}} R \widetilde{\mathcal{V}}^*.
\end{aligned}$$

Similarly we can get $|\widetilde{\Gamma}_1| = \widetilde{\mathcal{V}} R_1 \widetilde{\mathcal{V}}^*$. The fact that φ_1 is only a partial isometry does not spoil anything; since φ_1 commutes with R_1 and $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, we have $\varphi_1^* R_1 \varphi = R_1$, and the rest of the computations follows exactly as for the case of $|\Gamma|$.

Next we define a conjugation $\widetilde{\mathfrak{J}}_u$ on $(\operatorname{Ker} \Gamma)^\perp = \operatorname{Ran} \widetilde{\mathcal{V}}$ by setting $\widetilde{\mathfrak{J}}_u := \widetilde{\mathcal{V}} \mathfrak{J}_p \widetilde{\mathcal{V}}^*$. Easy to see that $\widetilde{\mathfrak{J}}_u$ commutes with $|\widetilde{\Gamma}| = \widetilde{\mathcal{V}} R \widetilde{\mathcal{V}}^*, |\widetilde{\Gamma}_1| = \widetilde{\mathcal{V}} R_1 \widetilde{\mathcal{V}}^*$ and preserves $\widetilde{\mathcal{V}} p$. Now we extend $\widetilde{\mathfrak{J}}_u$ to a conjugation $\mathfrak{J}_u$ defining on the whole space ℓ^2. This can be done by following the process stated in Remark 4.2.3 (we can set $\widetilde{\mathfrak{J}}'_u$ to be an arbitrary partial conjugation with support $\operatorname{Ker}\Gamma$, and then let $\mathfrak{J}_u = \widetilde{\mathfrak{J}}_u \oplus \widetilde{\mathfrak{J}}'_u$, thus we have $\mathfrak{J}_u$ commutes with $|\Gamma|, |\Gamma_1|$, and this $\mathfrak{J}_u$ also preserves $\widetilde{\mathcal{V}}p$).

Define $\widetilde{\phi} := \widetilde{\mathcal{V}}\varphi\widetilde{\mathcal{V}}^*$, $\widetilde{\phi}_1 := \widetilde{\mathcal{V}}\varphi_1\widetilde{\mathcal{V}}^*$. Clearly $\widetilde{\phi}$ is a unitary operator commuting with $|\widetilde{\Gamma}|$, and $\widetilde{\phi}_1$ is a partial isometry, $\operatorname{Ker}\widetilde{\phi}_1 = \operatorname{Ker}\widetilde{\Gamma}_1$ commuting with $|\Gamma_1|$. Now we can rewrite equations (5.2.3), (5.2.4) as

$$\widetilde{\Gamma} = \mathfrak{C}(\widetilde{\mathcal{V}}R\widetilde{\mathcal{V}}^*)(\widetilde{\mathcal{V}}\varphi\widetilde{\mathcal{V}}^*)(\widetilde{\mathcal{V}}\mathfrak{J}_p\widetilde{\mathcal{V}}^*) = \mathfrak{C}|\widetilde{\Gamma}|\widetilde{\phi}\widetilde{\mathfrak{J}}_u,$$
$$\widetilde{\Gamma}_1 = \mathfrak{C}(\widetilde{\mathcal{V}}R_1\widetilde{\mathcal{V}}^*)(\widetilde{\mathcal{V}}\varphi_1\widetilde{\mathcal{V}}^*)(\widetilde{\mathcal{V}}\mathfrak{J}_p\widetilde{\mathcal{V}}^*) = \mathfrak{C}|\widetilde{\Gamma}|\widetilde{\phi}_1\widetilde{\mathfrak{J}}_u,$$

which are exactly identities (5.1.5) (for the particular choice of $\widetilde{\mathfrak{J}}_u$, $\widetilde{\phi}$, $\widetilde{\phi}_1$).

Finally, let us notice that (5.2.5) is just the identity $u = \widetilde{\mathcal{V}}p$. □

5.3 Proof of Proposition 5.2.3

The whole proof consists of three different parts: existence, uniqueness and the trivial kernel condition.

5.3.1 Existence of Hankel Operator Γ

Proof. Let us first rewrite the equation $R^2 - R_1^2 = pp^*$. Since φ is unitary and commutes with R, we can write

$$R^2 = \varphi^* R^2 \varphi. \tag{5.3.1}$$

Similarly, since φ_1 is a partial isometry commuting with R_1, $\operatorname{Ker}\varphi_1 = \operatorname{Ker} R_1$, we have

$$R_1^2 = (R_1\varphi_1)^*(R_1\varphi_1) = \varphi^* R\mathcal{T}^*\mathcal{T}R\varphi, \tag{5.3.2}$$

where the last equality follows from the definition of $\mathcal{T}$. Denoting $\hat{q} := \mathfrak{J}_p q$, we have

$$p = \mathfrak{J}_p p = \mathfrak{J}_p R\varphi q = R\varphi^* \mathfrak{J}_p q = R\varphi^* \hat{q}. \tag{5.3.3}$$

Combining (5.3.1), (5.3.2) and (5.3.3), the equation $R^2 - R_1^2 = pp^*$ can be translated to

$$\varphi^* R(I - \mathcal{T}^*\mathcal{T})R\varphi = R\varphi^* \hat{q}\hat{q}^* R\varphi.$$

Since the operator $R\varphi$ has trivial kernel, we conclude that

$$I - \mathcal{T}^*\mathcal{T} = (\hat{q})(\hat{q})^*. \tag{5.3.4}$$

Applying both sides of (5.3.4) to x and taking inner product with x, we get

$$\|x\|^2 - \|\mathcal{T}x\|^2 = |\langle x, \hat{q}\rangle|^2.$$

Replacing x in the above identity by $\mathcal{T}x, (\mathcal{T})^2 x, ..., (\mathcal{T})^{n-1}x$, and summing up all n equations, we see that

$$\|x\|^2 - \|\mathcal{T}^n x\|^2 = \sum_{i=0}^{n-1} |\langle \hat{q}, \mathcal{T}^i x\rangle|^2.$$

Letting $n \to \infty$ and using the asymptotic stability of $\mathcal{T}$, we conclude that

$$\|x\|^2 = \sum_{n=0}^{\infty} |\langle \mathcal{T}^n x, \hat{q}\rangle|^2, \tag{5.3.5}$$

which implies that the operator $\mathcal{V} : H \to \ell^2$ defined by

$$\mathcal{V}x := \big(\langle x, \hat{q}\rangle, \langle \mathcal{T}x, \hat{q}\rangle, \langle \mathcal{T}^2 x, \hat{q}\rangle, \ldots\big) = \big(\langle \mathcal{T}^k x, \hat{q}\rangle\big)_{k=0}^{\infty} \tag{5.3.6}$$

is an isometry. The above identity (5.3.6) implies that

$$S^*\mathcal{V} = \mathcal{V}\mathcal{T}. \tag{5.3.7}$$

Denote $\tilde{\mathcal{V}}$ as the operator $\mathcal{V}$ with the restricted target space on $\operatorname{Ran}\mathcal{V}$; then the operator $\tilde{\mathcal{V}} : \mathcal{H} \to \operatorname{Ran}\mathcal{V}$ is unitary. Trivially (5.3.6) implies that $\operatorname{Ran}\mathcal{V}$ is S^*-invariant, so we can define $\widetilde{S}^* : \operatorname{Ran}\mathcal{V} \to \operatorname{Ran}\mathcal{V}$ as $\widetilde{S}^* := S^*|_{\operatorname{Ran}\mathcal{V}}$. Denote by $\widetilde{S}$ the adjoint of $\widetilde{S}^*$, $\widetilde{S} := (\widetilde{S}^*)^* = P_{\operatorname{Ran}\mathcal{V}} S|_{\operatorname{Ran}\mathcal{V}}$. Then the identity (5.3.7) implies that

$$\widetilde{S}^* = \widetilde{\mathcal{V}}\mathcal{T}\widetilde{\mathcal{V}}^* \qquad \text{and} \qquad \widetilde{S} = \widetilde{\mathcal{V}}\mathcal{T}^*\widetilde{\mathcal{V}}^*. \tag{5.3.8}$$

Now we define operator Γ and Γ_1 as following

$$\Gamma := \mathfrak{C}\mathcal{V}R\varphi\mathfrak{J}_p\mathcal{V}^* \qquad \Gamma_1 := \mathfrak{C}\mathcal{V}R_1\varphi_1\mathfrak{J}_p\mathcal{V}^*. \tag{5.3.9}$$

We will show that Γ is a Hankel operator by proving that $\Gamma S = \Gamma_1 = S^*\Gamma$.

To show that $\Gamma S = \Gamma_1$, we recall that $\mathfrak{T}^* = \varphi^* R^{-1} R_1 \varphi_1$ and that $\mathfrak{T}^*\mathfrak{J}_p = \mathfrak{J}_p\mathcal{T}^*$; together with identity $\mathcal{T}^*\mathcal{V}^* = \mathcal{V}^*S$ (which is just the adjoint of (5.3.7)) it gives us

$$\begin{aligned}
\Gamma_1 &= \mathfrak{C}\mathcal{V}R_1\varphi_1\mathfrak{J}_p\mathcal{V}^* = \mathfrak{C}\mathcal{V}R\varphi\mathfrak{T}^*\mathfrak{J}_p\mathcal{V}^* \\
&= \mathfrak{C}\mathcal{V}R\varphi\mathfrak{J}_p\mathcal{T}^*\mathcal{V}^* = \mathfrak{C}\mathcal{V}R\varphi\mathfrak{J}_p\mathcal{V}^*S \\
&= \Gamma S.
\end{aligned}$$

And for the identity $\Gamma_1 = S^*\Gamma$, recalling that $\mathcal{T} = R_1\varphi_1R^{-1}\varphi^*$ and using (5.3.7), we have

$$\begin{aligned}
\mathfrak{C}\Gamma_1 &= \mathcal{V}R_1\varphi_1\mathfrak{J}_p\mathcal{V}^* = \mathcal{V}\mathcal{T}R\varphi\mathfrak{J}_p\mathcal{V}^* \\
&= S^*\mathcal{V}R\varphi\mathfrak{J}_p\mathcal{V}^* = S^*\mathfrak{C}\Gamma.
\end{aligned}$$

Since $\mathfrak{C}$ commutes with S^*, we see that $\mathfrak{C}\Gamma_1 = \mathfrak{C}S^*\Gamma$, and left multiplying this identity by $\mathfrak{C}$ we get the desired result.

Thus we know Γ is a Hankel operator. It remains to show that Γ satisfies (5.2.3), (5.2.4) and (5.2.5).

Using the definition of Γ, Γ_1 in (5.3.9), we know $\operatorname{Ker}\Gamma = (\operatorname{Ran}\mathcal{V})^\perp$, hence we can write

$$\widetilde{\Gamma} := \Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp} = \mathfrak{C}\mathcal{V}R\varphi\mathfrak{J}_p\widetilde{\mathcal{V}}^* = \mathfrak{C}\widetilde{\mathcal{V}}R\varphi\mathfrak{J}_p\widetilde{\mathcal{V}}^*, \tag{5.3.10}$$

$$\widetilde{\Gamma}_1 := \Gamma_1\big|_{(\operatorname{Ker}\Gamma)^\perp} = \mathfrak{C}\widetilde{\mathcal{V}}R_1\varphi_1\mathfrak{J}_p\widetilde{\mathcal{V}}^*; \tag{5.3.11}$$

Thus (5.2.3) and (5.2.4) holds.

For (5.2.5), we first derive the expression for $\mathcal{V}^*$ and show that $\mathcal{V}^*e_0 = \hat{q}$. From the definition of $\mathcal{V}$ in (5.3.6), for $\forall x \in \mathcal{H}, \forall y \in \ell^2$, we have

$$\begin{aligned}\langle \mathcal{V}x, y\rangle &= \sum_{k=0}^{\infty} \overline{y_k}\langle \mathcal{T}^k x, \hat{q}\rangle \\ &= \langle x, \sum_{k=0}^{\infty} y_k(\mathcal{T}^*)^k\hat{q}\rangle,\end{aligned}$$

Hence $\mathcal{V}^*y = \sum_{k=0}^{\infty} y_k(\mathcal{T}^*)^k\hat{q}$, and we have $\mathcal{V}^*e_0 = \mathfrak{S}^0\hat{q} = \hat{q}$. Together with $\Gamma^* = \mathcal{V}\mathfrak{J}_p\varphi^*R\mathcal{V}^*\mathfrak{C}$ gained from (5.3.9), we have

$$\begin{aligned}\Gamma^*e_0 &= \mathcal{V}\mathfrak{J}_p\varphi^*R\mathcal{V}^*e_0 \\ &= \mathcal{V}\mathfrak{J}_p\varphi^*R\hat{q} \\ &= \mathcal{V}\mathfrak{J}_p p = \mathcal{V}p = \widetilde{\mathcal{V}}p.\end{aligned}$$

□

5.3.2 Uniqueness of Hankel Operator Γ

Proof. Now let's discuss the uniqueness by showing that the Hankel symbol $\gamma_k = \langle \Gamma e_k, e_0 \rangle$ must have representation (5.2.6), which is independent of $\mathcal{V}$. Assuming that we are given Hankel operator Γ and $\Gamma_1 = \Gamma S$ that satisfy (5.2.3), (5.2.4) and (5.2.5), we have already built up the unitary equivalence between the tuple $\langle |\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u \rangle$ and $\langle R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p \rangle$ in the proof of Proposition 5.2.4.

Now applying Proposition 5.1.4, we have $\gamma_k = \langle (\mathfrak{S}^*)^k u, v \rangle$, where $\mathfrak{S}^* = |\widetilde{\Gamma}_1| \widetilde{\phi}_1^* \widetilde{\phi} |\widetilde{\Gamma}|^{-1}$ is unitary equivalent to $\mathfrak{T} := \varphi_1^* R_1 R^{-1} \varphi$, and the definition of v is also given in Proposition 5.1.4.

We first show that $q = \widetilde{\mathcal{V}}^* v$. Indeed from (5.1.8), we have

$$\widetilde{\mathcal{V}} p = u = |\widetilde{\Gamma}| \widetilde{\varphi} v = \widetilde{\mathcal{V}} R \varphi \widetilde{\mathcal{V}}^* v,$$

thus $p = R\varphi \widetilde{\mathcal{V}}^* v$ and $q = \widetilde{\mathcal{V}}^* v$. Now from (5.1.6), we can write

$$\begin{aligned} \gamma_k &= \langle (\mathfrak{S}^*)^k u, v \rangle \\ &= \langle \widetilde{\mathcal{V}} \mathfrak{T}^k \widetilde{\mathcal{V}}^* \widetilde{\mathcal{V}} p, \widetilde{\mathcal{V}} q \rangle \\ &= \langle \mathfrak{T}^k p, q \rangle. \end{aligned} \tag{5.3.12}$$

Thus we get (5.2.6), and the coefficients $\{\gamma_k\}_{k=0}^{\infty}$ only depends on the tuple $\langle R, R_1, p, \varphi, \varphi_1 \rangle$, which is independent of $\widetilde{\mathcal{V}}$. The uniqueness is done. □

5.3.3 The Trivial Kernel Condition of Hankel Γ

Proof. As discussed above, $\operatorname{Ker} \Gamma = \{0\}$ if and only if $\mathcal{V} : \mathcal{H} \to \ell^2$ defined by (5.3.6) satisfies $\operatorname{Ran} \mathcal{V} = \ell^2$, and this is equivalent to $\mathcal{T}$ being unitary equivalent to S^*.

If $\operatorname{Ker} \Gamma = \{0\}$, then $\mathfrak{S}^*$ is unitarily equivalent to the backward shift S^*, and comparing identities

$$I - SS^* = e_0 e_0^*, \qquad I - \mathcal{T}^*\mathcal{T} = (\mathfrak{J}_p q)(\mathfrak{J}_p q)^*$$

we conclude that $\|q\| = \|\mathfrak{J}_q q\| = 1$. Now assuming $q \in \operatorname{Ran} R$, $q = Rx$, we will lead to a contradiction.

Take a vector $f = \varphi_1 \varphi^* \mathfrak{J}_p x$, thus $x = \mathfrak{J}_p \varphi \varphi_1^* f$ and $\mathfrak{J}_p q = R\varphi\varphi_1^* f$, Hence

$$\mathcal{T} R \varphi \varphi_1^* f = \mathcal{T}(\mathfrak{J}_p q) = S^* e_0 = 0.$$

But on the other hand we have

$$\mathcal{T} R \varphi \varphi_1^* = R_1 = \varphi_1 \varphi^* R \mathcal{T}^*,$$

where the last equality follows from R_1 is a self-adjoint operator. Hence we have

$$\varphi_1 \phi^* R \mathcal{T}^* f = 0,$$

so $\mathcal{T}^* f = 0$, which contradicts to the fact that $\mathcal{T}^*$ is an isometry. Hence $q \notin \operatorname{Ran} R$.

Now we prove the sufficiency part. Suppose $\|q\| = 1$ and $q \notin \operatorname{Ran} R$. We first show that $\operatorname{Ker} R_1 = \{0\}$.

Let $R_1 x = 0$ for a $x \neq 0$. Applying to x the identity

$$R_1^2 = R\Big(I - \langle \cdot, \varphi q \rangle \varphi q\Big) R,$$

we get that

$$Rx = \langle Rx, \varphi q \rangle \varphi q;$$

note that $Rx \neq 0$ because R has trivial kernel. This implies $R\varphi^* x = \varphi^* R x = \alpha q$,

$\alpha = \langle Rx, \varphi q\rangle^{-1}$ which contradicts the assumption $q \notin \operatorname{Ran} R$.

Now from the definition of $\mathfrak{S}^*$: $\varphi_1^* R_1 = \varphi^* R \mathcal{T}^*$, we know $\operatorname{Ker} \mathcal{T}^* = \{0\}$ since $\operatorname{Ker} R_1 = \{0\}$.

In addition, we apply $\mathfrak{J}_p q$ on both sides of (5.3.4) we get $\mathcal{T}^*\mathcal{T}\mathfrak{J}_p q = 0$, together with $\operatorname{Ker} \mathcal{T}^* = \{0\}$, we have $\mathcal{T}\mathfrak{J}_p q = 0$, hence by the definition of $\mathfrak{T} = R_1 \varphi_1^* \varphi R^{-1}$ we also get $\mathfrak{T} q = 0$.

Now left and right multiplying (5.3.4) by $\mathfrak{S}^*$ and $\mathfrak{S}$ respectively, we get

$$\mathcal{T}\mathcal{T}^* - \mathcal{T}\mathcal{T}^*\mathcal{T}\mathcal{T}^* = \mathcal{T}\Big(\langle \cdot, \mathfrak{J}_p q\rangle \mathfrak{J}_p q\Big)\mathcal{T}^* = 0,$$

hence $\mathcal{T}^*\mathcal{T}$ is a projection. Furthermore, since $\operatorname{Ker} \mathcal{T}^* = \{0\}$, we have $\mathcal{T}\mathcal{T}^* = I$ and $\mathcal{T}^*$ is an isometry.

By Wold Decomposition Theorem [26, Theorem 1.1, p. 3], there exists an orthogonal decomposition for the whole Hilbert space: $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_1$, such that $\mathfrak{S}|_{\mathcal{H}_0}$ is unitary and $\mathfrak{S}|_{\mathcal{H}_1}$ is a unilateral shift. Since $\mathfrak{S}^*$ is asymptotically stable proved in subsection 5.4 which has no unitary part, we have $\mathcal{H}_0 = \{0\}$ and thus there exists a wandering subspace $\mathcal{L}$, such that $\mathcal{H} = \oplus_{n=0}^{\infty} (\mathfrak{S})^n \mathcal{L}$.

To show that $\mathcal{T}$ unitary equivalent to backward shift operator S^*, it's suffices to show that $\dim \mathcal{L} = 1$.

Now apply vector $\mathcal{T}^* x$ for an arbitrary x into (5.3.4), we get

$$\langle \mathcal{T}^* x, \mathfrak{J}_p q\rangle \mathfrak{J}_p q = \mathcal{T}^* x - \mathcal{T}^*(\mathcal{T}\mathcal{T}^*) x = \mathcal{T}^* x - \mathcal{T}^* x = 0,$$

hence $\mathfrak{J}_p q \perp \operatorname{Ran} \mathcal{T}^*$, and $\mathcal{L}$ is the space spanned by $\mathfrak{J}_p q$, which is of dimension 1. This implies $\mathcal{T}$ is unitarily equivalent to S^*, and $\operatorname{Ker} \Gamma = \{0\}$. □

Chapter 6

EIGENSPACE STRUCTURE OF COMPACT RANK-ONE PERTURBATION

In chapter 2 and chapter 4, we have discussed the inverse spectral problem for self-adjoint Hankel operators and non self-adjoint Hankel operators as $\mathfrak{C}$-symmetric operators respectively. Starting from this chapter, we will mainly focus on the category of compact Hankel operators.

Recall that for a general Hankel operator Γ, Γ and $\Gamma_1 := \Gamma S$ satisfy a rank-one perturbation relation: $|\Gamma|^2 - |\Gamma_1|^2 = uu^*$. And also recall that in section 5.1, we construct a triple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \widetilde{\mathfrak{J}}_u)$. Thus in this chapter, we start with a triple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ with similar properties:

(i) R, R_1 are two positive, self-adjoint compact operator defined on a Hilbert space $\mathcal{H}$. In addition we have $\operatorname{Ker} R = \{0\}$;

(ii) $R^2 - R_1^2 = pp^*$ for a vector p with $\|R^{-1}p\| \leq 1$;

(iii) $\mathfrak{J}_p$ is a conjugation commutes with R, R_1 and preserves p.

(iv) φ is a $\mathfrak{J}_p$-symmetric unitary operator, which commutes with R;

(v) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry with $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, which commutes with R_1. In addition, we have $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is unitary (See Remark 5.2.1);

(vi) The contraction $\mathcal{T}$ defined as $\mathcal{T} := \varphi_1 R_1 R^{-1} \varphi^*$ is asymptotically stable.

In this chapter, we will mainly focus on the eigenspaces structure of R, R_1. Denoting $\mathcal{H}_0 = \overline{\text{Span}}\{R^n p | n \geq 0\}$, we start this chapter by stating an equivalent condition of $\mathcal{H}_0 = \mathcal{H}$, i.e., the cyclicity of p (see Lemma 6.1.1). Then in section 6.2, we first analyze the eigenspace structure of R, R_1 on $\mathcal{H}_0$, and then we derive the structure of eigenspaces of R, R_1 on the whole space $\mathcal{H}$.

Finally, we close this chapter by generating a canonical choice of φ, φ_1, which is given in Lemma 6.3.2. For the canonical choice of φ, φ_1, we can further analyze the behavior of φ, φ_1 acting on each eigenspaces of R, R_1 (see Proposition 7.6.3). We can also show that the conjugation $\mathfrak{J}_p$ will be uniquely determined if taking the canonical choice of $\varphi . \varphi_1$ (see Remark 7.6.6)

6.1 Setting and preparation

We start this section with a tuple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ which satisfies (i), (ii), (iii), (iv), (v), (vi) stated at the beginning of this chapter. Denoting $\mathcal{H}_0 := \overline{\text{Span}}\{R^n p | n \geq 0\}$, we first derive an equivalent condition of $\mathcal{H}_0 = \mathcal{H}$, i.e., the cyclicity of vector p.

Lemma 6.1.1. *With the assumptions above, p is cyclic with respect to R if and only if*

(i) *For every $\lambda \in \sigma(R)$, we have $\dim \text{Ker}(R - \lambda I) \leq 1$;*

(ii) $\text{Proj}_{\text{Ker}(R-\lambda I)} p \neq 0$.

Proof. If we have the cyclicity of vector p, we will prove (i) and (ii) by contradiction.

If $\dim \text{Ker}(R - \lambda I) \geq 2$ for a certain λ, then $H_0 \cap \text{Ker}(R - \lambda I)$ have at most dimension 1 spanned by $\text{Proj}_{\text{Ker}(R-\lambda I)} p$ (will be a trivial space if $\text{Proj}_{\text{Ker}(R-\lambda I)} p = 0$) , thus p can't be cyclic with respect to R.

For (ii), if $\text{Proj}_{\text{Ker}(R-\lambda I)} p = 0$, then $\mathcal{H}_0 \cap \text{Ker}(R - \lambda I) = \emptyset$, thus p can't be cyclic.

For the other direction, under (i) and (ii), we denote the spectrum of R as

$$\lambda_1 > \lambda_2 > ... > \lambda_n \to 0,$$

and we write $p = \sum_{k=1}^{\infty} p_k$ where $p_k \in \mathrm{Ker}(R - \lambda_k I)$. (The case when R is of finite-rank, i.e. finitely many eigenvalues, is trivially true, so we only consider case when $\dim \mathrm{Ran}\, R = \infty$)

We can show that

$$\lim_{n\to\infty} \big(\frac{1}{2\lambda_1}\big)^n (R + \lambda_1 I)^n p \rightharpoonup p_1.$$

Since $\mathcal{H}_0$ is weakly closed, thus we have $p_1 \in \mathcal{H}_0$. In fact we have

$$\big(\frac{1}{2\lambda_1}\big)^n (R + \lambda_1 I)^n p = \sum_{k=0}^{\infty} \big(\frac{1}{2} + \frac{\lambda_k}{2\lambda_1}\big)^n p_k,$$

thus

$$\lim_{n\to\infty} \langle \big(\frac{1}{2\lambda_1}\big)^n (R + \lambda_1 I)^n p, p_i \rangle = \begin{cases} \|p_1\|^2 & \text{if } i = 1 \\ 0 & \text{else} \end{cases}$$

so we have $p_1 \in \mathcal{H}_0$.

Similarly we can also show that

$$\lim_{n\to\infty} \big(\frac{1}{2\lambda_2}\big)^n \Big\{ (R + \lambda_2 I)^n p - (\lambda_1 + \lambda_2)^n p_1 \Big\} \rightharpoonup p_2,$$

and $p_2 \in \mathcal{H}_0$. Thus followed by the process of induction, if we have $p_1, p_2, ..., p_k \in \mathcal{H}_0$, then since

$$\lim_{n\to\infty} \big(\frac{1}{2\lambda_{k+1}}\big)^n \Big\{ (R + \lambda_{k+1} I)^n p - \sum_{i=1}^{k} (\lambda_i + \lambda_{k+1})^n p_i \Big\} \rightharpoonup p_{k+1},$$

then we will have $p_{k+1} \in \mathcal{H}_0$. Hence $p_k \in \mathcal{H}_0$ holds for all k and $\mathcal{H} = \overline{\text{Span}}\{p_k | k \geq 1\} \subset \mathcal{H}_0$, p is cyclic. □

Remark 6.1.2. In fact, Lemma 6.1.1 is also true for a general compact operator without given the trivial kernel condition. The statement can be modified as follow:

Proposition 6.1.3. *Let A be a compact operator on a Hilbert space $\mathcal{H}$ with simple eigenvalues, then there exists a vector $x \in A$, such that x is a cyclic vector for A in $\mathcal{H}$.*

In fact, if we write all eigenvalues $\{\lambda_k\}_{k=1}^{\infty}$ in the non-increasing order:

$$|\lambda_1| \geq |\lambda_2| \geq ....,$$

then we can take x to be

$$x = e_0 + \sum_{k=1}^{\infty} \frac{1}{k^2} e_k,$$

here e_k is a unit vector in $\text{Ker}(A - \lambda_k I)$, and e_0 is a unit vector in $\text{Ker}\, A$ if $0 \in \sigma_p(A)$, otherwise we take $e_0 = 0$. The proof is very much similar to the proof of Lemma 6.1.1.

In this section we discuss the case when operator R is not cyclic (that is, there doesn't exist any vector p such that p is cyclic with respect to R in $\mathcal{H}$.). Then according to Lemma 6.1.1 R have some non-simple singular values.

6.2 Non-simple Eigenvalues for Operators of Rank-one Perturbation

We begin this part with the following simple lemma.

Lemma 6.2.1. *For $\forall \lambda \in \mathbb{R}^+ \cup \{0\}$, we have*

$$\big| \dim \text{Ker}(R - \lambda I) - \dim \text{Ker}(R_1 - \lambda I) \big| \leq 1.$$

Proof. For all vector $x \in \operatorname{Ker}(R-\lambda I)\cap p^{\perp}$, we have $x \in \operatorname{Ker}(R_1^2-\lambda^2 I)$, thus $x \in \operatorname{Ker}(R_1-\lambda I)$ as R_1 is positive. Thus $\operatorname{Ker}(R-\lambda I)\cap p^{\perp} \subseteq \operatorname{Ker}(R_1-\lambda I)$, and we have

$$\dim \operatorname{Ker}(R-\lambda I) \leq \dim \operatorname{Ker}(R_1-\lambda I)+1.$$

Similarly, $\operatorname{Ker}(R_1-\lambda I)\cap p^{\perp} \subseteq \operatorname{Ker}(R-\lambda I)$ and we have

$$\dim \operatorname{Ker}(R_1-\lambda I) \leq \dim \operatorname{Ker}(R-\lambda I)+1.$$

So the dimension of two kernel space at most differ by 1. □

From Lemma 6.2.1, it's easily seen that $\dim \operatorname{Ker} R_1 \leq 1$.

