

Abstract of “Identifying potential mild traumatic brain injury events through inertial sensor based kinematics and continuum mechanics based strain calculations”

by Yang Wan, Ph.D., Brown University, February 2025.

Mild Traumatic Brain Injuries (mTBI) resulting from mechanical trauma remain a significant global public health concern. These injuries are often triggered by violent head motions or impacts. There is considerable interest in advancing methods to predict the resulting brain tissue strains and strain rates from such head motions. Accurately estimating the brain injury risk through these predictions is critical for developing effective prevention and mitigation strategies aimed at reducing the incidence and severity of mTBI. In this dissertation, we introduce a multiscale, physics-based approach to assess the risk of brain injury, establishing a connection between mechanical loading and brain deformation at the cellular level, where injury originates. The approach is composed of the following three key steps: *Step-1*: Estimate the kinematics of the individual’s head, including acceleration, angular velocity, and angular acceleration, using inertial measurement units in conjunction with motion prediction algorithms. We present an algorithm for determining the complete motion of the head using data from only four head mounted tri-axial accelerometers. The algorithm provides the rigid body’s acceleration field in both the body frame and the lab frame. Compared to other accelerometer-only algorithms, the presented algorithm is significantly less sensitive to bias type errors, such as those that arise from inaccurate measurement of sensor positions and orientations; *Step-2*: Estimate the strains and strain rates by inputting the kinematics and biometric information of the head into a computational continuum-mechanics-based model of the individual’s brain tissues. In this dissertation, we introduce two new idealized continuum-mechanics-based head models for predicting brain tissue strains and strain rates. One model assumes the brain behaves as an elastic material, while the other accounts for viscoelastic properties of brain tissue. Both models account for the head’s finite rotation, which is an improvement upon prior models that relied on the assumption of small rotations. Despite the simplicity of the models, we show that the proposed 2D finite rotation head models predict comparable strains to a more detailed finite element head model; *Step-3*: Determine the risk of injury by comparing the estimated strains with their critical values. We present a centrifugation based *in vitro* mTBI model that we developed in collaboration with Prof. Diane Hoffman-Kim’s group. This model can be used to estimate the critical strains.

**Identifying potential mild traumatic brain injury events
through inertial sensor based kinematics and continuum
mechanics based strain calculations**

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Refereed Journal Publications

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Yang Wan*, Rafael D. González-Cruz*, Diane Hoffman-Kim, and Haneesh Kesari. A mechanics theory for the exploration of a high-throughput, sterile 3D in vitro traumatic brain injury model. *Biomechanics and Modeling in Mechanobiology*, 23, 1179-1196, 2024.

Rafael D. González-Cruz*, **Yang Wan***, Amina Burgess, Dominick Calvao, William Renken, Francesca Vecchio, Christian Franck, Haneesh Kesari, and Diane Hoffman-Kim. Cortical spheroids show strain-dependent cell viability loss and neurite disruption following sustained compression injury. *PloS one*, 19(8):e0295086, 2024.

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DEDICATED TO MY FAMILY

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Chapter 1

Introduction

1.1 Background

Mild traumatic brain injury (mTBI), also referred to as concussion, is a common injury for both civilians and warfighters. Traumatic brain injury affects around 55 million people around the world each year and represents an ongoing global public health issue [1]. Its prevalence and incidence are higher than other common neurological diseases, including stroke, Alzheimer's and Parkinson's diseases [1]. Mild traumatic brain injury is associated with a spectrum of symptoms, including significant neurological and cognitive impairments, such as headaches, dizziness, memory problems, and mood disorders [2, 3, 4]. In the US and Europe, 190–225 patients die every day after suffering from a TBI event and tens of thousands suffer from chronic neurodegenerative diseases and complications resulting from the injury [5, 1].

Mild Traumatic Brain Injuries are caused by violent head motions, that may occur from intense blunt impacts to the head in contact sports, military exercises, motor vehicle crashes, falls following blasts, etc. Traumatic brain injury is a disease process rather than an event [6]. It leads to structural damage and functional impairments, which arise not only from the initial onset of the injury (primary injuries) but also from the subsequent progression (secondary injuries). In mTBI, the primary injury results from the violent motion of the head, leading to the deformation of brain tissue, stretching and tearing of blood vessels, neurons, glia, and axons, and triggering the onset of secondary injury

processes. Secondary injury evolves over minutes to months following the primary injury, driven by cascades of metabolic, cellular and molecular events that ultimately lead to brain cell death, tissue damage and atrophy [7, 8, 9, 10, 11, 12].

The high mortality and long-term disability associated with TBI highlight the need for further research into its treatment and diagnosis. Currently there are no FDA approved treatments [13] for TBI. Traumatic brain injury is often diagnosed using a combination of standard-of-care imaging techniques such as computed tomography (CT) [14] and magnetic resonance imaging (MRI) scans [15], as well as neurological scales assessing consciousness like the Glasgow Coma Scale [16]. For developing effective treatments and diagnosis, it is critical to understand both TBI's onset and its progression. As such, several *in vivo* as well as *in vitro* TBI models have been developed for delineating the various complex pathophysiological processes involved in its onset and progression. A literature review of TBI models is given in §1.2.3.

Despite the development of the medical treatment and diagnosis of mTBI, there is an urgent need for effective prevention and mitigation strategies of mTBI in mechanically traumatic events. Various strategies have been employed to prevent mTBI. In sports, new regulations have been implemented to modify player's behavior, aiming to reduce or eliminate exposure to blunt impacts [17]. Helmets and neck collars are examples of equipment that can alter the motion experienced by the head. A jugular vein compression collar aims to change the stiffness of the brain making it less susceptible to injury [18]. Most mTBI prevention strategies explicitly or implicitly rely on a "brain injury criterion" for their effective synthesis, implementation, and evaluation. The inputs to the brain injury criterion are some descriptors of the head's motion, such as acceleration, angular velocity, or brain's deformation following mechanically loading, like strains and strain rates. Based on these inputs, the brain injury criterion yields a prediction for that motion or brain deformation's potential for causing mTBI as the output. A comprehensive review of brain injury criteria is given in §1.1.1.

1.1.1 Approaches to assess the risk of traumatic brain injury

Several approaches have been put forward for assessing the risk of traumatic brain injury, including (i) empirical injury criteria, (ii) computational mechanics (CM) based injury criteria, and (iii) machine learning (ML) based injury criteria.

Empirical injury criteria An empirical injury criterion takes some descriptor of the head's motion as input. The head's motion can be thought of as a composition of translatory and rotatory motions. In initial brain injury criteria the focus was on the head's translatory motion. Two of the first published injury criteria are the Gadd Severity Index (SI) and the Head Injury Criterion (HIC) [19, 20]. Both SI and HIC ignore the head's rotations and take the head's translational acceleration as their input. Later, however, it was realized that in the context of mTBI the head's rotations play an even more important role in causing injury than its translations. The first brain injury criterion to take the rotational aspect of the head's motion into consideration was GAMBIT [21]. The input to GAMBIT is the tuple of center-of-mass-acceleration and angular-acceleration time series. Following the development of GAMBIT, brain injury criteria that use descriptors that only depend on the head's rotational motion as inputs have also been put forward. One such criteria is the Brain Injury Criteria (BrIC) [22]. Aiming to compliment HIC, BrIC only uses the head's angular velocity time series as input.

Empirical injury criteria are popular given their ease of use. They are easy to understand and apply since the calculations involve simple data processing and evaluation of simple algebraic mathematical expressions. They also have low to no computational expense and do not require detailed personalized information (e.g., head magnetic resonance images (MRI)). A major drawback of empirically based injury criteria is that they are overly simple, i.e., they do not include important physical quantities that are known to affect the brain tissue strains and strain rates. Both brain strains and strain rates have been shown to play an important role in deformation-induced neural injury, which is a common injury mechanism in mTBI [23, 24]. As a result, the effectiveness of these empirical injury criteria in predicting the risk of mTBI remains an open question.

Computational mechanics based injury criteria In contrast to empirical injury criteria, computational mechanics (CM) based injury criteria attempt to account for all relevant physics that lead to injury in a mTBI event. Computational mechanics based models of mTBI are implicitly based on the hypothesis that the risk of injury of a traumatic event is correlated with some measure of brain tissue strains and strain rates. The application of CM based injury criteria involves the numerical simulation of head motion of a traumatic event of interest followed by a comparison of the

predicted strain (ϵ) and strain rate ($\dot{\epsilon}$) from the model with critical values for injury. Some measures of strain that are widely used in CM based injury criteria are the peak *maximum principal strain* (MPS), *maximum axonal strain* (MAS) or *tract-oriented strain*, and the *cumulative strain damage measure* (CSDM) [25, 26, 27, 28, 29, 30]. Determining the measures of strain and strain rate that are most pertinent for mTBI and determining their critical values, i.e., the values at which the risk of injury becomes significant, is an active area of research [24, 31, 23]. A range of values have been proposed for various critical strain measures, with the majority of estimates (maximum principal Green-Lagrange strain [32], MAS [33], and CSDM [25]) falling between 10% and 25%. There have been relatively fewer proposals for the critical values of strain rate measures. A recent study proposed a critical *maximum axonal strain rate* (MASR) value of $40 - 90 \text{ s}^{-1}$ for traumatic axonal injury [23]. With time, both the measure used to predict injury as well as the accuracy of the critical values for that measure are expected to improve.

Machine learning based injury criteria The need to rapidly predict brain strains and strain rates to obtain real time estimates of mTBI risk has been a factor in the development of machine learning (ML) based methods as an alternative to CM based head models. In this approach, ML algorithms are trained using CM based models, and once trained, the ML algorithms are applied to rapidly predict the risk of injury for a given head impact event. Machine learning based methods have shown promise in predicting brain strain directly from head kinematic data [34, 35, 36]. However, ML derived injury measures still have some limitations. For example, there is a large cost upfront to develop ML based injury criteria. Furthermore, subject-specific CM based models now take personal details, such as head size and shape, into account whereas ML based injury criteria usually do not. Machine learning based injury criteria can, of course, always be updated so that the details of an individual are taken into account. This can be done, for example, by generating new data from CM based head models that have been updated by taking the individual's details into account and then training the ML based head model on that data. However, if such an approach is followed, then the development of ML based injury criteria will end up being even more expensive than the development of CM based injury criteria.

1.1.2 Motivation

Irrespective of whether a head-motion-based or brain-deformation-based brain injury criterion is used, both rely on the availability of a robust algorithm for accurately reconstructing head motion. Head-motion-based criteria, such as empirical injury metrics, use motion descriptors derived from measurable data as input. On the other hand, the brain-deformation-based criteria require the time history of head motion to serve as the loading conditions in computational head models in order to calculate the resulting strains and strain rates. Hence one of our objectives in the thesis is to develop an algorithm for determining the complete motion of the head, which is presented in Chapter 2.

Assuming the head kinematics is accurately captured, there is a critical need of a computational head model to map head kinematics to tissue-based deformation measures to assess injury risk. Identifying the distribution of the strains and strain rates within the brain will facilitate the development of point-of-care injury predictions. The effectiveness of a CM based head model will depend on how easily the strains and strain rates in the model can be calculated and how close those calculated values are to their respective values during the real traumatic event. The accuracy of the strain and strain rate estimates are dependent on several factors, such as the level of incorporated anatomical detail in the model, the numerical methods used, and the quality of the computational mesh.

After accurately predicting head motion and brain deformation, advancing brain injury criteria requires a deeper understanding of injury mechanisms at the cellular level, as the precise force magnitudes and directions that initiate TBI at this scale remain unknown. TBI models are essential tools for establishing tolerance thresholds—critical strain and strain rate values that lead to brain injury. Moreover, these models help us understand the onset and progression of traumatic brain injury, aiding in the development of effective medical interventions. In this dissertation, we propose a spheroid-based, *in vitro* TBI model, where spheroids are mechanically loaded using a centrifuge, as outlined in Chapter 5.

1.2 Our strategy for estimating the potential risk of traumatic brain injury due to mechanical loading

In this dissertation, we adopt a multiscale, physics-based approach to assess the risk of brain injury by linking mechanical loading to brain deformation at the cellular level, where injury originates. This approach is based on the hypothesis that the risk of injury is primarily a function of the tissues' strain and strain rate histories. Directed by the hypothesis, our strategy for predicting the risk of injuries from traumatic events consists of the following three steps:

- Step-1* Estimate the kinematics of the individual's head, including acceleration, angular velocity, and angular acceleration. We developed a wearable sensor system, Accelo-Hat, equipped with IMUs (shown in Fig. 1.1 (a)) to take inertial measurements over the head. For example, the Accelo-Hat is being used to continuously collect inertial data from a player's head during a soccer header motion in Fig. 1.1 (a). The inertial measurement data is then fed to a suite of algorithms, collectively referred to as the *AO-algorithms*, to reconstruct the motion of the head. For example, the reconstructed motion of the soccer header motion shown in Fig. 1.1 (a) is shown in Fig. 1.1 (b).
- Step-2* Estimate the strains and strain rates within the brain tissues. The reconstructed motion is used to synthesize the boundary and initial conditions for a computational continuum-mechanics-based model of the head to estimate the tissue strain and strain rate histories corresponding to that motion. For example, the reconstructed motion shown in Fig. 1.1 (b) is applied to a mechanics model to estimate the strain and strain rate histories in the player's brain for the soccer header motion shown in Fig. 1.1 (a). A time snapshot of the strains and strain rates estimation is shown in Fig. 1.1 (c).
- Step-3* Determine the critical values for the tissue strains and strain rates at which the risk of injury becomes significant via a TBI model. For example, the axonal injury measures (Fig. 1.1 (d)) based on the strain and strain rate histories are compared with their critical values to estimate the risk of injury from brain deformation shown in Fig. 1.1 (c).

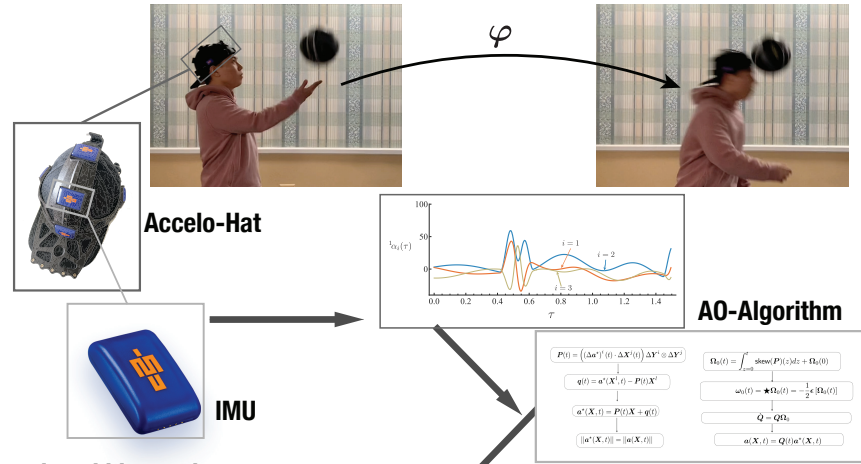
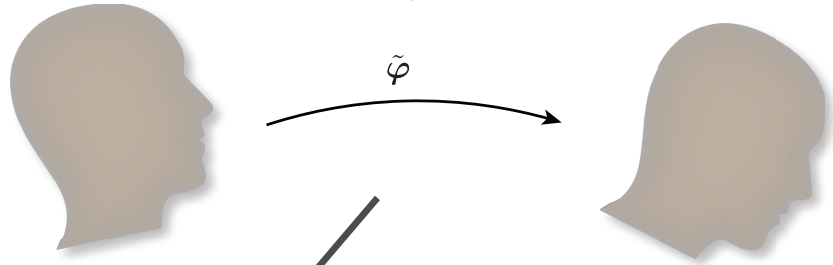
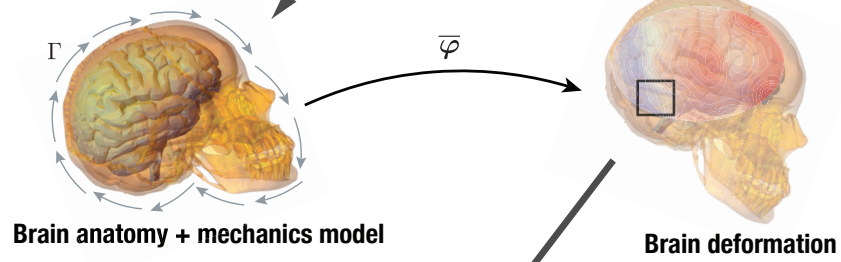
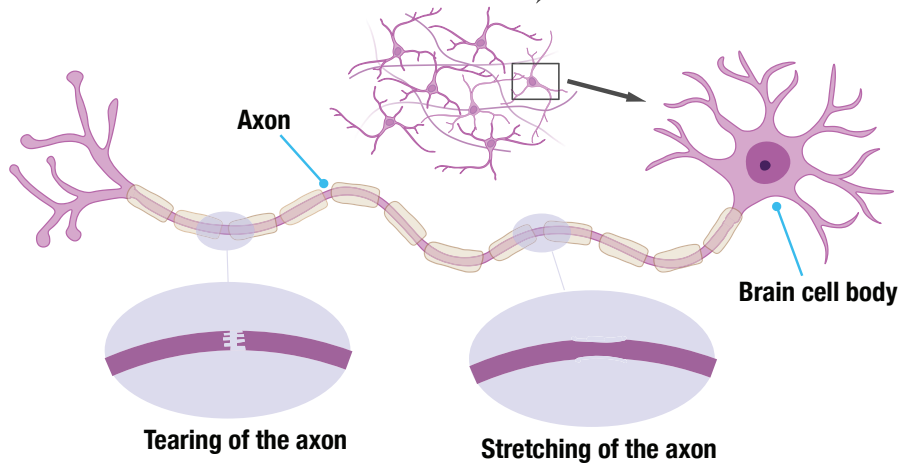
a Motion capture**b Reproduced kinematics****c Deformation prediction****d Cellular injury estimation**

Figure 1.1: Demonstration of our strategy for estimating the risk of mTBI during a soccer header. (a) Inertial measurements are collected on the outer surface of the head using an Accelo-Hat equipped with inertial measurement units (IMUs). (b) Through the AO-algorithm, rigid body kinematics of the head are reproduced from the inertial measurements. (c) The head kinematics are subsequently applied as boundary conditions to mechanical models of the brain, from which the tissue strain and strain rate histories are estimated at anatomical sites. (d) Estimate the critical strains and strain rates that lead to brain injuries, such as axonal stretching and tearing.

These three specific steps will significantly advance prediction (detection) and prevention of mTBI. We discuss *Step-1*, the algorithms to estimate the head motion in §1.2.1; *Step-2*, the computational head model to predict the strains and strain rates within the brain in §1.2.2; and *Step-3*, TBI models to determine the critical thresholds of strain and strain rates that result in cellular injuries in §1.2.3.

1.2.1 Algorithms to estimate the head motion

When we refer to any aspect of the head's motion, we, in fact, are referring to that aspect as it pertains to the skull's motion; since it is the skull's motion that is, at least currently, observable and quantifiable in the field, either using video recording equipment or inertial sensor systems. The Young's modulus of bone from human skulls generally lies in the 2–13 GPa range [37, 38, 39]. In comparison, brain tissue is extremely compliant. Recent indentation tests on brain slices that were kept hydrated show that the Young's modulus of brain tissue lies in the 1–2 kPa range (white matter 1.9 ± 0.6 kPa, and gray matter 1.4 ± 0.3 kPa) [40]. Due to this large disparity between the skull's and the brain's stiffnesses, in biomechanical investigations of mTBI the skull is usually modeled as a rigid body [41, 42].

There are existing algorithms capable of fully determining the motion of a rigid body, capturing both its translational and rotational dynamics. Several methods compute the rigid body motion using computational vision (CV) approaches that process visual inputs from cameras, such as video or image sequences of the moving object in 3D space [43, 44, 45]. The CV based approaches compute motion through a series of techniques and algorithms, including feature detection and extraction, optical flow, structure from motion, and more. Computational vision approaches have wide-ranging applications in robotics, augmented reality, biomechanics, and autonomous systems. However, they face limitations in capturing motion over large spaces due to the restricted field of view of the camera. In traumatic events, such as military training or contact sports, injuries can occur in various locations, making it essential to develop methods that can effectively monitor and analyze motion across expansive environments.

Methods that utilize sensor data circumvent the limitations of CV based approaches by employing inertial navigation technology, which determines the orientation and position of a rigid body using

measurements from inertial measurement units (IMUs). Typically, these units consist of a tri-axial gyroscope and a tri-axial accelerometer. IMUs can be conveniently mounted on personnel's head via wearable sensor system, enabling the measurement of kinematics over expansive spaces. These sensor systems, including instrumented custom mouthguards, earpieces, skin patches, and headbands, capture time series data of head kinematics [46, 47, 48].

In conjunction with the measurements from sensor systems, several sensor data processing algorithms have been developed to predict head kinematics, including rotation, translation, and the acceleration field of the head. Some of these algorithms utilize gyroscope measurements to estimate the acceleration of the head's center of mass. Gyroscopes can measure angular velocity directly, but determining accelerations using gyroscope measurement requires taking at least one time derivative of the time signals. However, it is well known that numerically differentiating data significantly amplifies noise, which can reduce the accuracy of the computed accelerations ([49, 50]). In addition, gyroscopes' power requirements are much higher than those of accelerometers (gyroscopes consume approximately 25 times more power than accelerometers [51]).

Several non-gyroscope-dependent algorithms exist for constructing head motion, which only make use of data from accelerometers. Though they avoid the noise amplification issue associated with gyroscopes, these algorithms have their own limits. For example, certain algorithms impose constraints on the locations and orientations of the accelerometers, requiring specific configurations to ensure accurate motion predictions. Padgaonkar *et al.* [52] require that accelerometers are oriented perpendicular to one another and located along each accelerometer's measurement axes. Zou and Angeles [53] require each accelerometer to be located at the centroid of each face of the polyhedron.

There exist several algorithms giving limited information (Padgaonkar *et al.* [52], Genin *et al.* [54] and Naunheim *et al.* [55]). For example, all these algorithms give the rigid body's acceleration field in terms of the body frame, which is a set of vectors that are attached to the rigid body, and hence move with it. These algorithms do not provide any information of how the body frame is oriented in space. However, that information is critical for constructing inputs for computational head models. Furthermore, such algorithms typically estimate the parameters related to the acceleration field through the application of non-linear fitting procedures (e.g. see [55]) or filtering and state estimation algorithms (e.g. see [56]). Both of those techniques involve numerical non-linear

minimization or equivalently numerical solution of a set of non-linear equations.

Rahaman *et al.* [57] developed an algorithm for determining the rigid body motion using data from only four tri-axial accelerometers, which is referred to as the AO (accelerometer-only) algorithm. The AO-algorithm provides the acceleration field both in terms of the body frame and in terms of the known, laboratory frame. That is, the AO-algorithm provides information about how the body frame is related to the laboratory frame as a function of time. The only requirement for the accelerometers' positions in the AO-algorithm is that all the accelerometers not lie on the same plane.

Despite its many advantages, the AO-algorithm has one critical limitation. It is quite sensitive to bias type errors in the accelerometer data (see details in Chapter 2). To address this limitation, we developed a novel accelerometer-only algorithm that determines rigid body motion from accelerometer data through the square-root of a negative semi-definite tensor, as presented in Chapter 2.

1.2.2 Estimation of the strains and strain rates within brain tissue

A large number of computational mechanics (CM) based head models have been developed, and they range from high resolution models that take into account the anisotropy of white matter and the detailed geometry of the brain to more simplified models that incorporate fewer anatomical details and have a lower spatial resolution [58, 59, 60, 41, 61]. Typically these models use finite element methods to solve the equations governing the deformation and stress responses to the mechanical loading on the head. The computational time involved in simulating a head impact event is directly related to the mesh resolution. The person-hours involved in preparing the mesh is also directly dependent on the quality and accuracy of the mesh. For these reasons, getting accurate calculations of strains and strain rates in CM based head models is currently very computationally expensive, and from a person-hours point of view, very time consuming, complex, and tedious. In the future, the time to create these models is expected to decrease with the development of robust algorithms for automatically segmenting and meshing the brain directly from medical imaging data [62, 63]. Even with these improvements, the computational expense of simulating traumatic events with high resolution, subject-specific models can still be prohibitive for some applications, given that these

simulations can take hours or days to run even with the use of high performance computers.

An alternative to CM based head models are idealized continuum mechanics (ICM) based head models. In most ICM based head models, the brain is simplified as a homogeneous, isotropic, incompressible, linear elastic or viscoelastic solid and the skull as a rigid solid; the geometry of the brain is modelled as a sphere or cylinder and the corresponding skull is modelled as a spherical shell or a hollow cylinder [64, 65, 66, 67, 68, 69, 70, 71, 72, 73, 74, 75, 76, 77, 78, 79]. Due to that simplicity, there is no need for a computational mesh, and there is a large reduction in computational expense.

2D ICM models There exist several ICM based head models in which the head is modeled as a cylinder [64, 65, 66, 67, 68, 69]. In all the 2D models, we surveyed the head was subjected to a rotational motion with no translations being involved. The rotations take place about a fixed central axis, which is an axis that is fixed in space w.r.t. time and is initially the cylinder's central axis. In the context of ICM models "2D" means that the displacements in the direction of the central axis are assumed to vanish, and that the other displacement components are assumed to not vary in the direction of the central axis.

In the work of Margulies and Thibault [65] the brain is modeled as an incompressible, homogeneous Kelvin-Voigt viscoelastic material, and the skull is subjected to a sudden time varying rotation. The rotation can be fairly arbitrary as long as the corresponding angular acceleration can be represented using a Fourier series. In the work of Massouros *et al.* [66, 67, 68] the brain is modeled as a Maxwell viscoelastic material, and the skull's rotation angle is a sinusoidal function of time. In the work of Bayly *et al.* [69] the brain is modeled as a three-parameter linear viscoelastic material. Initially the brain and the skull rotate with the same constant angular velocity, and the brain is curtailed from experiencing any deformation. The head is then loaded by subjecting the skull to an angular deceleration pulse, which has the shape of a half-sine pulse, and the brain is allowed to experience deformation. Bayly *et al.* solve their model analytically as well as numerically using finite element methods. They additionally validate their model by comparing its predicted strain fields with the strain fields they measure in a gelatin based experiment that was set up to closely resemble their ICM based head model.

3D ICM models There also exist several ICM based head models in which the brain is modeled as a sphere [77, 78, 64, 70, 74, 72, 73, 76, 71, 75, 79]. Leaving [77, 78], in all other models, the head is not subjected to any translatory motion; it is only subjected to a rotational motion about a fixed central axis, i.e., an axis that is fixed in space with time and that initially passes through the sphere's center.

In the work of Christensen and Gottenberg [77], the brain is modeled as a general linear viscoelastic material, and the skull is first subjected to a rotational motion about a fixed central axis and then later to a translatory motion. In the work of Chandran *et al.* [78], the brain is modeled as a linear elastic material, and the skull is subjected to purely translatory motion.

In the work of Lee and Advani [70], the brain is first modeled as a linear elastic material and then later as an arbitrary linear viscoelastic solid; the skull is subjected to a time varying rotation. The function that maps time to the rotation angle is such that its second derivative is a step function. Firoozbakhsh and DeSilva [71] and Bycroft [72] model the brain as a general linear viscoelastic solid. In the work of Firoozbakhsh and DeSilva, the function that maps time to the rotation angle can be fairly arbitrary, whereas in the work of Bycroft the function is a half sine pulse. Liu *et al.* [76] model the brain first as a linear elastic material and then as a linear Kelvin viscoelastic material; the skull is subjected to fairly arbitrary rotations. Liu *et al.* solved their ICM model using finite-difference methods. For the case of the brain being linear elastic, Liu *et al.*'s results [76] match those of Liu and Chandran [74]. The work of Liu and Chandran [74] is cited often in the context of 3D ICM models. However, we were not able to access this publication. From the comments regarding it in [76, 73, 75], we know that in this work the brain is modeled as a linear elastic solid and the skull is subjected to fairly arbitrary rotations.

The published ICM models that we surveyed are capable of capturing the wave dynamics in the brain due to sudden head motion, and provide a good first order approximation for the peak strains and strain rates in the brain. Despite the valuable preliminary physical insight that they provide, all the surveyed ICM models have a key limitation. Their derivation is based on the implicit assumption that the head rotations are small. This fact can be gleaned by noting that in the surveyed ICM models, the authors do not make a distinction between the reference and the deformed configurations. The head rotations in most mechanically traumatic events, however, are far from being small. In most

mechanically traumatic events, such as vehicle crashes, sports injuries and maneuvers, martial arts, etc., the head rotations quite routinely exceed 45° , and in some cases even exceed 90° [80, 81, 82, 83]. In this dissertation, we present two novel 2D ICM based head models that accounts for the head's finite rotations, as detailed in Chapters 3 and 4.

1.2.3 Traumatic brain injury model to understand cellular injuries from mechanical deformation

Several *in vivo* as well as *in vitro* TBI models have been developed for investigating cellular injury from mechanical deformation. *In vivo* TBI studies have relied extensively on rodent models. They include injury modalities such as controlled cortical impact (CCI) [84], fluid percussion injury (FPI) [85], weight drop [86], and sustained focal compression [87]. As a whole, *in vivo* injury models are very attractive because they are all-encompassing: they include the brain's vasculature and structural organization, the brain's multiple, distinct cell types, the blood-brain barrier, and access to blood circulation and peripheral immune cells. That is, they allow the study of complex multicellular, mechanistic, and systems-level responses to TBI, including axonal demyelination, blood brain barrier breakdown, peripheral immune cell-mediated inflammation, and neurocognitive impairment.

The *in vivo* injury models, however, can also have a few limitations. (1) The *in vivo* models' all-encompassing nature also makes them difficult to interpret. Specifically, in *in vivo* injury models it is difficult to delineate how the various cellular, biochemical, and biophysical processes affect each other. The various processes have a complex interdependence on each other, and form feedback loops that drive the disease progression. (2) The *in vivo* models can be expensive, and (3) difficult to use. (4) In *in vivo* models, it is difficult to visualize brain tissue deformation in real time and correlate injury severity to those deformations. Such correlations can potentially provide valuable information for developing inertial-sensor-system-based diagnosis techniques for mild TBI (mTBI) [88, 89, 90, 26].

In vitro TBI models in most cases circumvent the limitations (1)-(3) of the *in vivo* models. That is, they are easier to interpret, are less expensive, are easier to use, and pose fewer ethical concerns compared to their *in vivo* counterparts. The earliest *in vitro* studies involved subjecting neuronal and

glial monolayers (2D cell cultures) to higher pressures and monitoring plasma membrane damage and cell death [91]. In fact, the 2D *in vitro* models do not even suffer from limitation (4) of the *in vivo* models, since it is straightforward to monitor deformations in them using a time sequence of microscopy images and image processing algorithms. Despite their many attractive features, 2D *in vitro* models suffer from one major limitation: they may not be sophisticated enough to capture the primary pathophysiological processes involved in TBI.

Three dimensional (3D) cell culture models are a relatively recent development [92]. They are generally more sophisticated than 2D cell culture models. In particular, cortical spheroids—which are a special type of 3D cell culture model—replicate key aspects of *in vivo* brain tissue, such as the electrophysiology, dimensionality, physicochemical microenvironment, and the extracellular matrix composition observed [93, 94] *in vivo*. Hence, currently there is significant interest in developing *in vitro* TBI models that are based on 3D cell culture models. Some recent works in this direction involve compressing neuronal cells embedded in hydrogels made of specific extracellular matrix proteins or biomaterial composites [24, 95], and brain organoids [94]. The two main challenges in developing 3D *in vitro* TBI models are (1) designing a mechanical loading system that can be used to mimic the mechanical forces that *in vivo* tissue experiences during TBI while maintaining high throughput, and sterility in the experiment, and (2) being able to estimate the deformations experienced by the *in vitro* tissue during the experiment.

In this dissertation, we propose a new 3D *in vitro* TBI model in which mechanical loads are applied to cortical spheroids, via the aid of centrifugal forces, as presented in Chapter 5. The proposed model does not have the limitations (1)–(3) of the *in vivo* models, neither does it have the limitation of insufficient sophistication of the 2D *in vitro* models. The model is designed to be high throughput; maintain conditions that are as sterile as those present at the time the spheroids are grown; and allow loading of the spheroids in a reproducible manner.

1.3 Thesis structure

The thesis is organized as follows. In Chapter 2 we present an algorithm for determining the complete motion of the head using data from only four head mounted tri-axial accelerometers. We

present a procedure to implement the proposed algorithm and check the validity and robustness of the algorithm using virtual simulation data. In Chapter 3 we present a new idealized continuum mechanics based head model that accounts for the head's finite rotations, which is an improvement upon prior models that have been based on a small rotation assumption. The proposed head model assumes that the brain behaves as a linear elastic material. In Chapter 4 we extend the head model developed in Chapter 3 to account for the brain's viscosity. We compare the strains predicted by the head models developed in Chapter 3 and Chapter 4 with those obtained from a more detailed finite element head model. In Chapter 5 we introduce a spheroid-based, *in vitro* TBI model in which the spheroids are mechanically loaded by centrifugation. As part of this proposed TBI model, we develop a continuum mechanics-based theory to estimate the strains experienced by the spheroids when subjected to constant angular velocity during spinning. We make a few concluding remarks in Chapter 6.

Chapter 2

Determining rigid body motion from accelerometer data through the square-root of a negative semi-definite tensor

Note: A version of this chapter is published in *Computer Methods in Applied Mechanics and Engineering*. Data and figures have been used with all co-authors' consent.

Yang Wan, Alice Lux Fawzi, and Haneesh Kesari. Determining rigid body motion from accelerometer data through the square-root of a negative semi-definite tensor, with applications in mild traumatic brain injury. Computer Methods in Applied Mechanics and Engineering, 390:114271, 2021.

2.1 Introduction

Irrespective of which existing, or yet to be developed, brain injury criterion will be used in the future, its successful application will hinge on the availability of a robust algorithm for constructing the motion descriptor that the criterion takes as input from measurable data. Currently, different algorithms are used to obtain the descriptors taken by the injury criteria as inputs. The inputs to

GAMBIT can be obtained from the measurements of one tri-axial accelerometer and one tri-axial gyroscope mounted in a mouthguard [96] if the center-of-mass-acceleration and angular-acceleration are obtained by processing the data using the algorithm in [97]. In another example the input to BrIC (i.e., angular velocity) is prepared by numerically integrating the angular acceleration, which is determined by applying the 6DOF algorithm [98] to the data from 12 single-axis accelerometers mounted in a helmet [99]. Interestingly the inputs to most of the currently employed brain injury criteria can be prepared from the knowledge of a few key rigid body motion descriptors. To make this idea more concrete, consider the following equation, which is often used to describe rigid body motion,

$$\mathbf{x}(\tau) = \mathbf{Q}(\tau) \mathbf{X} + \mathbf{c}(\tau). \quad (2.1)$$

In (2.1) τ is a real number that denotes a non-dimensional time instant; $\mathbf{X}$ is a column matrix of real numbers that denotes the initial position vector of a rigid body material particle; $\mathbf{x}(\tau)$ is the column matrix of real numbers that denotes that material particle's position vector at the time instance τ ; $\mathbf{Q}(\tau)$ is a time dependent square matrix of real numbers with positive determinant whose transpose equals its inverse; and $\mathbf{c}(\tau)$ is a time dependent column matrix of real numbers. The matrix $\mathbf{Q}(\tau)$ quantifies the rotation or orientation of the rigid body at the time instance τ , while $\mathbf{c}(\tau)$ quantifies the rigid body's translation at that time instance. The inputs to most current brain injury criteria can be computed from a knowledge of the maps $\mathbf{Q}$ and $\mathbf{c}$ and their first and second-order time derivatives, i.e., $\mathbf{Q}'$, $\mathbf{c}'$, $\mathbf{Q}''$, $\mathbf{c}''$. In this manuscript we present an algorithm for determining these maps and their derivatives using data from only four tri-axial accelerometers. The only requirement on the accelerometers' positions is that all the accelerometers not lie on the same plane.

The presented algorithm has some similarities to the one recently presented by Rahaman *et al.* [57]¹, which is referred to as the AO (accelerometer-only) algorithm. For reasons that will become clear shortly, we refer to the algorithm that we present in this manuscript as the $\sqrt{\text{AO}}$ -algorithm. The AO-algorithm also presents a framework for completely determining the rigid body's motion, i.e., for constructing the maps $\mathbf{Q}$ and $\mathbf{c}$ and their time derivatives, using data only from four tri-axial

¹A graphical user interface for applying the AO algorithm to different types of data sets and visualizing its results is freely available [100]

accelerometers not lying on the same plane. The reasons why it is required that the accelerometers not lie on a plane in the $\sqrt{\text{AO}}$ and the AO algorithms are the same. They are discussed in [57, §2.1, §3.1, §5]. The $\sqrt{\text{AO}}$ -algorithm has all the advantages of the AO-algorithm.

There are existing algorithms for completely determining a rigid body's motion using sensor data. However, these algorithms take data from sensor systems called inertial measurement units (IMUs). These units contain one or more gyroscopes. One of the primary advantages of the AO and $\sqrt{\text{AO}}$ algorithms is their non-dependence on gyroscopes. For a detailed discussion on why accelerometers are preferable over gyroscopes in the context of mTBI please see §1 in [57]. Briefly, gyroscopes' power requirements are much higher than those of accelerometers (gyroscopes consume approximately 25 times more power than accelerometers [51]), and algorithms that aim to construct descriptors of a rigid body's acceleration using data from gyroscopes add a significant amount of noise to those descriptors [49, 50].

Several algorithms exist for constructing inputs to brain injury criteria that too only make use of data from accelerometers (Padgaonkar *et al.* [52], Genin *et al.* [54] and Naunheim *et al.* [55]). These algorithms, however, give much more limited information than is given by the AO and the $\sqrt{\text{AO}}$ algorithms. For example, all these algorithms give the rigid body's acceleration field in terms of the body frame, which is a set of vectors that are attached to the rigid body, and hence move with it. These algorithms do not provide any information of how the body frame is oriented in space. However, that information is critical for constructing inputs for the upcoming finite element based brain injury criteria. The AO and the $\sqrt{\text{AO}}$ algorithms provide complete information of how the body frame is oriented in space. See §1 in [57] for further discussion on the advantages of the AO and the $\sqrt{\text{AO}}$ algorithms over other algorithms that also make use of only accelerometer data.

Despite its many advantages we note that the AO-algorithm has one critical limitation. It is quite sensitive to bias type errors in the accelerometer data. Bias type errors are distinct from random errors in that they do not arise as a consequence of stochastic processes. For accelerometers, bias type errors can arise as a consequence of inaccurately defining sensor position and orientation (see Fig. 2.1). As we explain below, the advantage of the $\sqrt{\text{AO}}$ -algorithm over the AO-algorithm is that it is far less sensitive to bias type errors than the AO-algorithm.

One of the critical steps in the $\sqrt{\text{AO}}$ and the AO algorithms is the determination of the map

$\tau \mapsto \overline{\mathbf{W}}(\tau)$. Here $\overline{\mathbf{W}}(\tau)$ is time dependent skew-symmetric matrix of real numbers that is related to the rigid body's angular velocity. In the AO-algorithm $\overline{\mathbf{W}}$ is determined by numerically integrating the equation ([57, §3.12])

$$\overline{\mathbf{W}}'(\tau) = \text{skew part of } \mathbf{P}(\tau). \quad (2.2)$$

Here $\mathbf{P}(\tau)$ is a square matrix of real numbers that is to be computed from the accelerometers' data, relative locations, and orientations. Due to numerical integration any bias type errors in $\mathbf{P}$ will give rise to errors in $\overline{\mathbf{W}}$ that grow with time. In the $\sqrt{\text{AO}}$ -algorithm we alternatively determine $\overline{\mathbf{W}}$ by taking the square-root of the equation

$$\overline{\mathbf{W}}(\tau)\overline{\mathbf{W}}(\tau) = \text{symmetric part of } \mathbf{P}(\tau). \quad (2.3)$$

We derive (2.3) in A.2. Due to the elimination of the numerical integration step associated with the solution of (2.2), the $\sqrt{\text{AO}}$ -algorithm gives much better persistent accuracy over time when applied to the data containing bias type errors, compared to the AO-algorithm.

In §2.2 we present the mathematics and mechanics of rigid body motion from [57, §2] that is needed for the development of the $\sqrt{\text{AO}}$ -algorithm. In §2.3 we review the AO-algorithm as preparation for the development of the $\sqrt{\text{AO}}$ -algorithm. In §2.4 we detail the $\sqrt{\text{AO}}$ -algorithm and present a procedure for taking the square root of (2.3). In §2.5 we check the validity and robustness of the $\sqrt{\text{AO}}$ -algorithm. We do so by feeding in virtual accelerometer data, to which differing amounts of bias and noise type errors have been added, to both the $\sqrt{\text{AO}}$ and AO algorithms and comparing their predictions. Using those predictions in §2.6 we show that the $\sqrt{\text{AO}}$ -algorithm is less sensitive to bias type errors than the AO-algorithm. We make a few concluding remarks in §2.7.

2.2 Preliminary mathematics and kinematics of rigid body motion

In this section we briefly recapitulate the mathematics and kinematics of rigid body motion from [57, §2] that are needed for the development of the proposed $\sqrt{\text{AO}}$ -algorithm.

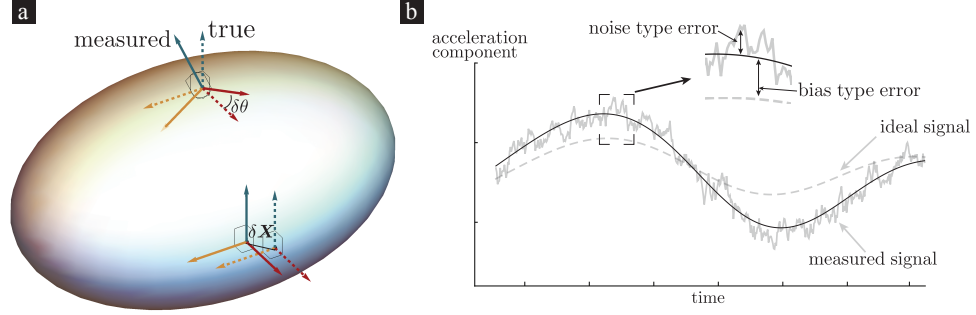


Figure 2.1: Bias and noise type errors in acceleration component measurements. When there are errors δX and $\delta \theta$ in accurately defining an accelerometer's position and orientation, respectively (see (a)), bias type errors can occur in the measurement of acceleration components (see (b)). Noise type errors in the acceleration component measurements are usually a consequence of seismic, electrical, and other types of noise.

2.2.1 Notation

Let $\mathbb{E}$ be a finite dimensional, oriented, Hilbert space, i.e., a Euclidean vector space. The Euclidean point space $\mathcal{E}$ has $\mathbb{E}$ as its associated vector space. Let $o \in \mathcal{E}$ be $\mathcal{E}$'s origin. The spaces $\mathbb{E}$ and $\mathcal{E}$ are related to each other such that for any point $x \in \mathcal{E}$ there exists a vector $\mathbf{x} \in \mathbb{E}$ such that $o + \mathbf{x} = x$. The topological space $\mathcal{B}$ serves as our model for a rigid body that executes its motion in $\mathcal{E}$. For that reason, we refer to $\mathbb{E}$ and $\mathcal{E}$ as the physical Euclidean vector space and point space, respectively. The spaces $\mathbb{E}_R$ and $\mathcal{E}_R$ are another pair of Euclidean vector and point spaces, respectively, that are related to each other in the same way that $\mathbb{E}$ and $\mathcal{E}$ are related to each other. We refer to $\mathbb{E}_R$ and $\mathcal{E}_R$ as the reference Euclidean vector and point spaces, respectively. The spaces $\mathbb{E}$, $\mathcal{E}$, $\mathbb{E}_R$, and $\mathcal{E}_R$ have the same dimension, which we denote as n_{sd} . The dimension of $\mathcal{B}$ is less than or equal to n_{sd} . We call a select continuous, injective map from $\mathcal{B}$ into $\mathbb{E}_R$ the reference configuration and denote it as κ_R . The elements of $\mathcal{B}$ are called material particles. We call $\mathbf{X} = \kappa_R(\mathcal{X})$, where $\mathcal{X} \in \mathcal{B}$, the reference position vector of the material particle $\mathcal{X}$, and we call the set $\kappa_R(\mathcal{B}) = \{\kappa_R(\mathcal{X}) \in \mathbb{E}_R \mid \mathcal{X} \in \mathcal{B}\}$ the reference body (see Fig. 2.2). When we refer to $\mathbf{X}$ as a material particle we in fact mean the material particle $\kappa_R^{-1}(\mathbf{X}) \in \mathcal{B}$. We model time as a one-dimensional normed vector space $\mathbb{T}$ and denote a typical element in it as $\boldsymbol{\tau} = \tau \mathbf{s}$, where $\tau \in \mathbb{R}$ and $\mathbf{s}$ is a fixed vector in $\mathbb{T}$ of unit norm. We model the rigid body's motion using the one-parameter family of maps $\mathbf{x}_\tau : \mathbb{E}_R \rightarrow \mathbb{E}$ (see Fig. 2.2). We call $\mathbf{x}_\tau$ the deformation map and $\mathbf{x} = \mathbf{x}_\tau(\mathbf{X})$ the material particle $\mathbf{X}$'s position vector at the time instance $\boldsymbol{\tau}$. The set $\kappa_\tau(\mathcal{B}) = \{\mathbf{x}_\tau(\mathbf{X}) \in \mathbb{E} \mid \mathbf{X} \in \kappa_R(\mathcal{B})\}$ (see Fig. 2.2) is called the current body.

2.2.2 Components

The sets $(\mathbf{E}_i)_{i \in \mathcal{I}}$ and $(\mathbf{e}_i)_{i \in \mathcal{I}}$, where $\mathcal{I} = (1, \dots, n_{\text{sd}})$, are orthonormal sets of basis vectors for $\mathbb{E}_{\mathbb{R}}$ and $\mathbb{E}$, respectively. By orthonormal we mean that the inner product between $\mathbf{E}_i$ and $\mathbf{E}_j$, or $\mathbf{e}_i$ and $\mathbf{e}_j$, where $i, j \in \mathcal{I}$, equals δ_{ij} , the Kronecker delta symbol, which equals unity iff $i = j$ and zero otherwise. We call X_i the component of $\mathbf{X}$ w.r.t. $\mathbf{E}_i$ iff $X_i = \mathbf{X} \cdot \mathbf{E}_i$, where the dot denotes the inner product in $\mathbb{E}_{\mathbb{R}}$. The dot in other expressions is to be similarly interpreted noting the space to which the vectors belong. We call the ordered set $(X_i)_{i \in \mathcal{I}}$ the component form of $\mathbf{X}$ w.r.t. $(\mathbf{E}_i)_{i \in \mathcal{I}}$ and denote it as $\mathbf{X}$ or $\mathcal{M}\mathbf{X}$. We denote the space of all $m \times n$ real matrices, where $m, n \in \mathbb{N}$, $\mathcal{M}_{m,n}(\mathbb{R})$; here $\mathbb{N}$ and $\mathbb{R}$ denote the set of natural numbers and the space of real numbers, respectively. Thus, $\mathbf{X} \in \mathcal{M}_{n_{\text{sd}},1}(\mathbb{R})$. We access the i^{th} component, where $i \in \mathcal{I}$, of $\mathbf{X}$, which of course is X_i , as $(\mathbf{X})_i$. Similarly, we denote the component of $\mathbf{x}$ w.r.t. $\mathbf{e}_i$ as x_i and call $\mathbf{x} = (x_i)_{i \in \mathcal{I}} \in \mathcal{M}_{n_{\text{sd}},1}(\mathbb{R})$ the component form of $\mathbf{x}$ w.r.t. $(\mathbf{e}_i)_{i \in \mathcal{I}}$.

Say $\mathbb{W}$ and $\mathbb{U}$ are two arbitrary, oriented, finite dimensional Hilbert spaces; for instance, they can be $\mathbb{E}_{\mathbb{R}}$ and $\mathbb{E}$. We denote the space of all linear maps (transformations/operators) from $\mathbb{W}$ to $\mathbb{U}$ as $\mathcal{L}(\mathbb{W}, \mathbb{U})$ ². We denote the norm of a vector $\mathbf{w}_1$ in $\mathbb{W}$ that is induced by $\mathbb{W}$'s inner product, i.e., $(\mathbf{w}_1 \cdot \mathbf{w}_1)^{1/2}$, as $\|\mathbf{w}_1\|$. For $\mathbf{u}_1 \in \mathbb{U}$, the expression $\mathbf{u}_1 \otimes \mathbf{w}_1$ denotes the linear map from $\mathbb{W}$ to $\mathbb{U}$ defined as

$$(\mathbf{u}_1 \otimes \mathbf{w}_1) \mathbf{w}_2 = \mathbf{u}_1 (\mathbf{w}_1 \cdot \mathbf{w}_2), \quad (2.4)$$

where $\mathbf{w}_2 \in \mathbb{W}$. If the sets $(\mathbf{u}_i)_{i \in \mathcal{I}}$ and $(\mathbf{w}_i)_{i \in \mathcal{I}}$ provide bases for $\mathbb{U}$ and $\mathbb{W}$, respectively, then it can be shown that $\left((\mathbf{u}_i \otimes \mathbf{w}_j)_{j \in \mathcal{I}} \right)_{i \in \mathcal{I}}$, which we will henceforth abbreviate as $(\mathbf{u}_i \otimes \mathbf{w}_j)_{i,j \in \mathcal{I}}$, provides a basis for $\mathcal{L}(\mathbb{W}, \mathbb{U})$. The number T_{ij} , where $i, j \in \mathcal{I}$, is called the component of $\mathbf{T} \in \mathcal{L}(\mathbb{W}, \mathbb{U})$ w.r.t. $\mathbf{u}_i \otimes \mathbf{w}_j$ iff $T_{ij} = \mathbf{u}_i \cdot (\mathbf{T} \mathbf{w}_j)$. We call the nested ordered set $(T_{ij})_{i,j \in \mathcal{I}}$ the component form of $\mathbf{T}$ w.r.t. $(\mathbf{u}_i \otimes \mathbf{w}_j)_{i,j \in \mathcal{I}}$, and denote it as $\mathcal{M}\mathbf{T}$, or, when possible, briefly as $\mathbf{T}$. We sometimes access the $i^{\text{th}}, j^{\text{th}}$ component of $\mathbf{T}$, where $i, j \in \mathcal{I}$, as $(\mathbf{T})_{ij}$.

From here on, unless otherwise specified, we will be following the Einstein summation convention. As per this convention a repeated index in a term will imply a sum over that term with the

²In our previous paper [57], we denoted the set of bounded linear operators from $\mathbb{U}$ to $\mathbb{W}$ as $B(\mathbb{U}, \mathbb{W})$. As a linear operator on a finite dimensional normed space is automatically bounded, here we use $\mathcal{L}(\mathbb{U}, \mathbb{W})$ instead of $B(\mathbb{U}, \mathbb{W})$ to denote the set of all linear operators from $\mathbb{U}$ to $\mathbb{W}$.

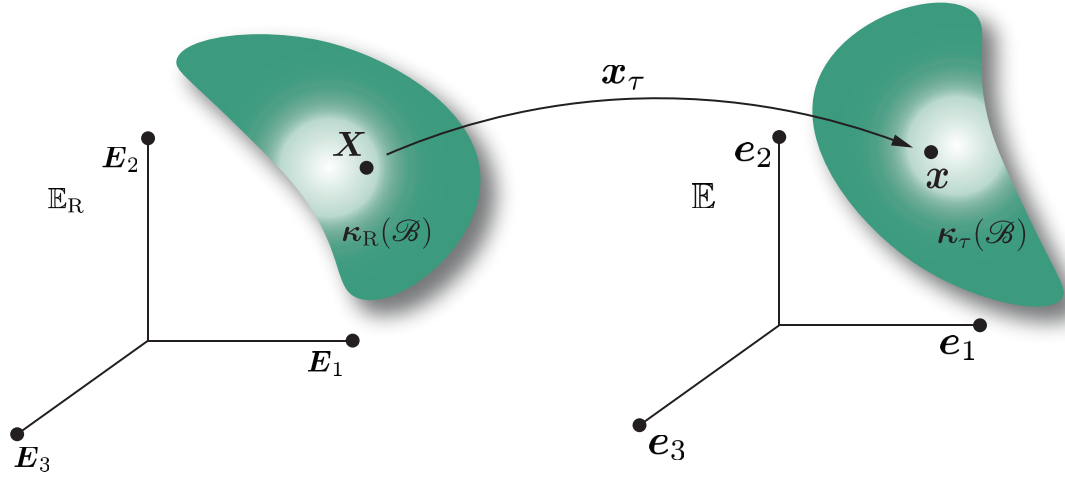


Figure 2.2: Some mathematical quantities used in the description of motion. Illustration of the reference Euclidean vector space $\mathbb{E}_R$, reference body $\kappa_R(\mathcal{B})$, a material particle X , the deformation map x_τ , current body $\kappa_\tau(\mathcal{B})$, the (physical) Euclidean vector space $\mathbb{E}$, and the location of the material particle X in $\mathbb{E}$, i.e., the material particle X 's spatial position vector x . See §2.2.1 for details.

repeated index taking values in $\mathcal{I}$. For example, the expression $X_i E_i$ represents the sum $\sum_{i \in \mathcal{I}} X_i E_i$.

And an unrepeated index in a term will signify a set of n_{sd} terms. For example, the term E_i represents the set $\{E_i \mid i \in \mathcal{I}\}$.