Now we first restrict $R^2 - R_1^2 = pp^*$ on $\mathcal{H}_0$:

$$R^2|_{\mathcal{H}_0} - R_1^2|_{\mathcal{H}_0} = pp^*.$$

Since p is cyclic with respect to $\mathcal{H}_0$, thus $R|_{\mathcal{H}_0}$ have simple singular values. For the special case, if $R|_{\mathcal{H}_0}$ has finite rank, then R_1 is also finite rank. For the following discussion, we assume that $\dim \operatorname{Ran} R|_{\mathcal{H}_0} = \infty$.

In the next section, we will show the following proposition, which gives the eigenspace structure for (R, R_1) restricted on $\mathcal{H}_0$.

Proposition 6.2.2. *There exists an intertwining sequence*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0, \tag{6.2.1}$$

such that the non-zero eigenvalues of $R|_{\mathcal{H}_0}, R_1|_{\mathcal{H}_0}$ are simple, and coincide with $\{\lambda_k\}_{k=1}^{\infty}$. $\{\mu_k\}_{k=1}^{\infty}$ respectively.

6.2.1 Eigenspace Structure on $\mathcal{H}_0$

We show Proposition 6.2.2 in the following steps.

Proposition 6.2.3. *The dimension of any eigenspace of* $R_1|_{\mathcal{H}_0}$ *is no more than 1, i.e.* $R_1|_{\mathcal{H}_0}$ *has simple singular values.*

Proof. For any λ, if $\dim \operatorname{Ker}(R_1 - \lambda I) \geq 3$, since we have

$$\operatorname{Ker}(R_1 - \lambda I) \cap p^{\perp} \subseteq \operatorname{Ker}(R - \lambda I), \tag{6.2.2}$$

this contradicts to the fact that R has simple singular values on $\mathcal{H}_0$.

If $\dim \operatorname{Ker}(R_1 - \lambda I) = 2$ for some λ, then by (6.2.2), we can take a non-zero vector x such that

$$x \in \operatorname{Ker}(R - \lambda I) \cap \operatorname{Ker}(R_1 - \lambda I) \cap \mathcal{H}_0,$$

then applying this x to $R^2 - R_1^2 = pp^*$ we get $\langle x, p \rangle = 0$, which contradicts to the property that $P_{\operatorname{Ker}(R - \lambda I)} p \neq 0$ given in Lemma 6.1.1. □

Proposition 6.2.4. *Denoting the non-zero eigenvalues of* R, R_1 *as* $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$ *respectively with*

$$\lambda_1 > \lambda_2 > ... \to 0, \qquad \mu_1 > \mu_2 > ... \to 0$$

then we have

$$\lambda_k \geq \mu_k, \qquad \mu_k \geq \lambda_{k+1}$$

holds for all k.

Proof. Here we use the mini-max definition for singular values. That is,

$$\lambda_k^2 = \min_{subspace \mathcal{F} \subseteq \mathcal{H}_0, codim \mathcal{F} = k-1} \max_{x \in \mathcal{F}, \|x\|=1} \langle R^2 x, x \rangle \tag{6.2.3}$$

$$\mu_k^2 = \min_{subspace \mathcal{G} \subseteq \mathcal{H}_0, codim \mathcal{G} = k-1} \max_{x \in \mathcal{G}, \|x\|=1} \langle R_1^2 x, x \rangle \tag{6.2.4}$$

To show $\lambda_k \geq \mu_k$, we take a subspace $\mathcal{F} \subseteq \mathcal{H}_0$ with *codim* $= k-1$ where (6.2.3) reaches its equality. Then we have

$$\mu_k^2 \leq \max_{x\in\mathcal{F},\|x\|=1} \langle R_1^2 x, x\rangle \leq \max_{x\in\mathcal{F},\|x\|=1} \langle R^2 x, x\rangle = \lambda_k^2$$

To show that $\mu_k \geq \lambda_{k+1}$, we take a subspace $\mathcal{G} \subseteq \mathcal{H}_0$ with *codim* $= k-1$ where (6.2.4) reaches its equality. Then denoting $\mathcal{F}_1 := \mathcal{G} \cap p^\perp$, then $\mathcal{F}_1$ has codim no more than k where R coincides with R_1 on $\mathcal{F}_1$. Thus we have

$$\mu_k^2 = \max_{x\in\mathcal{G},\|x\|=1} \langle R_1^2 x, x\rangle \geq \max_{x\in\mathcal{F}_1,\|x\|=1} \langle R_1^2 x, x\rangle = \max_{x\in\mathcal{F}_1,\|x\|=1} \langle R^2 x, x\rangle \geq \lambda_{k+1}^2$$

□

Proposition 6.2.5. (6.2.1) *holds for* $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$ *defined in Proposition 6.2.4. Equivalently saying, we have*

$$\lambda_k > \mu_k, \qquad \mu_k > \lambda_{k+1}$$

Proof. We prove the two inequalities by contradiction.

(i) If $\lambda_k = \mu_k$, we take a subspace $\mathcal{F}$ as

$$\mathcal{F} = \bigoplus_{i=k}^{\infty} \big(\operatorname{Ker}(R - \lambda_i I) \cap \mathcal{H}_0\big),$$

then *codim*$\mathcal{F} = k-1$ in $\mathcal{H}_0$ and $\max_{x\in\mathcal{F},\|x\|=1} \langle R^2 x, x\rangle = \lambda_k^2$. The maximum can only be reached when $x \in \operatorname{Ker}(R - \lambda_k I)$.

Now for any vector $x \in \mathcal{F}$, we have

$$\langle R_1^2 x, x\rangle = \langle R^2 x, x\rangle - |\langle x, p\rangle|^2 \leq \lambda_k^2\|x\|^2 - |\langle x, p\rangle|^2 \leq \lambda_k^2 \|x\|^2. \tag{6.2.5}$$

Thus if we have $\mu_k = \lambda_k$, then according to (6.2.5), there exists a vector $x \in \mathcal{F}$, such that $\|Rx\| = \lambda_k \|x\|$ and $x \perp p$. This contradicts to the fact that $\mathrm{Proj}_{\mathrm{Ker}(R-\lambda_k I)\cap \mathcal{H}_0}\, p \neq 0$, which is given in Lemma 6.1.1

(ii) If $\mu_k = \lambda_{k+1}$, we take a subspace $\mathcal{G} \subseteq \mathcal{H}_0$ as

$$\mathcal{G} = \mathrm{Ker}\, R_1 \bigoplus_{i=k}^{\infty} \big(\mathrm{Ker}(R_1 - \mu_i I) \cap \mathcal{H}_0\big),$$

then we have $\max\limits_{x\in\mathcal{G}, \|x\|=1} \langle R_1^2 x, x\rangle = \mu_k^2$, and the maximum can only be achieved when $x \in \mathrm{Ker}(R_1 - \mu_k)$.

Denote again $\mathcal{F}_1 := \mathcal{G} \cap p^{\perp}$, since

$$\mu_k^2 = \max_{x\in\mathcal{G}, \|x\|=1} \langle R_1^2 x, x\rangle \geq \max_{x\in\mathcal{F}_1, \|x\|=1} \langle R_1^2 x, x\rangle = \max_{x\in\mathcal{F}_1, \|x\|=1} \langle R^2 x, x\rangle \geq \lambda_{k+1}^2, \tag{6.2.6}$$

where here the maximum of first equality in (6.2.6) holds when $x \in \mathrm{Ker}(R_1 - \mu_k) \cap \mathcal{H}_0$, and the first inequality inside (6.2.6) is reached when this $x \perp p$. Thus if we have $\mu_k = \lambda_{k+1}$, we have $p \perp \mathrm{Ker}(R_1 - \mu_k I)$

Now take a vector $y \in \mathrm{Ker}(R_1 - \mu_k I)$, since $y \perp p$, we have $Ry = R_1 y = \mu_k y$, and μ_k is a simple singular value for both $R|_{\mathcal{H}_0}$ and $R_1|_{\mathcal{H}_0}$. Together with the property that $R = R_1$ on $\mathcal{H}_0^{\perp}$, we have

$$\mathrm{Ker}(R - \mu_k I) = \mathrm{Ker}(R_1 - \mu_k I) \qquad \text{on } \mathcal{H}.$$

Now apply a $x \in \mathrm{Ker}(R - \mu_k I)$ to the equation $R_1\varphi_1 = \mathcal{T} R\varphi$, we have $\varphi_1 x = \mathcal{T}\varphi x$. Since $\mathrm{Ker}(R - \mu_k I) = \mathrm{Ker}(R_1 - \mu_k I)$ is a reducing subspace for $R, R_1, \varphi, \varphi_1$, thus $\mathcal{T}$ maps unitarily on $\mathrm{Ker}(R - \mu_k I)$, which leads to a contradiction to the asymptotic stability of $\mathcal{T}$.

Remark 6.2.6. For the special case when $\mathcal{H}_0$ has finite dimension, we have the following result:

Proposition 6.2.7. *If* $\dim \mathcal{H}_0 < \infty$*, there exists an intertwining sequence*

$$\lambda_1 > \mu_1 > ... > \lambda_N > \mu_N \geq 0$$

such that $R\big|_{\mathcal{H}_0}, R_1\big|_{\mathcal{H}_0}$ *have simple eigenvalues, and their eigenvalues coincides with* $\{\lambda_k\}_{k=1}^N$. $\{\mu_k\}_{k=1}^N$ *respectively.*

□

6.2.2 Eigenspace Structure on $\mathcal{H}$

With the result in Proposition 6.2.2, we are ready to deep into the eigenspace structure of (R, R_1) on $\mathcal{H}$. We derive the structure of eigenspaces of R, R_1 by the following steps.

Lemma 6.2.8. *For a non-zero* $\tau \neq \lambda_k, \mu_k$ *for all* k*, we have*

$$\operatorname{Ker}(R - \tau I) = \operatorname{Ker}(R_1 - \tau I) = 0$$

Proof. We first show that $\operatorname{Ker}(R-\tau I) = \operatorname{Ker}(R_1-\tau I)$. For all $x \in \operatorname{Ker}(R-\tau I)$, we have $x \perp \operatorname{Ker}(R - \lambda_k I)$ for all k, thus $x \perp H_0$. $R = R_1$ on $\mathcal{H}_0^\perp$, we have $x \in \operatorname{Ker}(R_1 - \tau I)$. Thus $\operatorname{Ker}(R-\tau I) \subseteq \operatorname{Ker}(R_1-\tau I)$. Similarly for all $x \in \operatorname{Ker}(R_1-\tau I)$, we have $x \perp \mathcal{H}_0$ and $x \in \operatorname{Ker}(R - \tau I)$, thus

$$\operatorname{Ker}(R - \tau I) = \operatorname{Ker}(R_1 - \tau I).$$

Now we show that $\mathcal{T}$ can't be asymtotically stable if the kernel space is non-trivial. If $\operatorname{Ker}(R - \tau x) \neq \emptyset$, then apply $x \in \operatorname{Ker}(R - \tau I)$ into equation $\mathcal{T}\varphi R = \varphi_1 R_1$, we get $\mathcal{T}\varphi x = \varphi_1 x$. Since φ commutes with R, and φ_1 commutes R_1, thus we have $\operatorname{Ker}(R - \tau I)$ is a reducing subspace with respect to $R, R_1, \varphi, \varphi_1$, and $\mathcal{T}$ maps unitarily on $\operatorname{Ker}(R - \tau I)$, which contradicts to the asymptotic stability of $\mathcal{T}$. □

Now we know that R, R_1 has no eigenvalues other than $\{\lambda_k\}_{k=1}^{\infty}, \{\mu_k\}_{k=1}^{\infty}$ and 0. We denote

$$E_{\lambda_k} := \mathrm{Ker}(R - \lambda_k I), \qquad E_{\mu_k} := \mathrm{Ker}(R - \mu_k I); \tag{6.2.7}$$

$$E_{\lambda_k}^1 := \mathrm{Ker}(R_1 - \lambda_k I), \qquad E_{\mu_k}^1 := \mathrm{Ker}(R_1 - \mu_k I). \tag{6.2.8}$$

Note that all of those eigenspaces have finite dimension since R, R_1 are compact. In addition, Lemma 6.2.1 implies that the dimension between E_{λ_k} and $E_{\lambda_k}^1$, E_{μ_k} and $E_{\mu_k}^1$ at most differs by 1.

Proposition 6.2.9. $E_{\lambda_k}, E_{\lambda_k}^1, E_{\mu_k}, E_{\mu_k}^1$ *satisfy the following properties*

(i) $E_{\lambda_k}^1 \subseteq \mathcal{H}_0^{\perp}$, $E_{\mu_k} \subseteq \mathcal{H}_0^{\perp}$;

(ii) *Denote*

$$p_k := P_{\mathcal{H}_0 \cap \mathrm{Ker}(R - \lambda_k I)} p, \qquad p_k^1 := P_{\mathcal{H}_0 \cap \mathrm{Ker}(R_1 - \mu_k I)} p, \tag{6.2.9}$$

then we have

$$E_{\lambda_k} = E_{\lambda_k}^1 \oplus \mathrm{Span}\{p_k\}, \qquad E_{\mu_k}^1 = E_{\mu_k} \oplus \mathrm{Span}\{p_k^1\}.$$

Proof. For (i), since we have

$$\mathcal{H}_0 = \bigoplus_{k=1}^{\infty} \Big(\mathrm{Ker}(R - \lambda_k I) \cap \mathcal{H}_0 \Big) = \bigoplus_{k=1}^{\infty} \Big(\mathcal{H}_0 \cap \mathrm{Ker}(R_1 - \mu_k I) \Big) \bigoplus \Big(\mathcal{H}_0 \cap \mathrm{Ker}\, R_1 \Big),$$

and we have $\mathrm{Ker}(R_1 - \lambda_k I) \perp \mathrm{Ker}(R_1 - \mu_j I)$ for all j and $\mathrm{Ker}(R_1 - \lambda_k I) \perp \mathrm{Ker}\, R_1$, thus $\mathrm{Ker}(R_1 - \lambda_k I) \perp \mathcal{H}_0$. Similarly we also have $\mathrm{Ker}(R - \mu_k) \perp \mathcal{H}_0$ for all k.

Now for (ii), since $R = R_1$ on $H_0^{\perp}$, which implies $R = R_1$ on $E_{\lambda_k}^1$ and E_{μ_k} for all k. Thus $E_{\lambda_k}' \subseteq E_{\lambda_k}$, $E_{\mu_k} \subseteq E_{\mu_k}'$.

On the other hand, we also know that $p_k \in E_{\lambda_k}$ and $p_k^1 \in E_{\mu_k}^1$ (easy to see that

$p_k, p_k^1 \neq 0$ because p is cyclic with respect to R and R_1 on $\mathcal{H}_0$). Hence

$$\text{Span}\{p_k\} \oplus E^1_{\lambda_k} \subseteq E_{\lambda_k}, \qquad \text{Span}\{p^1_k\} \oplus E_{\mu_k} \subseteq E^1_{\mu_k}, \tag{6.2.10}$$

then applying Lemma 6.2.1 to (6.2.10), and we finish the proof of (ii). □

So far, we can describe the complete structure of all eigenspaces of R, R_1.

Proposition 6.2.10. *Let* $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ *be a tuple satisfying the setting in subsection 6.1, and we again denote* $\mathcal{H}_0 = \overline{\text{Span}}\{R^n p : n \geq 0\}$*. If* R *is not finite rank, then there exists an intertwining sequences*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > ... >\to 0,$$

such that

(i) $R|_{\mathcal{H}_0}, R_1|_{\mathcal{H}_0}$ *have simple eigenvalues* $\{\lambda_k\}_{k=1}^{\infty}, \{\mu_k\}_{k=1}^{\infty}$ *respectively;*

(ii)

$$\mathcal{H}_0^{\perp} = \left(\bigoplus_{k=1}^{\infty} E^1_{\lambda_k}\right) \bigoplus \left(\bigoplus_{k=1}^{\infty} E_{\mu_k}\right) \tag{6.2.11}$$

(iii)

$$E_{\lambda_k} = E^1_{\lambda_k} \oplus \text{Span}\{p_k\}, \qquad E^1_{\mu_k} = E_{\mu_k} \oplus \text{Span}\{p^1_k\},$$

where p_k, p^1_k *are defined in* (6.2.9).

For the special case when R, R_1 are finite rank operators, proposition 6.2.10 will be modified as follow:

Proposition 6.2.11. *Let* $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ *be a tuple satisfying the setting in subsection 6.1. In addition,* R *is a finite rank operator. Then there exists an intertwining* *sequence*

$$\lambda_1 > \mu_1 > ... > \lambda_N > \mu_N \geq 0,$$

such that

(i) $R|_{\mathcal{H}_0}, R_1|_{\mathcal{H}_0}$ *have simple eigenvalues* $\{\lambda_k\}_{k=1}^N, \{\mu_k\}_{k=1}^N$ *respectively;*

(ii) $\mathcal{H}_0^\perp = \left(\bigoplus_{k=1}^N E^1_{\lambda_k}\right) \bigoplus \left(\bigoplus_{k=1}^N E_{\mu_k}\right)$;

(iii)

$$E_{\lambda_k} = E^1_{\lambda_k} \oplus \operatorname{Span}\{p_k\}, \qquad E^1_{\mu_k} = E_{\mu_k} \oplus \operatorname{Span}\{p^1_k\},$$

where p_k, p^1_k *are defined in* (6.2.9).

For the special case when $\mu_N = 0$, *we have* $\dim E^1_{\mu_N} = 1$, *and* E_{μ_N} *is trivial.*

Remark 6.2.12. Since we have the decomposition of the whole space

$$\mathcal{H} = \mathcal{H}_0 \bigoplus \left(\bigoplus_{k=1}^N E^1_{\lambda_k}\right) \bigoplus \left(\bigoplus_{k=1}^N E_{\mu_k}\right)$$

implied by (6.2.11), we can also analyze the structure of φ, φ_1 on $\mathcal{H}$. In fact, since φ, φ_1 commutes with R, R_1 respectively, we have

(i) φ acts unitarily on E_{λ_k}, E_{μ_k} for all k.

(ii) In Remark 5.2.1, we already know $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is unitary. Since $E^1_{\lambda_k}, E^1_{\mu_k} \subseteq (\operatorname{Ker} R_1)^\perp$ are all invariant subspaces for φ_1. Thus we have φ_1 acts unitarily on $E^1_{\lambda_k}, E^1_{\mu_k}$. Another way to see this is that $\varphi_1|_{E^1_{\lambda_k}}, \varphi_1|_{E^1_{\mu_k}}$ are both finite-rank isometry, thus they are onto.

6.3 Canonical Choice for φ and φ_1

In this section, we generate a canonical choice for φ, φ_1 and $\mathfrak{J}_p$ on $\mathcal{H}$. As we have discussed in Remark 5.1.3, the choice of $\mathfrak{J}_u$ is not unique and we say two triples $(\widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1)$ and $(\widetilde{\mathfrak{J}}'_u, \widetilde{\phi}', \widetilde{\phi}'_1)$ are in the same equivalent class iff there exists a $\widetilde{\mathfrak{J}}_u-$symmetric unitary operator ψ such that

$$\widetilde{\mathfrak{J}}'_u = \psi\widetilde{\mathfrak{J}}_u, \qquad \widetilde{\phi}' = \widetilde{\phi}\psi^*, \qquad \widetilde{\phi}'_1 = \widetilde{\phi}_1\psi^*.$$

As we have stated in Proposition 5.2.4, we have built up unitary equivalence between tuples $(\mathfrak{J}_p, \varphi, \varphi_1)$ and $(\widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1)$. Now we define a similar equivalence class relation for the tuple $(\mathfrak{J}_p, \varphi, \varphi_1)$.

Definition 6.3.1. *We say two tuples* $(\mathfrak{J}_p, \varphi, \varphi_1)$ *and* $(\mathfrak{J}'_p, \varphi', \varphi'_1)$ *are in the same* equivalent class *if and only if there exists and* $\mathfrak{J}_p$*-symmetric unitary operator* ψ*, such that*

$$\mathfrak{J}'_p = \psi\mathfrak{J}_p, \qquad \varphi' = \varphi\psi^*, \qquad \varphi'_1 = \varphi_1\psi^*.$$

Easy to check that the two tuples $(\mathfrak{J}_p, \varphi, \varphi_1)$ and $(\mathfrak{J}'_p, \varphi', \varphi'_1)$ from the same equivalent class define the same Hankel Γ in Proposition 5.2.3 and the same contraction $\mathfrak{S}^* := \varphi_1 R_1 R^{-1}\varphi^*$.

The following lemma gives the canonical choice of φ, φ_1 from a given equivalent class.

Lemma 6.3.2. *In each equivalent class of* $(\mathfrak{J}_p, \varphi, \varphi_1)$*, there exists a unique triple* $(\mathfrak{J}'_p, \varphi', \varphi'_1)$ *such that*

$$\varphi'_1|_{E^1_{\lambda_k}} = I, \qquad \varphi'|_{E_{\mu_k}} = I.$$

Proof. As we have discussed in subsection 6.2, $\mathcal{H}_0^\perp$ can be decomposed as

$$\mathcal{H}_0^\perp = \left(\bigoplus_{k=1}^{\infty} E^1_{\lambda_k}\right) \bigoplus \left(\bigoplus_{k=1}^{\infty} E_{\mu_k}\right).$$

.

Now we define an operator ψ, such that

(i) $\psi = \varphi$ on E_{μ_k} for all $k \in \mathbb{N}$;

(ii) $\psi = \varphi_1$ on $E^1_{\lambda_k}$ for all $k \in \mathbb{N}$;

(iii) $\psi = I$ on $\mathcal{H}_0$.

Since we have assumed that φ, φ_1 are $\mathfrak{J}_p$-symmetric in subsection 6.1, and $E^1_{\lambda_k}, E_{\mu_k}$ are all reducing subspaces for $\mathfrak{J}_p$, we know ψ is $\mathfrak{J}_p$-symmetric. In addition, from Remark 6.2.12, we know that φ, φ_1 acts unitarily on $E_{\mu_k}, E^1_{\lambda_k}$ respectively, hence ψ is unitary, and the tuple $(\mathfrak{J}'_p, \varphi', \varphi'_1)$ defined as

$$\mathfrak{J}'_p := \psi\mathfrak{J}_p, \qquad \varphi' := \varphi\psi^*, \qquad \varphi'_1 := \varphi_1\psi^*$$

satisfy

$$\varphi'|_{E_{\mu_k}} = I, \qquad \varphi'_1|_{E^1_{\lambda_k}} = I.$$

Here the uniqueness of such triple follows from the uniqueness of ψ. □

We say the triple $(\mathfrak{J}_p, \varphi, \varphi_1)$ which satisfies

$$\varphi_1|_{E^1_{\lambda_k}} = I, \qquad \varphi|_{E_{\mu_k}} = I \qquad \text{hold for all k} \in \mathbb{N}$$

to be the *canonical choice of* $(\mathfrak{J}_p, \varphi, \varphi_1)$. Indeed, we can show that given the choice of φ, φ_1 and the asymptotic stability of $\mathcal{T}$, $\mathfrak{J}_p$ will be uniquely determined. The explicit representation of $\mathfrak{J}_p$ will be given in a later Remark 7.6.6.

Chapter 7

ASYMPTOTIC STABILITY OF CONTRACTION

In chapter 2 and chapter 4, we have studied the inverse spectral problem for self-adjoint Hankel operators and non self-adjoint Hankel operators as $\mathfrak{C}$-symmetric operators respectively. Notice that in both cases (Propostition 3.1.3 and Proposition 5.2.3) we require the asymptotic stability of a defined contraction $\mathcal{T}$ (In chapter 2, $\mathcal{T}$ is given as $\mathcal{T} = \mathcal{R}_1\mathcal{R}^{-1}$. And in chapter 4, $\mathcal{T}$ is given as $\mathcal{T} = R_1\varphi_1\varphi^* R^{-1}$).

However, finding out an equivalent condition for the asymptotic stability is not an easy thing. The following *stability test* is a criterion for asymptotic stability given in [12].

Lemma 7.0.1. *Let $\mathcal{T}$ be a contraction on a Hilbert space. If $\mathcal{T}$ has no eigenvalues on the unit circle $\mathcal{T}$ and the set $\sigma(\mathcal{T}) \cap \mathcal{T}$ is at most countable, then $\mathcal{T}$ is asymptotically stable.*

In addition, Nagy gave an equivalent condition for asymptotic stability in his book [26]. Here we first introduce the definition of *defect operators*, *defect indices*, and *minimal unitary dilation*, which will also be used in later chapters.

Definition 7.0.2. *Let $\mathcal{T}$ be a contraction on a Hilbert space $\mathcal{H}$ (thus $\mathcal{T}^*\mathcal{T} \leq I$ and $\mathcal{T}\mathcal{T}^* \leq I$), we define the operators*

$$D_{\mathcal{T}} = (I - \mathcal{T}^*\mathcal{T})^{1/2}, \qquad D_{\mathcal{T}^*} = (I - \mathcal{T}\mathcal{T}^*)^{1/2},$$

which are self-adjoint, and bounded by 0 *and* 1*. We call* $D_{\mathcal{T}}$, $D_{\mathcal{T}^*}$ *be the* defect operators *of* $\mathcal{T}$, $\mathcal{T}^*$ *respectively.*

In addition, we call $\mathcal{D}_A := \operatorname{Clos}\operatorname{Ran} D_A$ *to* $\mathcal{D}_{A^*} := \operatorname{Clos}\operatorname{Ran} D_{A^*}$ *to be the* defect spaces*, and*

$$\mathfrak{D}_{\mathcal{T}} := \dim \overline{D_{\mathcal{T}}\mathcal{H}}, \qquad \mathfrak{D}_{\mathcal{T}^*} := \dim \overline{D_{\mathcal{T}^*}\mathcal{H}}$$

to be the defect indices *of* $\mathcal{T}$ *and* $\mathcal{T}^*$ *respectively.*

Definition 7.0.3. *Let* $\mathcal{A}$ *be an operator on a Hilbert space* $\mathcal{H}_1$*, and* $\mathcal{B}$ *be an operatoar on a Hilbert space* $\mathcal{H}_2$ *which containing* $\mathcal{H}_1$*. We call* $\mathcal{B}$ *a dilation of* $\mathcal{A}$ *if*

$$\mathcal{A}^n = P_{\mathcal{H}_1}\mathcal{B}^n$$

Definition 7.0.4 (Theorem 4.2, Chapter 1,[26])**.** *For every contraction* $\mathcal{T}$ *on a Hilbert space* $\mathcal{H}_1$*, there exists a unitary dilation* U *on a space* $\mathcal{H}_2$ *containing* $\mathcal{H}_1$ *as a subspace, which is minimal, that is,*

$$\mathcal{H}_2 = \overline{\operatorname{Span}}\{U^n\mathcal{H}_1 | n \in \mathbb{Z}\}.$$

This minimal unitary dilation is determined up to isomorphism, and thus is called the minimal unitary dilation *of* $\mathcal{T}$*.*

With the preparations of those definitions above, Nagy gave an equivalent condition of asymptotic stability in his book [26], stated as below:.

Theorem 7.0.5 (Proposition 1.3, Chapter 2, [26])**.** *Let* $\mathcal{T}$ *be a contraction on a Hilbert space* $\mathcal{H}_1$*, and* U *be its minimal dilation on a Hilbert space* $\mathcal{H}_2$ *containing* $\mathcal{H}_1$*. Then* $\mathcal{T}$ *is asymptotically stable if and only if the following two properties hold:*

(i) *The defect index* $\mathfrak{D}_{\mathcal{T}}$ *is finite;*

(ii) *The minimal unitary dilation* U *is a bilateral shift of multiplicity equal to* $\mathfrak{D}_{\mathcal{T}}$. *That is, there exists a subspace* $\mathcal{L} \subseteq \mathcal{H}_2$ *(called* wandering space*) with dimension* $\mathfrak{D}_{\mathcal{T}}$, *satisfies*

$$\mathcal{H}_2 = \bigoplus_{n=-\infty}^{\infty} U^n \mathcal{L}$$

For the special case of complete non-unitary dilation (See Definition 7.3.1), the following result was stated:

Theorem 7.0.6 (Proposition 6.7, Chapter 2, [26]). *Let* $\mathcal{T}$ *be a completely non-unitary contraction, and suppose that the intersection of the spectrum of* $\mathcal{T}$ *with the unit circle* $\mathbb{T}$ *has Lebesgue measure* 0*, then both* $\mathcal{T}$ *and* $\mathcal{T}^*$ *are asymptotically stable.*

However, finding the spectral property of $\mathcal{T}$ and its minimal unitary dilation U is still not an easy thing, but things become easier if we assume that R, R_1 are compact. In this chapter, we mainly discuss the asymptotic stability of $\mathcal{T} := R_1\varphi_1\varphi^* R^{-1}$ under the condition that R, R_1 are compact.