2.2.3 Velocities and Accelerations

For the case of rigid body motion x_τ takes the form

$$x_\tau(X) = Q_\tau X + c(\tau), \quad (2.5)$$

where Q_τ is a proper (orientation preserving), linear isometry from $\mathbb{E}_R$ into $\mathbb{E}$ and $c(\tau) = c_i(\tau)e_i$, where c_i belongs to the space of twice continuously differentiable real valued functions over $\mathbb{R}$, i.e., to $C^2(\mathbb{R}, \mathbb{R})$. The operator Q_τ can be written as $Q_{ij}(\tau)e_i \otimes E_j$, where $Q_{ij} \in C^2(\mathbb{R}, \mathbb{R})$ and satisfy $Q_{ki}(\tau)Q_{kj}(\tau) = \delta_{ij}$ for all $\tau \in \mathbb{R}$. We abbreviate $(Q_{ij}(\tau))_{i,j \in \mathcal{I}} \in \mathcal{M}_{n_{sd}, n_{sd}}(\mathbb{R})$, $(c_i(\tau))_{i \in \mathcal{I}} \in \mathcal{M}_{n_{sd}, 1}(\mathbb{R})$, and $(\delta_{ij})_{i,j \in \mathcal{I}} \in \mathcal{M}_{n_{sd}, n_{sd}}(\mathbb{R})$ as $\mathbf{Q}(\tau)$, $\mathbf{c}(\tau)$, and $\mathbf{I}$, respectively. The component or non-dimensional form of (2.5) is (2.1). Since Q_τ is a proper isometry, it follows that $\mathbf{Q}(\tau)$, which we refer to as the rotation matrix, belongs to the special orthogonal group $SO(n_{sd})$. As a consequence of belonging to $SO(n_{sd})$ the matrix $\mathbf{Q}(\tau)$ satisfies the equations

$$\mathbf{Q}^\top(\tau) \mathbf{Q}(\tau) = \mathbf{I}, \quad (2.6a)$$

and

$$\mathbf{Q}(\tau) \mathbf{Q}^\top(\tau) = \mathbf{I}, \quad (2.6b)$$

where $\mathbf{Q}^\top(\tau)$ is the transpose of $\mathbf{Q}(\tau)$, i.e., $\mathbf{Q}^\top(\tau) = (\mathbf{Q}(\tau))^\top$.

We call $\mathcal{L}(\mathbb{T}, \mathbb{E})$ the physical velocity vector space and denote it as $\mathbb{V}$. It can be shown that the set $(\mathbf{v}_i)_{i \in I}$, where $\mathbf{v}_i \in \mathbb{V}$ and are defined such that $\mathbf{v}_i \tau = \tau \mathbf{e}_i$, provides an orthonormal basis for $\mathbb{V}$. The velocity of a material particle X executing its motion in $\mathbb{E}$ lies in $\mathbb{V}$. The velocity of the material particle X at the instant τ , which we denote as $V_\tau(X)$, equals the value of the Fréchet derivative³ of the map $\mathbb{T} \ni \tau \mapsto \mathbf{x}_X(\tau) \in \mathbb{E}$, where $\mathbf{x}_X(\tau) = \mathbf{x}_\tau(X)$, at the time instance τ . Thus, it follows from (2.5) that

$$V_\tau(X) = L_\tau X + \mathbf{c}'(\tau), \quad (2.7)$$

where $L_\tau := Q'_{ij}(\tau) \mathbf{v}_i \otimes \mathbf{E}_j$ and $\mathbf{c}'(\tau) := c'_i(\tau) \mathbf{v}_i$, and Q'_{ij} and c'_i are the derivatives of Q_{ij} and c_i , respectively. We abbreviate $(Q'_{ij}(\tau))_{i,j \in I} \in \mathcal{M}_{n_{sd}, n_{sd}}(\mathbb{R})$ and $(c'_i(\tau))_{i \in I} \in \mathcal{M}_{n_{sd}, 1}(\mathbb{R})$ as $\mathbf{Q}'(\tau)$ and $\mathbf{c}'(\tau)$, respectively. Using (2.7) and (2.5) it can be shown that the velocity at the time instance τ of the material particle occupying the spatial position $\mathbf{x} \in \mathbb{E}$ at the time instance τ is $\mathbf{W}_\tau(\mathbf{x} - \mathbf{c}(\tau)) + \mathbf{c}'(\tau)$, where the linear map $\mathbf{W}_\tau : \mathbb{E} \rightarrow \mathbb{V}$ is defined by the formula

$$\mathbf{W}_\tau \mathbf{x} = L_\tau \mathbf{Q}_\tau^* \mathbf{x}, \quad (2.8)$$

for all $\mathbf{x} \in \mathbb{E}$. The operator $\mathbf{Q}_\tau^*$ is the Hilbert-adjoint of $\mathbf{Q}_\tau$ and is equal to $Q_{ji}(\tau) \mathbf{E}_i \otimes \mathbf{e}_j$. Let the component form of $\mathbf{W}_\tau$ w.r.t. $(\mathbf{v}_i \otimes \mathbf{e}_j)_{i,j \in I}$ be $(W_{ij}(\tau))_{i,j \in I} \in \mathcal{M}_{n_{sd}, n_{sd}}(\mathbb{R})$, which we abbreviate as $\mathbf{W}(\tau)$. It follows from (2.8) that $W_{ij}(\tau) = Q'_{ik}(\tau) Q_{jk}(\tau)$, or equivalently,

$$\mathbf{W}(\tau) = \mathbf{Q}'(\tau) \mathbf{Q}^\top(\tau). \quad (2.9)$$

³For the definition of Fréchet derivative in the context of the current work see [57, §2.1]

We call $\mathcal{L}(\mathbb{T}, \mathbb{V})$ the physical acceleration vector space and denote it as $\mathbb{A}$. It can be shown that the set $(\mathbf{a}_i)_{i \in \mathcal{I}}$, where $\mathbf{a}_i \in \mathbb{A}$ and are defined such that $\mathbf{a}_i \tau = \tau \mathbf{v}_i$, provides an orthonormal basis for $\mathbb{A}$. The acceleration of a material particle $\mathbf{X}$ executing its motion in $\mathbb{E}$ lies in $\mathbb{A}$. The acceleration of $\mathbf{X}$ at the time instance τ equals the value of the Fréchet derivative of the map $\mathbb{T} \ni \tau \mapsto \mathbf{V}_{\mathbf{X}}(\tau) \in \mathbb{V}$, where $\mathbf{V}_{\mathbf{X}}(\tau) = \mathbf{V}_{\tau}(\mathbf{X})$, at the time instance τ . Thus, it follows from (2.7) that

$$\mathbf{A}_{\tau}(\mathbf{X}) = \mathbf{M}_{\tau} \mathbf{X} + \mathbf{c}''(\tau), \quad (2.10)$$

where the map $\mathbf{M}_{\tau} : \mathbb{E}_{\mathbb{R}} \rightarrow \mathbb{A}$ is defined by the equation

$$\mathbf{M}_{\tau} := \mathcal{Q}_{ij}''(\tau) \mathbf{a}_i \otimes \mathbf{E}_j \quad (2.11)$$

and $\mathbf{c}''(\tau) := c_i''(\tau) \mathbf{a}_i$, where $\mathcal{Q}_{ij}''$ and c_i'' are the derivatives of $\mathcal{Q}_{ij}'$ and c_i' , respectively. Let $A_{\tau i}(\mathbf{X})$ be the component of $\mathbf{A}_{\tau}(\mathbf{X})$ w.r.t. $\mathbf{a}_i$. We abbreviate the ordered sets $(A_{\tau i}(\mathbf{X}))_{i \in \mathcal{I}} \in \mathcal{M}_{n_{\text{sd}}, 1}(\mathbb{R})$, $(\mathcal{Q}_{ij}''(\tau))_{i, j \in \mathcal{I}} \in \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{R})$, and $(c_i''(\tau))_{i \in \mathcal{I}} \in \mathcal{M}_{n_{\text{sd}}, 1}(\mathbb{R})$ as $\mathbf{A}_{\tau}(\mathbf{X})$, $\mathbf{Q}''(\tau)$, and $\mathbf{c}''(\tau)$, respectively.

We will predominantly be presenting the ensuing results in component form. The component form can be converted into physical or dimensional form. Therefore, from here on we will be often omit explicitly using the qualification “is the component form of” when referring to the component form of a physical quantity. For example, instead of saying “ $\mathbf{A}_{\tau}(\mathbf{X})$ as the component form of the acceleration of the material particle $\mathbf{X}$ at the time instance τ ”, we will often write “ $\mathbf{A}_{\tau}(\mathbf{X})$ is the acceleration of the material particle $\mathbf{X}$ at the time instance τ ”. The acceleration $\mathbf{A}_{\tau}(\mathbf{X})$ can be interpreted as the value of the (non-dimensional) acceleration field $\mathbf{A}_{\tau} : B_{\mathbb{R}}(\mathcal{B}) \rightarrow \mathbb{R}^3$, where we call $B_{\mathbb{R}}(\mathcal{B}) := \{(X_1, X_2, X_3) \in \mathbb{R}^3 \mid X_i \mathbf{E}_i \in \kappa_{\mathbb{R}}(\mathcal{B})\}$ the non-dimensional reference body.

2.3 Review of the AO-algorithm

Let $\overline{\mathcal{Q}}_{\tau} : \mathbb{A} \rightarrow \mathbb{E}_{\mathbb{R}}$ be defined by the equation

$$\overline{\mathcal{Q}}_{\tau} = \mathcal{Q}_{ji}(\tau) \mathbf{E}_i \otimes \mathbf{a}_j, \quad (2.12)$$

then we call the map $\bar{\mathbf{A}}_\tau : \kappa_{\mathbb{R}}(\mathcal{B}) \rightarrow \mathbb{E}_{\mathbb{R}}$ defined by the equation

$$\bar{\mathbf{A}}_\tau(\mathbf{X}) = \bar{\mathbf{Q}}_\tau \mathbf{A}_\tau(\mathbf{X}) \quad (2.13)$$

the “Pseudo-acceleration field”. Say $\bar{A}_{\tau i}(\mathbf{X})$ is the component of $\bar{\mathbf{A}}_\tau(\mathbf{X})$ w.r.t. $\mathbf{E}_i$, then we abbreviate $\left(\bar{A}_{\tau i}(\mathbf{X})\right)_{i \in \mathcal{I}} \in \mathcal{M}_{n_{\text{sd}},1}(\mathbb{R})$, the component form of $\bar{\mathbf{A}}_\tau(\mathbf{X})$ w.r.t. $\mathbf{E}_i$, as $\bar{\mathbf{A}}_\tau(\mathbf{X})$. From the definitions of the pseudo acceleration field $\bar{\mathbf{A}}_\tau$ (2.13), and $\bar{\mathbf{A}}_\tau(\mathbf{X})$, and the definitions of $\mathbf{A}_\tau(\mathbf{X})$, and $\mathbf{Q}(\tau)$, which are given in §2.2.3, it follows that

$$\mathbf{A}_\tau(\mathbf{X}) = \mathbf{Q}(\tau) \bar{\mathbf{A}}_\tau(\mathbf{X}). \quad (2.14)$$

In [57, §2.1.1] it was shown that

$$\bar{\mathbf{A}}_\tau(\mathbf{X}) = \mathbf{P}(\tau) \mathbf{X} + \mathbf{q}(\tau), \quad (2.15)$$

where

$$\mathbf{P}(\tau) = \mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau) \quad (2.16)$$

is the component form of the linear map $\mathbf{P}_\tau := \bar{\mathbf{Q}}_\tau \circ \mathbf{M}_\tau$ w.r.t. $(\mathbf{E}_i \otimes \mathbf{E}_j)_{i,j \in \mathcal{I}}$, and $\mathbf{q}(\tau) \in \mathcal{M}_{n_{\text{sd}},1}(\mathbb{R})$

is the component form of

$$\mathbf{q}(\tau) := \bar{\mathbf{Q}}_\tau \mathbf{c}''(\tau) \quad (2.17)$$

w.r.t. $(\mathbf{E}_i)_{i \in \mathcal{I}}$. Thus, the acceleration field $\mathbf{A}_\tau$ is taken to be fully determined once $\mathbf{Q}(\tau)$, $\mathbf{P}(\tau)$, and $\mathbf{q}(\tau)$ have been computed.

Both the AO and the $\sqrt{\text{AO}}$ algorithms can be described as consisting of three primary steps. The AO-algorithm’s three steps can briefly be described as follows:

AO-Step 1 Compute (time discrete versions of) the maps $\tau \mapsto \mathbf{P}(\tau)$ and $\tau \mapsto \mathbf{q}(\tau)$ using the measurements and the geometry of the arrangement of the four tri-axial accelerometers.

AO-Step 2 Compute the map $\tau \mapsto \bar{\mathbf{W}}(\tau)$, where

$$\bar{\mathbf{W}}(\tau) := \mathbf{Q}^\top(\tau) \mathbf{W}(\tau) \mathbf{Q}(\tau), \quad (2.18)$$

using the map $\mathbf{P}$ computed in [AO-Step 1](#) and numerically integrating (2.2). From Lemma A.2.1 we have that the matrix $\overline{\mathbf{W}}(\tau)$ belongs to the space of $n_{sd} \times n_{sd}$ real skew-symmetric matrices, which we denote as $\mathfrak{so}(\mathbb{R}, n_{sd})$.

AO-Step 3 Compute the map $\tau \mapsto \mathbf{Q}(\tau)$ using the $\overline{\mathbf{W}}$ map computed in [AO-Step 2](#) and numerically integrating the equation

$$\mathbf{Q}'(\tau) = \mathbf{Q}(\tau)\overline{\mathbf{W}}(\tau). \quad (2.19)$$

Equation (2.19) is from [57], where it appears as equation 3.14.

Step one of the $\sqrt{\text{AO}}$ -algorithm has two sub-steps: the *predictor* step and the *corrector* step. The predictor step is the same as [AO-Step 1](#) of the AO-algorithm. The corrector step is necessary for carrying out step two of the $\sqrt{\text{AO}}$ -algorithm. In step two of the $\sqrt{\text{AO}}$ -algorithm instead of obtaining $\overline{\mathbf{W}}$ from (2.2), as is done in the AO-algorithm, we obtain it from (2.3). More precisely, in the $\sqrt{\text{AO}}$ -algorithm $\overline{\mathbf{W}}(\tau)$ is obtained as the square root of the symmetric part of $\mathbf{P}(\tau)$. We use $\text{sym}(\mathbf{P}(\tau))$ to denote the symmetric part of $\mathbf{P}(\tau)$. The derivation of (2.3) is presented in A.2. A procedure for determining $\overline{\mathbf{W}}(\tau)$ as the square root of $\text{sym}(\mathbf{P}(\tau))$, i.e., for solving (2.3) for $\overline{\mathbf{W}}(\tau)$ with given $\mathbf{P}(\tau)$, is presented in §2.4.2. The goal of step three of the $\sqrt{\text{AO}}$ -algorithm is to compute $\mathbf{Q}$ using the $\overline{\mathbf{W}}$ computed in step two. It involves using a slightly modified version of the numerical integration scheme described by equations 3.15, 3.16, and 3.17 in [57, §3.2] to solve (2.19). We discuss it in §2.4.3.

2.4 The $\sqrt{\text{AO}}$ -algorithm

As we mentioned in §2.3 the $\sqrt{\text{AO}}$ algorithm consists of three primary steps. Those steps are as follows:

Step 1 Compute (time discrete versions of) the maps $\tau \mapsto \mathbf{P}(\tau)$ and $\tau \mapsto \mathbf{q}(\tau)$ using the measurements and the geometry of the arrangement of the four tri-axial accelerometers (see §2.4.1 for details).

Step 2 Use (2.3) and the $\mathbf{P}$ map obtained from [Step 1](#) to solve for $\tau \mapsto \overline{\mathbf{W}}(\tau)$. That is, for each τ

in a discrete sequence of time instances, compute $\overline{\mathbf{W}}(\tau)$ as the square root of the symmetric part of $\mathbf{P}(\tau)$ (for details see §2.4.2).

Step 3 Compute (a time discrete version of) the map $\tau \mapsto \mathbf{Q}(\tau)$ using the $\overline{\mathbf{W}}$ map computed in Step 2 and numerically integrating (2.19) (details in §2.4.3).

2.4.1 $\sqrt{\text{AO}}$ -algorithm, Step 1 of 3

In §3.1 of [57] a method was presented to estimate $\mathbf{P}(\tau)$ and $\mathbf{q}(\tau)$ from the accelerometer measurements corresponding to the time instance τ . Applying that method for each τ in a discrete time sequence yields a numerical approximation for the maps $\tau \mapsto \mathbf{P}(\tau)$ and $\tau \mapsto \mathbf{q}(\tau)$. We present here an augmented version of that method for computing similar numerical approximations. The primary difference between our method and that presented in [57] is that the estimate for $\mathbf{P}(\tau)$ yielded by our method is certain to retain some of the mathematical properties that are expected of $\mathbf{P}(\tau)$ based on our theoretical analysis. Specifically, it follows from Lemmas A.3.1 and A.3.2 that $\text{sym}(\mathbf{P}(\tau))$ is a negative semidefinite matrix with its negative eigenvalues, if any, being of even algebraic multiplicities. These mathematical properties of $\mathbf{P}(\tau)$ are critical for carrying out Step 2 of the $\sqrt{\text{AO}}$ -algorithm. We found that experimental noise and errors can cause the estimate for $\mathbf{P}(\tau)$ provided by the method presented in [57] to lose the aforementioned mathematical properties. Our method, on the contrary, ensures that the symmetric part of the estimated $\mathbf{P}(\tau)$ is negative semidefinite and that its negative eigenvalues, when they exist, are of even algebraic multiplicities. Once $\mathbf{P}(\tau)$ is estimated, our method to estimate $\mathbf{q}(\tau)$ is exactly the same as that in [57]. We review it in 2.4.1.2.

2.4.1.1 Estimating $\mathbf{P}(\tau)$

Our method for estimating $\mathbf{P}(\tau)$ can be described as consisting of two steps: a *predictor step* and a *corrector step*. In the predictor step we use the method presented in [57, §3.1] for estimating $\mathbf{P}(\tau)$ to compute a prediction for $\mathbf{P}(\tau)$. We denote this prediction as $\mathbf{P}(\tau)^p$. In the corrector step we estimate $\mathbf{P}(\tau)$ as the sum of $\mathbf{P}(\tau)^p$ and a correction term, which we construct using $\mathbf{P}(\tau)^p$. The correction term is constructed such that the estimated $\mathbf{P}(\tau)$ is as close as possible to $\mathbf{P}(\tau)^p$ under the constraint

that the estimated $\mathbf{P}(\tau)$'s symmetric part is negative semidefinite and its negative eigenvalues (if they exist) are of even algebraic multiplicities.

Predictor step Say the four tri-axial accelerometers are attached to the rigid body $\mathcal{B}$ at the material particles $(^\ell\mathcal{X})_{\ell \in \mathcal{J}}$, where $\mathcal{J} := (1, \dots, 4)$, and let the position vectors of those particles in $\mathbb{E}_R$, respectively, be $(^\ell\mathbf{X})_{\ell \in \mathcal{J}}$ (see Fig. 2.3). A tri-axial accelerometer is capable of measuring the components of its acceleration in three mutually perpendicular directions. We refer to those directions as the accelerometer's measurement directions. The measurement directions are usually marked on the accelerometer package by the manufacturer as arrows that are labeled x , y , and z . As $\mathcal{B}$ moves in $\mathbb{E}$, the attached accelerometers move with it, and, therefore, the measurement directions (in $\mathbb{E}$) can change with time. For an accelerometer ℓ , where $\ell \in \mathcal{J}$, we denote its time varying measurement directions in $\mathbb{E}$ using the orthonormal set $(^\ell\mathbf{e}_{\tau i})_{i \in \mathcal{I}}$. Assuming that the accelerometers remain rigidly attached to $\mathcal{B}$, i.e., their positions and orientations w.r.t. $\mathcal{B}$ do not change as $\mathcal{B}$ moves in $\mathbb{E}$, it can be shown that $\mathbf{Q}_\tau^* {}^\ell\mathbf{e}_{\tau i}$, where $\ell \in \mathcal{J}, i \in \mathcal{I}$, is a constant vector in $\mathbb{E}_R$, which we denote as $^\ell\mathbf{E}_i$. The position vectors $(^\ell\mathbf{X})_{\ell \in \mathcal{J}}$ and the directions $(^\ell\mathbf{E}_i)_{\ell \in \mathcal{J}, i \in \mathcal{I}}$ are known from the arrangement and orientation of the accelerometers at the experiment's beginning.

For $\ell \in \mathcal{J}$, let $^\ell\bar{\mathbf{A}}(\tau) := (^\ell\alpha_j(\tau) {}^\ell\mathbf{E}_j \cdot \mathbf{E}_i)_{i \in \mathcal{I}}$ (no sum over ℓ), where $^\ell\alpha_i(\tau)$, $i \in \mathcal{I}$, is the measurement reported by accelerometer $^\ell\mathbf{X}$ for the (non-dimensional) component of its acceleration in the $^\ell\mathbf{e}_{\tau i}$ direction⁴ at the time instance τ . And let $^\ell\mathbf{X} := (^\ell\mathbf{X} \cdot \mathbf{E}_i)_{i \in \mathcal{I}}$. Then, we compute $\mathbf{P}(\tau)^p$ as $\mathbf{P}(\tau)$ is estimated in [57] using the equation

$$\mathbf{P}(\tau)^p = \left(\left({}^i\Delta\bar{\mathbf{A}}(\tau) \right)^\top \left({}^j\Delta\mathbf{X} \right) \right) \left(({}^i\Delta\mathbf{Y}) \left({}^j\Delta\mathbf{Y} \right)^\top \right), \quad (2.20)$$

where ${}^i\Delta\bar{\mathbf{A}}(\tau) := {}^{i+1}\bar{\mathbf{A}}(\tau) - {}^i\bar{\mathbf{A}}(\tau)$, ${}^i\Delta\mathbf{X} := {}^{i+1}\mathbf{X} - {}^i\mathbf{X}$. The ordered sets ${}^i\Delta\mathbf{Y}$ belong to $\mathcal{M}_{n_{sd}, 1}(\mathbb{R})$ and are defined by the equation

$$\left({}^1\Delta\mathbf{Y}, \dots, {}^{n_{sd}}\Delta\mathbf{Y} \right) = \left({}^1\Delta\mathbf{X}, \dots, {}^{n_{sd}}\Delta\mathbf{X} \right)^{-\top}, \quad (2.21)$$

where $(\cdot)^{-\top}$ is the operator that acts on an invertible element of $\mathcal{M}_{n_{sd}, n_{sd}}(\mathbb{R})$ and returns the transpose

⁴Or to be mathematically precise, in the $^\ell\mathbf{a}_{\tau i} \in \mathbb{A}$ direction that is defined such that $(^\ell\mathbf{a}_{\tau i} \mathbf{s}) \mathbf{s} = {}^\ell\mathbf{e}_{\tau i}$.

of its inverse.

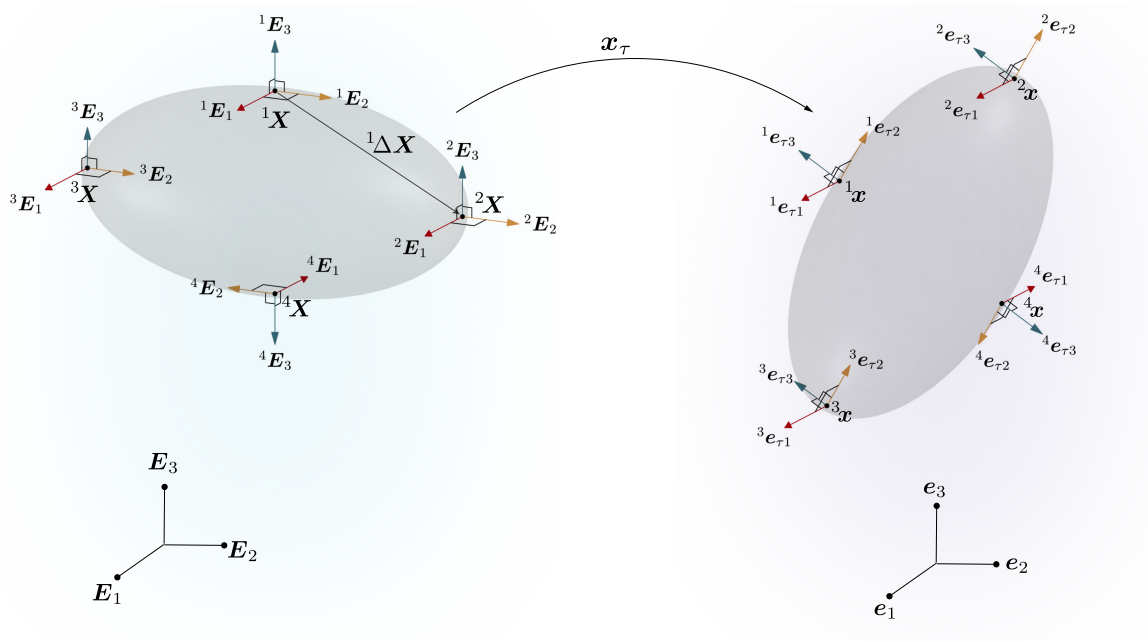


Figure 2.3: Schematic of the locations and orientations of four tri-axial accelerometers (left) and their motion (right) (modified from [57], copyright 2020, Elsevier).

Corrector step In A.4.1 we show that $\text{sym}(\mathbf{P}(\tau))$ allows itself to be decomposed as

$$\mathbf{N}(\tau) \mathbf{D}(\tau) \mathbf{N}^\top(\tau), \quad (2.22a)$$

where $\mathbf{N}(\tau) \in \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{R})$ is an orthogonal matrix, i.e.,

$$\mathbf{N}^\top(\tau) \mathbf{N}(\tau) = \mathbf{I}, \quad (2.22b)$$

and $\mathbf{D}(\tau) \in \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{R})$ is a diagonal matrix that for $n_{\text{sd}} = 2$ and 3 , respectively, has the form

$$\mathbf{D}(\tau) = \text{diag}(-\lambda(\tau)^2, -\lambda(\tau)^2) \text{ and } \text{diag}(0, -\lambda(\tau)^2, -\lambda(\tau)^2), \quad (2.22c)$$

where $\lambda(\tau) \in \mathbb{R}$ and the function $\text{diag}(\cdot) : \mathbb{F}^{n_{\text{sd}}} \rightarrow \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{F})$, where $\mathbb{F}$ is either $\mathbb{R}$ or $\mathbb{C}$, is defined such that $\text{diag}(a_1, \dots, a_{n_{\text{sd}}})$ is a diagonal matrix with diagonal entries $a_1, \dots, a_{n_{\text{sd}}}$.

The matrix $\text{sym}(\mathbf{P}(\tau))$ allowing the decomposition (2.22) is critical for carrying out Step 2 of the $\sqrt{\text{AO}}$ -algorithm. In an ideal scenario, in which there are no experimental errors or noise in the

accelerometer measurements, $\mathbf{P}(\tau)^p$ would be the same as $\mathbf{P}(\tau)$. However, due to the experimental noise and other errors $\mathbf{P}(\tau)^p$ will generally be different from $\mathbf{P}(\tau)$. In general, such a deviation would not be of much consequence, since, experimental measurements of physical quantities, more often than not, are different from the true values of those quantities. Thus, generally, we would, as done by [57], take $\mathbf{P}(\tau)^p$ to be the final estimate for $\mathbf{P}(\tau)$ and no longer distinguish between $\mathbf{P}(\tau)^p$ and $\mathbf{P}(\tau)$. However, in the present case the deviation of $\mathbf{P}(\tau)^p$ from $\mathbf{P}(\tau)$ has an important consequence which requires us to not take $\mathbf{P}(\tau)^p$ as $\mathbf{P}(\tau)$'s final estimate. The important consequence is that in general $\text{sym}(\mathbf{P}(\tau)^p)$ will not allow a decomposition of the form (2.22). In general, it will only allow itself to be decomposed as $\mathbf{N}_\rho(\tau) \text{diag}(\lambda_1(\tau), \dots, \lambda_{n_{sd}}(\tau)) \mathbf{N}_\rho^T(\tau)$, where $\mathbf{N}_\rho(\tau)$'s columns are the eigenvectors of $\text{sym}(\mathbf{P}(\tau)^p)$ that are chosen such that $\mathbf{N}_\rho(\tau)$ is orthogonal and their corresponding eigenvalues $\lambda_i(\tau) \in \mathbb{R}$ form a non-increasing sequence, i.e., $\lambda_1(\tau) \geq \dots \geq \lambda_{n_{sd}}(\tau)$. Therefore, instead of taking $\mathbf{P}(\tau)^p$ as the final estimate of $\mathbf{P}(\tau)$ we derive the final estimate for $\mathbf{P}(\tau)$, as we detail next, using $\mathbf{P}(\tau)^p$ so that its symmetric part does allow a decomposition of the form (2.22).