We first restate the setting. We are given a tuple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ satisfying

(i) R, R_1 are two positive, self-adjoint compact operators defined on a Hilbert space $\mathcal{H}$. In addition we have $\operatorname{Ker} R = \{0\}$;

(ii) $R^2 - R_1^2 = pp^*$ for a vector p with $\|R^{-1}p\| \leq 1$;

(iii) $\mathfrak{J}_p$ is a conjugation commutes with R, R_1 and preserves p, implied by Lemma 4.0.1 and Lemma 5.1.1;

(iv) φ is a $\mathfrak{J}_p$-symmetric unitary operator, which commutes with R;

(v) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry with $\operatorname{Ker}\varphi_1 = \operatorname{Ker} R_1$, which commutes with R_1. In addition, we have $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is unitary (See Remark 5.2.1);

In addition, $\mathcal{T}$ is the unique contraction which satisfies

$$R_1\varphi_1 = \mathcal{T}R\varphi, \tag{7.0.1}$$

implied by Douglas Lemma 3.1.1.

In this chapter, we will show that

(i) If $\mathcal{H}_0 = \mathcal{H}$, i.e., p is a cyclic vector for R. Then $\mathcal{T}$ is automatically asymptotically stable without any further assumptions;

(ii) If $\mathcal{H}_0 \subsetneq \mathcal{H}$, then the condition (ii) in Proposition 7.5.1 gives a criterion for the asymptotically stability of $\mathcal{T}$.

Finally, in section 7.6, we give an equivalent condition for the asymptotic stability of $\mathcal{T}$ in Proposition 7.6.3 when taking the canonical choice of φ, φ_1. With this proposition, we can further analyze the behavior of φ, φ_1 restricted on each eigenspaces of R, R_1, which will be discussed later in chapter 9.

7.1 Preparation

We first discuss the case when p is a cyclic vector with respect to R. We introduce the following lemma, which is a slight modification of [12, lemma 3.2].

Lemma 7.1.1. *Let $\|\mathcal{T}\| \leq 1$, and let K be a compact operator with dense range. Assume that an operator A satisfies*

$$\mathcal{T}K = KA. \tag{7.1.1}$$

If A is weakly asymptotically stable, meaning that $A^n \to 0$ in the weak operator topology (W.O.T) as $n \to \infty$, then T is asymptotically stable.

Proof. Iterating (7.1.1) we get that $\mathcal{T}^n K = KA^n$, $n \geq 1$. Take $x \in \mathcal{H}$. Since $A^n \to 0$ in W.O.T. and K is compact, we have that $\|KA^n x\| \to 0$.

So $\lim_{n\to\infty} \|\mathcal{T}^n y\| = 0$ for all $y \in \operatorname{Ran} K$. Thus, we have strong convergence on a dense set S.

For any $y \in \mathcal{H}$ and any positive ε, we can find a x such that $\|x - y\| \leq \frac{\varepsilon}{3}$. Then we can find a sufficiently large N, such that $\|\mathcal{T}^n x - x\| \leq \frac{\varepsilon}{3}$ holds for all $n \geq N$. Then we have

$$\|\mathcal{T}^n y - y\| \leq \|y - x\| + \|x - \mathcal{T}^n x\| + \|\mathcal{T}^n x - \mathcal{T}^n y\| \leq 2\|y - x\| + \|x - \mathcal{T}^n x\| \leq \varepsilon$$

Thus we conclude (by $\varepsilon/3$-Theorem) that $\mathcal{T}^n \to 0$ in the strong operator topology.

□

Recall that for an operator R (in a Hilbert space) its *modulus* $|R|$ is defined as $|R| := (R^* R)^{1/2}$

Lemma 7.1.2. *For the operators R and $\mathcal{T}$ from (7.0.1), there exists a unique contraction A, such that*

$$\mathcal{T} R^{1/2} = R^{1/2} A$$

Proof. Since $R_1 \varphi_1 = \mathcal{T} R \varphi$, we have

$$R_1^2 = (R_1\varphi_1)(R_1\varphi_1)^* = \mathcal{T} R \varphi \varphi^* R \mathcal{T}^* = \mathcal{T} R^2 \mathcal{T}^*.$$

Hence by $\mathcal{T}^*\mathcal{T} \leq I$, we have

$$R^2 \geq R_1^2 = \mathcal{T} R^2 \mathcal{T}^* \geq \mathcal{T} R \mathcal{T}^* \mathcal{T} R \mathcal{T}^* = (\mathcal{T} R \mathcal{T}^*)^2$$

This tells us $R \geq \mathcal{T} R \mathcal{T}^*$ and

$$\|R^{1/2} x\| \geq \|R^{1/2} \mathcal{T}^* x\| \qquad \text{holds for all} \qquad x \in H$$

Thus by Douglas Lemma 3.1.1, we can find a contraction denoted as A^*, satisfying

$$A^* R^{1/2} = R^{1/2} \mathcal{T}^*.$$

Taking the adjoint for the equation above, and we finish the proof of this lemma. □

By Lemma 7.1.1 and Lemma 7.1.2 we can see that, in order to show that $\mathcal{T}$ is asymptotically stable, it's sufficient to show that A is weakly asymptotically stable.

7.2 Case when p is Cyclic

To prove the weak asymptotic stability of A we need to investigate its structure in more detail.

We know that $R_1^2 = R^2 - pp^* \leq R^2$. By the Löwner–Heinz inequality with $\alpha = 1/2$ we have that $R_1 \leq R$, so by Lemma 3.1.1 there exists an unique contraction Q such that

$$R_1^{1/2} = Q R^{1/2} \tag{7.2.1}$$

The following simple proposition (modified from [12, Lemma 3.5]) gives an expression for operator A.

Proposition 7.2.1. *The operator A from Lemma 7.1.2 is given by*

$$A = Q^* \varphi_1 Q \varphi^*.$$

Proof. We calculate the representation of $R_1 \varphi_1$ in two different ways. First we have

$$R_1 \varphi_1 = \mathcal{T} R^{1/2} R^{1/2} \varphi = R^{1/2} A R^{1/2} \varphi.$$

On the other hand,

$$R_1\varphi_1 = R^{1/2}Q^*QR^{1/2}\varphi_1.$$

Since $\operatorname{Ker} R = \{0\}$, combining the two equations above, we get

$$A = Q^*QR^{1/2}\varphi_1\varphi^*R^{-1/2} = Q^*(R_1^{1/2}\varphi_1\varphi^*R^{-1/2}) = Q^*\varphi_1Q\varphi^*, \tag{7.2.2}$$

which finishes our proof. □

In addition, the following proposition gives the structure of Q.

Proposition 7.2.2. *Let $\mathcal{H}_0$ be the smallest invariant subspace of R that contains p. Then the operator Q with respect to the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ has the block structure*

$$Q = \begin{pmatrix} Q_0 & 0 \\ 0 & I \end{pmatrix},$$

where Q_0 defined on $\mathcal{H}_0$ is a strict contraction (i.e. $\|Q_0x\| < \|x\|$ for all $x \neq 0$).

Proof. We know that

$$R_1^2 = R^2 - pp^*, \tag{7.2.3}$$

so R^2 coincides with R_1^2 on $p^\perp$.

One can easily see that $\mathcal{H}_0$ is an invariant subspace for R^2 and for R_1^2, and therefore so is $\mathcal{H}_0^\perp$. That means $\mathcal{H}_0$ and $\mathcal{H}_0^\perp$ are reducing subspaces for both R^2 and R_1^2, i.e. that these operators in the decomposition $\mathcal{H} = \mathcal{H}_0 \oplus \mathcal{H}_0^\perp$ are block diagonal. Therefore, the same is true for $R^{1/2}$

Easy to see that $R^{1/2}$ and $R_1^{1/2}$ coincide on $\mathcal{H}_0^\perp$, which is a reducing space for both operators, so only need to show that Q_0 is a strict contraction.

Using (7.2.3) and the identity $R_1^{1/2} = QR^{1/2} = R^{1/2}Q^*$, we can write

$$R^2 - pp^* = R_1^2 = R^{1/2}Q^*QRQ^*QR^{1/2}.$$

Recalling that $p = R\varphi q$, we can rewrite the above identity as

$$R^{1/2}\left(R - (R^{1/2}\varphi q)(R^{1/2}\varphi q)^*\right)R^{1/2} = R^{1/2}Q^*QRQ^*QR^{1/2}.$$

Since $\operatorname{Ker} R = \{0\}$, we have

$$Q^*QRQ^*Q = R - (R^{1/2}\varphi q)(R^{1/2}\varphi q)^*. \tag{7.2.4}$$

Applying both sides to x, and taking the inner product with x, we get

$$(RQ^*Qx, Q^*Qx) = (Rx, x) - |(x, R^{1/2}\phi(R)q)|^2. \tag{7.2.5}$$

Now, take x such that $\|Qx\| = \|x\|$. Since $\|Q\| \le 1$, we have

$$\langle x, x\rangle = \langle Qx, Qx\rangle = \langle x, Q^*Qx\rangle \le \|x\|\|Q^*Qx\| \le \|x\|^2.$$

Thus we have $x = Q^*Qx$. The equation (7.2.5) can be rewritten in this case as

$$(Rx, x) = (Rx, x) - |(x, R^{1/2}\varphi q)|^2,$$

which implies that $x \perp R^{1/2}\varphi q$. Applying equation (7.2.4) to such x, and using again the fact that $Q^*Qx = x$, we get that

$$Q^*QRx = Rx.$$

Hence set $\mathcal{H}_1 := \{h \in \mathcal{H} : h \in \mathcal{H}, \|Qh\| = \|h\|\} = \operatorname{Ker}(I - Q^*Q)$ is an invariant subspace for R, which is orthogonal to $\varphi R^{1/2}q$. Therefore

$$\mathcal{H}_1 \perp \overline{\text{Span}}\{R^n R^{1/2}\varphi q : n \geq 0\} = \overline{\text{Span}}\{R^n p : n \geq 0\} = \mathcal{H}_0,$$

and so $Q_0 = Q|_{\mathcal{H}_0}$ is a strict contraction. □

Now from Proposition 7.2.2, we have Q is a pure contraction since p is a cyclic vector for R, thus by proposition 7.2.1 we know that A is also a pure contraction.

Remark 7.2.3. Indeed, we can also show that Q_0^* is also a pure contraction on $\mathcal{H}_0$. For all $x, y \in \mathcal{H}_0$, we have

$$\langle Q_0^* x, y\rangle = \langle x, Q_0 y\rangle \leq \|x\|\|Q_0 y\| < \|x\|\|y\|$$

Take $y = Q_0^* x$, then we have $\|Q_0^* x\| < \|x\|$.

7.3 Case when p is not Cyclic

When p is not cyclic with respect to R inside $\mathcal{H}$, instead we show that A is a completely non-unitary contraction under certain assumptions of φ, φ_1. Beforehead, we recall the definition of completely nonunitary (c.n.u) contraction.

Definition 7.3.1. *(From [26]) We call a contraction $\mathcal{T} : \mathcal{H} \to \mathcal{H}$ a* completely nonunitary contraction (c.n.u) *if and only if for no nonzero reducing subspace E for $\mathcal{T}$ is $\mathcal{T}|_E$ is a unitary operator.*

Proposition 7.3.2. *The contraction A is completely non-unitary if and only if $\varphi_1\varphi^*$ does not have any non-zero reducing subspace $E \subset \mathcal{H}_0^\perp$ such that $\varphi^* E \perp \mathcal{H}_0$.*

To prove this proposition we need the following simple observation.

Lemma 7.3.3. *Let A be a contraction and E be a subspace. The following statements are equivalent:*

(i) *The subspace E is a reducing subspace for A such that $A\big|_E$ is unitary;*

(ii) *The operator A acts isometrically on E (i.e. $\|Ax\| = \|x\|$ for all $x \in E$) and $AE = E$.*

Proof of lemma 7.3.3. The proof of this lemma is simple.

Suppose that we have (i), then we have $A\big|_E$ is isometric and onto, thus (ii) holds.

For the other direction, suppose that we have (ii). Since we already have $A\big|_E$ is isometric and onto, which implies that $A\big|_E$ is unitary. It's sufficient to show that $AE^\perp \subset E^\perp$.

In fact, for $\forall x \in E, y \in E^\perp$, we have

$$\langle Ax, Ay\rangle = \langle A^*Ax, y\rangle = \langle x, y\rangle = 0,$$

here the first equality results $A\big|_E$ is isometric. Thus E is reducing subspace for A, and (i) holds. □

Proof of Proposition 7.3.2. Assume that $E \subset \mathcal{H}_0^\perp$ is a reducing subspace for $\varphi_1\varphi^*$ such that $\varphi^*E \subset \mathcal{H}_0^\perp$.

Since $Q = Q^* = I$ on $\mathcal{H}_0^\perp$, we have for any $x \in E$,

$$Ax = Q^*\varphi_1 Q\varphi^* x = Q^*\varphi_1\varphi^* x = \varphi_1\varphi^* x; \tag{7.3.1}$$

in the second equality we used the fact that $\varphi^* x \in \mathcal{H}_0^\perp$, and in the last one the fact that $\varphi_1\varphi^* x \in E \subset \mathcal{H}_0^\perp$.

Since φ_1 acts isometrically on $\mathcal{H}_0^\perp$ (Ker φ_1 can only belongs to $\mathcal{H}_0$), we have

$$\|Ax\| = \|\varphi_1\varphi^* x\| = \|x\|,$$

i.e. A acts isometrically on E.

Also, this implies that $\varphi_1\varphi^*$ acts isometrically on E. In addition, φ is unitary, and (see Remark 5.2.1) we know $\varphi_1\big|_{(\mathrm{Ker}\, R_1)^\perp}$ is unitary, therefore $\varphi\varphi_1^*$ acts unitarily on its reducing subspace E. Now from (7.3.1) we have

$$AE = \varphi_1\varphi^* E = E.$$

Applying Lemma 7.3.3, we can see that E is a reducing subspace for A such that $A|_E$ is unitary, so A is not c.n.u.

Now let's move on to the sufficiency. If A is not c.n.u., then we can find a reducing subspace E for A, such that $A|_E = Q^*\varphi_1 Q\varphi^*|_E$ is unitary. Using that Q is a pure contraction on H_0, we have

$$\varphi^* E \subset \mathcal{H}_0^\perp, \qquad \varphi_1\varphi^* E \subset \mathcal{H}_0^\perp;$$
$$AE = \varphi_1\varphi^* E = E.$$

Thus $E = \varphi_1\varphi^* E \subset \mathcal{H}_0^\perp$. In addition, since $\varphi_1\varphi^* E^\perp \perp \varphi_1\varphi^* E$, thus $\varphi_1\varphi^* E^\perp \subseteq E^\perp$, and E is a reducing subspace for $\varphi_1\varphi^*$. This gives a contradiction, so we finish the proof. □

7.4 C.n.u Implies Weakly Asymptotic Stability

Recall that in Subsection 7.2, we have shown that the operator A is a pure contraction, and in Subsection 7.3 we have shown that A is c.n.u under certain assumptions for φ, φ_1. Since a pure contraction is certainly c.n.u from the Definition 7.3.1, it's sufficient to prove that completely non-unitary implies asymptotical stability.

The following proposition and its proof is inspired from [27].

Proposition 7.4.1. *Let A be a completely non unitary contraction on a Hilbert space $\mathcal{H}$, then A is weakly asymptotically stable.*

Proof. Denote set H_1 as

$$H_1 := \left\{ x \middle| \|A^n x\| = \|A^{*n} x\| = \|x\| \text{ holds for } \forall n \in \mathbb{N} \right\}, \tag{7.4.1}$$

we firstly show that $H_1 = \{0\}$.

For $\forall x \in H_1$ and $\forall n \in \mathbb{N}$,

$$\|x\|^2 = \|A^n x\| = \langle A^n x, A^n x \rangle = \langle A^{*n} A^n x, x \rangle \leq \|A^{*n} A^n x\| \|x\| \leq \|x\|^2,$$

thus we have $A^{*n} A^n x = x$ holds for all n, and similarly we have $A^n A^{*n} x = x$.

On the contrary, if a vector x satisfies that $A^{*n} A^n x = A^n A^{*n} x = x$ holds for all n, then we have

$$\|A^n x\|^2 = \langle A^{*n} A^n x, x \rangle = \langle x, x \rangle = \|x\|^2,$$

and similarly we have $\|A^{*n} x\| = \|x\|$, thus we have

$$H_1 = \left\{ x \middle| A^{*n} A^n x = A^n A^{*n} x = x \text{ holds for } \forall n \in \mathbb{N} \right\}. \tag{7.4.2}$$

Thus H_1 is a subspace. Furthermore, H_1 is a reducing subspace for A and A^*. In fact, for all $x \in H_1$, we have

$$\|A^n A x\| = \|x\|, \qquad \|A^{*n} A x\| = \|A^{*(n-1)} x\| = \|x\|$$
$$\|A^n A^* x\| = \|A^{n-1} x\| = \|x\|, \qquad \|A^{*n} A^* x\| = \|A^{*(n+1)} x\| = \|x\|$$

Thus by (7.4.1), we have $Ax, A^*x \in H_1$. Since we also have $A\big|_{H_1}, A^*\big|_{H_1}$ is onto due to (7.4.2), applying Lemma 7.3.3 we have $A\big|_{H_1}$ is unitary. Since we are given that A is c.n.u, we have $H_1 = \{0\}$.

Now we prove that A is weakly asymptotically stable by contradiction. Assume that A is not weakly asymptotically stable, then there exists $x, y \in \mathcal{H}$, a positive ε, and a subsequence $\{n_j\}_{j=1}^{\infty}$ such that

$$|\langle A^{n_j} x, y\rangle| > \varepsilon. \tag{7.4.3}$$

We know that the set $\{A^{n_j} x\}_{j=1}^{\infty}$ is bounded, and since $\mathcal{H}$ is reflexive and separable, thus weakly compactness implies sequentially weakly compactness, hence there exists a subsequence of $\{A^{n_j} x\}_{j=1}^{\infty}$ which is weakly convergent. Without lose of generality, we take this subsequence to be $\{A^{n_j} x\}_{j=1}^{\infty}$ itself. Denote $A^{n_j} x \rightharpoonup x_0$ for a x_0.

Thus for a fixed $k \in \mathbb{N}$ we have

$$A^{*k} A^k A^{n_j} x \rightharpoonup A^{*k} A^k x_0 \text{ when } n_j \to \infty. \tag{7.4.4}$$

On the other hand, we can show that $A^{*k} A^k A^{n_j} x \rightharpoonup x_0$. In fact we have

$$\begin{aligned} \|A^{*k} A^k A^{n_j} x - A^{n_j} x\|^2 &= \|A^{*k} A^k A^{n_j} x\|^2 + \|A^{n_j} x\|^2 - 2\langle A^{*k} A^k A^{n_j} x, A^{n_j} x\rangle \\ &\le \|A^{k+n_j} x\|^2 + \|A^{n_j} x\|^2 - 2\|A^{k+n_j} x\|^2 \\ &= \|A^{n_j} x\|^2 - \|A^{k+n_j} x\|^2. \end{aligned}$$

Here the right hand side of the above inequality goes to 0 when $n_j \to \infty$, because sequence $\|A^n x\|_{n=1}^{\infty}$ is monotonically decreasing, hence convergent. And since $A^{n_j} x \rightharpoonup x_0$, we have

$$A^{*k} A^k A^{n_j} x \rightharpoonup x_0 \text{ when } n_j \to \infty. \tag{7.4.5}$$

Together with (7.4.4), we have $A^{*k} A^k x_0 = x_0$. Similar to the process above, we can get $A^k A^{*k} x_0 = x_0$. By (7.4.2), we have $x_0 \in H_1$, which implies that $x_0 = 0$. Hence $A^{n_j} x \rightharpoonup 0$ weakly which contradicts to (7.4.3). □

Remark 7.4.2. We can also use functional model theory to shorten the proof of Proposition 7.4.1.

Since every completely non-unitary operator A is unitary equivalent to a so-called model operator M_θ defined on the functional space K_θ. Here $\theta(z)$ is called to be the *characteristic function* of A, which is an analytic function defined as

$$\theta_A(z) = \big(- A + zD_{A^*}(I - zA^*)^{-1}D_A\big)\big|_{\mathcal{D}_A}, \qquad z \in \text{ unit disk } D. \tag{7.4.6}$$

For a given z satisfying that $I - zA^*$ is bounded invertible, $\theta_A(z)$ is a bounded operator mapping $\mathcal{D}_A := \operatorname{Clos}\operatorname{Ran} D_A$ to $\mathcal{D}_{A^*} := \operatorname{Clos}\operatorname{Ran} D_{A^*}$.

And the model space K_θ is an appropriately constructed subspace of a weighted space $L^2(E^* \oplus E, W)$ on the unit circle $\mathbb{T}$ with respect to the normalized Lebesgue measure W on $\mathbb{T}$, where E^*, E are some Hilbert spaces with dimensions $\mathfrak{D}_{A^*}, \mathfrak{D}_A$ respectively (See Definition 7.0.2). And the model operator M_θ is the compression of the multiplication operator M_z onto K_θ. That is,

$$M_\theta = P_\theta M_z|_{K_\theta}, \tag{7.4.7}$$

Here P_θ is the orthogonal projection onto K_θ. The specific choice of K_θ can be found in [26, Proposition 2.1, Chapter VI]. (As for the foundation of functional model theory, details can be found in [26, Chapter VI] and [28, P. 109-115]).

From (7.4.7), we have

$$M_\theta^n = P_{K_\theta} M_z^n\big|_{K_\theta},$$

since $M_z^n \to 0$ when $n \to 0$ in the weak operator topology of $B(L^2(E^* \oplus E, W))$ (the set of all bounded operators on this space), we claim that $M_\theta^n \to 0$ when $n \to \infty$ in the weak operator topology of $B(K_\theta)$ as well, thus $A^n \to 0$ in the weak operator topology.

7.5 Main Result

Combining the results in subsections 6.1-6.4, we reach the following proposition.

Proposition 7.5.1. *$\mathcal{T}$ defined by $\mathcal{T} := \varphi_1 R_1 R^{-1} \varphi^*$ is asymptotically stable if one of the following two conditions hold*

(i) *$\mathcal{H}_0 = \mathcal{H}$, i.e. p is cyclic with respect to R in $\mathcal{H}$;*

(ii) *$\varphi_1 \varphi^*$ does not have any non-zero reducing subspace $E \subset \mathcal{H}_0^\perp$ such that $\varphi^* E \perp \mathcal{H}_0$.*

Proof of Proposition 7.3.3. To show that $\mathcal{T}$ is asymptotically stable, it suffices to show that the operator A defined by

$$\mathcal{T} R^{1/2} = R^{1/2} A$$

is weakly asymptotically stable according to Lemma 7.1.1 and Lemma 7.1.2. And furthermore by Lemma 7.4.1, it's sufficient to show that A is a completely non-unitary contraction.

When $\mathcal{H}_0 = \mathcal{H}$ holds, by Proposition 7.2.1 we have $A = Q^* \varphi_1 Q \varphi^*$. Since Q is a pure contraction according to Proposition 7.2.2, we have A is also a pure contraction, thus a completely non-unitary contraction.

When $\mathcal{H}_0 \subsetneq \mathcal{H}$, according to Proposition 7.3.2, we still have A is a c.n.u. So we finish the proof. □

7.6 Asymptotic Stability under the Canonical Choice

In this section, we consider the asymptotic stability of contraction $\mathcal{T}$ under the canonical choice of φ, φ_1. We will show that the requirement in Proposition 7.5.1 can be substituted by a sufficient and necessary condition.

Recall the definition of four eigenspaces E_{λ_k} and $E^1_{\lambda_k}$, E_{μ_k} and $E^1_{\mu_k}$ stated in (6.2.7), the definition of p_k, p^1_k given in (6.2.9), and the canonical choice of φ, φ_1 given in Lemma

6.3.2:

$$\varphi_1\big|_{E^1_{\lambda_k}} = I, \qquad \varphi\big|_{E_{\mu_k}} = I. \tag{7.6.1}$$

We first state the definition of star-cyclicity.

Definition 7.6.1. *Here we say a vector x is* *-cyclic *for A in $\mathcal{H}$ if and only if*

$$\overline{\text{Span}}\big\{A^n x, (A^*)^n x | n \geq 0\big\} = \mathcal{H}.$$

An equivalent definition in [29, Def 3.1,Chapter IX ,p. 268] is given as follow:

We say a vector x is *-cyclic *for A in $\mathcal{H}$ if and only if $\mathcal{H}$ is the smallest reducing subspace for A that contains x.*

Definition 7.6.2. *We say that an operator A on $\mathcal{H}$ is cyclic if and only if there exists a vector $x \in \mathcal{H}$, such that*

$$\overline{\text{Span}}\Big\{A^n x | n \geq 0\Big\} = \mathcal{H}.$$

*Similarly, we say an operator A on $\mathcal{H}$ is *-cyclic if and only if there exists a vector $x \in \mathcal{H}$, such that*

$$\overline{\text{Span}}\big\{A^n x, (A^*)^n x | n \geq 0\big\} = \mathcal{H}.$$

J. Bram proved in [30] that a normal operator A is *-cyclic if and only if A is cyclic.

The results below gives an equivalent condition of φ, φ_1 which guarantees the asymptotic stability of $\mathcal{T}$.

Proposition 7.6.3. *Under the choice of φ, φ_1 given in (7.6.1), $\mathcal{T}$ is asymptotically stable if and only if both of the two conditions below are satisfied:*

(i) *For all k the vector p_k is *-cyclic for the operator $\varphi\big|_{E_{\lambda_k}}$;*

(ii) *For all k the vector p_k^1 is *-cyclic for the operator $\varphi_1\big|_{E^1_{\mu_k}}$.*

Proof of Lemma 7.6.3. If for some k the vector p_k is not *-cyclic for $\varphi\big|_{E_{\lambda_k}}$, then there exists a subspace $H_k \subset E_{\lambda_k}$, $H_k \perp p_k$ (and so $H_k \subset E^1_{\lambda_k}$) which is a reducing subspace for $\varphi\big|_{E_{\lambda_k}}$ (and thus for φ).

We know that $R = R_1$ on $E^1_{\lambda_k}$, and that $\varphi_1|_{E^1_{\lambda_k}} = I$, so the representation $\mathcal{T} = \varphi_1 R_1 R^{-1} \varphi^*$ implies that

$$\mathcal{T}x = \varphi^* x \in H_k \qquad \forall x \in H_k.$$

Therefore H_k is an invariant subspace for $\mathcal{T}$ (in fact, one can show that it is a reducing subspace, but we do not need this in the proof) on which $\mathcal{T}$ acts isometrically, and so $\mathcal{T}$ cannot be asymptotically stable.

Similarly, if p_k^1 is not *-cyclic for $\varphi_1|_{E^1_{\mu_k}}$ for a certain k, then we can find a subspace of $E^1_{\mu_k}$ denoted as H^1_k, $H^1_k \perp p^1_k$ (simply set $H^1_k := E^1_{\mu_k} \ominus \overline{\operatorname{Span}}\{(\varphi_1^*)^n p^1_k, \varphi_1^n p^1_k : n \geq 0\}$), which is a reducing subspace for $\varphi_1|_{E^1_{\mu_k}}$.

Since $\varphi|_{E_{\mu_k}} = I$, applying any $x \in H^1_k$ to $R_1\varphi_1 = \mathcal{T} R \varphi$ implies that

$$\mathcal{T}x = \varphi_1 x \qquad \forall x \in H^1_k.$$

Hence H^1_k is an invariant subspace for $\mathcal{T}$. Noticing that φ_1 is an isometry restricted on $\mathcal{H}_0^\perp$, thus $\mathcal{T}$ can't be asymptotically stable.

So, we proved that the conditions (i), (ii) are necessary for the asymptotic stability of $\mathcal{T}$.

To prove the sufficiency of these conditions, we will show that under (i), (ii), $A = Q^*\varphi_1 Q \varphi^*$ is c.n.u. Then proposition 7.4.1 implies that A is weakly asymptotically stable, and so by Lemma 7.1.1 and Lemma 7.1.2, $\mathcal{T}$ is asymptotically stable.

So, let us assume that conditions (i), (ii) are satisfied, but A is not c.n.u. Then there exists a reducing subspace L where $A\big|_L$ is unitary. Since $A^* = \varphi Q^* \varphi_1^* Q$ acts isometrically on L, we have $L \subseteq \mathcal{H}_0^\perp$. From $A|_L = (Q^*\varphi_1 Q\varphi^*)|_L$ is unitary, and we also know from Proposition 7.2.2 and Remark 7.2.3 that both Q and Q^* are pure contractions on $\mathcal{H}_0$, thus

$$\varphi^* L \subseteq \mathcal{H}_0^\perp, \qquad \varphi_1 \varphi^* L = L. \tag{7.6.2}$$

Now take a vector $x \in L \subseteq \mathcal{H}_0^\perp$. We write $x = a + b$ where $a \in \bigoplus_{k=1}^{\infty} E^1_{\lambda_k}$ and $b \in \bigoplus_{k=1}^{\infty} E_{\mu_k}$. Then

$$\varphi^* x = \varphi^* a + \varphi^* b = \varphi^* a + b.$$

Here $\varphi^* a \in \bigoplus_k E_{\lambda_k} = \bigoplus_k E^1_{\lambda_k} \oplus \mathcal{H}_0$. On the other hand, from (7.6.2) we have $\varphi^* x \in \mathcal{H}_0^\perp$, thus $\varphi^* a \in \bigoplus_k E^1_{\lambda_k}$.

So now we have

$$Ax = \varphi_1 \varphi^* x = \varphi^* a + \varphi_1 b,$$

where here we use $\varphi_1|_{E^1_{\lambda_k}} = I$ in the second identity. Since $Ax \in L \subseteq \mathcal{H}_0^\perp$, we have $\varphi_1 b \in \bigoplus_{k=1}^{\infty} E_{\mu_k}$. Hence by induction, we will get

$$A^n x = (\varphi^*)^n a + (\varphi_1)^n b, \tag{7.6.3}$$

holds for all $n \in \mathbb{N}$, where $(\varphi^*)^n a \in \bigoplus_k E^1_{\lambda_k}, \varphi_1^n b \in \bigoplus_k E_{\mu_k}$.

And similar property holds for A^*, with the same notation of x, a, b, we have

$$(A^*)^n x = \varphi^n a + (\varphi_1^*)^n b, \tag{7.6.4}$$

where $\varphi^n a \in \bigoplus_k E^1_{\lambda_k}, (\varphi_1^*)^n b \in \bigoplus_k E_{\mu_k}$.

Now we need the following simple lemma.

Lemma 7.6.4. *Let U be an arbitrary operator on a Hilbert space $\mathcal{H}$, and x is a *-cyclic vector with respect to U on $\mathcal{H}$, then for $\forall y \in \mathcal{H}$, we have*

$$P_x \overline{\text{Span}}\big\{U^n y, (U^*)^n y | n \geq 0\big\} \neq 0.$$

Here P is the projection operator.

Proof of Lemma 7.6.4. If the projection is zero, then we have $x \perp U^n y, (U^*)^n y$ for all n. Thus accordingly, we have

$$y \perp U^n x, (U^*)^n x \qquad \text{for all } n,$$

from the *-cyclicity of x, we have $y \perp \mathcal{H}$, which gives a contradiction. □

Now back to the proof of Proposition 7.6.3. For the chosen vector $x \in L$, there exists a $k \in \mathbb{N}$ such that $x_k^1 := P_{E^1_{\lambda_k}} x \neq 0$ or $x_k := P_{E_{\mu_k}} x \neq 0$.

If $x_k^1 \neq 0$, then applying Lemma 7.6.4 we have

$$P_{p_k} \overline{\text{Span}}\big\{\varphi^n x_k^1, (\varphi^*)^n x_k^1 | n \geq 1\big\} \neq 0.$$

Thus we can find a $n \in \mathbb{N}$ such that $P_{p_k}(\varphi^n x_k^1) \neq 0$ or $P_{p_k}(\varphi^*)^n x_k^1 \neq 0$.

If $P_{p_k}(\varphi^n x_k^1) \neq 0$, then it contradicts to the equation (7.6.4) where we have $\varphi^n a \in \bigoplus_k E^1_{\lambda_k}$. If $P_{p_k}(\varphi^*)^n x_k^1 \neq 0$, then it contradicts to the equation (7.6.3) where we have $(\varphi^*)^n a \in \bigoplus_k E^1_{\lambda_k}$.

Similarly, if $x_k \neq 0$ for a k, applying Lemma 7.6.4 we have

$$P_{p_k^1} \overline{\text{Span}}\big\{\varphi_1^n x_k, (\varphi_1^*)^n x_k | n \geq 1\big\} \neq 0.$$

Then we can also find a $n \in \mathbb{N}$ such that $P_{p_k^1}(\varphi_1^n x_k) \neq 0$ or $P_{p_k^1}(\varphi_1^*)^n x_k \neq 0$. Thus we have $\varphi_1^n b \notin \bigoplus_k E_{\mu_k}$ or $(\varphi_1^*)^n b \notin \bigoplus_k E_{\mu_k}$, which contradicts to (7.6.3) and (7.6.4) respectively, and it will also leads to a contradiction.

So there doesn't exist such reducing subspace L satisfying $A\big|_L$ is unitary. Hence A is c.n.u, and we finish the proof of sufficiency part. $\square$

Remark 7.6.5. Since $\varphi\big|_{E_{\lambda_k}}$ is unitary and finite rank, with this restriction, we can show that only cyclicity (instead of $*-$cyclicity) is required. That is,

(i) For all k the vector p_k is cyclic for the operator $\varphi\big|_{E_{\lambda_k}}$ on E_{λ_k};

(ii) For all k the vector p_k^1 is cyclic for the operator $\varphi_1\big|_{E^1_{\mu_k}}$ on $E^1_{\mu_k}$.

The specific explanantion of this part can be found in Remark 10.1.4.

Remark 7.6.6. We need to mention here that the conjugation $\mathfrak{J}_p$ will be uniquely determined under the canonical choice of φ, φ_1. Since all eigenspaces of R, R_1 are invariant subspaces for $\mathfrak{J}_p$, and we have

$$\mathcal{H} = \Big(\bigoplus_{k=1}^{\infty} E_{\lambda_k}\Big) \oplus \Big(\bigoplus_{k=1}^{\infty} E_{\mu_k}\Big),$$

it's sufficient to determine value of $\mathfrak{J}_p$ on E_{λ_k} and $E^1_{\mu_k}$ (since $E_{\mu_k} \subset E^1_{\mu_k}$) for all k.

We first show that $\mathfrak{J}_p p_k = p_k, \mathfrak{J}_p p_k^1 = p_k^1$ hold for all k. In fact, since $\mathfrak{J}_p R = R\mathfrak{J}_p$, apply p_k on both sides we get

$$R\mathfrak{J}_p p_k = \mathfrak{J}_p R p_k = \lambda_k \mathfrak{J}_p p_k.$$

Since $\mathfrak{J}_p p_k \in \mathcal{H}_0$, we have $\mathfrak{J}_p p_k = \alpha_k p_k$ for a $\alpha_k \in \mathbb{C}$ with module 1. Hence we have

$$p = \mathfrak{J}_p p = \mathfrak{J}_p \sum_{k=1}^{\infty} p_k = \sum_{k=1}^{\infty} \overline{\alpha_k} p_k,$$

thus $\alpha_k = 1$ and $\mathfrak{J}_p p_k = p_k$ hold for all k. $\mathfrak{J}_p p_k^1 = p_k^1$ can be shown in a similar process.

Now since $\mathfrak{J}_p p_k = p_k$ and $\varphi|_{E_{\lambda_k}}$ is $\mathfrak{J}_p$-symmetric, thus for all $n \in \mathbb{N}$ we have

$$\mathfrak{J}_p \varphi^n p_k = (\varphi^*)^n \mathfrak{J}_p p_k = (\varphi^*)^n p_k,$$
$$\mathfrak{J}_p (\varphi^*)^n p_k = \varphi^n \mathfrak{J}_p p_k = \varphi^n p_k.$$

thus $\mathfrak{J}_p|_{E_{\lambda_k}}$ is uniquely determined implied by Proposition 7.6.3.

Similarly we have $\mathfrak{J}_p|_{E^1_{\mu_k}}$ is uniquely determined by

$$\mathfrak{J}_p(\varphi_1^n p_k^1) = (\varphi_1^*)^n p_k^1, \qquad \mathfrak{J}_p((\varphi_1^*)^n p_k^1) = \varphi_1^n p_k^1 \qquad \text{for all } n.$$

Thus the uniqueness of $\mathfrak{J}_p$ is proved.

Chapter 8

ABSTRACT BORG'S THEOREM

In a previous Proposition 6.2.2, we have shown that for a given rank-one perturbation $(R, R_1, p, \varphi, \varphi_1)$ with $R^2 - R_1^2 = pp^*$, under the cyclicity of vector p and the asymptotical stability of contraction $\mathcal{T}$, there exists an intertwining sequence

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0, \tag{8.0.1}$$

such that $R\big|_{\mathcal{H}_0}, R_1\big|_{\mathcal{H}_0}$ has simple eigenvalues as $\{\lambda_k\}_{k=1}^{\infty}$. $\{\mu_k\}_{k=1}^{\infty}$ respectively.

Thus in this chapter we consider the inverse problem. Given an intertwining real sequence

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0,$$

can we find a rank-one perturbation (R, R_1, p), such that R, R_1 has the corresponding eigenvalues? Fortunately the answer to this question is positive, and in this chapter we will prove the following so-called The Abstract Borg's Theorem:

Theorem 8.0.1 (Abstract Borg's Theorem)**.** *Given two sequences $\{\lambda_k^2\}_{k\geq 1}$ and $\{\mu_k^2\}_{k\geq 1}$ satisfying intertwining relations (8.0.1) and such that $\lambda_k^2 \to 0$ as $k \to \infty$, there exists an unique (up to unitary equivalence) triple (W, W_1, p), such that $W = W^* \geq 0$, $\operatorname{Ker} W = \{0\}$ is a compact operator with simple eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$, and the operator*

$W_1 = W - pp^*$ has $\{\mu_k^2\}_{k=1}^\infty$ as its (simple) non-zero eigenvalues (W_1 can also have a simple eigenvalue at 0).

Moreover, the two identities $\|W^{-1/2}p\| = 1$, $\|W^{-1}p\| = \infty$ hold if and only if:

$$\sum_{j=1}^\infty \left(1 - \frac{\mu_j^2}{\lambda_j^2}\right) = \infty, \tag{8.0.2}$$

$$\sum_{j=1}^\infty \left(\frac{\mu_j^2}{\lambda_{j+1}^2} - 1\right) = \infty. \tag{8.0.3}$$

Remark 8.0.2. The original Borg's theorem [32] states that the potential q of a Schrödinger operator L, $Ly = y'' + q(x)y$ on an interval is uniquely defined by the two sets of eigenvalues, corresponding to two specific boundary conditions. Later Levinson [33] extended this result by showing that essentially any non-degenerate pair of self-adjoint boundary conditions would work.

Changing boundary conditions for a Schrödinger operator is essentially a rank one perturbation (by an unbounded operator). Namely, if L_1 and L_2 are Schrödinger operators on an interval with the same potential, but with two different self-adjoint boundary conditions, then for any $\lambda \notin \sigma(L_1) \cup \sigma(L_2)$ the difference $(L_1 - \lambda I)^{-1} - (L_2 - \lambda I)^{-1}$ is a rank one operator (and the operators $(L_1 - \lambda I)^{-1}$, $(L_2 - \lambda I)^{-1}$ are compact). Thus, by picking a real λ the problem can be reduced to rank one perturbations of compact self-adjoint operators.

Our Theorem 8.0.1 deals with rank one perturbations of (abstract) compact self-adjoint operators, hence we give its name. We do not assume that our operators came from Schrödinger operators, so we only reconstruct the spectral measure, and are not concerned with the reconstruction of the potential. However, it is well known how to reconstruct the potential from the spectral measure, or, more precisely, from the Titchmarsh–Weyl m-function, so it should be possible to get the Borg's result from our abstract theorem.

Note also, that Theorem 8.0.1 gives not only uniqueness, but the existence as well.

Remark 8.0.3. Note also that the triple (W, W_1, p) we constructed in Theorem 8.0.1 satisfies that $P_{\mathrm{Ker}(W-\lambda_k^2 I)}p \neq 0$ for all k, hence by Lemma 6.1.1 the cyclicity of p with respect to W is automatically satisfied. In fact, if there is a k such that

$$p_k := P_{\mathrm{Ker}(W-\lambda_k^2 I)}p = 0$$

Then λ_k^2 is a common eigenvalue for W, W_1, which gives us contradiction.

Similarly we also have $P_{\mathrm{Ker}(W_1-\mu_k^2 I)}p \neq 0$ holds for all k.

In section 8.1, we prove the existence and uniqueness part of the abstract Borg's theorem. Inside this proof, we give an expression for the scalar spectral measure of W with respect to p (see definition in (8.1.3)), denoted as $\rho(s)$, and its coefficients has the form in (8.1.9).

In section 8.2, we will translate the trivial kernel condition of Γ we get from Proposition 3.1.3 and Proposition 5.2.3, in terms of $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$. Within this part, we also give an equivalent condition of $\mathrm{Ker}\, W_1 = \{0\}$, and furthermore, give an expression for the scalar spectral measure of W_1 with respect to p (see the definition in (8.1.4)), denoted as $\rho_1(s)$, and its coefficients has the form in Proposition 8.2.9.

In section 8.3, we discuss a special case, i.e., a finite-rank version for The Abstract Borg's Theorem. Here the proof for Theorem 8.3.1 is similar to the proof of Theorem 8.0.1, so we omit it.

Finally we close this chapter by considering the inverse problem for the case of rank-one perturbation (W, W_1, p), when p is not cyclic with respect to W. Given an intertwining real sequence

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0$$

with their corresponding geometric multiplicities, we can also reconstruct the triple (W, W_1, p). In this case, we will have the following result, which is a simple generalized version of Abstract Borg's theorem:

Theorem 8.0.4 (Generalized abstract Borg's theorem)**.** *Given positive sequences* $\{\lambda_k\}_{k=1}^{\infty}$ *and* $\{\mu_k\}_{k=1}^{\infty}$ *satisfying the following intertwining property:*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > ... > \lambda_k > \mu_k ... \to 0,$$

and also integer sequences $\{m(\lambda_k)\}_{k=1}^{\infty}$, $\{m(\mu_k)\}_{k=1}^{\infty}$ *satisfying for* $\forall k \geq 0$, $m(\lambda_k) \geq 1, m(\mu_k) \geq 0$, *then there exists a unique triple* (W, W_1, p) *satisfying:*

(i) W, W_1 *are positive, compact self-adjoint operators;*

(ii) $W - W_1 = pp^*$*;*

(iii) *For all* $k \in \mathbb{N}$, W *has singular values* λ_k^2 *with multiplicity* $m(\lambda_k)$, *and singular values* μ_k^2 *with multiplicity* $m(\mu_k)$*;*

(iv) *For all* $k \in \mathbb{N}$, W_1 *has singular values* λ_k^2 *with multiplicity* $m(\lambda_k) - 1$, *and singular values* μ_k^2 *with multiplicity* $m(\mu_k) + 1$*;*

We will prove this Theorem 8.0.4 in the last section 8.4 of this chapter.

8.1 Proof of Abstract Borg's Theorem: Existence and Uniqueness Part

8.1.1 Some Preparation Work: the Definition of Scalar Spectral Measure and Cauchy Transform

Since everything is defined up to unitary equivalence, by Von Neumann Theorem 1.2.7 we can then assume without loss of generality that W is the multiplication M_s by the independent variable s in the weighted space $L^2(\rho)$,and the vector p is represented by the function 1 in $L^2(\rho)$. Here $\rho(s)$ is called to be the scalar spectral measure with respect to the vector p, defined as following:

$$\big\langle (W - zI)^{-1}p, p \big\rangle = \int_{\sigma(W)} \frac{d\rho(s)}{s - z}. \tag{8.1.1}$$

We also call $F(z) := \int \frac{d\rho(s)}{s-z}$ to be the *Cauchy transform* of $\rho(s)$.

Since W is a compact operator with eigenvalues $\{\lambda_k^2\}_{k\geq 1}$, the measure ρ is purely atomic, that is:

$$\rho = \sum_{k\geq 1} a_k \delta_{\lambda_k^2}, \qquad a_k > 0. \tag{8.1.2}$$

Since the choice of a_k can uniquely determine the triple W, W_1, p up to unitary equivalence. We want to find out values of $\{a_k\}_{k=1}^{\infty}$ according to the given intertwining sequence.

The following proposition gives the Cauchy transform of an operator under a rank one perturbation, also stated in cf. [11, Chapter 9], [34, Theorem 5.8.1, p. 335]

Proposition 8.1.1 (Aronszajn-Krein formula)**.** *Assume that W_α is a rank-one perturbation of operator W: $W_\alpha = W - \alpha pp^*$, then the Borel transform of W and W^α defined as*

$$F(z) = \big\langle (W - zI)^{-1}p, p \big\rangle, \qquad F_\alpha(z) = \big\langle (W_\alpha - zI)^{-1}p, p \big\rangle$$

have the following relation

$$F_\alpha(z) = \frac{F(z)}{1 - \alpha F(z)}.$$

Back to our main problem. Denote the Cauchy transform of W and W_1 with respect to p to be

$$F(z) = \left\langle (W - zI)^{-1} p, p \right\rangle = \int_{\sigma(W)} \frac{d\rho(s)}{s - z}; \tag{8.1.3}$$

$$F_1(z) = \left\langle (W_1 - zI)^{-1} p, p \right\rangle = \int_{\sigma(W_1)} \frac{d\rho_1(s)}{s - z}. \tag{8.1.4}$$

Applying Proposition 8.1.1 and take $\alpha = 1$, we have

$$1 - F = \frac{F}{F_1}. \tag{8.1.5}$$

8.1.2 Guess and Proof for Function F

In order to find out the coefficients $\{a_k\}_{k=1}^{\infty}$ in the scalar spectral measure (8.1.2), we need to give a guess for the function F.

Denote $\sigma = \{\lambda_k\}_{k=1}^{\infty} \cup \{0\}$ and $\sigma_1 = \{\mu_k\}_{k=1}^{\infty} \cup \{0\}$, then we know $F(z)$ has simple poles at $\{\lambda_k\}_{k=1}^{\infty}$ and analytic at $\mathbb{C} \setminus \sigma$, $F_1(z)$ has simple poles at $\{\mu_k\}_{k=1}^{\infty}$ and analytic at $\mathbb{C} \setminus \sigma_1$. Thus $1 - F = \frac{F}{F_1}$ should be a function which is analytic function on $\mathbb{C} \setminus \sigma$, and has simple zeros at $\{\mu_k\}_{k=1}^{\infty}$ and simple poles at $\{\lambda_k\}_{k=1}^{\infty}$.

In the following part of this section, we will prove our guess for the function F, which is:

Proposition 8.1.2. *We have the following equation for function F hold*

$$1 - F(z) = \frac{F}{F_1} = \prod_{k=1}^{\infty} \left(\frac{z - \mu_k^2}{z - \lambda_k^2} \right). \tag{8.1.6}$$

We first prove that the right hand side of (8.1.6) converges uniformly on compact subset of $\mathbb{C} \setminus \sigma$.

Lemma 8.1.3. $\Phi_N(z) = \prod_{k=1}^{N} \left(\frac{z-\mu_k^2}{z-\lambda_k^2} \right)$ *converges uniformly on compact subset of* $\mathbb{C} \setminus \sigma$.

Proof. We use the trivial fact that the convergence of $\sum_{k=0}^{\infty} |f_k(z) - 1|$ implies the convergence of $\prod_{k=0}^{\infty} f_k(z)$ (convergence here always means the uniform convergence on compact subset). Since $\left|1 - \frac{z-\mu_k^2}{z-\lambda_k^2}\right| = \left|\frac{\mu_k^2-\lambda_k^2}{z-\lambda_k^2}\right|$, it is sufficient to show that $\sum_{k=1}^{\infty} \left|\frac{\mu_k^2-\lambda_k^2}{z-\lambda_k^2}\right|$ converges.

For z in a compact $K \subset \mathbb{C} \setminus \sigma$, and sufficiently large k (i.e. for all $k > N$)

$$\left|\frac{\mu_k^2 - \lambda_k^2}{z - \lambda_k^2}\right| \leq C(K,N)|\lambda_k^2 - \mu_k^2| \leq C(K,N)(\lambda_k^2 - \lambda_{k-1}^2),$$

thus:

$$\sum_{k=N}^{\infty} \left|\frac{\mu_k^2 - \lambda_k^2}{z - \lambda_k^2}\right| \leq C(K,N)\lambda_N^2 < \infty,$$

and the convergence follows trivially. □

Now denote $\lim_{N\to\infty} \Phi_N(z) \to \Phi(z)$ on $\mathbb{C} \setminus \sigma$. Before we introduce several properties for this function $\Phi(z)$, let us recall the definition of Nevanlinna function and its integral representation.

Definition 8.1.4. Nevanlinna function *is an analytic function on the open half plane and image has non-negative imaginary part.*

Theorem 8.1.5 (Integral representation)**.** *Every Nevanlinna function f admits a following integral representation*

$$f(z) = C + Dz + \int_R \left(\frac{1}{\lambda - z} - \frac{\lambda}{1+\lambda^2}\right) d\mu(\lambda) \tag{8.1.7}$$

where C is a real constant, D is non-negative, μ is a Borel measure on R satisfying $\int_R \frac{d\mu(\lambda)}{1+\lambda^2} < \infty$.

Conversely, if a function has this type of form, then it's a Nevanlinna function, and the representation is unique.

Now let us get back to discuss function $\Phi(z)$.

Lemma 8.1.6. *Function $\Phi(z)$ satisfies the following properties:*

(i) $\lim_{z\to\infty} \Phi(z) = 1$;

(ii) *function* $-\Phi(z)$ *is a Nevanlinna function, restricted on* $\mathbb{C}^+$;

(iii) $\Phi(z)$ *is symmetric, i.e.* $\overline{\Phi(z)} = \Phi(\overline{z})$, *in particular,* $\Phi(z)$ *is real for all* $x \in \mathbb{R} \setminus \sigma$;

(iv) $\Phi(z)$ *has simple poles at* $\{\lambda_k^2\}_{k=1}^{\infty}$, *simple zeros at* $\{\mu_k^2\}_{k=1}^{\infty}$.

Proof. (i) This is true because

$$\begin{aligned}\sum_{k=1}^{\infty} \log \left|\frac{z-\mu_k^2}{z-\lambda_k^2}\right| &\le \sum_{k=1}^{\infty} \frac{|\lambda_k^2 - \mu_k^2|}{|z-\lambda_k^2|} \\ &\le \sum_{k=1}^{\infty} \frac{|\lambda_k^2 - \lambda_{k+1}^2|}{|z| - |\lambda_1|^2} \\ &\le \frac{|\lambda_1^2|}{|z| - |\lambda_1|^2}\end{aligned}$$

goes to 0 when $z \to \infty$, hence $\lim_{z\to\infty} \Phi(z) = 1$.

(ii) It's equivalent to show that $\frac{1}{\Phi(z)} = \prod_{i=1}^{\infty} \left(\frac{z-\lambda_i^2}{z-\mu_i^2} \right)$ is a Nevanlinna function on $\mathbb{C}^+$. Only need to show that $0 < \arg \frac{1}{\Phi(z)} < \pi$.

Denoting Z to be the point z on the complex plane $\mathbb{C}^+$, and $A_1, A_2, ...$ to be the point of sequence $\{\lambda_n\}$ on the real axis, and $B_1, ..., B_n, ...$ to be $\{\mu_n\}$. Then $\arg \frac{1}{\Phi(z)}$ is given by

$$0 < \measuredangle B_1 Z A_1 + ... + \measuredangle B_n Z A_n < \pi,$$

While this is trivially true because $\{\lambda_n\}$ and $\{\mu_n\}$ are two interwining sequences, and all those angles don't intersect with each other.

Here (iii) and (iv) is trivial, so we omit it.

□

Now we prove the following property.

Lemma 8.1.7. *The function* $\Phi(z)$ *given above is the only function that satisfies* (i),(ii),(iii), *and* (iv) *in Lemma 8.1.6*

Proof. Assume that $\Phi_1(z)$ is another function that satisfies those properties. Denote their ratio to be: $\Psi := \Phi_1/\Phi$. Additionally, we have both functions have simple poles at μ_n and simple zeros at λ_n, hence $\Psi(z)$ is analytic and zero-free in $\mathbb{C} \setminus \{0\}$.

In addition, we have $\lim_{z\to\infty} \Psi(z) = 1$ since both $\Phi(z), \Phi_1(z)$ satisfy (i) in Lemma 8.1.6.

Moreover, for $x \in \mathbb{R} \setminus \{0\}$ we have $\Psi(x) > 0$. Indeed, on $\mathbb{R} \setminus \sigma \setminus \sigma_1$ functions Φ_1 and Φ are real and have the same sign (If $\Phi(x), \Phi_1(x)$ have different signs for a certain $x \in \mathbb{R}/\sigma/\sigma_1$, then $\lim_{x\to+\infty} \Phi(x)$ and $\lim_{x\to+\infty} \Phi(x)$ have different signs, which gives a contradiction to (i) in Lemma 8.1.6), so $\Psi(x) > 0$ on $\mathbb{R} \setminus \sigma \setminus \sigma_1$. Since Ψ is continuous and zero-free on $\mathbb{R} \setminus \{0\}$, this tells us that Ψ is positive on $\mathbb{R} \setminus \{0\}$.

Next, let us notice that $\Psi(z)$ does not take real negative values. If $\operatorname{Im} z > 0$, then according to (ii) in Lemma 8.1.6, $\operatorname{Im} \Phi_1(z) < 0, \operatorname{Im} \Phi(z) < 0$, so $\Psi(z) = \Phi_1(z)/\Phi(z)$ cannot be negative real. If $\operatorname{Im} z < 0$, the symmetry $\Psi(\overline{z}) = \overline{\Psi(z)}$ implies the same conclusion. And, as we just discussed above, on the real line Ψ takes positive real values.

So Ψ omits infinitely many points, therefore by the Picard's Theorem, 0 is not the essential singularity for Ψ. Trivial analysis shows that 0 cannot be a pole, otherwise $\frac{1}{\Psi}$ is analytic at 0, which also contradicts to the fact that Ψ can't take negative real values on real axis. Hence we have 0 is a removable singularity for function Ψ, and Ψ is an entire function. By Liouville's Theorem, condition $\Psi(\infty) = 1$ implies that $\Psi \equiv 1$ for all $z \in$ C, hence the lemma is proved.

□

Remark 8.1.8. For the last paragraph of the proof for Lemma 8.1.7, we can also consider the square root $\Psi^{1/2}$, where we take the principal branch of the square root (cut along the negative half-axis). Then $\operatorname{Re} \Psi(z)^{1/2} \geq 0$, so by the Casorati–Weierstrass Theorem, 0 cannot be the essential singularity for $\Psi^{1/2}$. Again, trivial reasoning shows that 0 cannot be a pole, so again, $\Psi^{1/2}$ is an entire function. The condition $\Psi^{1/2}(\infty) = 1$ then implies that $\Psi^{1/2}(z) \equiv 1$.

Now back to the proof of Proposition 8.1.2.

Proof of Proposition 8.1.2. At the beginning of this section, we have mentioned that $\frac{F}{F_1}$ is a function which is analytic on $\mathbb{C} \setminus \sigma$, with simple poles at $\{\lambda_k\}_{k=1}^{\infty}$, and simple zeros at $\{\mu_k\}_{k=1}^{\infty}$,and equals 1 at ∞, which corresponds to property (i), (iv) in Lemma 8.1.6.

Now we can show that function $\frac{F}{F_1}$ also satisfies the property (ii), (iii) in Lemma 8.1.6.

In fact, for property (ii), to show that $\frac{F}{F_1}$ maps $\mathbb{C}^+$ to $\mathbb{C}^-$, according to (8.1.5) it's equivalent to show that $F(z)$ maps $\mathbb{C}^+$ to $\mathbb{C}^+$. This is trivially true, because we have shown that

$$F(z) = \sum_{k\geq 1} \frac{a_k}{\lambda_k^2 - z} \tag{8.1.8}$$

is analytic on $\mathbb{C}/\sigma$, and each single term $\frac{a_k}{\lambda_k^2 - z}$ has positive imaginary part if $Im(z) > 0$.