We take the skew-symmetric part of our final estimate for $\mathbf{P}(\tau)$ to be the same as that of $\mathbf{P}(\tau)^p$. We take its symmetric part to be

$$\mathbf{N}_\rho(\tau) \check{\mathbf{D}}(\tau) \mathbf{N}_\rho^T(\tau), \quad (2.23a)$$

where

$$\check{\mathbf{D}}(\tau) := \text{diag}\left(0, -\check{\lambda}(\tau)^2, -\check{\lambda}(\tau)^2\right), \quad (2.23b)$$

with

$$\check{\lambda}(\tau) := \begin{cases} \sqrt{-\frac{\lambda_2(\tau) + \lambda_3(\tau)}{2}}, & \lambda_2(\tau) + \lambda_3(\tau) \leq 0, \\ 0, & \lambda_2(\tau) + \lambda_3(\tau) > 0, \end{cases} \quad (2.23c)$$

for $n_{\text{sd}} = 3$, and

$$\check{\mathbf{D}}(\tau) := \text{diag} \left(-\check{\lambda}(\tau)^2, -\check{\lambda}(\tau)^2 \right), \quad (2.23d)$$

with

$$\check{\lambda}(\tau) := \begin{cases} \sqrt{-\frac{\lambda_1(\tau) + \lambda_2(\tau)}{2}}, & \lambda_1(\tau) + \lambda_2(\tau) \leq 0, \\ 0, & \lambda_1(\tau) + \lambda_2(\tau) > 0, \end{cases} \quad (2.23e)$$

for $n_{\text{sd}} = 2$.

The orthogonal matrix $\mathbf{N}_{\rho}(\tau)$ and the eigenvalues $\lambda_i(\tau)$ can be obtained from the spectral or symmetric-Schur [101, §8] decomposition of $\text{sym}(\mathbf{P}(\tau)^{\text{p}})$. Since $\text{sym}(\mathbf{P}(\tau)^{\text{p}})$ is a real symmetric matrix, it is always possible to carry out $\text{sym}(\mathbf{P}(\tau)^{\text{p}})$'s spectral or symmetric Schur decomposition.

To summarize, we take $\mathbf{P}(\tau)$'s final estimate to be

$$\mathbf{P}^{\text{p}}(\tau) + \Delta\mathbf{P}(\tau), \quad (2.24a)$$

where

$$\Delta\mathbf{P}(\tau) := \text{sym} \left(\check{\mathbf{P}}(\tau) - \mathbf{P}^{\text{p}}(\tau) \right), \quad (2.24b)$$

$$\check{\mathbf{P}}(\tau) := \mathbf{N}_{\rho}(\tau) \check{\mathbf{D}}(\tau) \mathbf{N}_{\rho}^{\text{T}}(\tau). \quad (2.24c)$$

It can be ascertained that the symmetric part of our final estimate for $\mathbf{P}(\tau)$, namely $\check{\mathbf{P}}(\tau)$, allows a decomposition of the form (2.22). In fact, that decomposition is precisely the one given by (2.23).

For $n_{\text{sd}} = 2$ or 3 it can be shown that $\check{\mathbf{P}}(\tau)$ is the best approximation, in the Frobenius norm, to $\text{sym}(\mathbf{P}^p(\tau))$ in the set of $n_{\text{sd}} \times n_{\text{sd}}$ real symmetric negative-semidefinite matrices whose negative eigenvalues (when they exist) are of even algebraic multiplicities.

2.4.1.2 Estimating $\mathbf{q}(\tau)$

After estimating $\mathbf{P}(\tau)$ as described in §2.4.1.1 using (2.24) we estimate $\mathbf{q}(\tau)$ as

$$\ell \bar{\mathbf{A}}(\tau) - \mathbf{P}(\tau) \ell \mathbf{X}, \quad (2.25)$$

where ℓ is some particular integer in $\mathcal{J}$.

2.4.2 $\sqrt{\text{AO}}$ -algorithm, Step 2 of 3

In A.4.2 we show using $\mathbf{P}(\tau)$'s decomposition (2.22) that $\bar{\mathbf{W}}(\tau)$ can be computed from (2.3) as

$$\bar{\mathbf{W}}(\tau) = \pm \begin{cases} \mathbf{N}(\tau) \star (\lambda(\tau)) \mathbf{N}^T(\tau), & n_{\text{sd}} = 2, \\ \mathbf{N}(\tau) \star ((\lambda(\tau), 0, 0)) \mathbf{N}^T(\tau), & n_{\text{sd}} = 3, \end{cases} \quad (2.26)$$

where the map $\star(\cdot)$ is defined in A.1.

Using the decomposition (2.23) for the symmetric part of our final estimate for $\mathbf{P}(\tau)$ and similar calculations as those used in §A.4.2, it can be shown that if we compute our final estimate for $\bar{\mathbf{W}}(\tau)$ as

$$\pm \begin{cases} \mathbf{N}_{\rho}(\tau) \star (\check{\lambda}(\tau)) \mathbf{N}_{\rho}^T(\tau), & n_{\text{sd}} = 2, \\ \mathbf{N}_{\rho}(\tau) \star ((\check{\lambda}(\tau), 0, 0)) \mathbf{N}_{\rho}^T(\tau), & n_{\text{sd}} = 3, \end{cases} \quad (2.27)$$

then it and our final estimate for $\mathbf{P}(\tau)$ will satisfy (2.3).

We take the time discrete versions of the $\bar{\mathbf{W}}$ and $\mathbf{P}$ maps to be constant over each time interval $\Delta\tau_n := [n\Delta\tau, (n+1)\Delta\tau)$, where $n \in (0, 1, \dots)$ and $\Delta\tau \in \mathbb{R}$. We denote the values of these two maps over $\Delta\tau_n$ as $\bar{\mathbf{W}}(n)$ and $\mathbf{P}(n)$, respectively. The quantity $\bar{\mathbf{W}}(0)$ is known from initial conditions. For

$n > 0$ we compute $\overline{\mathbf{W}}(n)$ using (2.27). Using the positive and negative signs in (2.27) will give us two different estimates for $\overline{\mathbf{W}}(n)$. Among those two estimates we choose the one that is closer to $\overline{\mathbf{W}}$'s value over the previous time interval. To be precise, we choose the estimate that gives a lower value for the metric $m(\overline{\mathbf{W}}(n), \overline{\mathbf{W}}(n-1))$, where $m : \mathfrak{so}(\mathbb{R}, n_{\text{sd}})^2 \rightarrow \mathbb{R}$,

$$m(\overline{\mathbf{W}}(n), \overline{\mathbf{W}}(n-1)) = \arccos \left(\frac{\star(\overline{\mathbf{W}}(n)) \cdot \star(\overline{\mathbf{W}}(n-1))}{\|\star(\overline{\mathbf{W}}(n))\| \|\star(\overline{\mathbf{W}}(n-1))\|} \right). \quad (2.28)$$

The metric (2.28) is a measure of the difference between $\overline{\mathbf{W}}(n)$ and $\overline{\mathbf{W}}(n-1)$. Our criterion for choosing between the two estimates given by (2.27) is essentially based on the assumption that in most practical scenarios $\overline{\mathbf{W}}$ should be a continuous function of time, and on the ansatz that due to the continuity of $\overline{\mathbf{W}}$ the true $\overline{\mathbf{W}}(n)$ would be the one that is closer to $\overline{\mathbf{W}}(n-1)$ when $\|\star(\overline{\mathbf{W}}(n-1))\|$ is large.

If $\overline{\mathbf{W}}(n-1) = \mathbf{0}$ or when $\|\star(\overline{\mathbf{W}}(n-1))\|$ is small, then our above criterion for choosing between the two estimates for $\overline{\mathbf{W}}(n)$ cannot be used. In such cases we first derive a prediction for $\overline{\mathbf{W}}(n)$ by applying the AO-algorithm to the previous time interval and then choose the estimate that is closer to that prediction.

2.4.3 $\sqrt{\text{AO}}$ -algorithm, Step 3 of 3

We use a slightly modified version of the numerical integration scheme described by equations 3.15, 3.16, and 3.17 in [57, §3.2] to solve (2.19). As we did with $\overline{\mathbf{W}}$ and $\mathbf{P}$, we assume that the discrete version of $\mathbf{Q}$ remains constant over each time interval $\Delta\tau_n$ and denote its constant values as $\mathbf{Q}(n)$, where $n \in (0, 1, \dots)$. The matrix $\mathbf{Q}(0)$ is taken to be known from the initial conditions of the experiment. For $n > 0$ the matrix $\mathbf{Q}(n)$ is computed as

$$\mathbf{Q}(n) = \mathbf{Q}(n-1) e^{\Delta\tau \overline{\mathbf{W}}_{n-\frac{1}{2}}}, \quad (2.29a)$$

where the map $\mathbf{e}^{(\cdot)} : \mathfrak{so}(\mathbb{R}, n_{sd}) \rightarrow SO(n_{sd})$ is defined by the equation⁵

$$\mathbf{e}^{(\cdot)} = \mathbf{I} + \text{sinc}(\|\star(\cdot)\|)(\cdot) + \frac{1}{2} \left(\text{sinc}\left(\frac{\|\star(\cdot)\|}{2}\right) \right)^2 (\cdot)^2, \quad (2.29b)$$

and

$$\overline{\mathbf{W}}_{n-\frac{1}{2}} := \frac{1}{2} \left(\overline{\mathbf{W}}(n) + \overline{\mathbf{W}}(n-1) \right). \quad (2.29c)$$

The difference between the integration scheme (2.29) and that given by 3.15, 3.16, and 3.17 in [57, §3.2] is the manner in which $\overline{\mathbf{W}}_{n-\frac{1}{2}}$ is computed. In Rahaman *et al.*'s integration scheme, $\overline{\mathbf{W}}_{n-\frac{1}{2}}$ is computed as $\overline{\mathbf{W}}(n-1) + \frac{\Delta\tau}{2} \text{skew}(\mathbf{P}(n-1))$, whereas we compute it using (2.29c). Here we use $\text{skew}(\mathbf{P}(n-1))$ to denote the skew-symmetric part of the matrix $\mathbf{P}(n-1)$.

2.5 *In silico* validation, evaluation and comparison of the $\sqrt{\text{AO}}$ -algorithm

In this section, we check the validity and robustness of the $\sqrt{\text{AO}}$ -algorithm. We do that by feeding in virtual accelerometer data, to which differing amounts of bias and noise type errors have been added, to the $\sqrt{\text{AO}}$ and AO algorithms, and comparing their resulting predictions. We discuss the creation of the virtual accelerometer data in §2.5.1, and simulating bias and noise type errors in §2.5.2. We compare the predictions in §2.5.3.

2.5.1 Virtual accelerometer data from the simulation of a rigid ellipsoid impacting an elastic half-space

The virtual accelerometer data we use for the comparison is from the numerical simulation of a rigid ellipsoid impacting an elastic half-space. This simulation is presented and discussed in detail in [57], starting in §4. However, for the readers convenience we give a very brief description of that simulation here.

⁵This equation is the corrected version of equation 3.17 in [57, §3.2], which has two typos in it.

In the simulation an ellipsoid, $\mathcal{B}$, is dropped onto an elastic half-space, H , under the action of gravity with the initial angular and translational velocities prescribed (see Fig. 2.5). In the simulation the Euclidean point space $\mathcal{E}$, in which the ellipsoid and the half-space, respectively, execute their motion and deformation, is taken to be three dimensional, i.e., $n_{sd} = 3$. The vectors $\mathbf{E}_i$ and $\mathbf{e}_i$, $i \in \mathcal{I}$, are taken to have units of meters and s to have units of seconds. Hence $\mathbf{v}_i$ and $\mathbf{a}_i$, $i \in \mathcal{I}$, have units of meters-per-second and meters-per-second-squared, respectively.

The reference configuration of the ellipsoid is given in Fig. 2.4. In $\mathcal{E}_R$ the ellipsoid occupies the region $\{(X_1, X_2, X_3) \mid (X_1/a)^2 + (X_2/b)^2 + (X_3/c)^2 \leq 1\}$, where $(a, b, c) = (0.15, 0.10, 0.08)$. The half-space when it is undeformed in $\mathcal{E}$ occupies the region $x_3 < 0$. The initial location and orientation of $\mathcal{B}$ w.r.t. H in $\mathcal{E}$ are shown in Fig. 2.5. They correspond to the initial conditions $\mathbf{Q}(0) = \text{diag}(1, 1, 1)$ and $\mathbf{c}(0) = (0, 0, 0.75)$.

The mechanics of H is modeled using the theory of small deformation linear elasto-statics and taking H 's Young's modulus and Poisson's ratio to be 10^4 Pa and 0.3, respectively. The ellipsoid is rigid and homogeneous. Its density and total mass are 1989.44 kg/m^3 and 10 kg, respectively.

The ellipsoid's dynamics are obtained by numerically solving its linear and angular momentum balance equations. The force in those equations arises due to the action of gravity on $\mathcal{B}$ and $\mathcal{B}$'s interaction with H ; and the torque exclusively from $\mathcal{B}$'s interaction with H . The interaction between $\mathcal{B}$ and H is modeled using the Hertz contact theory, e.g., [102, 103]. For details on effecting a numerical solution to the balance equations see [57, §B1.1]. For more details regarding the contact modeling see [57, §B1.2].

Four virtual accelerometers are taken to be rigidly attached to the ellipsoid's material particles ${}^\ell\mathcal{X}$, $\ell \in \mathcal{J}$. The locations and orientations of those accelerometers w.r.t. $\mathcal{B}$ in $\mathcal{E}_R$ are shown in Fig. 2.4. Their position vectors ${}^\ell\mathbf{X}$ and orientations $({}^\ell\mathbf{E}_i)_{i \in \mathcal{I}}$, $\ell \in \mathcal{J}$, are given in the caption of Fig. 2.4. The acceleration of any of the ellipsoid's material particles can be obtained from the simulation results using the procedure outlined in [57, §B2]. For $i \in \mathcal{I}$, the values of ${}^1\alpha_i$, which is the component of ${}^1\mathbf{X}$'s acceleration in the ${}^1\mathbf{e}_{\tau i}$ direction, or to be more precise the ${}^1\mathbf{a}_{\tau i}$ direction⁴ (see Fig. 2.5), at a sequence of time instances are shown in Fig. 2.6(a).

2.5.2 Adding synthetic errors to virtual accelerometer data

The acceleration components ℓ_{α_i} , $\ell \in \mathcal{J}$, from the simulation do not contain any errors; other than, of course, the errors that arise due to numerical discretization of the balance equations, numerical round-off, etc. However, those type of errors are of insignificant magnitude. Using the error free virtual accelerometer data ℓ_{α_i} , $\ell \in \mathcal{J}$, from the simulation we generate virtual error-inclusive accelerometer data $\ell_{\alpha_i}^{\text{Error}}$, $\ell \in \mathcal{J}$, as

$$\ell_{\alpha_i}^{\text{Error}}(\tau) = \ell_{\alpha_i}(\tau) + \eta_{\tau}. \quad (2.30)$$

In equation (2.30) η_{τ} denotes a particular realization of the Ornstein-Uhlenbeck (OU) process [104]. We will describe shortly what we mean by a “realization”. The OU process is a continuous time and state stochastic process that is defined by the integral equation

$$\eta_{\tau_1+\tau_2} - \eta_{\tau_1} = \beta \int_{\tau_1}^{\tau_1+\tau_2} (\mu - \eta_{\tau}) d\tau + \sigma \int_{\tau_1}^{\tau_1+\tau_2} dW_{\tau}, \quad (2.31)$$

where the second integral on the right is an Itô integral and W_{τ} is the Wiener process [105]. The real number μ is called the mean value, $\sigma \geq 0$ the diffusion coefficient, and $\beta > 0$ the drift coefficient. The symbols τ_1 , τ_2 denote any two (non-dimensional) time instances. Since the OU process is a stochastic process, a given set of OU parameters, i.e., a particular set of μ , σ , and β values, define an entire family or population of real valued functions on $\mathbb{R}$. For a given OU parameter set, a particular realization of the OU process is obtained by drawing η_0 from a Gaussian distribution of mean μ and variance $\sigma^2/(2\beta)$ and solving (2.31). As a consequence of (2.30), $\ell_{\alpha_i}^{\text{Error}}$, $\ell \in \mathcal{J}$, too are stochastic processes.

For $i \in \mathcal{I}$ and $\ell \in \mathcal{J}$, when $\mu \neq 0$ and $\sigma = 0$, any particular realization of $\ell_{\alpha_i}^{\text{Error}}$ will contain only bias type errors. A representative realization of $\ell_{\alpha_2}^{\text{Error}}$ for $\mu = 5$, $\sigma = 0$, and $\beta = 10^3$ is shown in Fig. 2.6(b). Alternatively, when $\mu = 0$ and $\sigma \neq 0$ any particular realization of $\ell_{\alpha_i}^{\text{Error}}$ will only contain noise type errors. A representative realization of $\ell_{\alpha_2}^{\text{Error}}$ for $\mu = 0$, $\sigma = 10^2$, and $\beta = 10^3$ is shown in Fig. 2.6(c). In general when μ and σ are both non-zero, realizations of $\ell_{\alpha_i}^{\text{Error}}$ will contain both bias and noise type errors. A representative realization of $\ell_{\alpha_2}^{\text{Error}}$ for $\mu = 5$, $\sigma = 10^2$,

and $\beta = 10^3$ is shown in Fig. 2.6(d).

From here on unless otherwise specified the value of β will always be equal to 10^3 .

2.5.3 Comparison of $\sqrt{\text{AO}}$ and AO algorithms using virtual error-inclusive accelerometer data

We compare the predictions of the $\sqrt{\text{AO}}$ and AO algorithms for the following categories of OU parameter sets.

Category I Exclusively bias type errors: $\mu = 0, 0.1, 0.2, 0.5$, and 1 , and $\sigma = 0$ (see Table 2.1).

Category II Exclusively noise type errors: $\mu = 0$, and $\sigma = 0, 1, 10, 50$, and 10^2 (see Table 2.2).

Category III Both bias and noise type errors: $\mu = 0, 0.1, 0.2, 0.5$, and 1 , and $\sigma = 10$ (see Table 2.3).

For a given OU parameter set we generate a large number of $\ell_{\alpha_i^{\text{Error}}}$, $\ell \in \mathcal{J}$, realizations. We apply the $\sqrt{\text{AO}}$ and AO algorithms to each of those realizations and derive a population of predictions for the acceleration of the material particle ${}^5\mathcal{X}$ (see Fig. 2.4). We denote the error-free (non-dimensional) acceleration of ${}^5\mathcal{X}$, which we know from the rigid-ellipsoid-impact-simulation's results, at the time instance τ as ${}^5\mathbf{A}(\tau) \in \mathcal{M}_{3,1}(\mathbb{R})$. The components of ${}^5\mathbf{A}(\tau)$, i.e., $({}^5\mathbf{A}(\tau))_i, i \in \mathcal{I}$, for a sequence of time instances are, respectively, shown in subfigures (a), (b), and (c) in each of Figs. 2.7–2.9. They are shown using thick gray curves.

Since the predictions of the $\sqrt{\text{AO}}$ and AO algorithms are derived by, respectively, feeding the $\sqrt{\text{AO}}$ and AO algorithms the stochastic processes $\ell_{\alpha_i^{\text{Error}}}$, $\ell \in \mathcal{J}$, they too, in fact, are stochastic processes. Representative realizations of the predictions from the $\sqrt{\text{AO}}$ (resp. AO) algorithm for different OU parameter sets are, respectively, shown in Figs. 2.7–2.9 in green (resp. red).

2.6 Results and Discussion

2.6.1 Category I

Among the OU parameter sets belonging to *Category I* the set corresponding to the most amount of error is $(\mu, \sigma) = (1.0, 0.0)$. Representative realizations of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for this parameter set are, respectively, shown in green and red in Fig. 2.7. In Fig. 2.7 the realization of the $\sqrt{\text{AO}}$ -algorithm's prediction appears to be more accurate than that of the AO-algorithm's prediction, especially with increasing time. In order to make a more quantitative comparison between the $\sqrt{\text{AO}}$ and AO algorithms' predictions, we focus on the time interval $[0, 1]$ and make use of the the metrics

$$\epsilon_2(\sqrt{\text{AO}}) := \frac{\|\sqrt{\text{AO}}(\mathbf{A}) - \mathbf{A}\|_2}{\|\mathbf{A}\|_2}, \quad (2.32)$$

$$\epsilon_2(\text{AO}) := \frac{\|\text{AO}(\mathbf{A}) - \mathbf{A}\|_2}{\|\mathbf{A}\|_2}, \quad (2.33)$$

where $\|f\|_2 := \sqrt{\int_0^1 \|f(\tau)\|^2 d\tau}$; and $\mathbb{R} \ni \tau \mapsto \sqrt{\text{AO}}(\mathbf{A})(\tau) \in \mathcal{M}_{3,1}(\mathbb{R})$, and $\mathbb{R} \ni \tau \mapsto \text{AO}(\mathbf{A})(\tau) \in \mathcal{M}_{3,1}(\mathbb{R})$ are, respectively, particular realizations of the $\sqrt{\text{AO}}$ and AO algorithms' predictions for $\mathbb{R} \ni \tau \mapsto \mathbf{A}(\tau) \in \mathcal{M}_{3,1}(\mathbb{R})$. The metric $\epsilon_2(\sqrt{\text{AO}})$ (resp. $\epsilon_2(\text{AO})$) is constructed such that the smaller its value the more accurate the realization used in computing it. The values of $\epsilon_2(\sqrt{\text{AO}})$ and $\epsilon_2(\text{AO})$ for the realizations shown in Fig. 2.7 are, respectively, 8.79% and 47.11%. The metric $\epsilon_2(\sqrt{\text{AO}})$'s smaller value in comparison to that of $\epsilon_2(\text{AO})$ corroborates our earlier assertion that among the $\sqrt{\text{AO}}(\mathbf{A})$ and $\text{AO}(\mathbf{A})$ shown in Fig. 2.7 the realization $\sqrt{\text{AO}}(\mathbf{A})$ is more accurate. This comparison between the $\sqrt{\text{AO}}$ and AO algorithms' predictions' particular realizations prompts us to hypothesize that the $\sqrt{\text{AO}}$ algorithm is more accurate than the AO algorithm.

In order to compare the $\sqrt{\text{AO}}$ and AO algorithms' predictions in a more well-balanced and comprehensive manner we calculated $\epsilon_2(\sqrt{\text{AO}})$ and $\epsilon_2(\text{AO})$, respectively, for a large number of realizations (population size $N = 200$) of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms. The mean values of the thus generated populations of $\epsilon_2(\sqrt{\text{AO}})$ and $\epsilon_2(\text{AO})$ are 8.788% and 47.114%,

respectively (see row number 5 of Table 2.1). The mean value of the population of $\epsilon_2(\sqrt{\text{AO}})$ being lower than the mean value of the population of $\epsilon_2(\text{AO})$ further supports our earlier hypothesis that $\sqrt{\text{AO}}$ -algorithm is more accurate than the AO-algorithm.

To recall, the discussion so far in this section exclusively relates to the (μ, σ) parameter set $(1.0, 0.0)$. We performed analysis similar to the one discussed in the previous paragraph for the parameter sets $(0.0, 0.0)$, $(0.1, 0.0)$, $(0.2, 0.0)$, and $(0.5, 0.0)$ as well. The means of $\epsilon_2(\sqrt{\text{AO}})$'s and $\epsilon_2(\text{AO})$'s populations for these other parameter sets are, respectively, given in the first and second columns of Table 2.1. It can be seen from Table 2.1 that the means of the $\epsilon_2(\sqrt{\text{AO}})$ populations are consistently smaller than those of $\epsilon_2(\text{AO})$ populations across all the parameter sets considered. Furthermore, in Table 2.1 the difference between the means of a $\epsilon_2(\sqrt{\text{AO}})$ population and a $\epsilon_2(\text{AO})$ population corresponding to the same parameter set increases with the amount of error, i.e., with the magnitude of μ in the present category. Thus for the category of exclusively bias type errors, in addition to the $\sqrt{\text{AO}}$ -algorithm appearing to be more accurate than the AO-algorithm, it further appears that the $\sqrt{\text{AO}}$ -algorithm's performance over the AO-algorithm increases with increasing amount of error.

2.6.2 Category II

In *Category II* we consider the (μ, σ) parameter sets $(0.0, 0.0)$, $(0.0, 1.0)$, $(0.0, 10.0)$, $(0.0, 50.0)$, and $(0.0, 100.0)$. The means of $\epsilon_2(\sqrt{\text{AO}})$'s and $\epsilon_2(\text{AO})$'s populations for these parameter sets are, respectively, given in the first and second columns of Table 2.2. It can be seen from Table 2.2 that the means of the $\epsilon_2(\sqrt{\text{AO}})$ populations are approximately the same as those of $\epsilon_2(\text{AO})$ populations across all the parameter sets considered. Thus, for the category of exclusively noise type errors the $\sqrt{\text{AO}}$ -algorithm appears to perform on par with the AO-algorithm.

Among the OU parameter sets belonging to *Category II* the set corresponding to the most amount of error is $(\mu, \sigma) = (0.0, 100.0)$. Representative realizations of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for this parameter set are, respectively, shown in green and red in Fig. 2.8. In Fig. 2.8, at least at the earlier time instances, the $\sqrt{\text{AO}}$ and AO algorithms' predictions' realizations are almost indistinguishable from one another. However, at later time instances the $\sqrt{\text{AO}}$ -algorithm seems to be

performing better than the AO-algorithm (This feature is likely not reflected in the results presented in Table 2.2 because they are calculated only using data from the initial time instances, or, to be more precise, from the $[0, 1]$ time interval.). Based on this observation we venture to speculate that even when the errors are predominantly of the noise type, the $\sqrt{\text{AO}}$ -algorithm will eventually begin to outperform the AO-algorithm.

2.6.3 Category III

In *Category III* we consider the (μ, σ) parameter sets $(0.0, 10.0)$, $(0.1, 10.0)$, $(0.2, 10.0)$, $(0.5, 10.0)$, and $(1, 10.0)$. The means of $\epsilon_2(\sqrt{\text{AO}})$'s and $\epsilon_2(\text{AO})$'s populations for these parameter sets are, respectively, given in the first and second columns of Table 2.3. Among the OU parameter sets belonging to *Category III* the set corresponding to the most amount of error is $(\mu, \sigma) = (1.0, 10.0)$. Representative realizations of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for this parameter set are, respectively, shown in green and red in Fig. 2.9.

It can be seen from Table 2.3 that the means of the $\epsilon_2(\sqrt{\text{AO}})$ populations are consistently smaller than those of $\epsilon_2(\text{AO})$ populations across all the parameter sets considered. Furthermore, in Table 2.3 the difference between the means of a $\epsilon_2(\sqrt{\text{AO}})$ population and a $\epsilon_2(\text{AO})$ population corresponding to the same parameter set increases with the amount of error, i.e., with the magnitudes of σ and μ . Thus, in *Category III* the relative performance of the $\sqrt{\text{AO}}$ and AO algorithms is very similar to that in *Category I*.