As for property (iii), to show $\frac{F}{F_1}$ is symmetric, it is equivalent to show that $F(z)$ is symmetric, which is also trivially true from (8.1.8).

So far we have shown that $\frac{F}{F_1}$ and $\Phi(z)$ are two functions that satisfy all four properties (i), (ii), (iii), (iv) given in Lemma 8.1.6. Hence those two functions coincide, and we proved Proposition 8.1.2. □

8.1.3 Coefficients of the Scalar Spectral Measure $\rho(s)$

In this section, we calculate the coefficients for the scalar spectral measure $\rho(s)$ mentioned in (8.1.1). We will show the following lemma.

Lemma 8.1.9. *The function* $\Phi(z) = 1 - F(z) = \frac{F(z)}{F_1(z)}$ *defined by* (8.1.6) *can be decomposed as*

$$\Phi(z) = 1 - \sum_{n\geq 1} \frac{a_n}{\lambda_n^2 - z},$$

where

$$a_n = (\lambda_n^2 - \mu_n^2) \prod_{k\neq n} \left(\frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2} \right). \tag{8.1.9}$$

Proof. Consider functions Φ_N defined as

$$\Phi_N(z) = \prod_{k=1}^{N} \left(\frac{z - \mu_k^2}{z - \lambda_k^2} \right).$$

Trivially

$$\Phi_N(z) = 1 - \sum_{n\geq 1} \frac{a_n^N}{\lambda_n^2 - z}, \tag{8.1.10}$$

where

$$a_n^N = (\lambda_n^2 - \mu_n^2) \prod_{\substack{k=1 \\ k\neq n}}^{N} \left(\frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2} \right) > 0 \qquad \text{if } n \leq N, \tag{8.1.11}$$

and $a_n^N = 0$ if $n > N$. This is because the two functions on left hand side and right hand side have the same poles and the same residues, so their difference is a polynomial. Then let $z \to \infty$, we know that polynomial equals 0 at ∞. Hence those two functions are equal.

Next, let $N \to \infty$. We know, see Lemma 8.1.3 that $\Phi_N(z)$ converges uniformly to $\Phi(z)$ in any compact subset $K \subset \mathbb{C} \setminus \sigma$. Hence, to prove the lemma it remains to show that $1 - \sum_{n\geq 1} \frac{a_n^N}{\lambda_n^2 - z}$ converges to $1 - \sum_{n\geq 1} \frac{a_n}{\lambda_n^2 - z}$ uniformly.

Take $z = 0$ in (8.1.10). Then we have

$$1 - \sum_{n\geq 1} \frac{a_n^N}{\lambda_n^2} = \prod_{k=1}^{N} \left(\frac{\mu_k^2}{\lambda_k^2}\right) > 0,$$

so $\sum\limits_{n\geq 1} \frac{a_n^N}{\lambda_n^2} \leq 1$.

Notice that for any fixed n the sequence $a_n^N \nearrow a_n$ as $N \to \infty$, so $\sum\limits_{n\geq 1} \frac{a_n}{\lambda_n^2} \leq 1$.

Take an arbitrary compact $K \subset \mathbb{C} \setminus \sigma$. Clearly for any $z \in K$,

$$\left|\frac{a_n^N}{\lambda_n^2 - z}\right| \leq \frac{a_n^N}{\operatorname{dist}(K,\sigma)} \leq \frac{a_n}{\operatorname{dist}(K,\sigma)} \leq \frac{\lambda_1^2}{\operatorname{dist}(K,\sigma)} \cdot \frac{a_n}{\lambda_n^2},$$

so the condition $\sum_{n\geq 1} a_n/\lambda_n^2 \leq 1$ implies that $\sum\limits_{n\geq 1} \frac{a_n^N}{\lambda_n^2 - z}$ is uniformly bounded, thus by dominated convergence theorem, we have $\sum\limits_{n\geq 1} \frac{a_n^N}{\lambda_n^2 - z}$ converges uniformly on K, and

$$\lim_{N\to\infty} \sum_{n\geq 1} \frac{a_n^N}{\lambda_n^2 - z} = \sum_{n\geq 1} \lim_{N\to\infty} \frac{a_n^N}{\lambda_n^2 - z} = \sum_{n\geq 1} \frac{a_n}{\lambda_n^2 - z}$$

□

Remark 8.1.10. We also need to calculate the coefficients of $\rho_1(s)$ (which is the scalar spectral measure of W_1 with respect to p given in (8.1.4)) for a future use in Proposition 10.1.5. However, this is slightly different from calculating $\rho(s)$ since $\operatorname{Ker} W = \{0\}$ but $\operatorname{Ker} W_1$ can be non-trivial. We will go back to this calculation in Proposition 8.2.9.

8.1.4 Existence and uniqueness of the triple (W, W_1, p)

Proof. Define a measure $\rho(s)$ as $\rho = \sum\limits_{k=1}^{\infty} a_k \delta_{\lambda_k^2}$, where $\{a_k\}_{k=1}^{\infty}$ is defined by (8.1.9). Let W be the multiplication by independent variable in $L^2(\rho)$:

$$(Wf)(t) = tf(t),$$

and let $p \equiv 1$. Clearly W is a positive compact operator with simple eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$, and its Borel transform

$$F(z) = \left\langle (W - zI)^{-1} p, p \right\rangle$$

is given by

$$F(z) = \int_{\mathbb{R}} \frac{d\rho(s)}{s - z} = \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2 - z}.$$

Now take $W_1 = W - pp^*$, and let $F_1(z) = ((W_1 - zI)^{-1}p, p)$. Recall that by Proposition 8.1.1 we have

$$1 - F = \frac{F}{F_1}. \tag{8.1.12}$$

Thus applying Lemma 8.1.9, we get

$$1 - F(z) = \prod_{k \geq 1} \left(\frac{z - \mu_k^2}{z - \lambda_k^2} \right) = \frac{F}{F_1}.$$

Together with (8.1.12), we know that F_1 has simple poles exactly at points $\{\mu_k^2\}_{k\geq 1}$, so $\{\mu_k^2\}_{k\geq 1}$ are the non-zero eigenvalues of W_1. So the existence of the triple (W, W_1, p) is proved. □

The uniqueness follows immediately from Lemma 8.1.7.

8.2 Proof of Abstract Borg's Theorem: Trivial Kernel Condition

Recall that in Proposition 3.1.3 and Proposition 5.2.3, we have discussed the equivalent condition for triple (R, R_1, p) such that the constructed Hankel operator Γ has trivial kernel. That is,

$$\|q\| = \|R^{-1}p\| = 1, \qquad q \notin \operatorname{Ran} R. \tag{8.2.1}$$

Now comparing equation $R^2 - R_1^2 = pp^*$ with $W - W_1 = pp^*$, we need to derive an equivalent condition for:

$$\|W^{-1/2}p\| = 1, \qquad \|W^{-1}p\| = \infty. \tag{8.2.2}$$

In this section, we will translate the condition (8.2.1) in terms of $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$. Then we derive the equivalent condition for $\operatorname{Ker} W_1 = \{0\}$ (we will see that this condition is also closely related to the two identities in (8.2.2)). Finally, we give an expression for the coefficients of $\rho_1(s) = \sum_{k\geq 1} b_k \delta_{\mu_k^2}(s) + b_0\delta_0(s)$, which is defined as the scalar spectral measure of W_1 with respect to p:

$$\left\langle (W_1 - zI)^{-1}p, p\right\rangle = \int_{\mathbb{R}} \frac{d\rho_1(s)}{s - z}$$

8.2.1 The Equivalence of Two Identities in Abstract Borg's theorem

Proposition 8.2.1. *The trivial kernel condition* $\|W^{-1/2}p\| = 1$ *and* $\|W^{-1}p\| = \infty$ *is equivalent to the following*

(i) $\sum\limits_{j=1}^{\infty} \left(1 - \frac{\mu_j^2}{\lambda_j^2}\right) = \infty$;

(ii) $\sum_{j=1}^{\infty} \left(\frac{\mu_j^2}{\lambda_{j+1}^2} - 1\right) = \infty.$

Proof. We first show that

$$\|P_{Ker(W-\lambda_k^2 I)}p\| = \sqrt{a_k}. \tag{8.2.3}$$

In fact, since the scalar spectral measure $\rho(s)$ satisfies that

$$\int_{\mathbb{R}} f(s)\rho(s) = \langle p, f(W)p\rangle$$

for all $f \in C(\mathbb{R})$, thus we take a sequence of polynomials $\{f_n\}$ that converges to $\mathbb{1}_{\{x=\lambda_k^2\}}$, then we have

$$a_k = \langle p, P_{\text{Ker}(W-\lambda_k^2 I)} p\rangle = \|P_{Ker(W-\lambda_k^2 I)} p\|^2$$

Then from (8.2.3), condition $\|W^{-\frac{1}{2}}p\| = 1$ can be written as

$$\sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = 1. \tag{8.2.4}$$

Now recall equation (8.1.6)

$$\prod_{k=1}^{\infty} \left(\frac{z-\mu_k^2}{z-\lambda_k^2} \right) = 1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2 - z}, \tag{8.2.5}$$

we take real $z < 0$, $z \to 0-$, then by (8.2.5) and monotone convergence theorem, we have

$$1 - \sum_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = \prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_k^2}. \tag{8.2.6}$$

Hence $\sum\limits_{k=1}^{\infty} \frac{a_k}{\lambda_k^2} = 1$ is equivalent to

$$\prod_{k=1}^{\infty} \frac{\mu_i^2}{\lambda_i^2} = 0, \tag{8.2.7}$$

which is also equivalent to:

$$\sum_{k=1}^{\infty}\left(\frac{\mu_k^2}{\lambda_k^2}-1\right)=-\infty, \tag{8.2.8}$$

gives us the condition for (9.1.2).

Now for the second condition $\|W^{-1}p\|=\infty$, using (8.2.3) again, it's equivalent to

$$\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k^4}=\infty. \tag{8.2.9}$$

We rewrite (8.1.6) as

$$\begin{aligned}\prod_{k=1}^{\infty}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)&=1-\sum_{k=1}^{\infty}a_k\frac{(\lambda_k^2-z)+z}{\lambda_k^2(\lambda_k^2-z)}\\&=1-\sum_{k\geq 1}\frac{a_k}{\lambda_k^2}-\sum_{k\geq 1}\frac{a_k z}{\lambda_k^2(\lambda_k^2-z)},\end{aligned}$$

assuming that we already have $\sum\limits_{k=1}^{\infty}\frac{a_k}{\lambda_k^2}=1$, then we have

$$-\frac{1}{z}\prod_{k=1}^{\infty}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)=\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k^2(\lambda_k^2-z)}. \tag{8.2.10}$$

Now denoting the function of z in (8.2.10) as $G(z)$. We apply $z=-\lambda_N^2$ to $G(z)$, and let $N\to\infty$, then RHS of (8.2.10) increases monotonically to $\sum\limits_{k=1}^{\infty}a_k/\lambda_k^4$. Therefore condition (8.2.9) is equivalent to

$$\lim_{N\to\infty}G(-\lambda_N^2)=\lim_{N\to\infty}\frac{1}{\lambda_N^2}\prod_{k=1}^{\infty}\left(\frac{\lambda_N^2+\mu_k^2}{\lambda_N^2+\lambda_k^2}\right)=\infty. \tag{8.2.11}$$

Denote $G_N(z)=\frac{1}{\lambda_N^2}\prod\limits_{k=1}^{N-1}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)$, we will prove the following lemma.

Lemma 8.2.2. *The following statements are equivalent:*

(i) $\lim_{N\to\infty} G(-\lambda_N^2) = \infty$;

(ii) $\lim_{N\to\infty} G_N(-\lambda_N^2) = \infty$;

(iii) $\lim_{N\to\infty} G_N(0) = \infty$.

Proof of Lemma 8.2.2. (i) $\iff$ (ii): To prove this equivalence, it is sufficient to show that

$$0 < C_1 \leq \prod_{k=N}^{\infty} \left(\frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \right) \leq C_2 < \infty \tag{8.2.12}$$

with constants C_1, C_2 independent of N. Note that for $k \geq N$, we trivially have

$$\frac{1}{2} \leq \frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} \leq 1. \tag{8.2.13}$$

Here the upper bound trivially implies the upper bound in (8.2.12) with $C_2 = 1$.

To get the lower bound in (8.2.12), we use the estimate (8.2.13) and the following inequality

$$\ln x \geq (\ln 2)(x - 1), \qquad \forall\, x \in [1/2, 1].$$

Thus

$$\sum_{k=N}^{\infty} \ln 2 \left(\frac{\lambda_N^2 + \mu_k^2}{\lambda_N^2 + \lambda_k^2} - 1 \right) = \sum_{k=N}^{\infty} - \ln 2 \frac{\lambda_k^2 - \mu_k^2}{\lambda_N^2 + \lambda_k^2} \geq (-\ln 2) \sum_{k=N}^{\infty} \frac{\lambda_k^2 - \mu_k^2}{\lambda_N^2} \geq -\ln 2,$$

we see that the lower bound in (8.2.12) holds with $C_1 = \frac{1}{2}$.

(ii) $\iff$ (iii): The function G_N is analytic and zero free in the disc D_N of radius $2\lambda_N^2$ centered at $-\lambda_N^2$, since we have λ_N^2 is smaller than μ_{N-1}^2 and so the function $\ln |G_N(z)|$ is harmonic in this disc.

Now we can find a sufficiently large N, such that $\ln |G_n(z)|$ is non-negative on the disk D_n when $n \geq N$. In fact, if we goes from (ii) to (iii), then we find a sufficiently large N, such that $|G_n(-\lambda_n^2)| > 1$ when $n \geq N$. Then applying the maximum principle to function $\ln|G_n(z)|$, then we have $|G_n(z)| > 1$ will be hold on the whole disk Clos D_n. Similarly, we can also assume $\ln|G_n(z)|$ is non-negative when going from (iii) to (ii).

Now applying Harnack inequality for function $\ln|G_N(z)|$ on the disk D_N (see Remark 8.2.3)

$$\frac{1}{3}\ln|G_N(-\lambda_N^2)| \leq \ln|G_N(0)| \leq 3\ln|G_N(-\lambda_N^2)|, \tag{8.2.14}$$

which proves the equivalence (ii) $\iff$ (iii). □

Now get back to the equation (8.2.11). By lemma 8.2.2, condition (8.2.9) is equivalent to $\lim_{N\to\infty} G_N(0) = \infty$, hence

$$\lim_{N\to\infty} \frac{1}{\lambda_1^2} \prod_{k=1}^{N-1} \frac{\mu_k^2}{\lambda_{k+1}^2} = \lim_{N\to\infty} \frac{1}{\lambda_N^2} \prod_{k=1}^{N-1} \frac{\mu_k^2}{\lambda_k^2} = \infty,$$

And condition $\prod_{k=1}^{\infty} \frac{\mu_k^2}{\lambda_{k+1}^2} = \infty$ is equivalent to

$$\sum_{k=1}^{\infty} \left(\frac{\mu_k^2}{\lambda_{k+1}^2} - 1 \right) = \infty \tag{8.2.15}$$

Thus (8.2.8) together with (8.2.15) finish our proof for Proposition 8.2.1. □

Remark 8.2.3. Inside the proof of Lemma 8.2.2, we use Harnack inequality (see [35]) for (8.2.14), which can be stated as below:

Theorem 8.2.4 (Harnack inequality). *Let f be a non-negative function defined on a closed ball $B(x_0, R)$. If f is continuous on the closed ball and harmonic on its interior, then for every point x with $|x - x_0| = r < R$, we have*

$$\frac{1-(r/R)}{[1+(r/R)]^{n-1}} f(x_0) \leq f(x) \leq \frac{1+(r/R)}{[1-(r/R)]^{n-1}} f(x_0)$$

8.2.2 Trivial Kernel Condition for W_1 and the Spectral Measure $\rho_1(s)$

With the constructed triple (W, W_1, p) in Theorem 8.0.1, we discuss the trivial kernel condition of W_1 in this subsection, and in addition, we give an expression for the coefficients of $\rho_1(s) = \sum_{k\geq 1} b_k \delta_{\mu_k^2}(s) + b_0\delta_0(s)$, which is defined as the scalar spectral measure of W_1 with respect to p:

$$\left\langle (W_1 - zI)^{-1}p, p \right\rangle = \int_{\mathbb{R}} \frac{d\rho_1(s)}{s-z}.$$

Proposition 8.2.5. *If the coefficients $\{a_n\}_{n\geq 1}$ given in Lemma 8.1.9 satisfies $\sum_{n\geq 1} \frac{a_n}{\lambda_n^2} < 1$, then $b_0 = 0$, and the coefficients $\{b_n\}_{n\geq 1}$ can be written as:*

$$b_n = (\lambda_n^2 - \mu_n^2) \prod_{k\neq n} \left(\frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right) \tag{8.2.16}$$

Proof. From (8.1.5) we have

$$1 + F_1(z) = \frac{F_1}{F} = \prod_{i=1}^{\infty} \left(\frac{z - \lambda_i^2}{z - \mu_i^2} \right).$$

If $\prod_{i=1}^{\infty} \left(\frac{\lambda_i^2}{\mu_i^2} \right) < \infty$ (thus by (8.2.6) we have $\sum_{n\geq 1} \frac{a_n}{\lambda_n^2} < 1$), then follow a similar procedure as the proof in Lemma 8.1.9, we write

$$1 + \sum_n \frac{b_n^N}{\mu_n^2 - z} = \prod_{k=1}^{N} \left(\frac{z - \lambda_k^2}{z - \mu_k^2} \right),$$

then we have

$$b_n^N = (\lambda_n^2 - \mu_n^2) \prod_{k=1, k\neq n}^{N} \left(\frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right) \text{ if } n \geq N,$$

and $b_n^N = 0$ if $n > N$.

Now take a upper bound B for $\prod_{i=1}^{\infty}\left(\frac{\lambda_i^2}{\mu_i^2}\right)$, then we have

$$\sum_n \frac{b_n^N}{\mu_n^2} < B - 1.$$

Since $b_n^N > 0$ and $b_n^N \searrow b_n$ as $N \to \infty$, where b_n has the formula as

$$b_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left(\frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2} \right),$$

we have $\sum_{n \geq 1} \frac{b_n}{\mu_n^2} < B - 1$ and $\sum_{n \geq 1} \frac{b_n^N}{\mu_n^2 - z}$ converges uniformly to $\sum_{n \geq 1} \frac{b_n}{\mu_n^2 - z}$ on any compact $K \subset \sigma$.

Thus in this case we can write $\rho_1(s) = \sum_n b_n \delta_{\mu_k^2}(z)$ where b_n has the form in (8.2.16).

□

Proposition 8.2.6. *For the triple* (W, W_1, p) *constructed in Theorem 8.0.1, we have* $\operatorname{Ker} W_1 \neq \{0\}$ *iff the following two equations hold:*

$$\sum_{n \geq 1} \frac{a_n}{\lambda_n^2} = 1, \qquad \sum_{n \geq 1} \frac{a_n}{\lambda_n^4} < \infty. \tag{8.2.17}$$

Hence by Proposition 8.2.1, (8.2.17) *is also equivalent to:*

$$\prod_{k=1}^{\infty} (\frac{\lambda_k^2}{\mu_k^2}) = \infty, \qquad \prod_{k=1}^{\infty} (\frac{\mu_k^2}{\lambda_{k+1}^2}) = \infty. \tag{8.2.18}$$

Proof. The proof is very much similar to the proof in Proposition 3.1.3.

We define two positive self-adjoint operators R, R_1 as: $R := W^{1/2}, R_1 = W_1^{1/2}$, and also a contraction $\mathcal{T} := R_1 R^{-1}$ implied by Douglas Lemma 3.1.1, and also a vector $q := R^{-1}p$. Then the two equations in (8.2.17) are equivalent to:

$$\|q\| = 1, \qquad q \in \operatorname{Ran} R$$

(i) Under the condition that $\|q\| < 1$, we have shown in Proposition 8.2.5, that $\operatorname{Ker} W_1 = \{0\}$;

(ii) Suppose that $\|q\| = 1$ and $q \notin \operatorname{Ran} R$. Since we have

$$W_1 = R_1^2 = RT^*TR = R(I - qq^*)R \tag{8.2.19}$$

Since $\operatorname{Ker}(I - qq^*) = \operatorname{Span}\{q\}$ and $q \notin \operatorname{Ran} R$, we can see that $\operatorname{Ker} R(I - qq^*)R = \{0\}$, thus $\operatorname{Ker} W_1 = \{0\}$.

(iii) Suppose that $\|q\| = 1$ and $q \in \operatorname{Ran} R$. Using (8.2.19) again, and apply $u := R^{-1}q$ to (8.2.19), we have $(I - qq^*)Ru = 0$, thus $W_1 u = 0$. □

Now in contrast with the case when $\sum_{k\geq 1} \frac{a_k}{\lambda_k^2} < 1$, we consider the case when $\sum_{k\geq 1} \frac{a_k}{\lambda_k^2} = 1$. We first prove the following lemma:

Lemma 8.2.7. *For all $z \notin \{\frac{1}{\mu_j^2}\}_{j\geq 1}$, we have the following equation*

$$\prod_{j=1}^{\infty} \frac{1 - z\lambda_j^2}{1 - z\mu_j^2} = 1 - b_0 z + \sum_{k\geq 1} \frac{b_k z}{\mu_k^2 z - 1}$$

is well-defined and holds for some constant $b_0 \in \mathbb{C}$. Here coefficients $\{b_k\}_{k\geq 1}$ has the expression in (8.2.16).

Proof of Lemma 8.2.7. The first step is similar to the proof of Proposition 8.1.9. We consider rewriting a finite product

$$\prod_{j=1}^{N} \frac{1 - z\lambda_j^2}{1 - z\mu_j^2} = 1 + \sum_{k=1}^{N} \frac{b_k^N}{\mu_k^2 - \frac{1}{z}} = 1 + \sum_{k=1}^{N} \frac{b_k^N z}{\mu_k^2 z - 1}, \tag{8.2.20}$$

then we have the coefficients b_k^N has the expression

$$b_k^N = (\lambda_k^2 - \mu_k^2) \prod_{n=1, n\neq k}^{N} (\frac{\mu_k^2 - \lambda_n^2}{\mu_k^2 - \mu_n^2}),$$

here $b_k^N > 0$ and $b_n^N \searrow b_n$ as $N \to \infty$.

Now we rewrite (8.2.20) and define function $H_N(z)$ as:

$$H_N(z) := \frac{1}{z}\big(\prod_{j=1}^{N} \frac{1 - z\lambda_j^2}{1 - z\mu_j^2} - 1\big) = \sum_{k=1}^{N} \frac{b_k^N}{\mu_k^2 z - 1}. \tag{8.2.21}$$

Let $z = 0$, and we take the expansion of infinite product at $z = 0$ on LHS, we have the sum

$$\sum_{k=1}^{N} b_k^N = \sum_{j=1}^{N} (\lambda_j^2 - \mu_j^2)$$

is bounded from above, thus let $N \to \infty$, we have $\sum_{k=1}^{\infty} b_k < \infty$.

Now again for the function $H_N(z)$ defined in (8.2.21), we write

$$H_N(z) = H_N(0) + \int_0^z H_N'(t)dt. \tag{8.2.22}$$

Here $H_N'(t)$ has the expression

$$H_N'(t) = \sum_{j=1}^{N} \frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2}.$$

Thus for a compact set K which doesn't intersect with any of $\{\frac{1}{\mu_j^2}\}_{j\geq 1}$ and any $t \in K$, we have

$$\big|\frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2}\big| \leq C(K, \sigma) b_j^N.$$

Since $\sum_{j\geq 1} b_j < \infty$, we have

$$\sum_{j=1}^{N} \frac{-b_j^N \mu_j^2}{(\mu_j^2 t - 1)^2} \to \sum_{j=1}^{\infty} \frac{-b_j \mu_j^2}{(\mu_j^2 t - 1)^2}$$

uniformly on K. Hence we take $N \to \infty$ in (8.2.22), we get

$$\frac{1}{z}\big(\prod_{j=1}^{\infty} \frac{1 - z\lambda_j^2}{1 - z\mu_j^2} - 1\big) = \lim_{N\to\infty} H_N(0) + \int_0^z \sum_{j=1}^{\infty} \frac{-b_j\mu_j^2}{(\mu_j^2 t - 1)^2}$$

Hence

$$\frac{1}{z}\big(\prod_{j=1}^{\infty} \frac{1 - z\lambda_j^2}{1 - z\mu_j^2} - 1\big) = \sum_{j=1}^{\infty} \frac{b_j}{\mu_k^2 z - 1} + C$$

for some constant $C \in \mathbb{C}$. And we finish the proof of lemma 8.2.7. $\square$

With Lemma 8.2.7, we replace z by $\frac{1}{z}$, we get that for $z \notin \{\mu_j^2\}_{j\geq 1}$, we have the following equation

$$\prod_{j=1}^{\infty} \frac{\lambda_j^2 - z}{\mu_j^2 - z} = 1 - b_0 \frac{1}{z} + \sum_{k\geq 1} \frac{b_k}{\mu_k^2 - z},$$

where $\{b_k\}_{j\geq 1}$ has the form as (8.2.16). The only remaining thing to do is to find the value of b_0.

Lemma 8.2.8. *With the assumptions that* $\sum_{k\geq 1} \frac{a_k}{\lambda_k^2} = 1$ *and* $\sum_{k\geq 1} \frac{a_k}{\lambda_k^4} < \infty$*, we have*

$$b_0 = \big(\sum_{k\geq 1} \frac{a_k}{\lambda_k^4}\big)^{-1}$$

Proof. If we denote the scalar spectral measure $\rho_1(s) = \sum_{k\geq 1} \delta_{\mu_k^2}(s) + b_0\delta_0(s)$, and substitute it into

$$1 + F_1(z) = \frac{F_1(z)}{F(z)} = \prod_{k=1}^{\infty} \big(\frac{z - \lambda_k^2}{z - \mu_k^2}\big) = 1 + \sum_{k\geq 1} \frac{b_k}{\mu_k^2 - z} - \frac{b_0}{z}.$$

Hence we have

$$b_0 = \lim_{z\to 0} \Big[- z \prod_{k=1}^{\infty} \big(\frac{z - \lambda_k^2}{z - \mu_k^2}\big)\Big]$$

On the other hand, from (8.2.10) we have

$$-\frac{1}{z}\prod_{k=1}^{\infty}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)=\sum_{k=1}^{\infty}\frac{a_k}{\lambda_k^2(\lambda_k^2-z)}. \tag{8.2.23}$$

Let $z \to 0$, then we get $\frac{1}{b_0}=\sum_{k\geq 1}\frac{a_k}{\lambda_k^4}$, hence we finish the proof of Lemma 8.2.8. □

Thus combining the results we get in Proposition 8.2.5, Lemma 8.2.7 and Lemma 8.2.8, we can give an expression for all coefficients of $\rho_1(s)$.

Proposition 8.2.9. *Denote the scalar spectral measure of W_1 with respect to p defined by (8.1.4) to be $\rho_1(s)$. Then $\rho_1(s)=\sum_{k\geq 1} b_k\delta_{\mu_k^2}(s)+b_0\delta_0(s)$ where:*

$$b_n=(\lambda_n^2-\mu_n^2)\prod_{k\neq n}\left(\frac{\mu_n^2-\lambda_k^2}{\mu_n^2-\mu_k^2}\right), \qquad \textit{when } n\geq 1$$

and

$$b_0=\begin{cases} 0 & \textit{if } \sum_n \frac{a_n}{\lambda_n^2}<1, \ \textit{or } \sum_n \frac{a_n}{\lambda_n^2}=1 \ \textit{and } \sum_n \frac{a_n}{\lambda_n^4}=\infty. \\ \left(\sum_n \frac{a_n}{\lambda_n^4}\right)^{-1} & \textit{if } \sum_n \frac{a_n}{\lambda_n^2}=1 \ \textit{and } \sum_n \frac{a_n}{\lambda_n^4}<\infty. \end{cases} \tag{8.2.24}$$

We can show that the coefficients $\{a_k\}_{k\geq 1}$ in (8.1.9), and the coefficients $\{b_k\}_{k\geq 0}$ in Proposition 8.2.9 satisfy the following identities.