From the discussion in §2.6.1 we know that the AO-algorithm is more sensitive to bias type errors than the $\sqrt{\text{AO}}$ -algorithm and from the discussion in §2.6.2 we know that the $\sqrt{\text{AO}}$ and AO algorithms are, approximately, equally sensitive to noise type errors. From the results in this section it appears that the $\sqrt{\text{AO}}$ -algorithm outperforms the AO-algorithm as long as the errors have some bias type component in them, irrespective of what the amount of the noise type component in them is.

Table 2.1: The mean and standard deviation of the error measure ϵ_2 for 200 realizations of the accelerometer data only containing bias type error with $\sigma = 0$. In this case, as the value of standard deviation is quite small compared to the value of mean, we do not show the value in the table

	$\epsilon_2 \times 10^3$ (mean $\pm$ std)	
μ	$\sqrt{\text{AO}}$ -algorithm	AO-algorithm
0	0.01	0.02
0.1	8.87	41.02
0.2	17.73	83.63
0.5	44.22	220.28
1	87.88	471.14

Table 2.2: The mean and standard deviation of the error measure ϵ_2 for 200 realizations of the accelerometer data only containing noise type error with $\mu = 0$

	$\epsilon_2 \times 10^3$ (mean $\pm$ std)	
σ	$\sqrt{\text{AO}}$ -algorithm	AO-algorithm
0	0.01	0.02
1	1.03 $\pm$ 0.09	1.16 $\pm$ 0.21
10	10.22 $\pm$ 0.64	11.46 $\pm$ 1.80
50	52.79 $\pm$ 3.33	56.87 $\pm$ 7.79
100	115.85 $\pm$ 10.11	112.83 $\pm$ 15.54

Table 2.3: The mean and standard deviation of the error measure ϵ_2 for 200 realizations of the accelerometer data containing bias and noise type errors with $\sigma = 10$

	$\epsilon_2 \times 10^3$ (mean $\pm$ std)	
μ	$\sqrt{\text{AO}}$ -algorithm	AO-algorithm
0	10.22 $\pm$ 0.64	11.46 $\pm$ 1.80
0.1	13.42 $\pm$ 1.14	42.61 $\pm$ 5.05
0.2	20.19 $\pm$ 1.43	84.52 $\pm$ 5.57
0.5	45.18 $\pm$ 1.47	221.08 $\pm$ 6.02
1	88.27 $\pm$ 1.44	472.26 $\pm$ 6.47

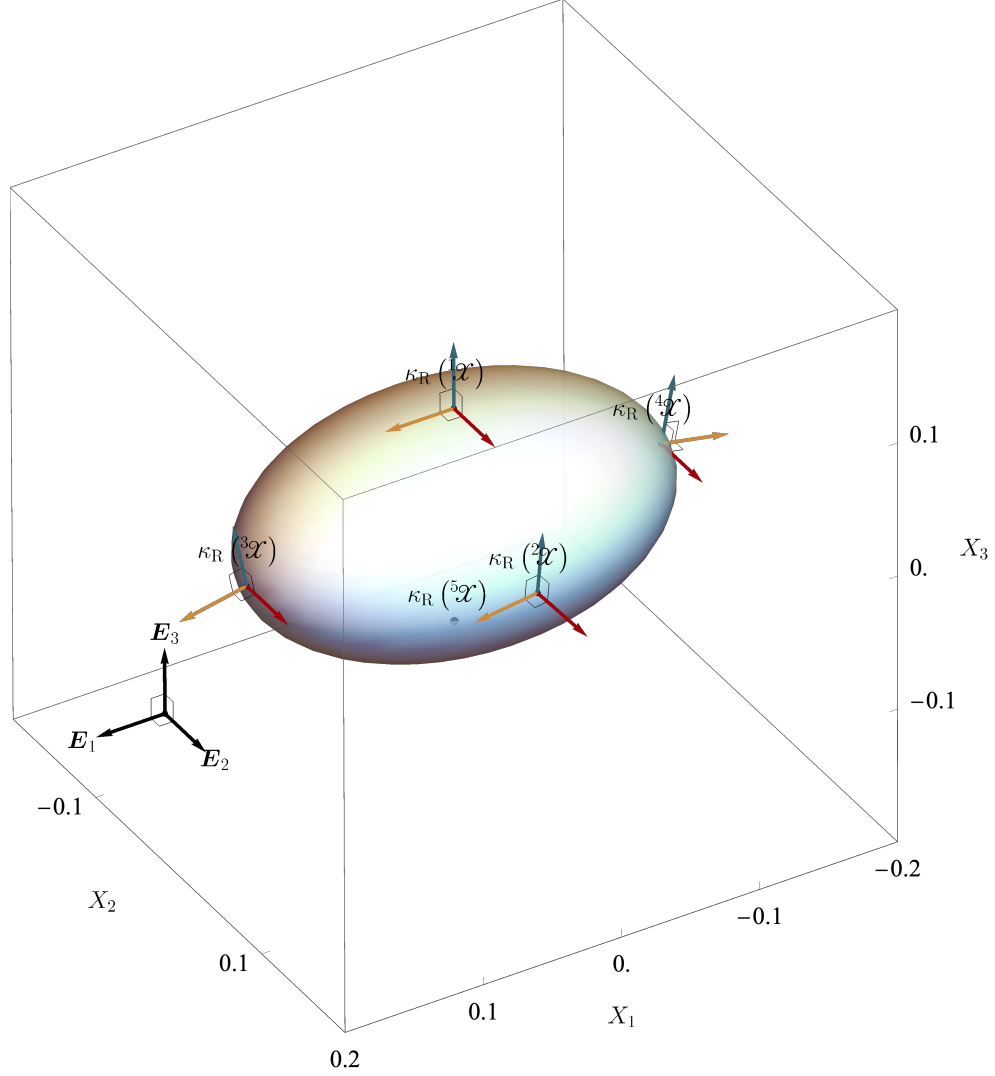


Figure 2.4: Accelerometer arrangement and orientation in numerical simulation of a rigid ellipsoid impacting an elastic half-space (see §2.5.1 for details). In the reference point space $\mathcal{E}_R$ the ellipsoid, $\mathcal{B}$, occupies the region $\{(X_1, X_2, X_3) \mid (X_1/a)^2 + (X_2/b)^2 + (X_3/c)^2 \leq 1\}$, where $(a, b, c) = (0.15, 0.10, 0.08)$. Four virtual accelerometers are, respectively, attached to the ellipsoid's material particles ${}^\ell\mathcal{X}$, $\ell \in \mathcal{J}$. The reference position vectors of ${}^\ell\mathcal{X}$, $\ell \in \mathcal{J}$, are, respectively, $c\mathbf{E}_3$, $b\mathbf{E}_2$, $a\mathbf{E}_1$, and $-a\mathbf{E}_1$, where $(\mathbf{E}_i)_{i \in I}$ are shown in the figure as well. The accelerometers' orientations are given by $({}^\ell\mathbf{E}_i)_{i \in I}$, $\ell \in \mathcal{J}$. The component representation of $({}^1\mathbf{E}_i)_{i \in I}$ w.r.t. $(\mathbf{E}_i)_{i \in I}$ is $((0, 1, 0), (1, 0, 0), (0, 0, 1))$; of $({}^2\mathbf{E}_i)_{i \in I}$ is $((-\frac{2}{\sqrt{229}}, \frac{225}{229}, -\frac{30}{229}), (\frac{15}{\sqrt{229}}, \frac{30}{229}, -\frac{4}{229}), (0, \frac{2}{\sqrt{229}}, \frac{15}{\sqrt{229}}))$; of $({}^3\mathbf{E}_i)_{i \in I}$ is $((0, 1, 0), (\frac{5}{\sqrt{26}}, 0, -\frac{1}{\sqrt{26}}), (\frac{1}{\sqrt{26}}, 0, \frac{5}{\sqrt{26}}))$; and of $({}^4\mathbf{E}_i)_{i \in I}$ is $((0, 1, 0), (-\frac{5}{\sqrt{26}}, 0, -\frac{1}{\sqrt{26}}), (-\frac{1}{\sqrt{26}}, 0, \frac{5}{\sqrt{26}}))$. We apply the $\sqrt{\text{AO}}$ -algorithm to the accelerometer data from the four virtual accelerometers ${}^\ell\mathcal{X}$, $\ell \in \mathcal{J}$ to predict the acceleration of the material particle ${}^5\mathcal{X}$. The reference position vector of ${}^5\mathcal{X}$ is $-c\mathbf{E}_3$. (modified from [57], copyright 2020, Elsevier)

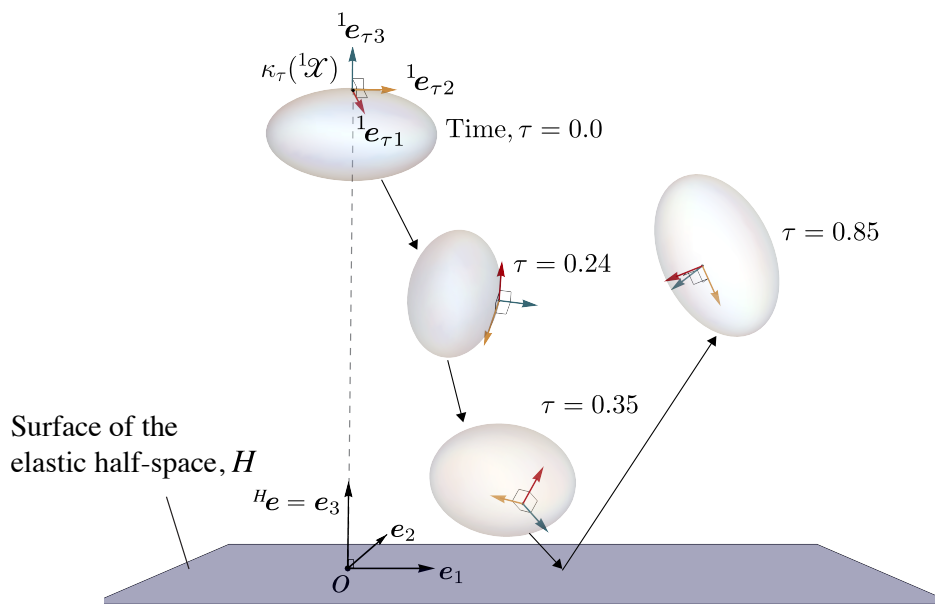


Figure 2.5: Configuration of the rigid ellipsoid at different time instances in the simulation of it impacting an elastic half space (see §2.5.1 for details). In the simulation, the ellipsoid, $\mathcal{B}$, is dropped onto an elastic half-space, H , under the action of gravity with the initial angular and translational velocities prescribed. The ellipsoid's initial position in $\mathcal{E}$ is prescribed by taking $\mathbf{c}(0) = (0, 0, 0.75)$, and $\mathbf{Q}(0) = \text{diag}(1, 1, 1)$. Its initial velocities are prescribed by setting $\star(\mathbf{W}(0)) = (5, 5, 5)$, and $\mathbf{c}'(0) = (0.75, 0, 0)$. (modified from [57], copyright 2020, Elsevier)

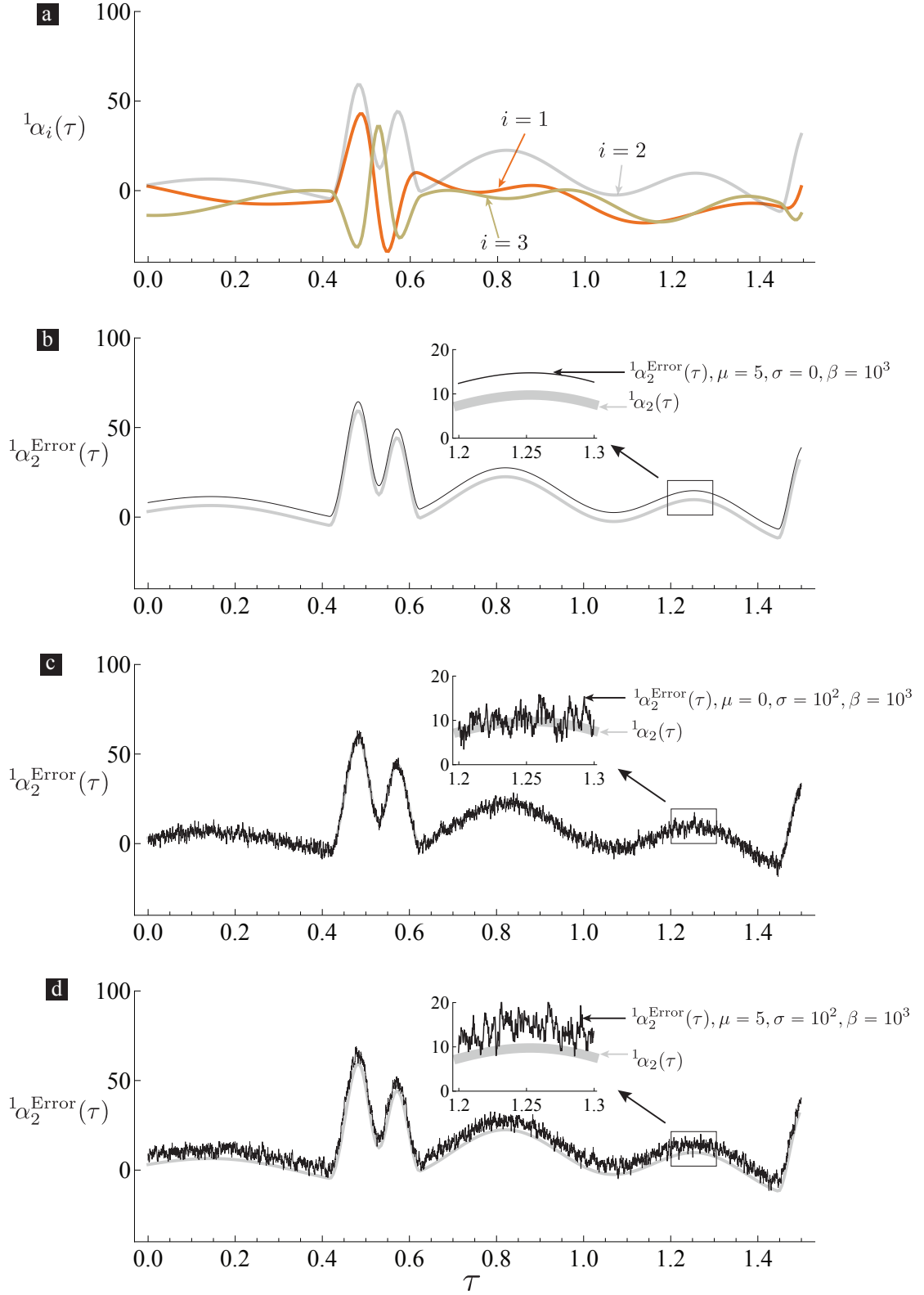


Figure 2.6: The acceleration components ${}^1\alpha_i(\tau)$, $i \in \mathcal{I}$, of the virtual accelerometer ${}^1\mathcal{X}$ before and after addition of synthetic errors (see §2.5.2 for details). (a) shows the acceleration components before the addition of synthetic errors. (b)–(d) show the error-inclusive acceleration component ${}^1\alpha_2^{\text{Error}}$, which is generated by adding different errors to the acceleration component ${}^1\alpha_2$. In (b), (c), and (d) the error time signals are particular realizations of the OU process for the OU parameter sets $(\mu, \sigma, \beta) = (5, 0, 10^3)$, $(0, 10^2, 10^3)$, and $(5, 10^2, 10^3)$, respectively. The error in (b) corresponds to *Category I* (exclusively bias type errors); in (c) to *Category II* (exclusively noise type errors); and in (d) to *Category III* (a combination of bias and noise type errors).

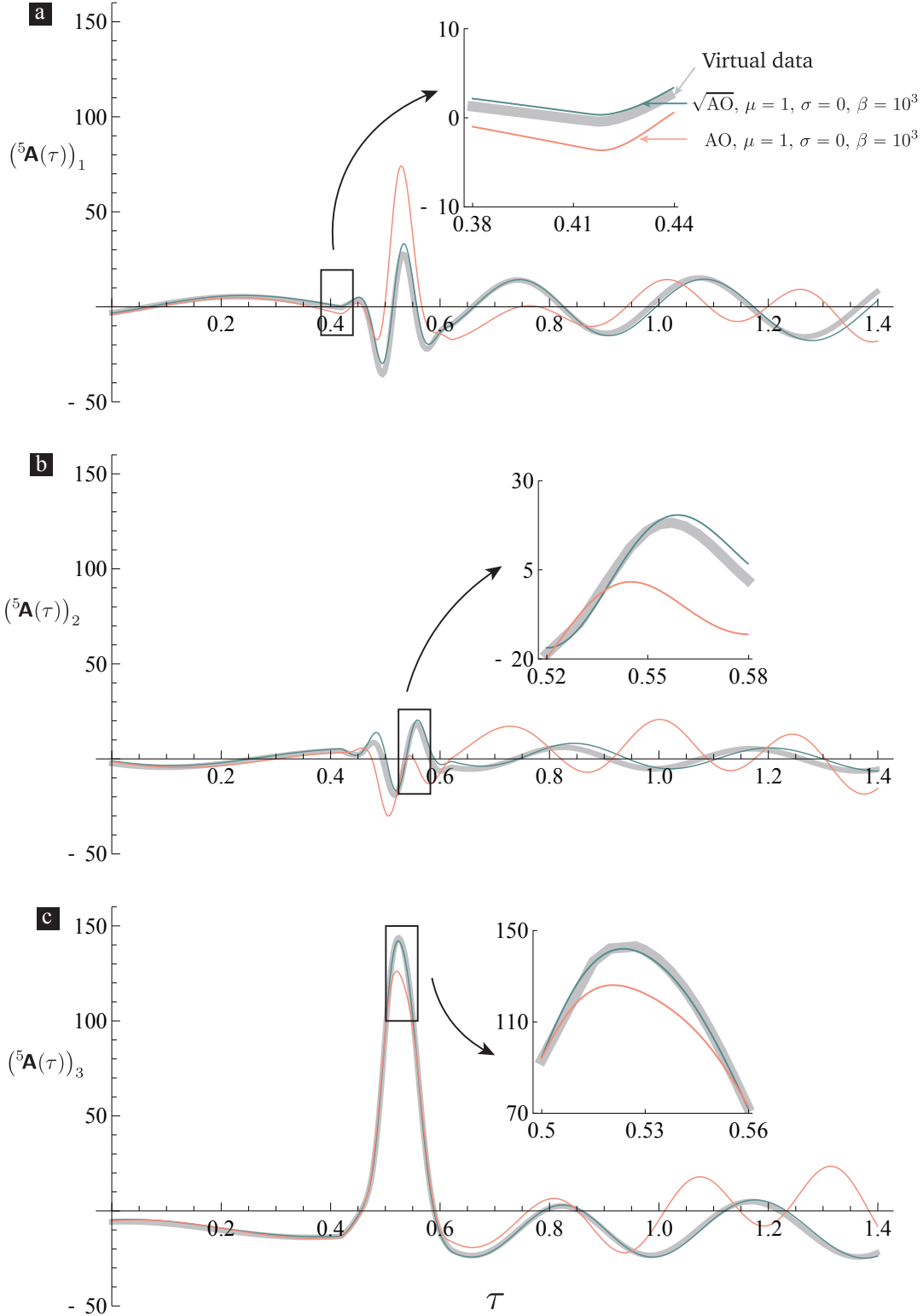


Figure 2.7: Comparison of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for the acceleration of the material particle 5X (see Fig. 2.4) in the rigid ellipsoid impact simulation (see §2.5.1 for details). Both the $\sqrt{\text{AO}}$ and AO algorithms were fed the same virtual error-inclusive accelerometer data. The data was generated by adding a particular realization of the OU process to the virtual accelerometer data from the rigid ellipsoid impact simulation. The OU realization corresponded to the OU parameter set $(\mu, \sigma, \beta) = (1, 0, 10^3)$. Subfigures (a), (b), and (c), respectively, show the comparison for the component of 5X 's acceleration in the $\mathbf{e}_i, i \in \mathcal{I}$, directions.

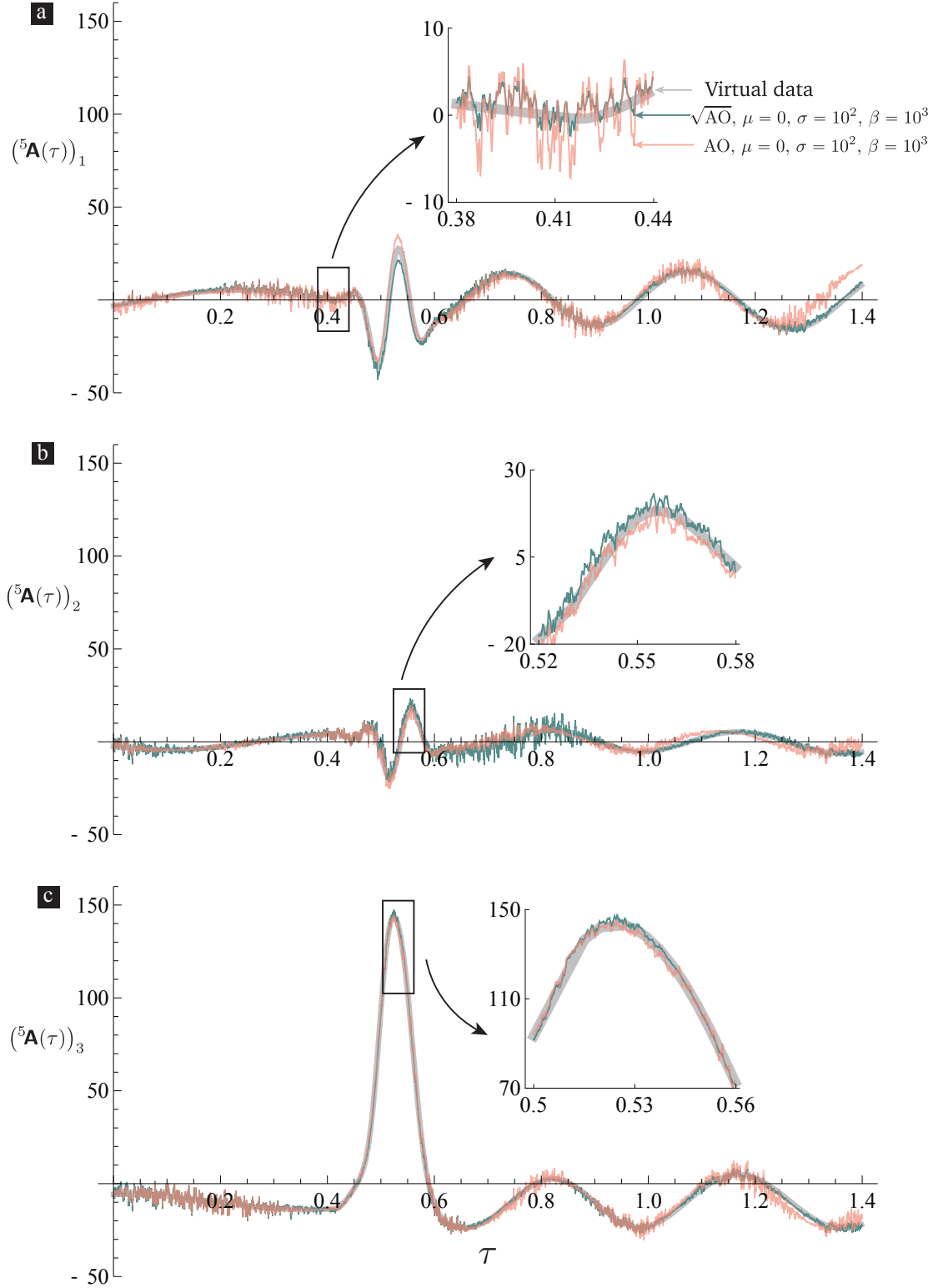


Figure 2.8: Comparison of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for the acceleration of the material particle ${}^5\chi$ (see Fig. 2.4) in the rigid ellipsoid impact simulation (see §2.5.1 for details). Both the $\sqrt{\text{AO}}$ and AO algorithms were fed the same virtual error-inclusive accelerometer data. The data was generated by adding a particular realization of the OU process to the virtual accelerometer data from the rigid ellipsoid impact simulation. The OU realization corresponded to the OU parameter set $(\mu, \sigma, \beta) = (0, 10^2, 10^3)$. Subfigures (a), (b), and (c), respectively, show the comparison for the component of ${}^5\chi$'s acceleration in the $\mathbf{e}_i, i \in \mathcal{I}$, directions.

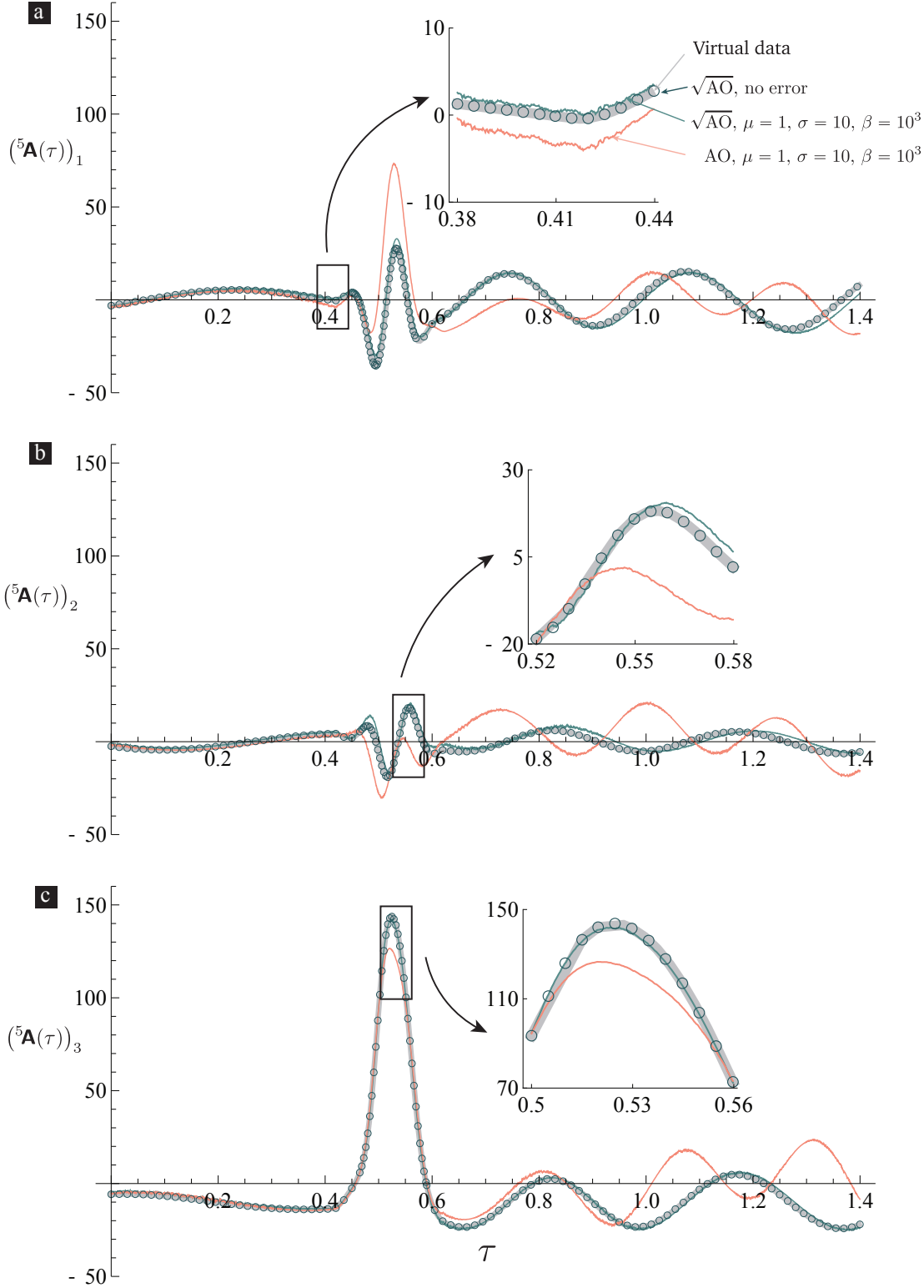


Figure 2.9: Comparison of the predictions from the $\sqrt{\text{AO}}$ and AO algorithms for the acceleration of the material particle ${}^5\mathcal{X}$ (see Fig. 2.4) in the rigid ellipsoid impact simulation (see §2.5.1 for details). Both the $\sqrt{\text{AO}}$ and AO algorithms were fed the same virtual error-inclusive accelerometer data. The data was generated by adding a particular realization of the OU process to the virtual accelerometer data from the rigid ellipsoid impact simulation. The OU realization corresponded to the OU parameter set $(\mu, \sigma, \beta) = (1, 10, 10^3)$. Subfigures (a), (b), and (c), respectively, show the comparison for the component of ${}^5\mathcal{X}$'s acceleration in the $\mathbf{e}_i$, $i \in \mathcal{I}$, directions. The predictions of the $\sqrt{\text{AO}}$ algorithm when fed just the virtual accelerometer data, i.e., with no added errors, is also shown in (a), (b), and (c) using black open circles.