Proposition 8.2.10 (Modified from Corollary 1, [36])**.** *For all $m, p\geq 1$, we have*

$$\sum_{j\geq 1}\frac{a_j}{\lambda_j^2-\mu_m^2}=1, \tag{8.2.25}$$

$$\sum_{j\geq 1}\frac{b_j}{\lambda_m^2-\mu_j^2}=1-\frac{b_0}{\lambda_m^2}, \tag{8.2.26}$$

$$\sum_{j\geq 1}\frac{a_j}{(\lambda_j^2-\mu_m^2)(\lambda_j^2-\mu_p^2)}=\frac{\delta_{mp}}{b_m}, \tag{8.2.27}$$

$$\sum_{j\geq 1}\frac{b_j}{(\mu_j^2-\lambda_m^2)(\mu_j^2-\lambda_p^2)}=\frac{\delta_{mp}}{a_m}-\frac{b_0}{\lambda_m^2\lambda_p^2}. \tag{8.2.28}$$

Here $\delta_{mp} = 1$ if $m = p$, and equals 0 otherwise.

Proof of Proposition 8.2.10. Recall that in previous calculation, the coefficients $\{a_k\}_{k\geq 0}$, $\{b_k\}_{k\geq 0}$ is given by

$$1 - \sum_{k\geq 1} \frac{a_k}{\lambda_k^2 - z} = \prod_{k\geq 1} (\frac{z - \mu_k^2}{z - \lambda_k^2}) \tag{8.2.29}$$

$$1 + \sum_{k\geq 1} \frac{b_k}{\mu_k^2 - z} - \frac{b_0}{z} = \prod_{k\geq 1} (\frac{z - \lambda_k^2}{z - \mu_k^2}) \tag{8.2.30}$$

Then (8.2.25) can be achieved by setting $z = \mu_m^2$ in (8.2.29), and (8.2.26) can be achieved by setting $z = \lambda_m^2$ in (8.2.30).

For the case of $m \neq p$ in (8.2.27), we have

$$\begin{aligned}
&\sum_{j\geq 1} \frac{a_j}{(\lambda_j^2 - \mu_m^2)(\lambda_j^2 - \mu_p^2)} \\
=&\sum_{j\geq 1} a_j \Big[\frac{1}{\lambda_j^2 - \mu_m^2} - \frac{1}{\lambda_j^2 - \mu_p^2}\Big] \frac{1}{\mu_m^2 - \mu_p^2} \\
=&\frac{1}{\mu_m^2 - \mu_p^2} \Big[\sum_{j\geq 1} \frac{a_j}{\lambda_j^2 - \mu_m^2} - \sum_{j\geq 1} \frac{a_j}{\lambda_j^2 - \mu_p^2}\Big] = 0.
\end{aligned}$$

Here the last identity follows from (8.2.25).

For the case of $m = p$ in (8.2.27), we take the differentiation on both sides of (8.2.29), and then we have

$$\sum_{k\geq 1} -\frac{a_k}{(\lambda_k^2 - z)^2} = \sum_{k\geq 1} \frac{\mu_k^2 - \lambda_k^2}{(z - \lambda_k^2)^2} \prod_{n\neq k} (\frac{z - \mu_n^2}{z - \lambda_n^2}).$$

By setting $z = \mu_m^2$, we get

$$\sum_{k\geq 1} \frac{a_k}{(\lambda_k^2 - \mu_m^2)^2} = \frac{1}{\lambda_m^2 - \mu_m^2} \prod_{n\neq m} (\frac{\mu_m^2 - \mu_n^2}{\mu_m^2 - \lambda_n^2}) = \frac{1}{b_m}.$$

For the case of $m \neq p$ in (8.2.28), we have

$$\begin{aligned}
&\sum_{j\geq 1}\frac{b_j}{(\mu_j^2-\lambda_m^2)(\mu_j^2-\lambda_p^2)}\\
=&\sum_{j\geq 1}b_j\Big[\frac{1}{\mu_j^2-\lambda_m^2}-\frac{1}{\mu_j^2-\lambda_p^2}\Big]\frac{1}{\lambda_m^2-\lambda_p^2}\\
=&\frac{1}{\lambda_m^2-\lambda_p^2}\Big[\sum_{j\geq 1}\frac{b_j}{\mu_j^2-\lambda_m^2}-\sum_{j\geq 1}\frac{b_j}{\mu_j^2-\lambda_p^2}\Big]\\
=&\frac{1}{\lambda_m^2-\lambda_p^2}\Big[(\frac{b_0}{\lambda_m^2}-1)-(\frac{b_0}{\lambda_p^2}-1)\Big]=-\frac{b_0}{\lambda_m^2\lambda_p^2}
\end{aligned}$$

Here the third identity follows from (8.2.26).

For the case of $m = p$ in (8.2.28), we take the differentiation on both sides of (8.2.26), and then we get

$$\sum_{k\geq 1}\frac{b_k}{(\mu_k^2-z)^2}+\frac{b_0}{z^2}=\sum_{k\geq 1}\frac{\lambda_k^2-\mu_k^2}{(z-\mu_k^2)^2}\prod_{n\neq k}(\frac{z-\lambda_n^2}{z-\mu_n^2}).$$

By taking $z = \lambda_m^2$, we get

$$\sum_{k\geq 1}\frac{b_k}{(\mu_k^2-\lambda_m^2)^2}=\frac{1}{\lambda_m^2-\mu_m^2}\prod_{n\neq m}(\frac{\lambda_m^2-\lambda_n^2}{\lambda_m^2-\mu_n^2})-\frac{b_0}{\lambda_m^4}=\frac{1}{a_m}-\frac{b_0}{\lambda_m^4}$$

□

8.3 Abstract Borg's Theorem: Finite Rank Case

Similar to the description in Theorem 8.0.1, we can also write the finite rank version of Abstract Borg's Theorem, which can be stated as below:

Theorem 8.3.1 (Abstract Borg's Thoerem, Finite Rank Version)**.** *Given two sequences $\{\lambda_k^2\}_{k=1}^N$, $\{\mu_k^2\}_{k=1}^N$ satisfy the following intertwining relation*

$$\lambda_1^2>\mu_1^2>...>\lambda_N^2>\mu_N^2\geq 0,$$

then there exists an unique (up to unitary equivalence) triple (W, W_1, p), *such that* W, W_1 *are positive self-adjoint compact operators with trivial kernel, and also satisfy* $W_1 = W - pp^*$. *In addition, the eigenvalues of* W, W_1 *are simple, and coincide with* $\{\lambda_k^2\}_{k=1}^N$, $\{\mu_k^2\}_{k=1}^N$ *respectively.*

Remark 8.3.2. If $\mu_N^2 = 0$, then we have the constructed operator W_1 has a kernel of dimension 1; if $\mu_N^2 > 0$, then we have the constructed operator W_1 also has trivial kernel.

Remark 8.3.3. The proof for this Abstract Borg's theorem in finite-rank version is trivial, because the infinite product $\prod\limits_{k=1}^{\infty}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)$ now becomes a finite product $\prod\limits_{k=1}^{N}\left(\frac{z-\mu_k^2}{z-\lambda_k^2}\right)$.

Under the setting in Theorem 8.3.1, if we write down the scalar spectral measure of W, W_1 with respect to p:

$$F(z) = \left\langle (W - zI)^{-1}p, p \right\rangle = \int_{\sigma(W)} \frac{d\rho(s)}{s - z},$$
$$F_1(z) = \left\langle (W_1 - zI)^{-1}p, p \right\rangle = \int_{\sigma(W_1)} \frac{d\rho_1(s)}{s - z},$$

we still have $1 - F = \frac{F}{F_1}$, and we have the coefficients of $\rho(s) = \sum\limits_{n=1}^{N} a_n \delta_{\lambda_n^2}(s)$, $\rho_1(s) = \sum\limits_{n=1}^{N} b_n \delta_{\mu_n^2}(s)$ has the following representation:

$$a_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left(\frac{\lambda_n^2 - \mu_k^2}{\lambda_n^2 - \lambda_k^2}\right),$$
$$b_n = (\lambda_n^2 - \mu_n^2) \prod_{k \neq n} \left(\frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2}\right).$$

Under this case, we have

$$\|W^{-1/2}p\| = 1 \qquad \Leftrightarrow \qquad \sum_{k=1}^{N} \frac{a_k}{\lambda_k^2} = 1 \qquad \Leftrightarrow \qquad \mu_N = 0;$$

as for the second identity $\|W^{-1}p\| = \infty$ in the trivial kernel condition, it can't be true because $\sum_{k=1}^{N} \frac{a_k}{\lambda_k^4}$ is always finite.

8.4 Proof of Generalized Abstract Borg's Theorem

Proof. First by using the result of Abstract Borg Theorem 8.0.1, we can find a triple (W, W_1, p) on a Hilbert space $\widetilde{\mathcal{H}}$, such that

(i) W, W_1 are self-adjoint compact, positive operators. $\operatorname{Ker} W = \{0\}$;

(ii) $W - W_1 = pp^*$;

(iii) p is cyclic with respect to W on $\widetilde{\mathcal{H}}$;

(iv) $\{\lambda_k^2\}_{k=1}^{\infty}, \{\mu_k^2\}_{k=1}^{\infty}$ are the simple eigevalues of W, W_1 respectively.

Now define operators $\widetilde{R}, \widetilde{R}_1$ as $\widetilde{R} = W^{1/2}, \widetilde{R}_1 = W_1^{1/2}$, we have $\widetilde{R}, \widetilde{R}_1$ have simple eigenvalues $\{\lambda_k\}_{k=1}^{\infty}, \{\mu_k\}_{k=1}^{\infty}$ respectively.

Now we define an extended Hilbert space

$$\mathcal{H} = \widetilde{\mathcal{H}} \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 \oplus ...\mathcal{H}_{2k-1} \oplus \mathcal{H}_{2k}...$$

Here $\mathcal{H}_{2k-1}$ is a Hilbert space with dimension $m(\lambda_k) - 1$, $\mathcal{H}_{2k}$ is a Hilbert space with dimension $m(\mu_k)$.

And we also extend $\widetilde{R}, \widetilde{R}_1$ to two operators R, R_1 defined on $\mathcal{H}$. We let

(i) For all $k \in \mathbb{N}$, We take $R = R_1 = \lambda_k I$ on $\mathcal{H}_{2k-1}$;

(ii) For all $k \in \mathbb{N}$, we take $R = R_1 = \mu_k I$ on $\mathcal{H}_{2k}$.

From (i) and (ii), we can see

$$\dim \operatorname{Ker}(R-\lambda_k I) = \dim \mathcal{H}_{2k-1}+1 = m(\lambda_k);$$
$$\dim \operatorname{Ker}(R_1-\lambda_k I) = \dim \mathcal{H}_{2k-1} = m(\lambda_k)-1;$$
$$\dim \operatorname{Ker}(R-\mu_k I) = \dim \mathcal{H}_{2k} = m(\mu_k);$$
$$\dim \operatorname{Ker}(R_1-\mu_k I) = \dim \mathcal{H}_{2k}+1 = m(\mu_k)+1;$$

Thus (R, R_1, p) is a satisfied triple on $\mathcal{H}$.

Now we move on to the uniqueness part. Assume that (W, W_1, p), (W', W_1', p') are two satisfied tuples defined on spaces $\mathcal{H}, \mathcal{H}'$ respectively. Then we denote

$$\mathcal{H}_0 := \overline{\operatorname{Span}}\big\{W^n p \big| n \geq 0\big\};$$
$$\mathcal{E}_{\lambda_k} := \operatorname{Ker}(W-\lambda^2 I), \qquad \mathcal{E}_{\mu_k} := \operatorname{Ker}(W-\mu_k^2 I);$$
$$\mathcal{E}^1_{\lambda_k} := \operatorname{Ker}(W_1-\lambda_k^2 I), \qquad \mathcal{E}^1_{\mu_k} := \operatorname{Ker}(W_1-\mu_k^2 I);$$
$$\widetilde{W} = W\big|_{\mathcal{H}_0}, \qquad \widetilde{W_1} = W_1\big|_{\mathcal{H}_0}$$

And we define $\mathcal{H}_0', \mathfrak{E}_{\lambda_k}, \mathfrak{E}_{\mu_k}, \mathfrak{E}^1_{\lambda_k}, \mathfrak{E}^1_{\mu_k}, \widetilde{W'}, \widetilde{W_1'}$ similarly.

Then followed by Proposition 6.2.10, we have $(\widetilde{W}, \widetilde{W_1}, p)$ and $(\widetilde{W'}, \widetilde{W_1'}, p')$ are two tuples that have simple eigenvalues $\{\lambda_k^2\}_{k=1}^\infty$, $\{\mu_k^2\}_{k=1}^\infty$ respectively, which also satisfy all requirements given in Theorem 8.0.1, Hence applying Theorem 8.0.1, we have those two tuples are unitary equivalent, i.e.

$$\widetilde{W'} = \widetilde{V}\widetilde{W}\widetilde{V}^*, \qquad \widetilde{W_1} = \widetilde{V}\widetilde{W_1}\widetilde{V}^*, \qquad p' = \widetilde{V}p$$

for an unitary $\widetilde{V} : \mathcal{H}_0 \to \widetilde{\mathcal{H}}_0$.

Now using Proposition 6.2.10 again, we have

$$\mathcal{H} \ominus \mathcal{H}_0 = \left(\bigoplus_{k=1}^{\infty} \mathcal{E}^1_{\lambda_k} \right) \bigoplus \left(\bigoplus_{k=1}^{\infty} \mathcal{E}_{\mu_k} \right)$$
$$\mathcal{H}' \ominus \mathcal{H}'_0 = \left(\bigoplus_{k=1}^{\infty} \mathfrak{E}^1_{\lambda_k} \right) \bigoplus \left(\bigoplus_{k=1}^{\infty} \mathfrak{E}_{\mu_k} \right).$$

In addition we have $W = W_1$ on $\mathcal{H} \ominus \mathcal{H}_0$, and $W' = W'_1$ on $\mathcal{H}' \ominus \mathcal{H}'_0$; i.e. we have

$$W = W_1 = \lambda_k I \text{ on } \mathcal{E}^1_{\lambda_k} \text{ for all } k, \qquad W = W_1 = \mu_k I \text{ on } \mathcal{E}_{\mu_k} \text{ for all } k$$
$$W' = W'_1 = \lambda_k I \text{ on } \mathfrak{E}^1_{\lambda_k} \text{ for all } k, \qquad W' = W'_1 = \mu_k I \text{ on } \mathfrak{E}_{\mu_k} \text{ for all } k$$

Since $\dim \mathcal{E}^1_{\lambda_k} = \dim \mathfrak{E}^1_{\lambda_k} = m(\lambda_k) - 1$, $\dim \mathcal{E}_{\mu_k} = \dim \mathfrak{E}_{\mu_k} = m(\mu_k)$, we can define ϕ^1_k to be an arbitrary unitary operator mapping $\mathcal{E}^1_{\lambda_k}$ to $\mathfrak{E}^1_{\lambda_k}$; and ϕ_k to be an arbitrary unitary operator mapping $\mathcal{E}_{\mu_k}$ to $\mathfrak{E}_{\mu_k}$. Then we can extend $\widetilde{V}$ to an unitary $V : \mathcal{H} \to \mathcal{H}'$ such that

$$V = \widetilde{V} \text{ on } \mathcal{H}_0,$$
$$V = \phi^1_k \text{ on } \mathcal{E}^1_{\lambda_k} \text{ for all } k, \qquad V = \phi_k \text{ on } \mathcal{E}_{\mu_k} \text{ for all } k.$$

Then we have

$$W = VWV^*, \qquad W_1 = VW_1V^*, \qquad p' = Vp.$$

Thus the uniqueness part is done. □

Chapter 9

RESULT FOR COMPACT HANKEL OPERATORS WITH SIMPLE EIGENVALUES

With the Abstract Borg's Theorem 8.0.1, we are ready to solve the question that what type of spectral data can uniquely determine a Hankel operator Γ when Γ is compact with simple singular values.

In section 9.1, we will show that under the self-adjoint case, a compact Hankel operator with simple singular values will be uniquely determined by two sequences of real numbers $\{\lambda_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$ satisfying an intertwining relation:

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0 \tag{9.0.1}$$

In section 9.2, we will show that under the non-self-adjoint case, a compact Hankel operator with simple singular values will be uniquely determined by two sequences of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$, whose modulus part satisfy an intertwining relation:

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0 \tag{9.0.2}$$

In section 9.3, we generate the result for finite rank Hankel operators with simple singular values, under both self-adjoint case and non self-adjoint case.

9.1 Result for Self-adjoint Compact Hankel Operators with Simple Eigenvalues

Theorem 9.1.1. *Given two sequences of real numbers* $\{\lambda_n\}_{n=1}^{\infty}$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *satisfying intertwining relations*

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0, \tag{9.1.1}$$

there exists a unique self-adjoint compact Hankel operator Γ *such that non-zero eigenvalues of* Γ *and* ΓS *are simple, and coincide with* $\{\lambda_n\}_{n=1}^{\infty}$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *respectively.*

Moreover, $\operatorname{Ker}\Gamma = \{0\}$ *if and only if both of the following identities hold:*

$$\sum_{j=1}^{\infty}\left(1-\frac{\mu_j^2}{\lambda_j^2}\right)=\infty, \tag{9.1.2}$$

$$\sum_{j=1}^{\infty}\left(\frac{\mu_j^2}{\lambda_{j+1}^2}-1\right)=\infty. \tag{9.1.3}$$

Proof. By Theorem 8.0.1, we can find a triple (W, W_1, p) on a Hilbert space $\mathcal{H}$ with the relation: $W_1 = W - pp^*$, here W and W_1 are positive, compact self-adjoint operators with simple eigenvalues $\{\lambda_k^2\}_{k=1}^{\infty}$, $\{\mu_k^2\}_{k=1}^{\infty}$ respectively. In addition, the triple is unique determined up to unitary equivalence.

Denote that the eigenvector of W corresponding to λ_k^2 is u_k, and the eigenvector of W_1 corresponding to μ_k^2 is v_k;

$$Wu_k = \lambda_k^2 u_k, \qquad W_1 v_k = \mu_k^2 v_k.$$

Since

$$\mathcal{H} = \bigoplus_{k=1}^{\infty}\overline{\operatorname{Span}}\{u_k\} = \operatorname{Ker} W_1 \oplus \Big(\bigoplus_{k=1}^{\infty}\overline{\operatorname{Span}}\{v_k\}\Big),$$

Then we define self-adjoint $\mathcal{R}$ and $\mathcal{R}_1$ on $\mathcal{H}$ by

$$\mathcal{R}u_k := \lambda_k u_k;$$
$$\mathcal{R}_1 v_k := \mu_k v_k, \qquad \mathcal{R}_1|_{\operatorname{Ker} W_1} = 0.$$

Here we have $\operatorname{Ker}\mathcal{R} = \{0\}$ and $\operatorname{Ker}\mathcal{R}_1 = \operatorname{Ker} W_1$ with dimension at most 1, and we can express $\mathcal{R}$ and $\mathcal{R}_1$ by

$$\mathcal{R}x = \sum_{k=1}^{\infty} \lambda_k P_{u_k} x, \qquad \mathcal{R}_1 x = \sum_{k=1}^{\infty} \mu_k P_{v_k} x,$$

Hence we get the unique square roots of W and W_1 according to the given signs of $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$.

Now we have the triple $(\mathcal{R}, \mathcal{R}_1, p)$ satisfies the relation (3.1.5), and the contraction $\mathcal{T}$ defined by $\mathcal{R}_1 = \mathcal{T}\mathcal{R}$ is asymptotically stable. Hence by Proposition 3.1.3, there exists an unique self-adjoint Hankel operator Γ such that the triple $\Gamma|_{(\operatorname{Ker}\Gamma)^\perp}$, $\Gamma_1|_{(\operatorname{Ker}\Gamma)^\perp}$, $u := \Gamma e_0 = \Gamma P_{(\operatorname{Ker}\Gamma)^\perp} e_0$ is unitarily equivalent to the triple $(\mathcal{R}, \mathcal{R}_1, p)$. This means that the non-zero eigenvalues of Γ and Γ_1 are simple and coincide with $\{\lambda_k\}_{k=1}^{\infty}$ and $\{\mu_k\}_{k=1}^{\infty}$ respectively.

As for the uniqueness, assume that $(\Gamma, \Gamma_1, u := \Gamma e_0)$ and $(\Gamma', \Gamma_1', u' := \Gamma' e_0)$ are two different set of triples satisfied the conditions. Then by Proposition 3.1.3, there are two different triples $(\mathcal{R}, \mathcal{R}_1, p), (\mathcal{R}', \mathcal{R}_1', p)$ defined on the same Hilbert space such that (Γ, Γ_1, u) unitary equivalent to $(\mathcal{R}, \mathcal{R}_1, p)$, and (Γ', Γ_1', u') unitary equivalent to $(\mathcal{R}', \mathcal{R}_1', p)$, where $(\mathcal{R}, \mathcal{R}_1)$ and $(\mathcal{R}', \mathcal{R}_1')$ share the same spectral characteristics.

From the equation

$$\mathcal{R}^2 - \mathcal{R}_1^2 = \mathcal{R}'^2 - \mathcal{R}_1'^2 = pp^*,$$

and the uniqueness in the abstract Borg's theorem 8.0.1, we have $\mathcal{R}^2 = \mathcal{R}'^2 = W, \mathcal{R}_1^2 = \mathcal{R}_1'^2 = W_1$. If $\mathcal{R} \neq \mathcal{R}'$, then there exists a k, such that $\operatorname{Ker}(R - \lambda I) \neq \operatorname{Ker}(R' - \lambda I)$, thus $\dim \operatorname{Ker}(W - \lambda^2 I) \geq 2$, which gives a contradiction.

Similarly we have $R_1 = R_1'$, which indicates the uniqueness of such Hankel operator.

As for the trivial kernel condition, we have mentioned in Proposition 3.1.3 that $\operatorname{Ker}\Gamma = \{0\}$ if and only if $\|q\| = 1$ and $q \notin \operatorname{Ran}\{\mathcal{R}\}$, where $q := \mathcal{R}^{-1}p$.

Here the first condition $\|q\| = 1$ can be rewritten as $\|W^{-1/2}p\| = 1$, and we have shown in Theorem 8.0.1 that this identity is equivalent to

$$\sum_{k=1}^{\infty}\left(\frac{\mu_k^2}{\lambda_k^2} - 1\right) = -\infty \qquad \Leftrightarrow \qquad \prod_{k=1}^{\infty}\frac{\mu_k^2}{\lambda_k^2} = 0$$

And the second condition $q \notin \operatorname{Ran}\mathcal{R}$ can be rewritten as $\|W^{-1}p\| = \infty$, again implied by Theorem 8.0.1 that this identity is equivalent to

$$\sum_{k=1}^{\infty}\left(\frac{\mu_k^2}{\lambda_{k+1}^2} - 1\right) = \infty \qquad \Leftrightarrow \qquad \prod_{k=1}^{\infty}\frac{\mu_k^2}{\lambda_{k+1}^2} = \infty.$$

□

Remark 9.1.2. With the discussion in subsection 8.2.2, we can also get an equivalent condition for $\operatorname{Ker}\Gamma_1/\operatorname{Ker}\Gamma = \{0\}$ for the constructed Hankel operator, which is:

$$\prod_{k=1}^{\infty}\frac{\mu_k^2}{\lambda_k^2} = 0, \qquad \prod_{k=1}^{\infty}\frac{\mu_k^2}{\lambda_{k+1}^2} < \infty$$

9.2 Result for Compact Hankel Operators with Simple Singular Values as Complex Symmetric Operators

Theorem 9.2.1. *Given complex sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$, where $\lambda_k = s_k e^{i\theta_k}$, $\mu_k = t_k e^{i\theta_k'}$, and the modulus part satisfies the following intertwining relation*

$$s_1 > t_1 > s_2 > t_2 > \ldots \to 0, \tag{9.2.1}$$

then there exists a unique compact Hankel operator Γ, *such that we can find a conjugation* $\mathfrak{J}_u$ *commutes with* $|\Gamma|$ *and preserves* $u := \Gamma^* e_0$ *(implied by Lemma 4.0.1) with* $\widetilde{\phi}, \widetilde{\phi}_1$ *defined in* (5.1.5), *satisfying that non-zero eigenvalues of* $|\widetilde{\Gamma}|\widetilde{\phi}$, $|\widetilde{\Gamma}_1|\widetilde{\phi}_1$ *are simple, and coincide with* $\{\lambda_n\}_{n=1}^{\infty}$ *and* $\{\mu_n\}_{n=1}^{\infty}$ *respectively.*

Moreover, $\operatorname{Ker}\Gamma = \{0\}$ *if and only if both of the following identities hold:*

$$\sum_{j=1}^{\infty}\left(1 - \frac{t_j^2}{s_j^2}\right) = \infty,$$
$$\sum_{j=1}^{\infty}\left(\frac{s_j^2}{t_{j+1}^2} - 1\right) = \infty.$$

Proof. By Theorem 8.0.1, we can find a triple (W, W_1, p) on a Hilbert space $\mathcal{H}$ (unique up to equivalence) satisfying

(i) $W_1 = W - pp^*$, where W, W_1 are positive, self-adjoint operators;

(ii) W has trivial kernels, p is a cyclic vector for W;

(iii) W, W_1 have simple nonzero eigenvalues $\{s_k^2\}_{k=1}^{\infty}$, $\{t_k^2\}_{k=1}^{\infty}$ respectively.

Now denote the eigenvectors of W corresponding to s_k^2 is u_k, and eigenvectors of W_1 corresponding to t_k^2 is v_k. Since

$$\mathcal{H} = \bigoplus_{k=1}^{\infty}\overline{\operatorname{Span}}\{u_k\} = \operatorname{Ker} W_1 \oplus \big(\bigoplus_{k=1}^{\infty}\overline{\operatorname{Span}}\{v_k\}\big),$$

We define positive self-adjoint operators R, R_1 on $\mathcal{H}$ as

$$Rx := \sum_{k=1}^{\infty} s_k P_{u_k} x, \qquad R_1 x := \sum_{k=1}^{\infty} t_k P_{v_k} x \tag{9.2.2}$$

Now we also define a unitary φ and a partial isometry φ_1 as

$$\varphi x := \sum_{k=1}^{\infty} e^{i\theta_k} P_{u_k} x, \qquad \varphi_1 x := \sum_{k=1}^{\infty} e^{i\theta_k'} P_{v_k} x \tag{9.2.3}$$

Easy to see that φ, φ_1 commutes with R, R_1 respectively, and share the same corresponding one-dimensional eigenspaces. Here $R\varphi, R_1\varphi_1$ are compact operators with simple eigenvalues $\{\lambda_n\}_{n=1}^{\infty}$, $\{\mu_n\}_{n=1}^{\infty}$ respectively.

Now we define a conjugation $\mathfrak{J}_p$ on $\mathcal{H}$. Since $\mathcal{H} = \overline{\text{Span}}\{R^n p | n \geq 0\}$, we define $\mathfrak{J}_p$ by

$$\mathfrak{J}_p R^n p = R^n p \qquad \text{for all } n.$$

This is the same as defining $\mathfrak{J}_p$ by

$$\mathfrak{J}_p p_k = p_k \qquad \text{for all } k,$$

where

$$p_k = P\big|_{\text{Ker}(W-\lambda_k^2 I)} p.$$

In addition, we can check that φ, φ_1 are $\mathfrak{J}_p$-symmetric. In fact, for all k we have

$$\mathfrak{J}_p \varphi p_k = \mathfrak{J}_p e^{i\theta_k} p_k = e^{-i\theta_k} \mathfrak{J}_p p_k = \varphi^* \mathfrak{J}_p p_k.$$

Thus $\mathfrak{J}_p \varphi = \varphi^* \mathfrak{J}_p$ on $\mathcal{H}$, and a similar result holds for φ_1.

Now all the assumptions in Proposition 5.2.4 are satisfied, and we apply the tuple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ to Proposition 5.2.4. Thus there exists a unique Hankel Γ, such that we can find a conjugation $\mathfrak{J}_u$ commuting with $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ and preserves $u := \Gamma^* e_0$, satisfying that the tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1, u)$ and $(R, R_1, \mathfrak{J}_p, \varphi, \varphi_1, p)$ are unitary equivalent. This implies that $|\widetilde{\Gamma}|\widetilde{\phi}, |\widetilde{\Gamma}_1|\widetilde{\phi}_1$ are compact, and has simple non-zero eigenvalues as $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ respectively. And we finish the proof of existence part.

Now we move on to uniqueness. Suppose that there are two different Hankel Γ, Γ' that satisfies all requirements in Proposition 9.2.1. We will show that Γ, Γ' have the same Hankel coefficients $\{\gamma_k\}_{k=1}^{\infty}$, hence they are the same.