2.7 Concluding remarks

1. The results discussed in §2.6 show that the $\sqrt{\text{AO}}$ -algorithm provides a valid approach to determine the complete motion of a rigid body using only data from four tri-axial accelerometers. We plan on carrying out more simulations in the future for gauging the robustness of the $\sqrt{\text{AO}}$ algorithm. Those simulations will be more complicated than the one we considered in this paper, e.g., in the future simulations we will consider non-symmetric geometries for the rigid body and non-planar surfaces for the elastic solid. The $\sqrt{\text{AO}}$ -algorithm's practical validity in the field too still remains to be explored. In the future, we plan to conduct an experimental evaluation of the $\sqrt{\text{AO}}$ -algorithm to compliment its *in silico* validation that we presented in this paper.
2. The comparison in §2.6 shows that for the cases we considered the $\sqrt{\text{AO}}$ -algorithm is less sensitive to bias type errors compared to the AO-Algorithm. However, we have not provided a mathematical proof that the $\sqrt{\text{AO}}$ -algorithm is better than the AO-algorithm with regard to bias type errors. Thus, though the comparison presented in §2.6 provides strong support to the hypothesis that the $\sqrt{\text{AO}}$ -algorithm is less sensitive to bias type errors than the AO-algorithm, it by no means provides a proof for the hypothesis. A definitive resolution to the question of whether the hypothesis is true requires an error analysis of both the $\sqrt{\text{AO}}$ -algorithm as well as the AO-algorithm. We currently have not carried out such analyses. Nevertheless, irrespective of the relative merit of the $\sqrt{\text{AO}}$ over the AO algorithm, it is quite clear from its derivation and the results discussed in §2.6 that it provides a valid approach for determining the complete motion of a rigid body from accelerometer data.
3. The $\sqrt{\text{AO}}$ -algorithm retains all the benefits of the AO-algorithm. Both algorithms provide the complete motion of the rigid body in the fixed laboratory frame. Without integration or differentiation, both algorithms are able to determine the pseudo acceleration field, providing the magnitude of acceleration for all material particles. Both algorithms can be applied to any arrangement of four tri-axial accelerometers as long as they do not lie in the same plane. There is no restriction on the orientation of the tri-axial accelerometers.
4. In the *in silico* validation, evaluation, and comparison of the $\sqrt{\text{AO}}$ -algorithm that we set up in

§2.5 we used the OU process to model experimental errors. Even more specifically, we took the magnitude of the parameter μ in the OU process as a measure of the bias type errors in the OU process' realizations. There of course exist bias type errors that cannot be modeled in this manner. Thus, our evaluation of the relative sensitivities of the $\sqrt{\text{AO}}$ and the AO algorithms to bias type errors was carried out using a limited form of bias type errors. A more general method to represent bias type errors in the virtual accelerometer data would provide a more comprehensive comparison of the relative sensitivities of the $\sqrt{\text{AO}}$ and the AO algorithms to bias type errors.

5. We envision the $\sqrt{\text{AO}}$ algorithm being used in design efforts that are aimed at providing better protection against mTBI. Design efforts that are aimed at providing better protection against mTBI, typically make use of *brain injury criteria*. Currently, several finite element (FE) based brain injury criteria are being developed with an aim of providing a better estimate of a motion's potential for causing injury, e.g., see [26]. Strain and strain rate have been shown to play an important role in brain injury. The FE based injury criteria connect the risk of injury to time histories of different stress and strain fields within the head. Application of the FE based injury criteria requires simulating the motion that is being evaluated on a computational head model and determining the time histories of the stress and strains fields within it. The developed $\sqrt{\text{AO}}$ algorithm provides a means for accurately simulating varieties of motions that are important in the context of mTBI. Since, it makes it possible to measure even 3D motions that are difficult to reproduce out of field, such as the landing of a paratrooper.

Chapter 3

A finite rotation, small strain 2D elastic head model

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3.1 Introduction

In this paper, we present a new 2D ICM based head model that accounts for the head's finite rotations (see §3.3). Like all the surveyed ICM head models, the strains in it are still small; and like all the surveyed 2D ICM head models the loading in it is purely rotational, with no translations. In it the loading on the brain is specified by prescribing the head's (skull's) angular velocity as a function of time¹. We refer to this new ICM based head model as the finite rotations head model. On accounting

¹In this new model we assume the skull to be an infinitely rigid body (see assumption (A.i)(e) in §3.3.1). Consequently, in this model the loading effect of all the forces and torques acting on the head is completely communicated to the brain through the skull's rigid body motion. In this work we only focus on purely rotational motion in 2D (see assumption (A.ii) in §3.3.1). Thus, the loading on the brain in this model is completely specified by specifying how the head rotates as a function of time; or more specifically, by specifying the function $\Omega[\cdot]$, which gives the head's rotation angle as a function of time (see section *Deformation mapping* of §3.2.3 for details). However, as we will see in §3.3.2.2, only the derivatives of $\Omega[\cdot]$ appear in the equations governing the motion of the brain (e.g., see (3.33)). Hence, in our model the loading is completely specified by specifying the head's angular velocity as a function of time.

for the head's finite rotations and distinguishing between the reference and deformed configurations, the equations (see (3.71)) governing the motion of the brain in it come out to be quite different from those arrived at in the surveyed ICM models (see (3.72)); as we mentioned previously, the surveyed ICM models do not account for the head's finite rotations. Using our model, we found that the estimates for various stress and strain measures, such as maximum principal logarithmic strain, can contain quite significant errors if the head's finite rotations are not taken into account (see §3.5.3.2). This, however, is not to say that our ICM model is better than the surveyed ICM models. In fact, our model is simpler than some of the surveyed ICM models in some aspects. For example, it is 2D in nature, and takes the brain to be a perfectly elastic solid. Thus, our model's key contribution in the context of surveyed ICM models is in that it highlights the important role played by the finiteness of the head's rotations in dictating the internal brain strains and strain rates. We believe that the presented finite rotations head model will prove useful for introducing finite rotations into more sophisticated head models, such as those that take the head's 3D nature and the brain's viscoelastic behavior into account.

This chapter is organized as follows. In §3.2, we present the mathematical and mechanics preliminaries needed for the development of our finite rotations head model. In §3.3, we present the equations that govern the mechanics of the finite rotations head model. In §3.4, we provide a semi-analytical solution for the governing equations and derive expressions for the strains and strain rates in the finite rotations head model. In §3.5.1, we compare the strains from the finite rotations head model with those from a recently presented finite element head model [26]. In §3.5.3, we compare the finite rotations head model with the surveyed ICM head models. We make a few concluding remarks in §3.6.

3.2 Mechanics and mathematical preliminaries

In this section we briefly present the mechanical and mathematical preliminaries from [89, 88, §2.1] that are needed for the development of the proposed head model.

3.2.1 Notation

We denote the space of real numbers as $\mathbb{R}$, the set of natural numbers as $\mathbb{N}$, the set of non-negative real numbers as $\mathbb{R}_{\geq 0}$, the set of positive real numbers as $\mathbb{R}_{> 0}$, and the set of non-negative integers as $\mathbb{Z}_{\geq 0}$.

An n -dimensional multi-index is an n -tuple, $n \in \mathbb{N}$, defined as

$$\alpha = (\alpha_1, \alpha_2, \dots, \alpha_n), \quad (3.1)$$

where $\alpha_i \in \mathbb{Z}_{\geq 0}$, $i \in (1, 2, \dots, n)$. The partial derivative of a function $(x_1, x_2, \dots, x_n) \mapsto f[x_1, x_2, \dots, x_n]$ w.r.t. α is defined as

$$\partial^\alpha f = \partial_1^{\alpha_1} \partial_2^{\alpha_2} \dots \partial_n^{\alpha_n} f, \quad (3.2)$$

where $\partial_i^{\alpha_i} := \partial^{\alpha_i} / \partial x_i^{\alpha_i}$.

3.2.2 Geometry of the abstract and physical spaces in our model

Let $\mathbb{E}_R$ be an oriented Euclidean vector space, i.e., an oriented finite dimensional, real, inner product space, and let the affine point space $\mathcal{E}_R$ have $\mathbb{E}_R$ as its associated vector translation space. We refer to $\mathbb{E}_R$ and $\mathcal{E}_R$ as the reference Euclidean vector and point space, respectively. Let $\mathbb{E}$ and $\mathcal{E}$ be another pair of Euclidean vector and affine point space, respectively. The topological space $\mathcal{B}$ serves as our model for the brain that executes its motion in $\mathcal{E}$. For that reason, we refer to $\mathbb{E}$ and $\mathcal{E}$ as the physical Euclidean vector space and point space, respectively.

We call a select continuous, injective map from $\mathcal{B}$ into $\mathbb{E}_R$ the reference configuration and denoted it as κ_R . The elements of $\mathcal{B}$ are called material particles. We call $X \equiv \kappa_R[X]$ the particle X 's reference position vector and $\kappa_R[\mathcal{B}]$ the reference body (see Fig. 3.1). Taking some arbitrary point $O_R \in \mathcal{E}_R$ to be $\mathcal{E}_R$'s origin, to κ_R we associate the map $\kappa_R : \mathcal{B} \rightarrow \mathcal{E}_R$ such that $O_R + \kappa_R[X] = \kappa_R[X]$. We call $X \equiv \kappa_R[X]$ the particle X 's reference point.

Cartesian basis vectors The sets $(\mathbf{E}_i)_{i \in \mathcal{I}}$ and $(\mathbf{e}_i)_{i \in \mathcal{I}}$, where $\mathcal{I} := (1, 2, 3)$, are orthonormal sets of basis vectors for $\mathbb{E}_R$ and $\mathbb{E}$, respectively. By orthonormal we mean that the inner product between $\mathbf{E}_i$ and $\mathbf{E}_j$, or $\mathbf{e}_i$ and $\mathbf{e}_j$, where $i, j \in \mathcal{I}$, equals δ_{ij} , the Kronecker delta symbol, which equals unity iff $i = j$ and zero otherwise. In our problem, we take $\mathbf{E}_i$ and $\mathbf{e}_i$, $i \in \mathcal{I}$, to have the units of meters. The Cartesian co-ordinates of X which we denote as $\check{X}[X] = (\check{X}_i[X])_{i \in \mathcal{I}}$, are components of X w.r.t. $\mathbf{E}_i$, that is $\check{X}_i[X] = X_i$, where $X_i := X \cdot \mathbf{E}_i$. For simplicity, $\mathbf{X} \equiv (X_1, X_2, X_3)$. We denote the space of all $m \times n$ real nested ordered sets, where $m, n \in \mathbb{N}$, $\mathcal{M}_{m \times n}(\mathbb{R})$. Thus $\check{X}[X] \in \mathcal{M}_{3 \times 1}(\mathbb{R})$. We call the map $\mathcal{E}_R \ni X \mapsto \check{X}[X] \in \mathcal{M}_{3 \times 1}(\mathbb{R})$ the Cartesian co-ordinate map. When we refer to $X \in \mathbb{E}_R$, $\mathbf{X} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$, or $X \in \mathcal{E}_R$ as a material particle we in fact mean the material particle $X \in \mathcal{B}$.

Cylindrical basis vectors The cylindrical co-ordinates of $\mathbf{X}$, which we denote as $(\bar{r}[\mathbf{X}], \bar{\theta}[\mathbf{X}], \bar{z}[\mathbf{X}])$, are defined in the standard manner using the Cartesian co-ordinates $\mathbf{X}$,

$$\bar{r}[\mathbf{X}] = \sqrt{X_1^2 + X_2^2}, \quad (3.3a)$$

$$\bar{\theta}[\mathbf{X}] = \text{atan2}[X_2, X_1], \quad (3.3b)$$

$$\bar{z}[\mathbf{X}] = X_3. \quad (3.3c)$$

Then the cylindrical basis vectors $(\bar{\mathcal{E}}_i[\mathbf{X}])_{i \in \mathcal{I}}$ for $\mathbb{E}_R$ are defined as

$$(\bar{\mathcal{E}}_i[\mathbf{X}])_{i \in \mathcal{I}} = \bar{\mathbf{R}}[\mathbf{X}] (\mathbf{E}_i)_{i \in \mathcal{I}}, \quad (3.4a)$$

where

$$\bar{\mathbf{R}}[\mathbf{X}] = \frac{1}{\sqrt{X_1^2 + X_2^2}} \begin{pmatrix} X_1 & X_2 & 0 \\ -X_2 & X_1 & 0 \\ 0 & 0 & \sqrt{X_1^2 + X_2^2} \end{pmatrix}. \quad (3.4b)$$

Say $\mathbb{W}$ and $\mathbb{U}$ are two arbitrary, oriented, finite dimensional Hilbert spaces; for instance, they can be $\mathbb{E}_R$ and $\mathbb{E}$. We denote the space of all linear maps (transformations/operators) from $\mathbb{W}$ to $\mathbb{U}$ as $\mathcal{L}(\mathbb{W}, \mathbb{U})$. We denote the norm of a vector $\mathbf{w}_1$ in $\mathbb{W}$ that is induced by $\mathbb{W}$'s inner product, i.e., $(\mathbf{w}_1 \cdot_{\mathbb{W}} \mathbf{w}_1)^{1/2}$, as $\|\mathbf{w}_1\|_{\mathbb{W}}$. For $\mathbf{u}_1 \in \mathbb{U}$, the expression $\mathbf{u}_1 \otimes \mathbf{w}_1$ denotes the linear map from $\mathbb{W}$ to $\mathbb{U}$

of $\|\cdot\|$.

3.2.3 Kinematics

Select a continuous injective map $\kappa_R : \mathcal{B} \rightarrow \mathcal{E}_R$ such that $\kappa_R [\mathcal{B}]$ is a finite right cylinder of radius of r_0 meters, where $r_0 \in \mathbb{R}_{>0}$, with its central axis ℓ passing through O_R and parallel to $\mathbf{E}_3$ direction (see Fig. 3.1).

We model time as a one-dimensional normed vector space $\mathbb{T}$ and denote a typical element in it as $\tau = \tau s$, where $\tau \in \mathbb{R}$ and s is a fixed vector which has units of seconds. We model the body's motion using the one-parameter family of maps $\mathbf{x}_\tau^2 : \mathbb{E}_R \rightarrow \mathbb{E}$ (see Fig. 3.1). We call $\mathbf{x}_\tau$ the deformation map and $\mathbf{x} \equiv \mathbf{x}_\tau [X]$ the material particle X 's position vector at the time instance τ . The set $\kappa_\tau [\mathcal{B}] = \{\mathbf{x}_\tau [X] \in \mathbb{E} \mid X \in \kappa_R [\mathcal{B}]\}$ is called the current body.

Deformation mapping It can be shown that for our problem for $\tau \geq 0$,

$$\mathbf{x}_\tau = \mathbf{Q}_\tau \circ \mathbf{T}_\tau, \quad (3.7)$$

where the map $\mathbf{T}_\tau : \mathbb{E}_R \rightarrow \mathbb{E}$ is defined by the equation $\mathbf{T}_\tau [X] = \mathbf{I} (X + \mathbf{U}_\tau [X])$. Here $\mathbf{I} := \sum_{i \in I} \mathbf{e}_i \otimes \mathbf{E}_i$ and we call the map $\mathbf{U}_\tau : \mathbb{E}_R \rightarrow \mathbb{E}_R$ the intermediate displacement field of $\mathcal{B}$ at the time instance τ . The set $\kappa^\star [\mathcal{B}] = \{\mathbf{T}_\tau [X] \in \mathbb{E} \mid X \in \kappa_R [\mathcal{B}]\}$ is called the intermediate body.

The operator $\mathbf{Q}_\tau$ is a proper (orientation preserving), linear isometry from $\mathbb{E}$ into $\mathbb{E}$. The operator $\mathbf{Q}_\tau$ can be written as $\sum_{i,j \in I} Q_{ij} [\tau] \mathbf{e}_i \otimes \mathbf{e}_j$, where $Q_{ij} \in C^2(\mathbb{R}_{\geq 0}, \mathbb{R})$. We abbreviate $(Q_{ij} [\tau])_{i,j \in I} \in \mathcal{M}_{3 \times 3}(\mathbb{R})$ as $\mathbf{Q} [\tau]$. We call $\mathbf{Q} [\tau]$ the non-dimensional form of $\mathbf{Q}_\tau$ w.r.t. $(\mathbf{e}_i \otimes \mathbf{e}_j)_{i,j \in I}$. Since $\mathbf{Q}_\tau$ is a proper isometry, it follows that $\mathbf{Q} [\tau]$, which we refer to as the rotation matrix, belongs to the special orthogonal group $SO(3)$. As a consequence of belonging to $SO(3)$ the matrix $\mathbf{Q} [\tau]$ satisfies the equations

$$\mathbf{Q}^T [\tau] \mathbf{Q} [\tau] = \mathbf{I}, \quad (3.8a)$$

²In [89, §2.1] and [88, §2.1] $\mathbf{x}_\tau$ (the subscript is set in bold font) appears as $\mathbf{x}_\tau$ (the subscript is set in regular font).

and

$$\mathbf{Q}[\tau] \mathbf{Q}^\top[\tau] = \mathbf{I}, \quad (3.8b)$$

where $\mathbf{Q}^\top[\tau]$ is the transpose of $\mathbf{Q}[\tau]$, i.e., $\mathbf{Q}^\top[\tau] = (\mathbf{Q}[\tau])^\top$ and $\mathbf{I} = (\delta_{ij})_{i,j \in I} \in \mathcal{M}_{3 \times 3}(\mathbb{R})$.

In our problem, the head rotates about $\mathbf{e}_3$ by the angle $\Omega[\tau]$. It can be shown that

$$\mathbf{Q}[\tau] = \begin{pmatrix} \cos \circ \Omega[\tau] & -\sin \circ \Omega[\tau] & 0 \\ \sin \circ \Omega[\tau] & \cos \circ \Omega[\tau] & 0 \\ 0 & 0 & 1 \end{pmatrix}. \quad (3.9)$$

We assume the function $\Omega : [0, \infty) \rightarrow \mathbb{R}$ to be at least once continuously differentiable, and for concreteness that

$$\Omega[0] = 0, \quad \Omega'[0] = 0. \quad (3.10a, 3.10b)$$

Displacements From (3.7), the deformation mapping of the body for $\tau \geq 0$ is described by

$$\mathbf{x}_\tau[\mathbf{X}] = \overline{\mathbf{Q}}_\tau(\mathbf{X} + \mathbf{U}_\tau[\mathbf{X}]), \quad (3.11)$$

where $\overline{\mathbf{Q}}_\tau : \mathbb{E}_R \rightarrow \mathbb{E}$ is defined by the equation $\overline{\mathbf{Q}}_\tau = \mathbf{Q}_\tau \mathbf{I} = \sum_{i,j \in I} Q_{ij}[\tau] \mathbf{e}_i \otimes \mathbf{E}_j$. In a non-dimensional form (3.11) reads as

$$\mathbf{x}_\tau[\mathbf{X}] = \mathbf{Q}[\tau](\mathbf{X} + \mathbf{U}_\tau[\mathbf{X}]), \quad (3.12)$$

where $\mathbf{x}_\tau[\mathbf{X}] := (x_{\tau i}[\mathbf{X}])_{i \in I}$, $\mathbf{U}_\tau[\mathbf{X}] := (U_{\tau i}[\mathbf{X}])_{i \in I}$,

$$x_{\tau i}[\mathbf{X}] = \mathbf{x}_\tau \left[\sum_{j \in I} X_j \mathbf{E}_j \right] \cdot \mathbf{e}_i \quad (3.13)$$

$$U_{\tau i}[\mathbf{X}] = \mathbf{U}_\tau \left[\sum_{j \in I} X_j \mathbf{E}_j \right] \cdot \mathbf{E}_i. \quad (3.14)$$

The displacement field is the map $\mathbf{u}_\tau [\cdot] : \mathbb{E}_R \rightarrow \mathbb{E}$,

$$\mathbf{u}_\tau [X] = \mathbf{x}_\tau [X] - \mathbf{I}X. \quad (3.15a)$$

In a non-dimensional form (3.15a) reads

$$\mathbf{u}_\tau [X] = \mathbf{x}_\tau [X] - X, \quad (3.15b)$$

where $\mathbf{u}_\tau [X] := (u_{\tau i} [X])_{i \in I}$ and

$$u_{\tau i} [X] := \mathbf{u}_\tau \left[\sum_{j \in I} X_j \mathbf{E}_j \right] \cdot \mathbf{e}_i. \quad (3.15c)$$

Velocities We call $\mathcal{L}(\mathbb{T}, \mathbb{E})$ the physical velocity vector space and denote it as $\mathbb{V}$. It can be shown that the set $(\mathbf{v}_i)_{i \in I}$, where $\mathbf{v}_i \in \mathbb{V}$ and are defined such that $\mathbf{v}_i \boldsymbol{\tau} = \boldsymbol{\tau} \mathbf{e}_i$, that is $\mathbf{v}_i := \mathbf{e}_i \otimes s^*$, where s^* is the dual of s , provides an orthonormal basis for $\mathbb{V}$. The velocity of a material particle X executing its motion in $\mathbb{E}$ lies in $\mathbb{V}$. The velocity of the material particle X at the instant τ , which we denote as $\mathbf{V}_\tau [X]$, equals the value of the Fréchet derivative of the map $\mathbb{T} \ni \tau \mapsto \mathbf{x}_X [\tau] \in \mathbb{E}$, where $\mathbf{x}_X [\tau] = \mathbf{x}_\tau [X]$, at the time instance τ . Thus, it follows from (3.11) that for $\tau \geq 0$

$$\mathbf{V}_\tau [X] = \sum_{i,j \in I} \left(\mathcal{Q}'_{ij} [\tau] X_j + \mathcal{Q}'_{ij} [\tau] U_{Xj} [\tau] + \mathcal{Q}_{ij} [\tau] U'_{Xj} [\tau] \right) \mathbf{v}_i. \quad (3.16)$$

The functions $U_{Xj} : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ in (3.16) are defined as $U_{Xj} [\tau] = \mathbf{U}_X [\tau] \cdot \mathbf{E}_j$, where $\mathbf{U}_X [\tau] = \mathbf{U}_\tau [X]$, and $\mathcal{Q}'_{ij}$ and U'_{Xj} are derivatives of $\mathcal{Q}_{ij}$ and U_{Xj} , respectively. Defining $\mathbf{V}_{\tau i} [X] = \mathbf{V}_\tau [\sum_{j \in I} X_j \mathbf{E}_j] \cdot \mathbf{v}_i$, and abbreviating $(\mathbf{V}_{\tau i} [X])_{i \in I} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$, $(\mathcal{Q}'_{ij} [\tau])_{i,j \in I} \in \mathcal{M}_{3 \times 3}(\mathbb{R})$, $(U_{Xi} [\tau])_{i \in I} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$, and $(U'_{Xi} [\tau])_{i \in I} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$ as $\mathbf{V}_\tau [X]$, $\mathbf{Q}' [\tau]$, $\mathbf{U}_X [\tau]$, and $\mathbf{U}'_X [\tau]$, respectively, (3.16) in a non-dimensional form reads

$$\mathbf{V}_\tau [X] = \mathbf{Q}' [\tau] X + \mathbf{Q}' [\tau] \mathbf{U}_X [\tau] + \mathbf{Q} [\tau] \mathbf{U}'_X [\tau]. \quad (3.17)$$

Accelerations We call $\mathcal{L}(\mathbb{T}, \mathbb{V})$ the physical acceleration vector space and denote it as $\mathbb{A}$. It can be shown that the set $(\mathbf{a}_i)_{i \in I}$, where $\mathbf{a}_i \in \mathbb{A}$ and are defined such that $\mathbf{a}_i \tau = \tau \mathbf{v}_i$, i.e., $\mathbf{a}_i = \mathbf{v}_i \otimes \mathbf{s}^*$, provides an orthonormal basis for $\mathbb{A}$. The acceleration of a material particle X executing its motion in $\mathbb{E}$ lies in $\mathbb{A}$. The acceleration of X at the time instance τ equals the value of the Fréchet derivative of the map $\mathbb{T} \ni \tau \mapsto \mathbf{V}_X(\tau) \in \mathbb{V}$, where $\mathbf{V}_X(\tau) = \mathbf{V}_\tau(X)$, at the time instance τ . Thus, it follows from (3.16) that for $\tau \geq 0$

$$\mathbf{A}_\tau[X] = \sum_{i,j \in I} \left(Q''_{ij}[\tau] X_j + Q''_{ij}[\tau] U_{Xj}[\tau] + 2Q'_{ij}[\tau] U'_{Xj}[\tau] + Q_{ij}[\tau] U''_{Xj}[\tau] \right) \mathbf{a}_i, \quad (3.18)$$

where Q''_{ij} and U''_{Xj} are derivatives of Q'_{ij} and U'_{Xj} , respectively. Defining $A_{\tau i}[X] = \mathbf{A}_\tau[\sum_{j \in I} X_j \mathbf{E}_j] \cdot \mathbf{a}_i$, and abbreviating $(A_{\tau i}[X])_{i \in I} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$, $(Q''_{ij}[\tau])_{i,j \in I} \in \mathcal{M}_{3 \times 3}(\mathbb{R})$, and $(U''_{Xi}[\tau])_{i \in I} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$ as $\mathbf{A}_\tau[X]$, $\mathbf{Q}''[\tau]$, and $\mathbf{U}''_X[\tau]$, respectively, (3.18) can be written in a non-dimensional form as

$$\mathbf{A}_\tau[X] = \mathbf{Q}''[\tau] X + \mathbf{Q}''[\tau] \mathbf{U}_X[\tau] + 2\mathbf{Q}'[\tau] \mathbf{U}'_X[\tau] + \mathbf{Q}[\tau] \mathbf{U}''_X[\tau]. \quad (3.19)$$

3.3 Analytical model for head motion and brain deformation

3.3.1 Problem statement

(A.i) We model the brain as a (a) homogeneous, (b) isotropic, (c) incompressible, (d) elastic solid, and (e) the skull as a rigid solid. (A.ii) We will be carrying out a 2D analysis, by which we mean that the components of the displacements in the directions of the rotation axis vanish, the displacements do not vary in the direction of the rotation axis, and the rotation axis remains fixed in space with time. (A.iii) The displacements and deformations in the brain with respect to the skull are taken to be small. (A.iv) The displacements of the brain w.r.t. the skull are taken to be axisymmetric. The meaning of the assumptions A.iii and A.iv will be made precise in §3.3.2. (A.v) With regard to geometry, we model the brain's cross-sections perpendicular to the rotation axis as disks. (A.vi) We model the interaction between the brain and the skull by positing that the brain's outer surface is rigidly bonded to the skull's interior surface³. (A.vii) The brain and the skull are assumed to be

³Although *in vivo* experiments show that the brain is not rigidly connected to the skull, e.g., in [106], this is a common assumption made in ICM head models to simplify the problem.

initially at rest or rotating with a constant angular velocity. (A.viii) Initially the brain is assumed to not have any displacements with respect to the skull, and (A.ix) to be stress free.