Denote their corresponding tuples to be $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1)$ and $(|\widetilde{\Gamma}'|, |\widetilde{\Gamma}'_1|, u', \widetilde{\phi}', \widetilde{\phi}'_1)$. We have

$$|\widetilde{\Gamma}|^2 - |\widetilde{\Gamma}_1|^2 = uu^*, \qquad |\widetilde{\Gamma}'|^2 - |\widetilde{\Gamma}'_1|^2 = u'(u')^*.$$

Here $|\widetilde{\Gamma}|, |\widetilde{\Gamma}'|$ have simple eigenvalues $\{s_k\}_{k=1}^{\infty}$, and $|\widetilde{\Gamma}_1|, |\widetilde{\Gamma}'_1|$ have simple eigenvalues $\{t_k\}_{k=1}^{\infty}$. Then by the uniqueness in Theorem 8.0.1, those two triples $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u)$ and $(|\widetilde{\Gamma}'|, |\widetilde{\Gamma}'_1|, u')$ are unitary equivalent. That is, there exists a unitary $\widetilde{\mathcal{V}} : \ell^2 \to \ell^2$ such that

$$|\widetilde{\Gamma}| = \widetilde{\mathcal{V}}|\widetilde{\Gamma}'|\widetilde{\mathcal{V}}^*, \qquad |\widetilde{\Gamma}_1| = \widetilde{\mathcal{V}}|\widetilde{\Gamma}'_1|\widetilde{\mathcal{V}}^*, \qquad u = \widetilde{\mathcal{V}}u' \tag{9.2.4}$$

We will show that

$$\widetilde{\phi} = \widetilde{\mathcal{V}}\widetilde{\phi}'\mathcal{V}^*, \qquad \widetilde{\phi}_1 = \widetilde{\mathcal{V}}\widetilde{\phi}'_1\widetilde{\mathcal{V}}^*. \tag{9.2.5}$$

In fact, if we denote the eigenvectors of $|\widetilde{\Gamma}|$ corresponding to $\{s_k\}_{k=1}^{\infty}$ as $\{u_k\}_{k=1}^{\infty}$, i.e.

$$|\widetilde{\Gamma}|u_k = s_k u_k \qquad \text{holds for all } k.$$

Then since $|\widetilde{\Gamma}|$ commutes with $\widetilde{\phi}$, we have $\{u_k\}_{k=1}^{\infty}$ are also eigenvectors for $\widetilde{\phi}$:

$$\widetilde{\phi}u_k = e^{i\theta_k}u_k \qquad \text{holds for all } k.$$

Now from $|\widetilde{\Gamma}| = \widetilde{\mathcal{V}}|\widetilde{\Gamma}'|\widetilde{\mathcal{V}}^*$, we have $\widetilde{\mathcal{V}}^*u_k$ is an eigenvector for $|\widetilde{\Gamma}'|$. In fact,

$$|\widetilde{\Gamma}'|\widetilde{\mathcal{V}}^*u_k = \widetilde{\mathcal{V}}^*|\widetilde{\Gamma}|u_k = s_k\widetilde{\mathcal{V}}^*u_k$$

Since $|\widetilde{\Gamma}'|$ commutes with $\widetilde{\phi}'$. we have $\{\widetilde{\mathcal{V}}^* u_k\}_{k=1}^{\infty}$ are eigenvectors for $\widetilde{\phi}'$:

$$\widetilde{\phi}'\widetilde{\mathcal{V}}^* u_k = e^{i\theta_k}\mathcal{V}^* u_k \qquad \text{holds for all } k.$$

Thus for all k we have

$$\widetilde{\mathcal{V}}\widetilde{\phi}'\widetilde{\mathcal{V}}^* u_k = e^{i\theta_k}\widetilde{\mathcal{V}}\widetilde{\mathcal{V}}^* u_k = e^{i\theta_k} u_k = \widetilde{\phi} u_k.$$

This proves that $\widetilde{\mathcal{V}}\widetilde{\phi}'\widetilde{\mathcal{V}}^* = \widetilde{\phi}$. For a similar reason, we have $\widetilde{\mathcal{V}}\widetilde{\phi}_1'\widetilde{\mathcal{V}}^* = \widetilde{\phi}_1$. Thus tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1)$ and $(|\widetilde{\Gamma}'|, |\widetilde{\Gamma}_1'|, u', \widetilde{\phi}', \widetilde{\phi}_1')$ are unitary equivalent.

Now we apply Proposition 5.1.4, we have the Hankel coefficients of Γ can be represented by:

$$\gamma_k = \langle (\mathfrak{S}^*)^k u, v \rangle, \tag{9.2.6}$$

where $\mathfrak{S}^* = |\widetilde{\Gamma}_1|\widetilde{\phi}_1^*\widetilde{\phi}|\widetilde{\Gamma}|^{-1}$, and $v = |\widetilde{\Gamma}|^{-1}\widetilde{\phi}^* u$ (see equation 5.1.8).

Since the two tuples for Γ, Γ_1, $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1)$ and $(|\widetilde{\Gamma}'|, |\widetilde{\Gamma}_1'|, u', \widetilde{\phi}', \widetilde{\phi}_1')$ are unitary equivalent. Thus by (9.2.6), Γ, Γ' have the same Hankel coefficients, thus they are the same.

As for the trivial kernel condition, we have $\operatorname{Ker}\Gamma = \{0\}$ is equivalent to $\|q\| = 1$ and $q \notin \operatorname{Ran} R$. Again, with the construction of $(W, W_1, p, \varphi, \varphi_1)$ in the existence part, $\|q\| = 1$ and $q \notin \operatorname{Ran} R$ are equivalent to:

$$\|q\| = 1 \Leftrightarrow \|\varphi^* R^{-1} p\| = 1 \Leftrightarrow \|W^{-1/2} p\| = 1$$
$$q \notin \operatorname{Ran} R \Leftrightarrow \|R^{-1} q\| = \infty \Leftrightarrow \|R^{-2} p\| = \|W^{-1} p\| = \infty$$

Thus by Theorem 8.0.1, $\operatorname{Ker}\Gamma = \{0\}$ if and only if

$$\sum_{j=1}^{\infty}\left(1 - \frac{t_j^2}{s_j^2}\right) = \infty, \qquad \sum_{j=1}^{\infty}\left(\frac{s_j^2}{t_{j+1}^2} - 1\right) = \infty.$$

□

Remark 9.2.2. Note that we use notation $(\mathcal{R}, \mathcal{R}_1)$ in section 8.1, meaning that $\mathcal{R}, \mathcal{R}_1$ are self-adjoint. While in section 8.2 we use notation (R, R_1) implying that R, R_1 are also positive. In other words, in the self-adjoint case if we write the polar decomposition form of $\mathcal{R}, \mathcal{R}_1$:

$$\mathcal{R} = R\varphi, \qquad \mathcal{R}_1 = R_1\varphi_1,$$

then $\sigma(\varphi) = \sigma_p(\varphi) \subseteq \{\pm 1\}$, $\sigma(\varphi_1) = \sigma_p(\varphi_1) \subseteq \{0, \pm 1\}$ and also $\dim \operatorname{Ker} \varphi_1 \leq 1$.

Remark 9.2.3. In the definition of unitary φ and partial isometry φ_1 (see (9.2.3)), we can also write $\varphi = f(R), \varphi_1 = f_1(R_1)$ for some unimodular measurable functions f, f_1. Here f, f_1 has the following expression:

$$f(x) = \sum_{k=1}^{\infty} \mathbb{1}_{\{x=s_k\}} e^{i\theta_k} + \mathbb{1}_{\mathbb{R}/\{s_k\}_{k=1}^{\infty}}, \tag{9.2.7}$$

$$f_1(x) = \sum_{k=1}^{\infty} \mathbb{1}_{\{x=t_k\}} e^{i\theta'_k} + \mathbb{1}_{\mathbb{R}/\{t_k\}_{k=1}^{\infty}}. \tag{9.2.8}$$

Remark 9.2.4. With the discussion in subsection 8.2.2, we can also get an equivalent condition for $\operatorname{Ker} \Gamma_1 / \operatorname{Ker} \Gamma = \{0\}$ for the constructed Hankel operator, which is:

$$\prod_{k=1}^{\infty} \frac{t_k^2}{s_k^2} = 0, \qquad \prod_{k=1}^{\infty} \frac{t_k^2}{s_{k+1}^2} < \infty.$$

9.3 Result for Finite Rank Hankel Operators with Simple Singular Values

Using the finite rank version of Abstract Borg's Theorem 8.3.1 given in section 8.3, we can also find out the spectral data that can uniquely determine a finite rank Hankel operator Γ with simple singular values.

The following result is for the self-adjoint case.

Theorem 9.3.1. *Given two sequences of real numbers* $\{\lambda_k\}_{k=1}^N$, $\{\mu_k\}_{k=1}^N$, *satisfying the following intertwining relation*

$$|\lambda_1| > |\mu_1| > ... > |\lambda_N| > |\mu_N| \geq 0,$$

there exists a unique self-adjoint finite-rank Hankel operator Γ *with simple singular values, such that the non-zero eigenvalues of* $\Gamma\big|_{(\operatorname{Ker}\Gamma)^\perp}$ *and* $\Gamma_1\big|_{(\operatorname{Ker}\Gamma)^\perp}$ *are simple, and coincide with* $\{\lambda_k\}_{k=1}^N$, $\{\mu_k\}_{k=1}^N$ *respectively.*

In addition, for the finite-rank case, we have that the constructed Γ *must have a infinite dimensional kernel. And* $\operatorname{Ker}\Gamma_1/\operatorname{Ker}\Gamma = \{0\}$ *if and only if* $\mu_N > 0$.

The following result is for the non self-adjoint case.

Theorem 9.3.2. *Given complex sequences* $\{\lambda_n\}_{n=1}^N$ *and* $\{\mu_n\}_{n=1}^N$, *where the modulus part* $s_k := |\lambda_k|, t_k := |\mu_k|$ *satisfy the intertwining relation*

$$s_1 > t_1 > s_2 > t_2 > ... > s_N > t_N \geq 0,$$

then there exists a unique compact Hankel operator Γ *with simple singular values, such that we can find a conjugation* $\mathfrak{J}_u$ *commutes with* $|\Gamma|$ *and preserves* $u := \Gamma^* e_0$ *(implied by Lemma 4.0.1) with* $\widetilde{\phi}, \widetilde{\phi}_1$ *defined in* (5.1.5), *satisfying that non-zero eigenvalues of* $|\widetilde{\Gamma}|\widetilde{\phi}$, $|\widetilde{\Gamma}_1|\widetilde{\phi}_1$ *are simple, and coincide with* $\{\lambda_n\}_{n=1}^\infty$ *and* $\{\mu_n\}_{n=1}^\infty$ *respectively.*

Moreover, the constructed Hankel operator Γ *must have an infinite dimensional kernel* Γ, *and* $\operatorname{Ker}\Gamma_1/\operatorname{Ker}\Gamma = \{0\}$ *if and only if* $\mu_N = 0$.

The proof to these two theorems are very much similar to the ones for Theorem 9.1.1 and Theorem 9.2.1, so we omit them.

9.4 Conclusion

In this chapter, we further translate the spectral data of two operators that can uniquely determine a compact Hankel operator with simple singular values.

(i) If the Hankel operator Γ is self-adjoint, Theorem 9.1.1 indicates that Γ will be uniquely determined by two sequences of real numbers $\{\lambda_n\}_{n=1}^{\infty}$,$\{\mu_n\}_{n=1}^{\infty}$, satisfying an intertwining relations

$$|\lambda_1| > |\mu_1| > |\lambda_2| > |\mu_2| > .... > |\lambda_n| > |\mu_n| > ... \to 0. \tag{9.4.1}$$

If we write the Borel transform of Γ^2 with respect to u:

$$\left((\Gamma^2 - zI)^{-1}u, u\right) = \int \frac{d\rho(s)}{s-z},$$

then the coefficients of the scalar spectral measure $\rho(s) = \sum_{k\geq 1} a_k \delta_{\lambda_k^2}(s)$ is uniquely determined with the expression in (8.1.9).

(ii) For the case when Hankel operator is not self-adjoint, Theorem 9.2.1 implies that Γ can be uniquely determined by two sequences of complex numbers $\{\lambda_n\}_{n=1}^{\infty}$,$\{\mu_n\}_{n=1}^{\infty}$, whose modulus part satisfy an intertwining relation (9.2.1).

$$|\lambda_1| > |\mu_1| > |\lambda_2| > ... \to 0.$$

In other words, with the constructed conjugation $\widetilde{\mathfrak{J}}_\Gamma$ which commutes with $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ and preserves $u := \Gamma^* e_0$, and the induced unitary $\widetilde{\phi}$ and partial isometry $\widetilde{\phi}_1$ given in (5.1.5). We have the non-zero eigenvalues of the two operators $|\widetilde{\Gamma}|\widetilde{\phi}$, $|\widetilde{\Gamma}_1|\widetilde{\phi}_1$ are simple, and coincide with $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$ respectively.

For this second case, we can also say that the Hankel operator Γ can be uniquely determined by three factors:

(a) A discrete measure $\rho(s)$, which is the scalar spectral measure of $|\Gamma|^2$ with respect to $u = \Gamma^* e_0$:

$$\left((|\Gamma|^2 - zI)^{-1}u, u\right) = \int \frac{d\rho(s)}{s - z}.$$

The coefficients of this scalar measure is uniquely determined by the modulus part of sequences $\{\lambda_n\}_{n=1}^{\infty}$ and $\{\mu_n\}_{n=1}^{\infty}$.

(b) A unimodular measurable function $f : \mathbb{R} \to \mathbb{C}$ (with the expression in (9.2.7)) , which is determined by the phase part of complex sequences $\{\lambda_n\}_{n=1}^{\infty}$. This function induces an unitary operator $f(|\widetilde{\Gamma}|)$ commutes with $|\Gamma|$ and share the same eigenvectors with $|\widetilde{\Gamma}|$.

(c) A unimodular measurable function $f_1 : \mathbb{R} \to \mathbb{C}$ (with the expression in (9.2.8)) , which is determined by the phase part of complex sequences $\{\mu_n\}_{n=1}^{\infty}$. This function induces an operator $f_1(|\widetilde{\Gamma}_1|)$ commutes with $|\widetilde{\Gamma}_1|$ and share the same eigenvectors with $|\widetilde{\Gamma}_1|$.

Notice that for both situations, we require the cyclicity of vector p to guarantee the asymptotic stability of a contraction $\mathcal{T} := R_1\varphi_1\varphi^* R$ in the description of Proposition 3.1.3 and Proposition 5.2.3. Thus the geometric multiplicities of singular values of constructed Γ must be no more than 1.

Chapter 10

RESULT FOR COMPACT HANKEL OPERATORS WITH NON-SIMPLE EIGENVALUES

In this chapter, we will further discuss the spectral data we need that can uniquely determine a compact Hankel operator with non-simple singular values. The main result in this chapter is given in Theorem 10.1.5, saying that a compact Hankel operator with non-simple singular values can be uniquely determined by two sequences of positive real numbers $\{\lambda_k\}_{k=1}^{\infty}$, $\{\mu_k\}_{k=1}^{\infty}$, and two sequences of positive discrete probability measure $\{\widetilde{\rho}_k\}_{k=1}^{\infty}$, $\{\widetilde{\rho}_k^1\}_{k=1}^{\infty}$. Here the two sequences of real numbers satisfy the following intertwining relation:

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0,$$

In section 10.1, we recall the setting in previous chapter and do some preparation work. And in section 10.2, we prove the main result (Theorem 10.1.5) in this chapter.

10.1 Preparation Work and Main Result

10.1.1 The setting of Tuple from Previous Chapters

We first recall the setting of tuple $(R, R_1, p, \varphi, \varphi_1, \mathfrak{J}_p)$ defined in chapter 5:

(i) R, R_1 are two positive, self-adjoint compact operator defined on a Hilbert space $\mathcal{H}$. In addition we have $\operatorname{Ker} R = \{0\}$;

(ii) $R^2 - R_1^2 = pp^*$ for a vector p with $\|R^{-1}p\| \leq 1$;

(iii) $\mathfrak{J}_p$ is a conjugation commutes with R, R_1 and preserves p, implied by Lemma 4.0.1 and Lemma 5.1.1;

(iv) φ is a $\mathfrak{J}_p$-symmetric unitary operator, which commutes with R;

(v) φ_1 is a $\mathfrak{J}_p$-symmetric partial isometry with $\operatorname{Ker} \varphi_1 = \operatorname{Ker} R_1$, which commutes with R_1. In addition, we have $\varphi_1|_{(\operatorname{Ker} R_1)^\perp}$ is unitary (See Remark 5.2.1);

(vi) The contraction $\mathcal{T}$ defined as $\mathcal{T} := \varphi_1 R_1 R^{-1} \varphi^*$ is asymptotically stable.

We again use the definition $\mathcal{H}_0 := \overline{\operatorname{Span}}\{R^n p | n \geq 0\}$. Then in Proposition 6.2.2, we have shown that there exists an intertwining real sequence

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0, \tag{10.1.1}$$

such that $R|_{\mathcal{H}_0}, R_1|_{\mathcal{H}_0}$ have $\{\lambda_k\}_{k=1}^\infty$. $\{\mu_k\}_{k=1}^\infty$ as their non-zero eigenvalues respectively.

Afterwards, Proposition 6.2.10 gives the complete structure of the eigenspaces of R, R_1. From Lemma 6.2.8 we know that R, R_1 doesn't have any eigenvalues other than $\{\lambda_k\}_{k=1}^\infty, \{\mu_k\}_{k=1}^\infty$ and 0. Using the same notation $E_{\lambda_k}, E^1_{\lambda_k}, E_{\mu_k}, E^1_{\mu_k}$ defined in (6.2.7), then we have

(i)

$$\mathcal{H}_0^\perp = \left(\bigoplus_{k=1}^\infty E^1_{\lambda_k}\right) \bigoplus \left(\bigoplus_{k=1}^\infty E_{\mu_k}\right)$$

(ii)

$$E_{\lambda_k} = E^1_{\lambda_k} \oplus \operatorname{Span}\{p_k\}, \qquad E^1_{\mu_k} = E_{\mu_k} \oplus \operatorname{Span}\{p^1_k\},$$

where p_k, p_k^1 are defined in (6.2.9).

Furthermore, if we take the canonical choice of φ, φ_1 given in Lemma 6.3.2:

$$\varphi_1|_{E^1_{\lambda_k}} = I, \qquad \varphi|_{E_{\mu_k}} = I,$$

then by Proposition 7.6.3, the asymptotic stability of $\mathcal{T}$ implies that:

(i) p_k is *-cyclic for φ restricted in E_{λ_k};

(ii) p_k^1 is *-cyclic for φ_1 restricted in $E^1_{\mu_k}$.

Remark 10.1.1. (i) We should notice that all $E_{\lambda_k}, E^1_{\lambda_k}, E_{\mu_k}, E^1_{\mu_k}$ are eigenspaces of compact operators, so they must have finite dimensions.

(ii) From the asymptotic stability assumption, we have for all k

$$E_{\lambda_k} = \overline{\text{Span}}\{\varphi^n p_k | n \in \mathbb{Z}\}.$$

Here we denote $\varphi^n = (\varphi^*)^{-n}$ for a negative n.

Actually, we can show that $E_{\lambda_k} = \overline{\text{Span}}\{\varphi^n p_k | |n| \leq N_k, n \in \mathbb{Z}\}$ holds for a finite N_k.

In fact, if we define $E_i = \overline{\text{Span}}\{\varphi^n p_k | |n| \leq i, n \in \mathbb{Z}\}$, then $E_1 \subseteq E_2 \subseteq ...$ is an increasing sequence. Since $\dim E_{\lambda_k} < \infty$, thus the sequence must stay at a constant subspace from a certain E_i.

10.1.2 Behavior of φ, φ_1 on the Eigenspaces

Now for the given $p_k \in E_{\lambda_k}$ and the unitary operator $\varphi\big|_{E_{\lambda_k}}$, we consider all functions $f(z)$ defined on $\mathbb{T}$ with the following form:

$$f(z) = \sum_{n=-N}^{N} a_n z^n \qquad N \text{ is finite }, a_n \in \mathbb{C}.$$

Denote the set that consists of all such f to be S. For all $f \in S$, we have

$$f \to \langle p_k, f(\varphi)p_k \rangle \tag{10.1.2}$$

is a linear functional on S. According to Stone Weierstrass Theorem, $P(\mathbb{T})$ (the set of polynomials on $\mathbb{T}$ with complex coefficients) is dense in $C(\mathbb{T})$ (the set of continuous functions on $\mathbb{T}$), thus S is also dense in $C(\mathbb{T})$. And (10.1.2) can be extended to all $f \in C(\mathbb{T})$.

Now applying Riesz Representation Theorem, we can find a complex Borel measure on $\mathbb{T}$ (We have $\sigma(\varphi) \subseteq \mathbb{T}$ since φ is unitary), denoted as ρ_k, such that

$$\int_{\mathbb{T}} f(s) d\rho_k(s) = \langle p_k, f(\varphi)p_k \rangle. \tag{10.1.3}$$

This measure ρ_k is also known as the *scalar spectral measure* of $\varphi|_{E_{\lambda_k}}$ with respect to p_k.

In addition, the *Borel transform* of $\varphi|_{E_{\lambda_k}}$ with respect to p_k is given by

$$F_k(z) = \langle (\varphi - zI)^{-1} p_k, p_k \rangle = \int_{\mathbb{T}} \frac{d\rho_k(s)}{s - z}.$$

Proposition 10.1.2. *The unitary operator $\varphi\big|_{E_{\lambda_k}}$ and the scalar spectral measure $\rho_k(s)$ in (10.1.3) satisfies the following properties:*

(i) $\rho_k(s) = \sum_{i=1}^{N} a_i \delta_{\beta_i}(s)$ *for some $a_i > 0, |\beta_i| = 1$;*

(ii) $\varphi\big|_{E_{\lambda_k}} = \sum_{i=1}^{N} \beta_i P_{\mathrm{Ker}(\varphi|_{E_{\lambda_k}} - \beta_i I)}$

(iii) $\dim \mathrm{Ker}(\varphi|_{E_{\lambda_k}} - \beta_i I) = 1$ *holds for all i. Hence $N = \dim E_{\lambda_k}$;*

(iv) $\sum_{i=1}^{N} a_i = \rho_k(\mathbb{T}) = \|p_k\|^2$. *For each i, we have*

$$a_i = \|P_{\mathrm{Ker}(\varphi|_{E_{\lambda_k}} - \beta_i I)} p_k\|^2$$

Proof of Proposition 10.1.2. First notice that $\varphi\big|_{E_{\lambda_k}}$ is unitary and finite rank, thus by the spectral theorem for compact normal operators (See [37]), we can write

$$\varphi\big|_{E_{\lambda_k}} = \sum_{i=1}^{N} \beta_i P_i. \tag{10.1.4}$$

Here $\{\beta_i\}_{i=1}^N$ are all eigenvalues of $\varphi\big|_{E_{\lambda_k}}$. and P_i is the projection operator onto $\mathrm{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I)$. Since $\varphi\big|_{E_{\lambda_k}}$ is unitary, we have all β_i has module 1.

Then the scalar spectral measure $\rho_k(s)$ can be written as

$$\rho_k(s) = \sum_{i=1}^{N} a_i \delta_{\beta_i}(s)$$

for some $a_i \in \mathbb{C}$.

Now we write $p_k = \sum_{i=1}^{N} p_{ki}$, where $p_{ki} \in \mathrm{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I)$. We will show that

$$p_{ki} \neq 0, \qquad \dim \mathrm{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I) = 1.$$

In fact, from (10.1.4), we can write $\varphi^*\big|_{E_{\lambda_k}}$ as

$$\varphi^*\big|_{E_{\lambda_k}} = \sum_{i=1}^{N} \overline{\beta_i} P_i. \tag{10.1.5}$$

If $p_{ki} = 0$, then we have

$$\overline{\mathrm{Span}}\{\varphi^n p_k, (\varphi^*)^n p_k\} \perp \mathrm{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I),$$

which contradicts to the *-cyclicity of p_k.

On the other hand, we know from the representation of φ, φ^* in (10.1.4) and

(10.1.5), we know that the intersection

$$\overline{\text{Span}}\{\varphi^n p_k, (\varphi^*)^n p_k\} \cap \text{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I)$$

is at most a one-dimensional space. Thus if $\dim \text{Ker}(\varphi|_{E_{\lambda_k}} - \beta_i I) \geq 2$ for some i, it will also contradict to the *-cyclicity of p_k. Hence we finish the proof of (iii).

Now for (iv), we apply a sequence of polynomials $\{f_n\}$ that converges to $\mathbb{1}_{\{x=\beta_i\}}$ to (10.1.3), then we get

$$a_i = \langle p_k, P_{\text{Ker}(\varphi|_{E_{\lambda_k}} - \beta_i I)} p_k \rangle = \|p_{ki}\|^2 > 0.$$

Thus we have

$$\rho_k(\mathbb{T}) = \sum_{i=1}^{N} a_i = \sum_{i=1}^{N} \|p_{ki}\|^2 = \|p_k\|^2.$$

We can also get this identity by simply setting $f = 1$ into (10.1.3) □

Remark 10.1.3. Note that equation (10.1.4) implies that $\varphi\big|_{E_{\lambda_k}}$ is diagonalizable. Thus for each eigenvalue β_i, the algebraic multiplicity of β_i equals to the geometric multiplicity of β_i. This is not necessarily true for a general compact operator, where algebraic multiplicity is usually bigger than the geometric multiplicity.

Remark 10.1.4. Denoting $N = \dim E_{\lambda_k}$, we can show that

$$\overline{\text{Span}}\{\varphi^n p_k \big| 0 \leq n \leq N-1\} = E_{\lambda_k},$$

which is stronger than the condition in Proposition 7.6.3.

In fact, this can immediately be resulted from the fact that the projection of p_k on each eigenspace $\mathrm{Ker}(\varphi\big|_{E_{\lambda_k}} - \beta_i I)$ is non-zero, and the Vandermonde matrix

$$M_{ij} = \begin{pmatrix} 1 & 1 & ... & 1 \\ \beta_1 & \beta_2 & ... & \beta_N \\ ... & & & \\ \beta_1^{N-1} & \beta_2^{N-1} & ... & \beta_N^{N-1} \end{pmatrix}$$

has non-zero determinant

$$|\big(M_{ij}\big)_{1\le i,j\le N}| = \prod_{i<j} \big(\beta_j - \beta_i\big).$$

Now similarly for p_k^1 and $\varphi_1\big|_{E^1_{\mu_k}}$, we can define the scalar spectrum measure ρ_k^1 on $\mathbb{T}$ by:

$$\int_{\mathbb{T}} f(s) d\rho_k^1(s) = \langle p_k^1, f(\varphi_1) p_k^1 \rangle,$$

and the Borel transform of $\varphi_1\big|_{E^1_{\mu_k}}$ with respect to p_k^1

$$F_k^1(z) = \langle (\varphi_1 - zI)^{-1} p_k^1, p_k^1 \rangle = \int_{\mathbb{T}} \frac{d\rho_k^1(s)}{s - z}.$$

Similar to the proof in Proposition 10.1.2, $\rho_k^1(s)$ is also a positive discrete measure, which concentrate on finitely many points on $\mathbb{T}$. And we also have

$$\rho_k^1(\mathbb{T}) = \|p_k^1\|^2.$$

We can see that the total measure of $\rho_k(s), \rho_k^1(s)$ equal to $\|p_k\|^2, \|p_k^1\|^2$, which is not independent on the intertwining sequence $\{\lambda_k\}_{k=1}^\infty, \{\mu_k\}_{k=1}^\infty$. Hence if we normalize ρ_k, ρ_k^1 by changing p_k, p_k^1 to $\frac{p_k}{\|p_k\|}, \frac{p_k^1}{p_k^1}$ respectively, that is, we denote $\widetilde{\rho}_k, \widetilde{\rho}_k^1$ to be the scalar spectral measure of $\varphi|_{E_{\lambda_k}}, \varphi_1|_{E^1_{\mu_k}}$ with respect to $\frac{p_k}{\|p_k\|}, \frac{p^1_k}{\|p^1_k\|}$ respectively:

$$\int_{\mathbb{T}} f(s)d\widetilde{\rho}_k(s) = \langle \frac{p_k}{\|p_k\|}, f(\varphi)\frac{p_k}{\|p_k\|}\rangle, \qquad \int_{\mathbb{T}} f(s)d\widetilde{\rho}^1_k(s) = \langle \frac{p^1_k}{\|p^1_k\|}, f(\varphi_1)\frac{p^1_k}{\|p^1_k\|}\rangle,$$

then we have $\widetilde{\rho}_k(\mathbb{T}) = \widetilde{\rho}^1_k(\mathbb{T}) = 1$.