3.3.2 Governing equations

3.3.2.1 Displacement-deformation relationship

It follows from our assumptions (A.ii) (2D deformations) and (A.iv) (axi-symmetric deformations) in §3.3.1 that there exist maps $U_i^C, i \in \mathcal{I}$, from $[0, r_0] \times [0, \infty)$ to $\mathbb{R}$ such that

$$U_\tau \left[\sum_{i \in \mathcal{I}} X_i E_i \right] = U_1^C [\bar{r} [X], \tau] \bar{\mathcal{C}}_1 [X] + U_2^C [\bar{r} [X], \tau] \bar{\mathcal{C}}_2 [X] + U_3^C [\bar{r} [X], \tau] \bar{\mathcal{C}}_3 [X]. \quad (3.20)$$

We denote the non-dimensional form of the deformation gradient of the map that transforms $\kappa_R [\mathcal{B}]$ to $\kappa^\star [\mathcal{B}]$ as $F^\star [X, \tau]$, or $F^\star [X]$ for short. And the non-dimensional form of the deformation gradient of the map that transforms $\kappa_R [\mathcal{B}]$ to $\kappa_\tau [\mathcal{B}]$ as $F [X, \tau]$, or $F_\tau [X]$ for short. It can be shown that

$$F^\star [X, \tau] = I + H_\tau^\star [X], \quad (3.21a)$$

$$F [X, \tau] = Q[\tau] F^\star [X, \tau]. \quad (3.21b)$$

In (3.21a), $H_\tau^\star [X] := \text{Grad}_X [U_\tau]$; here $\text{Grad}_X [U_\tau]$ is the gradient of U_τ at the material particle X . More explicitly,

$$H_\tau^\star [X]_{i,j} = \frac{\partial U_{\tau i}}{\partial X_j} [X]. \quad (3.22)$$

In §3.3.1, we stated that we assume the displacements and deformations of the brain with respect to the skull are small, by which we mean that we assume $U_\tau [X]$ and $H_\tau^\star [X]$ to be small.

It follows from our assumption A.ii (2D analysis, see §3.3.1) and (3.11) that

$$U_{\tau 3} [X] = U_3^C [\bar{r} [X], \tau] = 0. \quad (3.23)$$

Our assumptions that $\mathcal{B}$ is incompressible (assumption A.i.c in §3.3.1), and $H_\tau^\star [X]$ is small;

equation 3.23; and our assumption of axisymmetric deformation (assumption A.iv in §3.3.1), i.e., that $U_\tau [X]$ has the form given in (3.20) lead us to the result that $U_1^C = 0$ (see §B.1 for details). It then follows from (3.20) and (3.23) that

$$U_\tau \left[\sum_{i \in I} X_i E_i \right] = U_2^C [\bar{r} [X], \tau] \bar{\mathcal{C}}_2 [X], \quad (3.24)$$

or equivalently that

$$U_\tau [X] = U_2^C [\bar{r} [X], \tau] c_2 [X]. \quad (3.25)$$

The vector $c_2 [X]$ appearing in (3.25) is defined through the equations

$$c_i [X] = \left(\bar{\mathcal{C}}_i [X] \cdot E_j \right)_{j \in I}, \quad (3.26)$$

$i \in I$.

3.3.2.2 Equation of motion

Let $P^*[X, \tau]$ and $P[X, \tau]$ be the non-dimensional form of 1st Piola-Kirchhoff stress tensors in the body when it, respectively, assumes the configurations κ^* and κ_τ . It follows from the *principle of material frame indifference* (e.g., see [107, §21]) that

$$P[X, \tau] = Q[\tau] P^*[X, \tau]. \quad (3.27)$$

From Hamilton's principle [108] and (3.27) we get $\mathcal{B}$'s equation of motion to be

$$Q^T [\tau] (\text{Div}_X [Q[\tau] P^*[\tau]] - \rho_0 A_\tau [X]) \cdot c_2 [X] = 0, \quad (3.28)$$

where Div is the divergence operator, and $\rho_0 \in \mathbb{R}_{>0}$ is defined such that $\rho_0 \text{ kg/m}^3$ is the density of $\mathcal{B}$ in the reference configuration⁴.

⁴We do not provide the details of the derivation of (3.28) from Hamilton's principle. The primary novelty in our derivation is that in it

Let $\bar{\mathbf{P}}_X : \mathcal{M}_{3 \times 3}(\mathbb{R}) \rightarrow \mathcal{M}_{3 \times 3}(\mathbb{R})$ be the material's constitutive equation, that is $\mathbf{P}^\star[X, \tau] = \bar{\mathbf{P}}_X[\mathbf{F}^\star[X, \tau]]$. We assume that the reference configuration $\kappa_R[\mathcal{B}]$ is stress free (assumption A.ix in §3.3.1). It then follows that as $\mathbf{H} \in \mathcal{M}_{3 \times 3}(\mathbb{R}) \rightarrow \mathbf{0}$, the zero element in $\mathcal{M}_{3 \times 3}(\mathbb{R})$,

$$\bar{\mathbf{P}}_X[\mathbf{I} + \mathbf{H}] = \mathbf{C}_X \mathbf{H} + o(\mathbf{H}), \quad (3.29)$$

where $\mathbf{C}_X \in \mathcal{M}_{3 \times 3 \times 3 \times 3}(\mathbb{R})$ is a non-dimensional form of the *elasticity tensor*. In (3.29) “ o ” is the Landau “little- o ” symbol that implies that (3.29) is equivalent to

$$\lim_{\mathbf{H} \rightarrow \mathbf{0}} \frac{\|\bar{\mathbf{P}}_X[\mathbf{I} + \mathbf{H}] - \mathbf{C}_X \mathbf{H}\|}{\|\mathbf{H}\|} = 0. \quad (3.30)$$

Thus, the linear map $\mathbf{C}_X$ is the Fréchet derivative of $\bar{\mathbf{P}}_X$ at $\mathbf{I}$. In this chapter we model the brain as a homogenous solid. This implies that the elasticity tensor does not depend on X . Therefore, from here on we denote $\mathbf{C}_X$ simply as $\mathbf{C}$.

Assuming $\mathbf{H}_\tau^\star$ to be uniformly small, i.e, in the limit of the deformations of the brain with respect to the skull vanishing uniformly over the brain, we get from (3.28) and (3.30) that

$$\mathbf{Q}^\top[\tau] (\text{Div}_X [\mathbf{Q}[\tau] \mathbf{C} \mathbf{H}_\tau^\star] - \rho_0 \mathbf{A}_\tau[X]) \cdot \mathbf{c}_2[X] = 0. \quad (3.31)$$

As we stated in §3.3.1, we take $\mathcal{B}$ to be isotropic. For isotropic materials it can be shown that

$$\mathbf{C} = (C_{ijkl})_{i,j,k,l \in I}, \quad (3.32a)$$

$$C_{ijkl} = \lambda \delta_{ij} \delta_{kl} + \mu (\delta_{ik} \delta_{jl} + \delta_{il} \delta_{jk}), \quad (3.32b)$$

where λ, μ are defined such that $\lambda \text{ N/m}^2$ and $\mu \text{ N/m}^2$ are $\mathcal{B}$'s Lamé parameters.

Writing $\mathbf{C}$ in (3.31) in terms of λ and μ using (3.32); $\mathbf{H}_\tau^\star$ in (3.31) in terms of U_2^C using (3.22) and (3.25); $\mathbf{A}_\tau[X]$ in (3.31) in terms of $\mathbf{Q}[\tau]$ and $\mathbf{U}_X[\tau]$ using (3.19); replacing $\mathbf{U}_X[\tau]$ in the resulting equation with $\mathbf{U}_\tau[X]$ and then writing $\mathbf{U}_\tau[X]$ in terms of $U_2^C[\bar{\mathbf{r}}[X], \tau]$ and $\mathbf{c}_2[X]$ using (3.25); writing $\mathbf{Q}[\tau]$ in the resulting equation in terms of $\Omega[\tau]$ using (3.9); and finally writing $\mathbf{c}_2[X]$ in

the space of admissible displacements and admissible variations are different from those in standard application of Hamilton's principle in finite elasticity as a consequence of our displacements having to satisfy the constraint (3.25).

terms of X_1, X_2 in the resulting equation using (3.26) and simplifying we get that

$$\mu \left(\partial^{(2,0)} U_2^C [r, \tau] + \frac{\partial^{(1,0)} U_2^C [r, \tau]}{r} - \frac{U_2^C [r, \tau]}{r^2} \right) = \rho_0 r \Omega'' [\tau] + \rho_0 \partial^{(0,2)} U_2^C [r, \tau] - \rho_0 \Omega' [\tau]^2 U_2^C [r, \tau]. \quad (3.33)$$

3.3.2.3 Boundary and initial conditions

From (3.12), (3.15b), and (3.25) it follows that the displacements

$$u_\tau [X] = \sum_{i=1}^2 u_i^C [\bar{r} [X], \tau] c_i [X], \quad (3.34)$$

where

$$u_1^C [r, \tau] := r (\cos \circ \Omega [\tau] - 1) - \sin \circ \Omega [\tau] U_2^C [r, \tau], \quad (3.35a)$$

$$u_2^C [r, \tau] := r \sin \circ \Omega [\tau] + \cos \circ \Omega [\tau] U_2^C [r, \tau]. \quad (3.35b)$$

It follows from our assumptions *A.viii* (no initial displacements of the brain w.r.t. the skull) that

$$U_2^C [r, \tau = 0] = 0. \quad (3.36a)$$

It follows from our assumption that *A.vii* (the brain and skull are initially at rest) and (3.34) that at $\tau = 0$, $\partial^{(0,1)} u_i^C [r, \tau]$, where $i = 1, 2$, vanish. It therefore follows from (3.35) and (3.10) that

$$\partial^{(0,1)} U_2^C [r, \tau = 0] = 0. \quad (3.36b)$$

It follows from our assumptions *A.i.e* (skull is a rigid solid), and *A.vi* (brain is rigidly connected to the skull) and (3.25) that

$$U_2^C [r = r_0, \tau] = 0, \quad (3.37)$$

where recall that r_0 is the non-dimensional radius of the brain, which we have modeled as a finite

cylinder (see Fig. 3.1).

3.4 Semi-analytical solution

3.4.1 Scaling of the equation

We scale the equations (3.33), (3.36), and (3.37) by using one spatial constant and three time constants. The spatial constant is r_0 , where recall that r_0 meters is the radius of our cylindrical head model. There are three intrinsic time scales in the problem. Two of them are related to the loading, $\Omega [\cdot]$. These are $\tau_1, \tau_2 \in \mathbb{R}_{>0}$, where τ_1 is the non-dimensional time at which $\Omega' [\cdot]$ attains its maximum absolute value, and τ_2 is the inverse of $\Omega' [\cdot]$'s maximum absolute value. The time constants τ_1 and τ_2 are illustrated for a model $\Omega [\cdot]$ in Fig. 3.2(a). The derivative of that $\Omega [\cdot]$ is sketched in Fig. 3.2(b). The last time scale τ_s seconds is related to the elasticity and dimensions of our head model. It is defined as

$$\tau_s = \frac{r_0}{\sqrt{\mu/\rho_0}}, \quad (3.38)$$

where, recall that, μ N/m² is the brain's shear modulus in our model (cf. (3.32b)), and ρ_0 kg/m³ is the brain's density (cf. (3.28)). The time τ_s seconds is the time taken by a shear wave to travel from the brain's outer surface to its center in our model.

We define the scaled solution $\hat{U}_2^C [\cdot, \cdot] : [0, 1] \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ as

$$\hat{U}_2^C [\hat{r}, \hat{\tau}] = U_2^C [\hat{r}r_0, \hat{\tau}\tau_1] / r_0. \quad (3.39)$$

Similarly the scaled loading $\hat{\omega} [\cdot] : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ is defined as

$$\hat{\omega} [\hat{\tau}] = \tau_2 \Omega' [\hat{\tau}\tau_1]. \quad (3.40)$$

The $\hat{\omega} [\cdot]$ (resp. $\hat{\omega}' [\cdot]$) that corresponds to the representative model $\Omega' [\cdot]$ (resp. $\Omega'' [\cdot]$) shown with a black line in Fig. 3.2(a) (resp. (b)) is sketched in Fig. 3.2(c) (resp. (d)).

In terms of $\hat{U}_2^C [\cdot, \cdot]$, $\hat{r}$, and $\hat{\tau}$ the equation of motion (3.33) reads

$$\partial^{(2,0)} \hat{U}_2^C [\hat{r}, \hat{\tau}] + \frac{\partial^{(1,0)} \hat{U}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}} - \frac{\hat{U}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}^2} = \frac{\tau_s^2}{\tau_1^2} \left(\frac{\tau_1}{\tau_2} \hat{r} \hat{\omega}' [\hat{\tau}] + \partial^{(0,2)} \hat{U}_2^C [\hat{r}, \hat{\tau}] - \left(\frac{\tau_1}{\tau_2} \right)^2 \hat{\omega}^2 [\hat{\tau}] \hat{U}_2^C [\hat{r}, \hat{\tau}] \right); \quad (3.41a)$$

the boundary condition (3.37) reads

$$\hat{U}_2^C [\hat{r} = 1, \hat{\tau}] = 0; \quad (3.41b)$$

and the initial conditions (3.36a), (3.36b), respectively, read

$$\hat{U}_2^C [\hat{r}, \hat{\tau} = 0] = 0, \quad \partial^{(0,1)} \hat{U}_2^C [\hat{r}, \hat{\tau} = 0] = 0. \quad (3.41c, 3.41d)$$

3.4.2 Solution using eigenfunction expansions and Sturm-Liouville theory

In this section we solve for $\hat{U}_2^C [\cdot, \cdot]$ using the method of eigenfunction expansion. We postulate that $\hat{U}_2^C [\cdot, \cdot]$ can be expressed as,

$$\sum_{n=1}^{\infty} f_n [\hat{\tau}] g_n [\hat{r}], \quad (3.42)$$

where $f_n [\cdot] \in C^2(\mathbb{R}_{\geq 0}, \mathbb{R})$,

$$g_n [\hat{r}] := \sqrt{2} \frac{J_1 [j_{1,n} \hat{r}]}{J_0 [j_{1,n}]}. \quad (3.43)$$

In equation (3.43), $J_0 [\cdot]$ and $J_1 [\cdot]$ are Bessel functions of the first kind of zeroth and first order, respectively. The symbol $j_{1,n}$, $n \in \mathbb{N}$, denotes the n -th zero of J_1 . That is, $j_{1,n}$ is defined by the conditions that $J_1 [j_{1,n}] = 0$ and $j_{1,n} \neq j_{1,m}$ for $m < n$. The first few $j_{1,n}$ are illustrated in Fig. 3.3(a), and the first few $g_n [\cdot]$ are illustrated in Fig. 3.3(b).

It can be verified from $g_n [\cdot]$'s definition (3.43) that $g_n [1] = 0$, it therefore follows that the form (3.42) postulated for $\hat{U}_2^C [\cdot, \cdot]$ satisfies the boundary condition (3.41b). In order for the form (3.42)

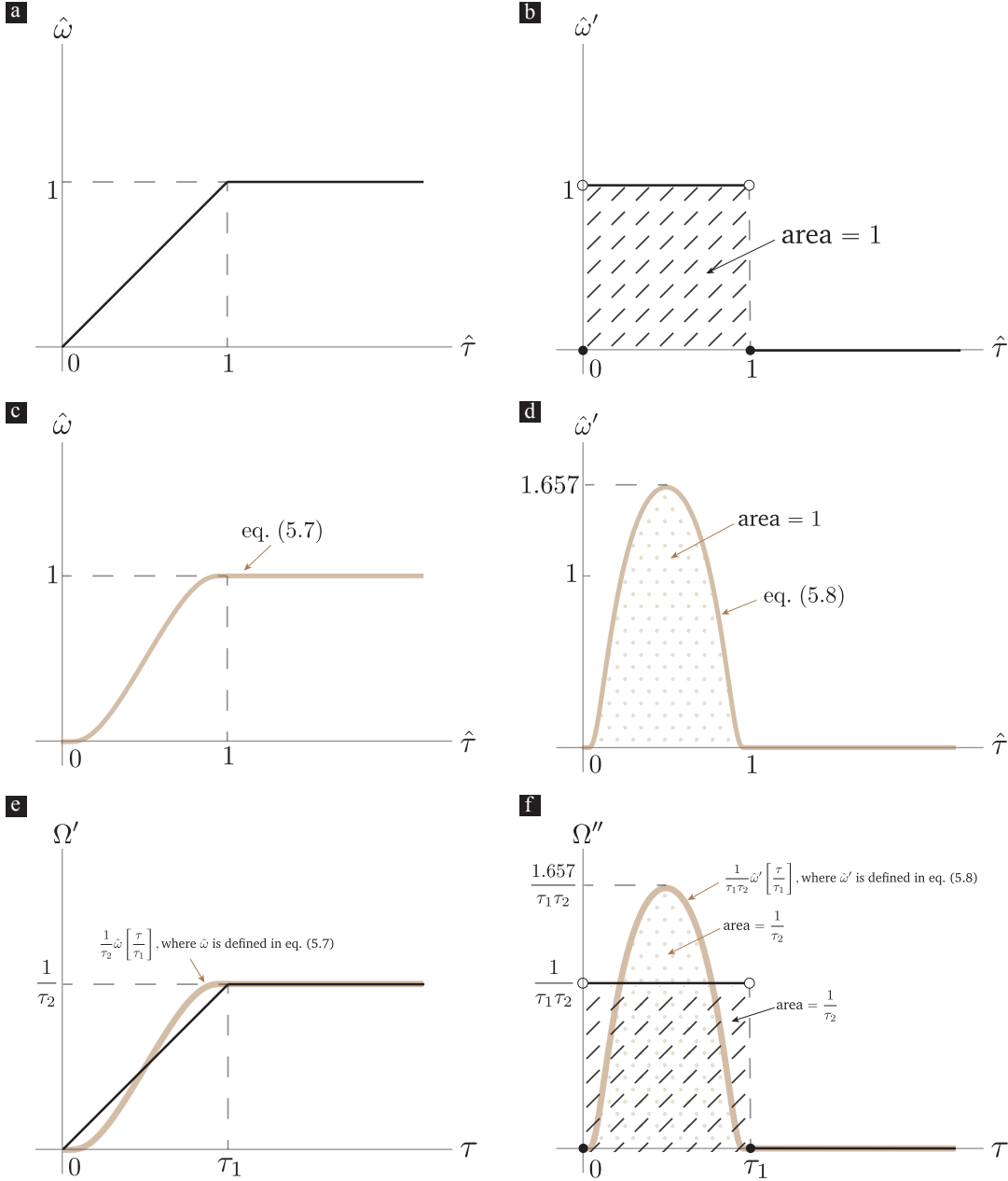


Figure 3.2: The subfigures (a) and (c) show two examples for the scaled loading function $\hat{\omega}[\cdot]$. The scaled loading function is defined in (3.40), in terms of the derivative of the loading function $\Omega[\cdot]$. The function $\Omega[\cdot]$ is defined and discussed in §3.2.3. The subfigure (a) shows the graph of the function $[0, \infty) \ni x \mapsto H[x-1](1-x) + xH[x] \in \mathbb{R}$, where $H[\cdot]$ is the unit step function, while subfigure (c) shows the graph of the function (3.64). The subfigure (b) (resp. (d)) shows the graph of the derivative of the function plotted in subfigure (a) (resp. (c)). The subfigure (e) shows sketches of the graphs of the $\Omega'[\cdot]$ that correspond to the scaled loading functions $\hat{\omega}[\cdot]$ plotted in subfigures (a) and (c). The curve in black corresponds to the function plotted in subfigure (a), while the curve in yellow/gold corresponds to the function plotted in subfigure (c). The black curve (resp. yellow/gold curve) in subfigure (f) is a sketch of the graph of the derivative of the function plotted in subfigure (e) in black (resp. yellow/gold).

to satisfy the initial conditions (3.41c) and (3.41d) it is necessary and sufficient that

$$f_n[0] = 0, \quad (3.44a)$$

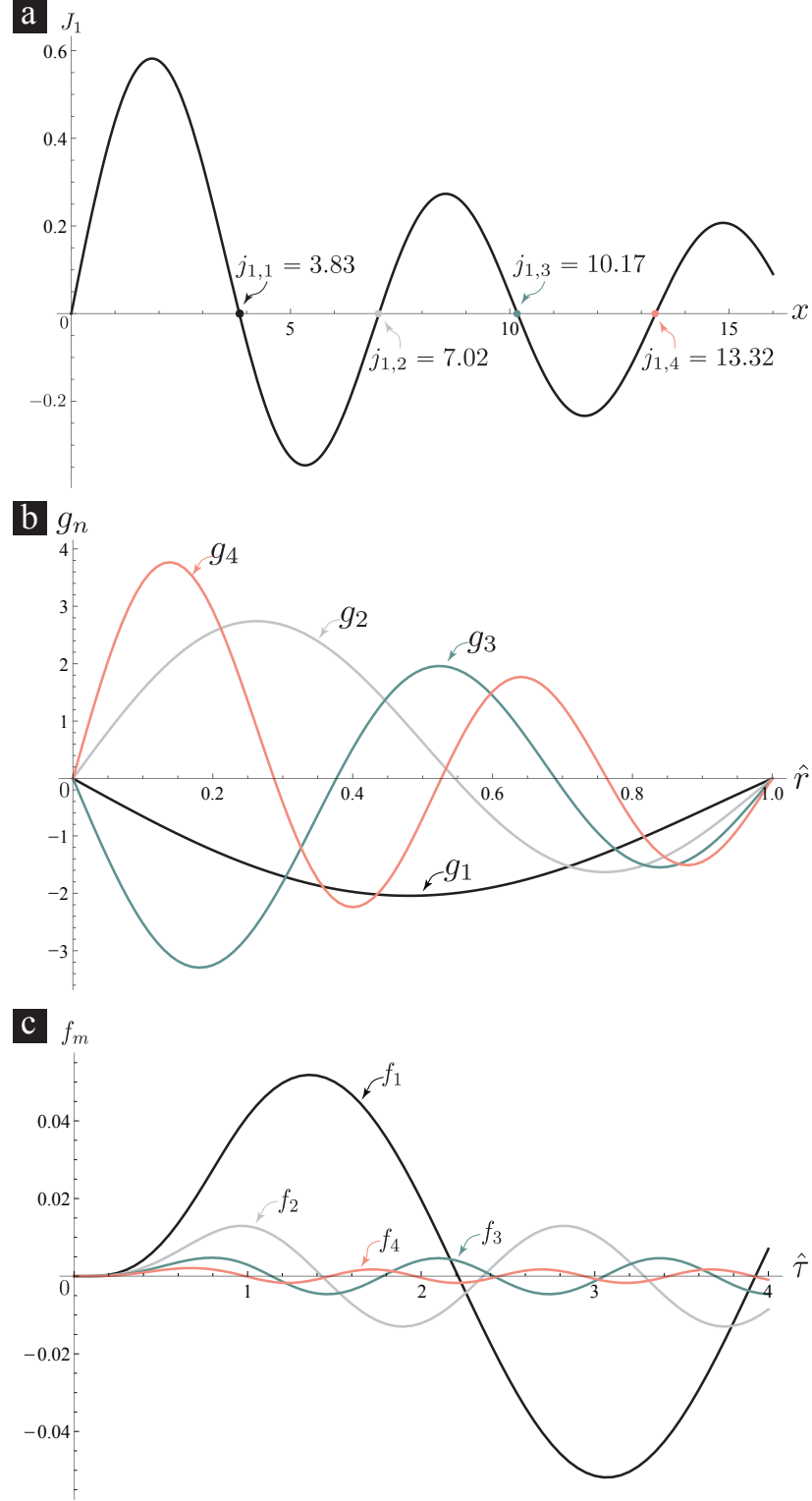


Figure 3.3: (a) Graph of the Bessel function of the first kind of the first order, J_1 [·]. The first few zeros of J_1 [·], i.e., $j_{1,n}$, $n = 1, 2, 3, 4$, are marked and labeled in the plot. (b) Graphs of the first few g_n [·], $n = 1, 2, 3, 4$. The functions g_n [·] are defined in (3.43). (c) Illustration of the first few f_m [·], $m = 1, 2, 3, 4$, that correspond to the material and geometry parameters (3.63); the loading function $\hat{\omega}$ [·] given in (3.64); and $(\tau_1, \tau_2) = (5.5 \times 10^{-3}, 20 \times 10^{-3})$ (equivalently $(\Omega'_{\max}, \Omega''_{\max}) = (50, 15000)$). These f_n [·] were obtained through a numerical solution of (3.52) with the initial conditions (3.44).

$$f'_n [0] = 0. \quad (3.44b)$$

Next we derive a governing equation for $f_n [\cdot]$, which in conjunction with the initial conditions (3.44) will determine $f_n [\cdot]$.

Let

$$H_r = \left\{ f : (0, 1) \rightarrow \mathbb{R} \mid \int_0^1 r f [r]^2 \, dr < \infty \right\}. \quad (3.45)$$

Defining the inner product $(\cdot, \cdot)_r : H_r \times H_r \rightarrow \mathbb{R}$,

$$(f, g)_r = \int_0^1 \xi f [\xi] g [\xi] \, d\xi, \quad (3.46)$$

on H_r , it can be shown that $(H_r, (\cdot, \cdot)_r)$ is a Hilbert Space. Noting that

$$(g_n, g_m)_r = \delta_{nm}, \quad (3.47)$$

it can be shown that $(g_n [\cdot])_{n \in \mathbb{N}}$ provide an orthonormal basis for H_r .