In this section, we will show that a Hankel operator can be uniquely determined by the choice of $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$, $\{\widetilde{\rho}_k\}_{k=1}^\infty$, $\{\widetilde{\rho}^1_k\}_{k=1}^\infty$; i.e. we can take $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$, $\{\widetilde{\rho}_k\}_{k=1}^\infty$, $\{\widetilde{\rho}^1_k\}_{k=1}^\infty$ as the spectral data of a compact Γ.

Theorem 10.1.5. *Given an intertwining sequence*

$$\lambda_1 > \mu_1 > \lambda_2 > \mu_2 > .... \to 0,$$

and two sequences of normalized measure on $\mathbb{T}$, denoted as $\{\widetilde{\rho}_k\}_{k=1}^\infty$, $\{\widetilde{\rho}^1_k\}_{k=1}^\infty$, satisfy

(i) $\widetilde{\rho}_k(\mathbb{T}) = 1, \widetilde{\rho}^1_k(\mathbb{T}) = 1$*;*

(ii) ρ_k, ρ^1_k *are positive discrete measure concentrated on finitely many points on $\mathbb{T}$.*

then there exists a unique Hankel operator Γ, such that

(a) $|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|$ *are compact, and have no other non-zero eigenvalues than $\{\lambda_k\}_{k=1}^\infty$, $\{\mu_k\}_{k=1}^\infty$. Furthermore, if denoting*

$$\mathfrak{E}^1_{\lambda_k} := \mathrm{Ker}(|\widetilde{\Gamma}_1| - \lambda_k I), \qquad \mathfrak{E}_{\lambda_k} := \mathrm{Ker}(|\widetilde{\Gamma}| - \lambda_k I);$$
$$\mathfrak{E}^1_{\mu_k} := \mathrm{Ker}(|\widetilde{\Gamma}_1| - \mu_k I), \qquad \mathfrak{E}_{\mu_k} := \mathrm{Ker}(|\widetilde{\Gamma}| - \mu I),$$

then we have

$$\mathfrak{E}^1_{\lambda_k} \oplus \mathrm{span}\{v^k_1\} = \mathfrak{E}_{\lambda_k}, \qquad \mathfrak{E}_{\mu_k} \oplus \mathrm{span}\{v^k_2\} \subseteq \mathfrak{E}^1_{\mu_k}$$

for some vectors v^k_1, v^k_2.

(b) Under the canonical choice of tuple $\langle \widetilde{\mathfrak{J}}_u, \widetilde{\phi}, \widetilde{\phi}_1 \rangle$ *(See* (5.1.5) *and Lemma 6.3.2), i.e.*

$$\widetilde{\phi}|_{\mathfrak{E}_{\mu_k}} = I, \qquad \widetilde{\phi}_1|_{\mathfrak{E}^1_{\lambda_k}} = I,$$

we have the spectral measure of $\widetilde{\phi}|_{E_{\lambda_k}}$ *with respect to* v_1^k*, and the spectral measure of* $\widetilde{\phi}_1|_{E^1_{\mu_k}}$ *with respect to* v_2^k*, are just scalar multiplication of* $\widetilde{\rho}_k$*,* $\widetilde{\rho}_k^1$ *respectively.*

We will prove this proposition in next section.

10.2 Proof of Theorem 10.1.5

10.2.1 Proof of Existence

Proof. According to Proposition 5.2.4, to prove the existence of such Hankel Γ, it's sufficient to construct a tuple $(R, R_1, p, \varphi, \varphi_1)$ that satisfies the same property stated in (a),(b).

First by applying The abstract Borg's theorem 8.0.1, there exists an unique (up to unitary equivalence) triple (W, W_1, p) defined on a Hilbert space $\mathcal{H}_0$, such that W, W_1 are positive compact operators, $\operatorname{Ker} W = \{0\}$ satisfying

(i) $W = W_1 + pp^*$;

(ii) W, W_1 have simple non-zero eigenvalues as $\{\lambda_k^2\}_{k=1}^\infty$, $\{\mu_k^2\}_{k=1}^\infty$ respectively.

Thus if we denoting the eigenvectors of W, W_1 as

$$Wx_k = \lambda_k^2 x_k, \qquad W_1 y_k = \mu_k^2 y_k;$$

Then we can construct two operators $\widetilde{R}, \widetilde{R}_1$ on $\mathcal{H}_0$ defined as

$$\widetilde{R}x_k := \lambda_k x_k, \qquad \widetilde{R}_1 y_k = \mu_k y_k.$$

Here we can express the norm of p_k, p_k^1 in terms of $\{\lambda_k^2\}_{k=1}^{\infty}$, $\{\mu_k^2\}_{k=1}^{\infty}$, which are defined as

$$p_k := P_{\mathrm{Ker}(R-\lambda_k I)} p, \qquad p_k^1 := P_{\mathrm{Ker}(R_1-\mu_k)} p.$$

In fact, if we denote μ, μ_1 as the scalar spectral measure of R^2, R_1^2 with respect to p, and $g(z), g_1(z)$ to be the Cauchy transform of R^2, R_1^2 with respect to p. That is,

$$\int_{\mathbb{R}} f(s)d\mu(s) = \langle p, f(R^2)p\rangle \text{ holds for all } f \in C(\mathbb{R}) \tag{10.2.1}$$

$$g(z) = \langle (R^2 - zI)^{-1}p, p\rangle = \int \frac{d\mu(s)}{s-z} \tag{10.2.2}$$

$$\int_{\mathbb{R}} f(s)d\mu_1(s) = \langle p, f(R_1^2)p\rangle \text{ holds for all } f \in C(\mathbb{R}) \tag{10.2.3}$$

$$g_1(z) = \langle (R_1^2 - zI)^{-1}p, p\rangle = \int \frac{d\mu_1(s)}{s-z} \tag{10.2.4}$$

Then we have $1 - g = \frac{g}{g_1}$ from Proposition 8.1.1, and we can write

$$\mu(z) = \sum_k a_k \delta_{\lambda_k^2}(z), \qquad \mu_1(z) = \sum_k b_k \delta_{\mu_k^2}(z) + b_0\delta_0(z).$$

According to Lemma 8.1.9 and Proposition 8.2.9, the coefficients $\{a_k\}_{k=1}^{\infty}, \{b_k\}_{k=1}^{\infty}$ has the following representation:

$$a_k = (\lambda_k^2 - \mu_k^2)\prod_{i\neq k}\left(\frac{\lambda_k^2 - \mu_i^2}{\lambda_k^2 - \lambda_i^2}\right); \tag{10.2.5}$$

$$b_n = (\lambda_n^2 - \mu_n^2)\prod_{k\neq n}\left(\frac{\mu_n^2 - \lambda_k^2}{\mu_n^2 - \mu_k^2}\right) \tag{10.2.6}$$

$$b_0 = \begin{cases} 0 & \text{if } \sum_n \frac{a_n}{\lambda_n^2} < 1, \text{ or } \sum_n \frac{a_n}{\lambda_n^2} = 1 \text{ and } \sum_n \frac{a_n}{\lambda_n^4} = \infty. \\ \left(\sum_n \frac{a_n}{\lambda_n^4}\right)^{-1} & \text{if } \sum_n \frac{a_n}{\lambda_n^2} = 1 \text{ and } \sum_n \frac{a_n}{\lambda_n^4} < \infty. \end{cases} \tag{10.2.7}$$

Here according to the proof in section 8.2,

$$\sum_n \frac{a_n}{\lambda_n^2} = 1, \qquad \Leftrightarrow \|W^{-1/2}p\| = 1, \qquad \Leftrightarrow \sum_{j=1}^{\infty} \left(1 - \frac{\mu_j^2}{\lambda_j^2}\right) = \infty,$$
$$\sum_n \frac{a_n}{\lambda_n^4} = \infty, \qquad \Leftrightarrow \|W^{-1}p\| = \infty, \qquad \Leftrightarrow \sum_{j=1}^{\infty} \left(\frac{\mu_j^2}{\lambda_{j+1}^2} - 1\right) = \infty$$

Now we apply a sequence of polynomials $\{f_n\}$ which converges to $\mathbb{1}_{\{x=\lambda_k^2\}}$ into (10.2.1), then we will have

$$a_k = \langle p, P_{\mathrm{Ker}(R^2 - \lambda_k^2 I)} p \rangle = \|p_k\|^2,$$

thus

$$\|p_k\| = \sqrt{a_k} = \sqrt{(\lambda_k^2 - \mu_k^2) \prod_{i \neq k} \left(\frac{\lambda_k^2 - \mu_i^2}{\lambda_k^2 - \lambda_i^2}\right)}. \tag{10.2.8}$$

Similarly we have

$$\|p_k^1\| = \sqrt{b_k} = \sqrt{(\lambda_k^2 - \mu_k^2) \prod_{i \neq k} \left(\frac{\mu_k^2 - \lambda_i^2}{\mu_k^2 - \mu_i^2}\right)}. \tag{10.2.9}$$

Now denoting $\widetilde{p}_k = \frac{p_k}{\|p_k\|}, \widetilde{p}_k^1 = \frac{p_k^1}{\|p_k^1\|}$, we want to construct a sequence of orthogonal Hilbert space $\mathcal{H}_1, \mathcal{H}_2, ...\mathcal{H}_{2k}$ which are all orthogonal to $\mathcal{H}_0$, and two sequences of unitary operators $\{\varphi_k\}_{k=1}^{\infty}$, $\{\varphi_k^1\}_{k=1}^{\infty}$ such that

(i) $\varphi_k : \mathcal{H}_{2k-1} \oplus \mathrm{span}\{p_k\} \to \mathcal{H}_{2k-1} \oplus \mathrm{span}\{p_k\}$ has $\widetilde{\rho}_k$ as its spectral measure with respect to $\widetilde{p}_k$;

(ii) $\varphi_k^1 : \mathcal{H}_{2k} \oplus \text{span}\{p_k^1\} \to \mathcal{H}_{2k} \oplus \text{span}\{p_k^1\}$ has $\widetilde{\rho}_k^1$ as its spectral measure with respect to $\widetilde{p}_k^1$;

We first need to determine the dimension of each $\mathcal{H}_k$. This can be easily seen from Proposition 10.1.2. If $\widetilde{\rho}_k$ concentrates on $m+1$ discrete points on $\mathbb{T}$, then we have $\dim \mathcal{H}_{2k-1} = m$. And a similar result holds for $\widetilde{\rho}_k^1$ and $\mathcal{H}_{2k}$.

Then we choose an orthogonal basis for $\mathcal{H}_{2k-1} \oplus \text{span}\{p_k\}$, denoted as $\widetilde{p}_k, e_1, ..., e_m$. We construct the unitary φ_k such that

$$\overline{\text{Span}}\{\widetilde{p}_k, \varphi_k^j \widetilde{p}_k \big| 1 \le j \le i\} = \overline{\text{Span}}\{\widetilde{p}_k, e_1, ..., e_i\} \qquad \text{holds for all } 1 \le i \le m$$

Notice that the inner product between any two of $\widetilde{p}_k, \varphi_k \widetilde{p}_k, ...\varphi_k^m \widetilde{p}_k$ can be uniquely determined by $\widetilde{\rho}_k(s)$. In fact,

$$\langle (\varphi_k)^i \widetilde{p}_k, (\varphi_k)^j \widetilde{p}_k \rangle = \langle \widetilde{p}_k, (\varphi_k)^{j-i} \widetilde{p}_k \rangle = \int_{\mathbb{T}} s^{j-i} d\widetilde{\rho}_k(s) \text{ for all integers } i, j. \qquad (10.2.10)$$

Thus if we write $\varphi_k \widetilde{p}_k = \beta_{10} \widetilde{p}_k + \beta_{11} e_1$, we have $\beta_{10} = \langle \varphi_k \widetilde{p}_k, \widetilde{p}_k \rangle$, and we take $\beta_{11} = \sqrt{1 - \beta_{10}^2}$. Since in Remark 10.1.4, we have stated that $\widetilde{p}_k, \varphi_k \widetilde{p}_k, ...\varphi_k^m \widetilde{p}_k$ are linearly independent, thus $\beta_{11} \neq 0$.

Similarly, if we write $\varphi_k^i \widetilde{p}_k = \beta_{i0} \widetilde{p}_k + ... + \beta_{ii} e_i$. Then the coefficients $\beta_{i0}, ..., \beta_{i(i-1)}$ will be uniquely determined by the inner product between $\varphi_k^i \widetilde{p}_k$ and $\widetilde{p}_k, \varphi_k \widetilde{p}_k, ..., \varphi_k^{i-1} \widetilde{p}_k$. And then we take $\beta_{ii} = \sqrt{1 - \sum_{j=0}^{i-1} \beta_{ij}^2}$. Proposition 10.1.2 guarantees that $\beta_{ii} = 0$.

So now we have defined $\varphi^i \widetilde{p}_k$ $(1 \le i \le m)$ in terms of basis $\widetilde{p}_k, e_1, ..., e_m$. Since $\overline{\text{Span}}\{\widetilde{p}_k, \varphi_k^j \widetilde{p}_k \big| 1 \le j \le m\} = \overline{\text{Span}}\{\widetilde{p}_k, e_1, ..., e_m\}$, we have φ_k well-defined on $\mathcal{H}_{2k-1} \oplus \text{span}\{p_k\}$.

Similarly, we find an orthogonal basis on $\mathcal{H}_{2k}$, and then calculate the coordinates of $(\varphi_k^1)^i \widetilde{p}_k^1$ in terms of this set of basis and $\widetilde{p}_k^1$, where $1 \le i \le \dim \mathcal{H}_{2k}$.

Now we denote

$$\mathcal{H} := \mathcal{H}_0 \oplus \mathcal{H}_1 \oplus \mathcal{H}_2 ...,$$

We extend $\widetilde{R}, \widetilde{R}_1$ to R, R_1 defined from $\mathcal{H}$ to $\mathcal{H}$:

$$R|_{\mathcal{H}_0} = \widetilde{R}, \qquad R|_{\mathcal{H}_{2k-1}} = \lambda_k I, \qquad R|_{\mathcal{H}_{2k}} = \mu_k I;$$
$$R_1|_{\mathcal{H}_0} = \widetilde{R}_1, \qquad R_1|_{\mathcal{H}_{2k-1}} = \lambda_k I, \qquad R_1|_{\mathcal{H}_{2k}} = \mu_k I.$$

Now we define an unitary $\varphi : \mathcal{H} \to \mathcal{H}$ as:

$$\varphi|_{\mathcal{H}_{2k-1} \oplus \text{span}\{p_k\}} = \varphi_k, \qquad \varphi|_{\mathcal{H}_{2k}} = I \text{ for all } k;$$

and a partial isometry $\varphi_1 : \mathcal{H} \to \mathcal{H}$ as:

$$\varphi_1|_{\mathcal{H}_{2k} \oplus \text{span}\{p_k^1\}} = \varphi_k^1, \qquad \varphi_1|_{\mathcal{H}_{2k-1}} = I \text{ for all } k; \qquad \varphi_1|_{\mathcal{H}_0 \cap \operatorname{Ker} R_1} = 0.$$

Thus we have constructed a tuple $(R, R_1, p, \varphi, \varphi_1)$ on $\mathcal{H}$. To determine the conjugation $\mathfrak{J}_p$ on $\mathcal{H}$, we can define $\mathfrak{J}_p$ on $\mathcal{H}_0$ by:

$$\mathfrak{J}_p p = p, \qquad \mathfrak{J}_p(R^n p) = R^n p \text{ for all } n.$$

The value of $\mathfrak{J}_p|_{\mathcal{H}_{2k-1} \oplus \text{span}\{p_k\}}$ can be defined by

$$\mathfrak{J}_p p_k = p_k, \qquad \mathfrak{J}_p(\varphi^n p_k) = (\varphi^*)^n p_k, \qquad \mathfrak{J}_p((\varphi^*)^n p_k) = \varphi^n p_k \text{ for all } n.$$

Similarly the value of $\mathfrak{J}_p|_{\mathcal{H}_{2k} \oplus \text{span}\{p_k^1\}}$ can be defined by

$$\mathfrak{J}_p p_k^1 = p_k^1, \qquad \mathfrak{J}_p(\varphi_1^n p_k^1) = (\varphi_1^*)^n p_k^1, \qquad \mathfrak{J}_p((\varphi_1^*)^n p_k^1) = \varphi_1^n p_k^1 \text{ for all } n$$

So now we have defined a tuple $(R, R_1, p, \mathfrak{J}_p, \varphi, \varphi_1)$, such that the contraction $\mathcal{T} := R_1\varphi_1\varphi^* R^{-1}$ is asymptotically stable implied by Proposition 7.6.3. Thus by Proposition 5.2.4, we can find a Hankel operator Γ, such that there exists a conjugation $\mathfrak{J}_u$ commuting with $|\Gamma|, |\Gamma_1|$ and preserves $\Gamma^* e_0$, and the tuple $(|\widetilde{\Gamma}|, |\widetilde{\Gamma}_1|, u, \widetilde{\phi}, \widetilde{\phi}_1, \widetilde{\mathfrak{J}}_u)$ given in (5.1.5) is unitarily equivalent to $(R, R_1, p, \mathfrak{J}_p, \varphi, \varphi_1, \mathfrak{J}_p)$, and Γ satisfies properties (a),(b) given in Proposition 10.1.5.

□

10.2.2 Proof of Uniqueness

Proof. According to Proposition 5.2.4, we only need to show the uniqueness of tuple $(R, R_1, p, \varphi, \varphi_1)$ (the unique choice of $\mathfrak{J}_p$ is given in Remark 7.6.6).

Suppose that there are two tuples $(R, R_1, p, \varphi, \varphi_1)$ and $(R', R_1', p', \varphi', \varphi_1')$ that both satisfy the condition (a),(b) in Proposition 10.1.5 (Here $\varphi, \varphi_1.\varphi', \varphi_1'$ are all the canonical choices for their equivalent classes). We will show that they are unitary equivalent.

We also reuse the definition inside the proof of Theorem 8.0.4. That is,

$$\mathcal{H}_0 := \overline{\text{Span}}\left\{R^n p \middle| n \geq 0\right\};$$

$$\mathcal{E}_{\lambda_k} := \text{Ker}(R - \lambda_k I), \qquad \mathcal{E}_{\mu_k} := \text{Ker}(R - \mu_k I);$$

$$\mathcal{E}^1_{\lambda_k} := \text{Ker}(R_1 - \lambda_k I), \qquad \mathcal{E}^1_{\mu_k} := \text{Ker}(R - \mu_k I);$$

$$\widetilde{R} = R\big|_{\mathcal{H}_0}, \qquad \widetilde{R}_1 = R_1\big|_{\mathcal{H}_0}$$

$$\mathcal{H}_0' := \overline{\text{Span}}\left\{(R')^n p' \middle| n \geq 0\right\};$$

$$\mathfrak{E}_{\lambda_k} := \text{Ker}(R' - \lambda_k I), \qquad \mathfrak{E}_{\mu_k} := \text{Ker}(R' - \mu_k I);$$

$$\mathfrak{E}^1_{\lambda_k} := \text{Ker}(R_1' - \lambda_k I), \qquad \mathfrak{E}^1_{\mu_k} := \text{Ker}(R_1' - \mu_k I);$$

$$\widetilde{R}' = R'\big|_{\mathcal{H}_0'}, \qquad \widetilde{R}_1' = R_1'\big|_{\mathcal{H}_0'}$$

And we also define the projection vector p_k, p_k^1 as

$$p_k := P_{\mathcal{H}_0 \cap \mathrm{Ker}(R - \lambda_k I)} p, \qquad p_k^1 := P_{\mathcal{H}_0 \cap \mathrm{Ker}(R_1 - \mu_k I)} p,$$

$p'_k, (p_k^1)'$ are defined similarly.

Now by Theorem 8.0.1, there exists a unitary equivalence between the tuple $(R\big|_{\mathcal{H}_0}, R_1\big|_{\mathcal{H}_0}, p)$ and $(R'\big|_{\mathcal{H}'_0}, R'_1\big|_{\mathcal{H}_0}, p')$:

$$\widetilde{R}' = \widetilde{V}\widetilde{R}\widetilde{V}^*, \qquad \widetilde{R}'_1 = \widetilde{V}\widetilde{R}_1\widetilde{V}^*, \qquad p' = \widetilde{V}p$$

We can easily see that

$$p'_k = \widetilde{V}p_k, \qquad (p_k^1)' = \widetilde{V}(p_k^1),$$

holds for all k.

Now also recall that in the proof of Generalized Borg's Theorem 8.0.4, that we extend $\widetilde{V}$ to a unitary $V : \mathcal{H} \to \mathcal{H}'$ by defining an arbitrary unitary ϕ_k on $\mathcal{E}_{\mu_k}$ and an arbitrary unitary ϕ_k^1 on $\mathcal{E}^1_{\lambda_k}$. and then construct the unitary equivalence between (R, R_1, p) and (R', R'_1, p'). Now for the proof of this theorem, we build up the unitary equivalence between (φ, φ_1) and (φ', φ'_1) by carefully choosing ϕ_k and ϕ_k^1.

We know that p_k is a *-cyclic vector for $\varphi\big|_{\mathcal{E}_{\lambda_k}}$ on $\mathcal{E}_{\lambda_k}$. Denote $N = \dim \mathcal{E}_{\lambda_k}$. In Proposition 10.1.2, we have shown that $\varphi\big|_{\mathcal{E}_{\lambda_k}}$ have N different eigenvalues on $\mathbb{T}$, and the projection of p_k on each eigenspace is non-zero. Denoting that the eigenvalues to be $\beta_1, ..., \beta_N$, and the projection of p_k onto the corresponding eigenspace to be $p_{k1}, ..., p_{kN}$.

Now we consider p'_k and the unitary $\varphi'\big|_{\mathfrak{E}_{\lambda_k}}$. Since the scalar spectral measure for two different tuples are the same, we have $\varphi'\big|_{\mathfrak{E}_{\lambda_k}}$ also have simple eigenvalues as $\beta_1, ..., \beta_N$. In addition, the projection of p'_k onto the corresponding eigenspaces, defined as $p'_{k1}, ..., p'_{kN}$, have the same norm as $p_{k1}, ..., p_{kN}$ respectively.

Thus we define a unitary $\phi_k : \mathcal{E}_{\lambda_k} \to \mathfrak{E}_{\lambda_k}$ by:

$$\phi_k p_{k1} = p'_{k1}, \qquad ..., \qquad \phi_k p_{kN} = p'_{kN}. \tag{10.2.11}$$

We can show that $\phi_k(\varphi^n p_k) = (\varphi')^n p'_k$ for all n. In fact,

$$\begin{aligned} \phi_k(\varphi^n p_k) &= \phi_k\big(\beta_1^n p_{k1} + ... + \beta_N^n p_{kN}\big) \\ &= \beta_1^n p'_{k1} + ...\beta_N^n p'_{kN} = (\varphi')^n p'_k \end{aligned} \tag{10.2.12}$$

Since in Remark 10.1.4, we have shown that $\overline{\text{Span}}\{\varphi^n p_k \big| 0 \le n \le N-1\} = \mathcal{E}_{\lambda_k}$. We can also use (10.2.12) for the definition of ϕ_k. Easy to see that for this ϕ_k, we have $\phi_k p_k = p'_k$ and ϕ_k maps $\mathcal{E}_{\lambda_k}$ to $\mathfrak{E}_{\lambda_k}$.

Similarly we define ϕ_k^1 be a unitary operator mapping $\mathcal{E}^1_{\mu_k}$ to $\mathfrak{E}^1_{\mu_k}$, such that

$$\phi_k^1(\varphi_1^n p_k^1) = (\varphi_1')^n (p_k^1)'$$

holds for all $n \in \mathbb{N}$.

Now we can define a unitary $V : \mathcal{H} \to \mathcal{H}'$ by:

$$V = \widetilde{V} \text{ on } \mathcal{H}_0,$$
$$V = \phi_k \text{ on } \mathcal{E}^1_{\lambda_k}, \qquad V = \phi_k^1 \text{ on } \mathcal{E}_{\mu_k}.$$

We show that $\varphi' = V\varphi V^*$ (the other equality $\varphi_1' = V\varphi_1 V^*$ can be shown similarly). For $\forall x \in \mathcal{E}_{\mu_k}$, we have

$$V\varphi x = Vx = \phi_k^1 x, \qquad \varphi' V x = \varphi' \phi_k^1 x = \phi_k^1 x.$$

Here $\varphi x = x$ and $\varphi'(\phi_k^1 x) = \phi_k^1 x$ follows from $\varphi\big|_{\mathcal{E}_{\mu_k}} = I$ and $\varphi'\big|_{\mathfrak{E}_{\mu_k}} = I$. Thus $V\varphi = \varphi' V$ on $\mathcal{E}_{\mu_k}$.

Now we show $V\varphi = \varphi' V$ is also true on $\mathcal{E}_{\lambda_k}$. For vector $(\varphi^n p_k)$, using (10.2.12), we have

$$\varphi' V(\varphi^n p_k) = \varphi'(\varphi')^n p_k^1 = (\varphi')^{n+1} p_k^1$$
$$V\varphi(\varphi^n p_k) = V(\varphi^{n+1} p_k) = (\varphi')^{n+1} p_k^1$$

Since $\mathcal{E}_{\lambda_k} = \overline{\text{Span}}\{\varphi^n p_k \big| n \geq 0\}$ given by Remark 10.1.4, we have $V\varphi = \varphi' V$ is true on $\mathcal{E}_{\lambda_k}$, hence holds on the whole space $\mathcal{H}$.

Similarly, we can show that $\varphi_1' = V\varphi_1 V^*$ also holds on $\mathcal{H}$. Thus we build up the unitary equivalence between $(R, R_1, p, \varphi, \varphi_1)$ and $(R', R_1', p', \varphi', \varphi_1')$. And the proof of uniqueness is done. □

10.3 Result for Finite Rank Hankel Operators with Multiple Singular Values

In this section, we will discuss the spectral data that can uniquely determine a finite rank Hankel operators with multiple singular values.

Recall that Proposition 6.2.11, given finite rank operators R, R_1, we have shown that on a subspace $\mathcal{H}_0 := \overline{\text{Span}}\{R^n p | n \geq 0\}$, there exists an intertwining sequence

$$\lambda_1 > \mu_1 > ... > \lambda_N > \mu_N \geq 0,$$

such that $R\big|_{\mathcal{H}_0}, R_1\big|_{\mathcal{H}_0}$ has simple eigenvalues as $\{\lambda_k\}_{k=1}^N, \{\mu_k\}_{k=1}^N$ respectively.

Condition 1: $\mu_N > 0$, then from a similar proof to Proposition 7.6.3, we can show that for $1 \leq k \leq N$, p_k is a *-cyclic vector for E_{λ_k}, and p_k^1 is a *-cyclic vector for $E^1_{\mu_k}$.

Hence for $1 \leq k \leq N$, we can define the scalar spectral measure ρ_k, ρ_k^1 on $\mathbb{T}$ by

$$\langle (\varphi - zI)^{-1} p_k, p_k \rangle = \int_{\mathbb{T}} \frac{d\rho_k(s)}{s - z}, \qquad \langle (\varphi_1 - zI)^{-1} p_k^1, p_k^1 \rangle = \int_{\mathbb{T}} \frac{d\rho_k^1(s)}{s - z}.$$

By normalizing $\{\rho_k\}_{k=1}^N, \{\rho_k^1\}_{k=1}^N$ as probability measure $\{\widetilde{\rho}_k\}_{k=1}^N, \{\widetilde{\rho}_k^1\}_{k=1}^N$, we can also prove that $\{\widetilde{\rho}_k\}_{k=1}^N, \{\widetilde{\rho}_k^1\}_{k=1}^N$ are discrete measure, and concentrated on finitely many points on $\mathbb{T}$. We can show that the four sequences $\{\lambda_k\}_{k=1}^N, \{\mu_k\}_{k=1}^N$, $\{\widetilde{\rho}_k\}_{k=1}^N, \{\widetilde{\rho}_k^1\}_{k=1}^N$ will uniquely determine a finite rank Hankel operator.

Condition 2: $\mu_N = 0$, then we have $\dim E_{\mu_N}^1 = 1$ and E_{μ_N} is trivial. Under this case, we have $\varphi_1\big|_{E_{\mu_N}^1} = 0$, and

(i) For $1 \leq k \leq N$, p_k is a *-cyclic vector for E_{λ_k};

(ii) For $1 \leq k \leq N-1$, p_k^1 is a *-cyclic vector for $E_{\mu_k}^1$.

We can show that the four sequences $\{\lambda_k\}_{k=1}^N, \{\mu_k\}_{k=1}^N$, $\{\widetilde{\rho}_k\}_{k=1}^N, \{\widetilde{\rho}_k^1\}_{k=1}^{N-1}$ will uniquely determine a finite rank Hankel operator.

The proof to both conditions are typically similar to the proof for theorem 10.1.5, so we omit it.

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