Replacing $\hat{U}_2^C [\hat{r}, \hat{\tau}]$ in (3.41a) with the form given in (3.42), replacing the function $(0, 1) \ni \hat{r} \mapsto \hat{r}$ that appears in the first term on the right hand side of (3.41a) with its expansion in the $g_n [\cdot]$ basis, i.e., writing $\hat{r}$ as $\sum_{n \in \mathbb{N}} p_n g_n [\hat{r}]$, where

$$p_n := \int_0^1 \xi^2 g_n [\xi] \, d\xi, \quad (3.48)$$

in the first term on the right hand side of (3.41a), we get that

$$\sum_{n=1}^{\infty} f_n [\hat{\tau}] (\mathcal{D} g_n) [\hat{r}] = \sum_{n=1}^{\infty} \frac{\tau_s^2}{\tau_1 \tau_2} p_n g_n [\hat{r}] \hat{\omega}' [\hat{\tau}] + \sum_{n=1}^{\infty} \frac{\tau_s^2}{\tau_1^2} f_n'' [\hat{\tau}] g_n [\hat{r}] - \sum_{n=1}^{\infty} \frac{\tau_s^2}{\tau_2^2} \hat{\omega}^2 [\hat{\tau}] f_n [\hat{\tau}] g_n [\hat{r}], \quad (3.49)$$

where f_n'' is the derivative of f_n' , and the operator $\mathcal{D}$ is defined such that

$$(\mathcal{D} g_n) [\hat{r}] = g_n'' [\hat{r}] + g_n' [\hat{r}] / \hat{r} - g_n [\hat{r}] / \hat{r}^2. \quad (3.50)$$

Noting that

$$(\mathcal{D}g_n) [\hat{r}] = -j_{1,n}^2 g_n [\hat{r}] \quad (3.51)$$

and substituting $(\mathcal{D}g_n) [\hat{r}]$ in equation (3.49) with the right side of (3.51); taking inner product (as defined in (3.46)) of the resulting equation with $g_m [\cdot]$, $m \in \mathbb{N}$; and simplifying the resulting equation using (3.47) we get a governing equation of $f_m [\cdot]$ to be

$$f_m'' [\hat{\tau}] + \frac{\tau_1^2}{\tau_s^2} \left(j_{1,m}^2 - \frac{\tau_s^2}{\tau_2^2} \hat{\omega}^2 [\hat{\tau}] \right) f_m [\hat{\tau}] = -\frac{\tau_1}{\tau_2} p_m \hat{\omega}' [\hat{\tau}]. \quad (3.52)$$

The first few $f_m [\cdot]$ for the $\hat{\omega} [\cdot]$ given in (3.64) are illustrated in Fig. 3.3(c). We computed those $f_m [\cdot]$ by numerically integrating (3.52) with the initial conditions (3.44). In these numerical calculations $\tau_1 = 0.0055$, $\tau_2 = 0.02$, and $\tau_s = 0.0114$. The p_n 's, of course are independent of the loading. We computed the p_n by evaluating the integral in (3.48). The first few p_n are $p_1 = -0.37$, $p_2 = -0.20$, $p_3 = -0.14$, and $p_4 = -0.11$.

3.4.3 Displacements, strains, and strain rates

In this section we outline procedures for computing displacements (§3.4.3.1), strains (§3.4.3.2), and strain rates (§3.4.3.3) in our head model. We show the displacements, strains, and strain rates that we computed using these procedures for the representative geometry and material properties given in (3.63), and the loading function $\hat{\omega} [\cdot]$ given in (3.64), in Figs. 3.4, 3.5, and 3.6, respectively. For the geometry and material properties given in (3.63) the time scale $\tau_s = 11.4 \times 10^{-3}$. We carried out the calculations for the loading related time scales $(\tau_1, \tau_2) = (5.5 \times 10^{-3}, 20 \times 10^{-3})$ (equivalently $(\Omega'_{\max}, \Omega''_{\max}) = (50, 15000)$). In each of the Figs. 3.4, 3.5, and 3.6, we show the calculations for the time instances $\tau = 5 \times 10^{-3}$, 10×10^{-3} , and 15×10^{-3} .

3.4.3.1 Displacements

We outline a procedure for computing displacements in our head model in procedure 1.

Procedure 1 Computing displacements in the head model

1. Given an individual's head's biometric and other details, e.g., from MRI scans, select values for the characteristic length scale r_0 , and the characteristic time scale τ_s .
 2. Given a $\Omega [\cdot]$, select the loading times scales τ_1, τ_2 based on the characteristics of $\Omega [\cdot]$.
 3. Construct non-dimensional scaled loading functions $\hat{\omega} [\cdot]$ and $\hat{\omega}' [\cdot]$ using the given $\Omega [\cdot]$, the τ_1 and τ_2 from the previous step, and (3.40).
 4. Using (3.43) and (3.48) compute $p_n, n \in \mathbb{N}$.
 5. Using the $\tau_s, \tau_1, \tau_2, \hat{\omega}, \hat{\omega}'$, and p_n from the previous steps calculate $f_m, m \in \mathbb{N}$, by numerically integrating (3.52) with the initial conditions given in (3.44).
 6. Using the calculated f_m and the g_m given by (3.43), construct $\hat{U}_2^C [\cdot, \cdot]$ using (3.42).
 7. Using the $\hat{U}_2^C [\cdot, \cdot]$ from the previous step and (3.39) construct $U_2^C [\cdot, \cdot]$.
 8. Compute the (non-dimensional version of the) intermediate displacement field, $\mathbf{U}_\tau$, using $U_2^C [\cdot, \cdot]$ from the previous step and (3.25).
 9. Compute $\mathbf{x}_\tau$ using the $\mathbf{U}_\tau$ from the previous step and (3.12), and the displacement of the material particle $\mathbf{X}$ at the time instance τ , as $\mathbf{x}_\tau [\mathbf{X}] - \mathbf{X}$.
-

3.4.3.2 Strains

For a given loading, head geometry, and material properties, the field $U_2^C [\cdot, \cdot]$ can be computed by following the first seven steps of procedure 1. Using that $U_2^C [\cdot, \cdot]$ the intermediate displacement gradients in our head model can be computed as

$$\mathbf{H}_\tau^* [\mathbf{X}] = \begin{pmatrix} \frac{X_1 X_2}{\bar{r}^2 [\mathbf{X}]} \left(\frac{U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau]}{\bar{r} [\mathbf{X}]} - \partial^{(1,0)} U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau] \right) & \frac{-1}{\bar{r}^2 [\mathbf{X}]} \left(\frac{X_1^2 U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau]}{\bar{r} [\mathbf{X}]} + X_2^2 \partial^{(1,0)} U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau] \right) & 0 \\ \frac{1}{\bar{r}^2 [\mathbf{X}]} \left(\frac{X_2^2 U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau]}{\bar{r} [\mathbf{X}]} + X_1^2 \partial^{(1,0)} U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau] \right) & \frac{X_1 X_2}{\bar{r}^2 [\mathbf{X}]} \left(-\frac{U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau]}{\bar{r} [\mathbf{X}]} + \partial^{(1,0)} U_2^{\mathcal{G}} [\bar{r} [\mathbf{X}], \tau] \right) & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.53)$$

Various different strain measures can be computed in our head model using the displacement gradients from (3.53). For example, using (3.53) it can be shown that the Green-Lagrange strains,

$$\mathbf{E}[\mathbf{X}, \tau] := \left(\mathbf{H}_\tau^\star[\mathbf{X}] + \mathbf{H}_\tau^{\star\top}[\mathbf{X}] + \mathbf{H}_\tau^{\star\top}[\mathbf{X}] \mathbf{H}_\tau^\star[\mathbf{X}] \right) / 2, \quad (3.54)$$

in our head model in terms of $U_2^C[\cdot, \cdot]$ read

$$\mathbf{E}[\mathbf{X}, \tau] = \bar{\mathbf{R}}^\top[\mathbf{X}] \mathbf{E}^\mathcal{E}[\bar{\mathbf{r}}[\mathbf{X}], \tau] \bar{\mathbf{R}}[\mathbf{X}], \quad (3.55a)$$

where

$$\mathbf{E}^\mathcal{E}[r, \tau] := \frac{1}{2} \begin{pmatrix} \left(\partial^{(1,0)} U_2^\mathcal{E}[r, \tau] \right)^2 & \partial^{(1,0)} U_2^\mathcal{E}[r, \tau] - \frac{U_2^\mathcal{E}[r, \tau]}{r} & 0 \\ \partial^{(1,0)} U_2^\mathcal{E}[r, \tau] - \frac{U_2^\mathcal{E}[r, \tau]}{r} & \left(\frac{U_2^\mathcal{E}[r, \tau]}{r} \right)^2 & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.55b)$$

We give details for computing the spatial logarithmic strain tensor in our head model using the $\mathbf{H}_\tau^\star$ given in (3.53) in §3.5.1.1.

3.4.3.3 Strain rates

Various strain rate measures can be computed using the rate of deformation tensor (or rate of strain tensor). The rate of deformation tensor is the symmetric part of the spatial velocity gradient tensor. Let $\mathbf{D}[\mathbf{X}, \tau]$ be the non-dimensional form of the rate of deformation tensor of a material particle $\mathbf{X}$ at time instance τ w.r.t. $(\mathbf{e}_i \otimes \mathbf{s}^* \otimes \mathbf{e}_j)_{i,j \in I}$. After computing the field $U_2^C[\cdot, \cdot]$ by following the first seven steps of procedure 1 we can compute $\mathbf{D}[\mathbf{X}, \tau]$ in our head model as

$$\mathbf{D}[\mathbf{X}, \tau] = \mathbf{Q}[\tau] \bar{\mathbf{R}}^\top[\mathbf{X}] \mathbf{D}^\mathcal{E}[\bar{\mathbf{r}}[\mathbf{X}], \tau] \bar{\mathbf{R}}[\mathbf{X}] \mathbf{Q}^\top[\tau], \quad (3.56a)$$

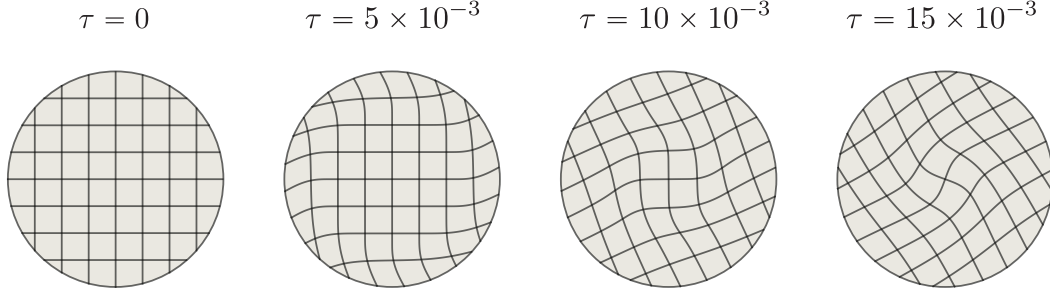


Figure 3.4: Configurations of the brain from the motion corresponding to the material and geometry parameters (3.63); the loading function $\hat{\omega}[\cdot]$ given in (3.64); and $(\tau_1, \tau_2) = (5.5 \times 10^{-3}, 20 \times 10^{-3})$ (equivalently $(\Omega'_{\max}, \Omega''_{\max}) = (50, 15000)$). The four configurations shown are for the time instances $\tau = 0, 5 \times 10^{-3}, 10 \times 10^{-3}$, and 15×10^{-3} . These configurations were computed by applying the procedure 1.

where

$$\mathbf{D}^{\mathcal{E}}[r, \tau] := \begin{pmatrix} \frac{\partial^{(1,0)} U_2^{\mathcal{E}}[r, \tau] \partial^{(0,1)} U_2^{\mathcal{E}}[r, \tau]}{r + U_2^{\mathcal{E}}[r, \tau] \partial^{(1,0)} U_2^{\mathcal{E}}[r, \tau]} & \frac{r \partial^{(1,1)} U_2^{\mathcal{E}}[r, \tau] - \partial^{(0,1)} U_2^{\mathcal{E}}[r, \tau]}{2(r + U_2^{\mathcal{E}}[r, \tau] \partial^{(1,0)} U_2^{\mathcal{E}}[r, \tau])} & 0 \\ \frac{r \partial^{(1,1)} U_2^{\mathcal{E}}[r, \tau] - \partial^{(0,1)} U_2^{\mathcal{E}}[r, \tau]}{2(r + U_2^{\mathcal{E}}[r, \tau] \partial^{(1,0)} U_2^{\mathcal{E}}[r, \tau])} & \frac{U_2^{\mathcal{E}}[r, \tau] \partial^{(1,1)} U_2^{\mathcal{E}}[r, \tau]}{r + U_2^{\mathcal{E}}[r, \tau] \partial^{(1,0)} U_2^{\mathcal{E}}[r, \tau]} & 0 \\ 0 & 0 & 0 \end{pmatrix}. \quad (3.56b)$$

3.5 Discussion

3.5.1 Comparison with finite element solutions

We compared “maximum 95th percentile MPLS”, which we will define in §3.5.1.1–§3.5.1.2, values in our head model with those in a related, finite element analysis (FEA) based, 2D head model [26]. The term “MPLS” stands for maximum principal logarithmic strain. The MPLS of a material particle $\mathbf{X}$ at a time instance τ , which we denoted as $\phi_{\tau 1}[\mathbf{X}]$, is defined as the maximum eigenvalue of $\ln \mathbf{V}_{\tau}[\mathbf{X}]$, the non-dimensional form of (spatial) logarithmic strain tensor at $\mathbf{X}$. We recall the definition of $\ln \mathbf{V}_{\tau}[\mathbf{X}]$ and more precisely define $\phi_{\tau 1}[\mathbf{X}]$ in §3.5.1.1. In §3.5.1.1 we also provide the details for computing $\phi_{\tau 1}[\mathbf{X}]$ in our head model. We define the 95th percentile MPLS at the time

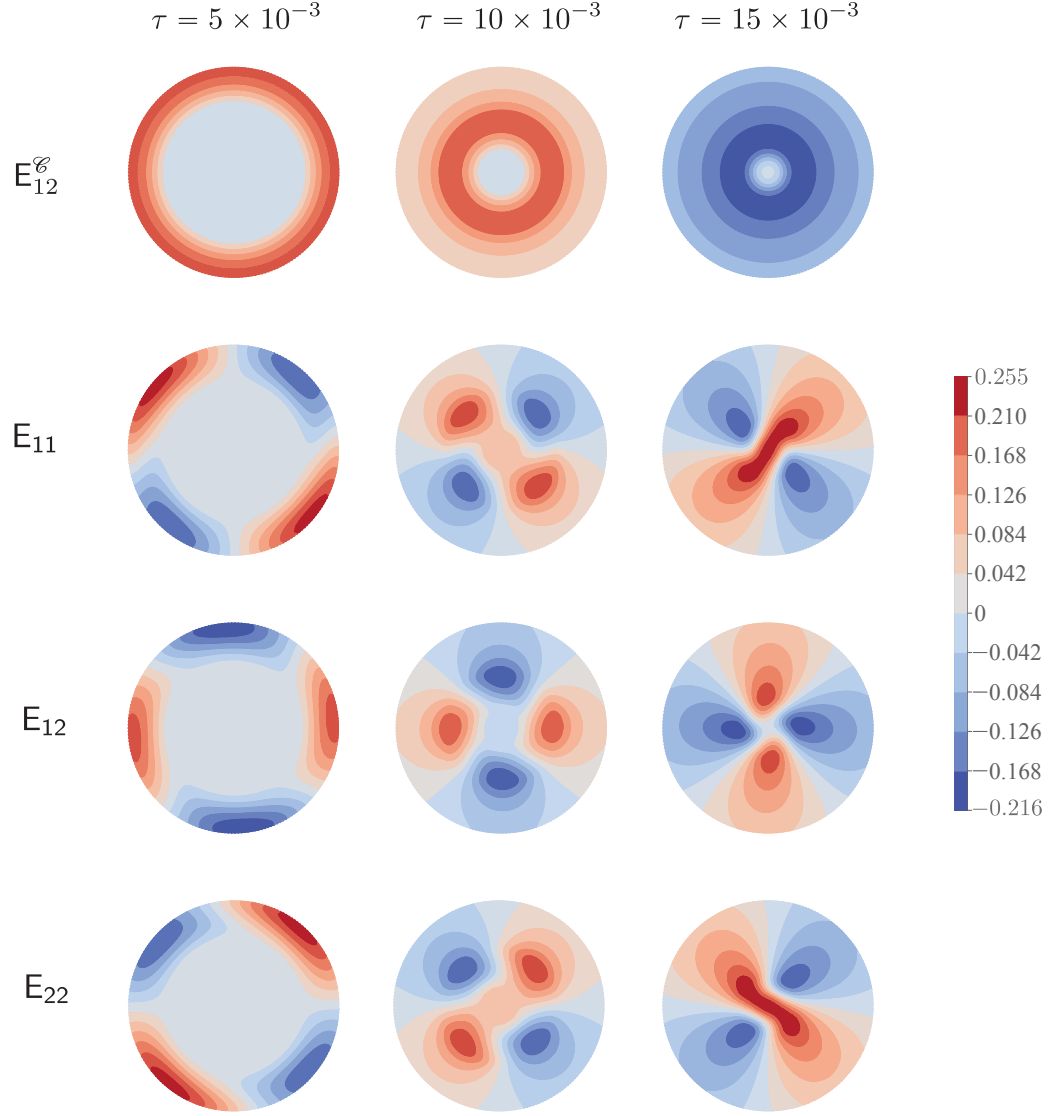


Figure 3.5: Contour plots of the components of the Green-Lagrange strain tensor that corresponds to the motion shown Fig. 3.4. The first row shows the contour plot of $E_{12}^{\mathcal{C}}[\cdot, \cdot]$ (see (3.55b)), which is the 1-2 component of the Green-Lagrange strain tensor w.r.t. the $(\bar{\mathcal{C}}_i[X] \otimes \bar{\mathcal{C}}_j[X])_{i,j \in I}$ basis. The second, third, and fourth rows, respectively, show the contour plots of $E_{11}[\cdot, \cdot]$, $E_{12}[\cdot, \cdot]$, and $E_{22}[\cdot, \cdot]$, which are the Cartesian components of the Green-Lagrange strain tensor (see (3.55a)). The first, second, and third columns correspond to the time instances $\tau = 5 \times 10^{-3}$, 10×10^{-3} , and 15×10^{-3} .

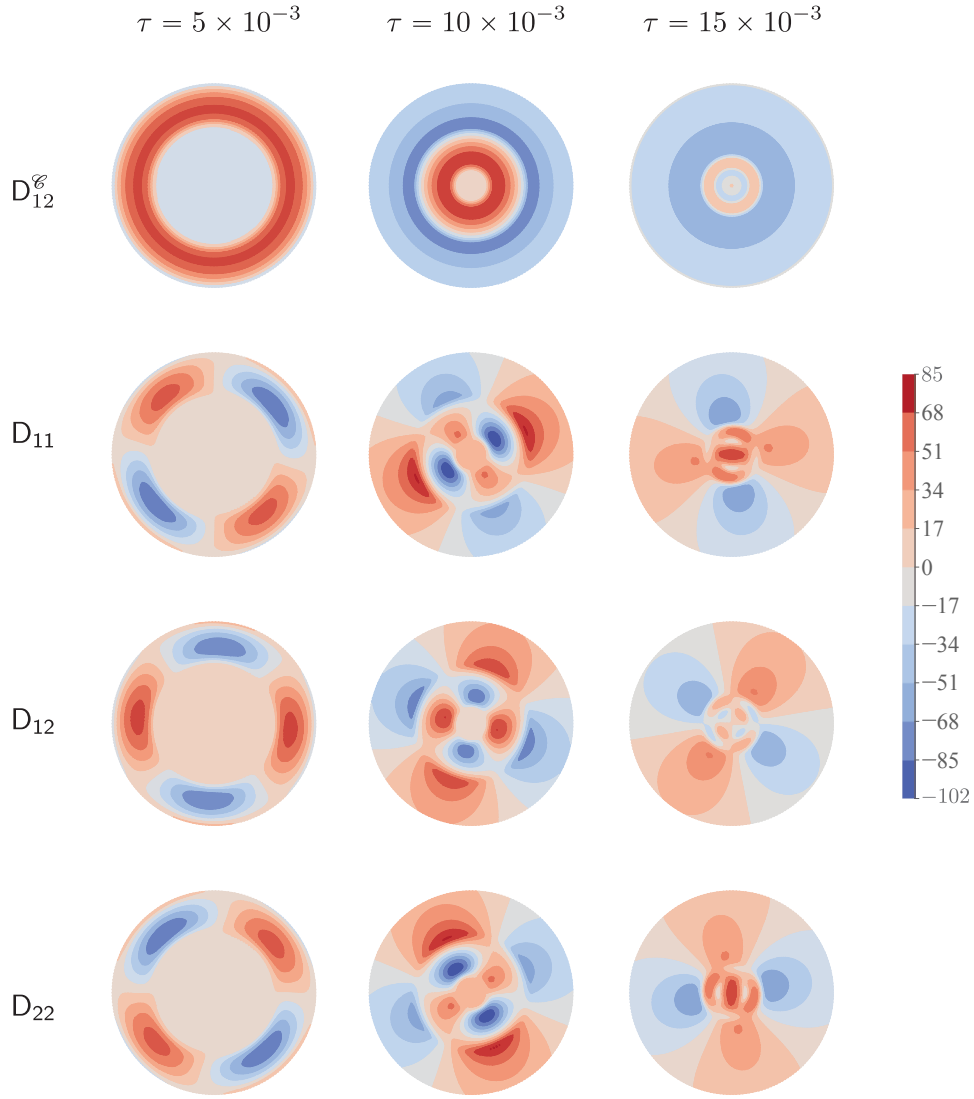


Figure 3.6: Contour plots of the components of the rate of deformation tensor that corresponds to the motion shown in Fig. 3.4. The first row shows the contour plot of $D_{12}^E[\cdot, \cdot]$, which is the 1-2 component of the non-dimensional rate of deformation tensor $D^E[\cdot, \cdot]$ (see (3.56b)). The second, third, and fourth rows, respectively, show the contour plots of $D_{11}[\cdot, \cdot]$, $D_{12}[\cdot, \cdot]$, and $D_{22}[\cdot, \cdot]$, which are components of the non-dimensional rate of deformation tensor $D[\cdot, \cdot]$ (see (3.56a)). The first, second, and third columns correspond to the time instances $\tau = 5 \times 10^{-3}$, 10×10^{-3} , and 15×10^{-3} .

instance τ , which we denote as $\phi^{95}[\tau]$, in §3.5.1.2 using $\phi_{\tau 1}[X]$. The maximum 95th percentile MPLS, which we denote as, $\phi_{\max}^{95}$, is also defined in §3.5.1.2 using $\phi^{95}[\tau]$.

We computed the $\phi_{\max}^{95}$ values in our head model for the loading, geometry, and material properties that are similar to those used in the finite element head model simulations. We give their details in §3.5.1.3. The $\phi_{\max}^{95}$ values computed in our model are compared with those in the finite element head model in Fig. (3.8). As can be seen in Fig. (3.8), the qualitative dependence of $\phi_{\max}^{95}$ values on the two loading parameters $\Omega'_{\max}$ and $\Omega''_{\max}$, which we will detail in §3.5.1.3, in our head model is very similar to that seen in the finite element head model. As will be detailed in §3.5.1.3, the finite element head model is far more detailed and sophisticated than our ICM based head model. For example, it includes information about the brain's spatially heterogenous and anisotropic material behavior, as well as information about the brain's viscous and non-linear elastic behavior. What we find remarkable is that despite the far less sophistication and detail in our model, it is able to predict strains values that are as close to the finite element ones as they are in Fig. (3.8).

3.5.1.1 Maximum principle logarithmic strain (MPLS, $\phi_{\tau 1}[X]$)

When $F_{\tau}[X]$ is invertible it follows from the *polar decomposition theorem* e.g., see [107, §2] that there exists a unique symmetric positive definite tensor $V_{\tau}[X]$, called the left stretch tensor, such that $V_{\tau}[X] V_{\tau}[X] = B_{\tau}[X]$, where

$$B_{\tau}[X] := F_{\tau}[X] (F_{\tau}[X])^T \quad (3.57)$$

is called the left Cauchy-Green deformation tensor. It follows from the definition of $V_{\tau}[X]$ and the *square root theorem*, e.g., see [107, §2] that

$$V_{\tau}[X] = \sum_{i \in I} \sqrt{\varphi_{\tau i}[X]} n_{\tau i}[X] \otimes n_{\tau i}[X], \quad (3.58)$$

where $n_{\tau i}[X] \in \mathcal{M}_{3 \times 1}(\mathbb{R})$ are the eigenvectors of $B_{\tau}[X]$ such that $n_{\tau i}[X] \cdot n_{\tau j}[X] = \delta_{ij}$ (which exist owing to the *spectral theorem* [107, §2], since $B_{\tau}[X]$ is symmetric), and $\varphi_{\tau i}[X]$ are $B_{\tau}[X]$'s eigenvalues corresponding to $n_{\tau i}[X]$. The (spatial) logarithmic strain tensor $\ln V_{\tau}[X]$ is defined to

be the natural logarithm of $V_\tau [X]$. It can be shown that

$$\ln V_\tau [X] = \sum_{i \in \mathcal{I}} \left(\ln \left[\sqrt{\varphi_{\tau i} [X]} \right] \right) n_{\tau i} [X] \otimes n_{\tau i} [X]. \quad (3.59)$$

It follows from (3.59) and the spectral theorem that $\ln \left[\sqrt{\varphi_{\tau i} [X]} \right]$ are $\ln V_\tau [X]$'s eigenvalues. Thus, the MPLS, $\phi_{\tau 1} [X]$, is the maximum of the $\ln \left[\sqrt{\varphi_{\tau i} [X]} \right]$, $i \in \mathcal{I}$.

Writing $H_\tau^* [X]$ in our head model in terms of $U_2^C [\cdot, \cdot]$ using (3.53); using that $H_\tau^* [X]$ and computing $F_\tau [X]$ using (3.21); computing the $B_\tau [X]$ corresponding to that $F_\tau [X]$ using (3.57); computing that $B_\tau [X]$'s eigenvalues and then computing the logarithm of the square root of the maximum of those eigenvalues we get that in our head model

$$\phi_{\tau 1} [X] = \phi_1^C [\bar{r} [X] / r_0, \tau / \tau_1], \quad (3.60)$$

where

$$\begin{aligned} \phi_1^C [\hat{r}, \hat{\tau}] = \frac{1}{2} \ln & \left[1 + \frac{1}{2\hat{r}^2} \left(\left(\hat{U}_2^C [\hat{r}, \hat{\tau}] \right)^2 + \hat{r}^2 \left(\partial^{(1,0)} \hat{U}_2^C [\hat{r}, \hat{\tau}] \right)^2 \right. \right. \\ & \left. \left. + \left| \hat{U}_2^C [\hat{r}, \hat{\tau}] - \hat{r} \partial^{(1,0)} \hat{U}_2^C [\hat{r}, \hat{\tau}] \right| \sqrt{4\hat{r}^2 + \left(\hat{U}_2^C [\hat{r}, \hat{\tau}] + \hat{r} \partial^{(1,0)} \hat{U}_2^C [\hat{r}, \hat{\tau}] \right)^2} \right) \right]. \end{aligned} \quad (3.61)$$

3.5.1.2 Ninety fifth percentile maximum principal logarithmic strain, $\phi^{95} [\cdot]$, and its maximum value $\phi_{\max}^{95}$

The 95th percentile MPLS at the time instance τ is defined to be the MPLS value such that at time instance τ the MPLS values over 95% of the brain are smaller than it. We give a more precise definition below, which has encoded in it the details to compute it.

Let $\mathcal{G} := \left\{ X \in \mathcal{B} \mid \check{X}_3 \circ \kappa_R [X] = 0 \right\}$ (for an illustration see Fig. 3.1), and let $\phi_{\tau, \inf}$ and $\phi_{\tau, \sup}$ be

the infimum and supremum of $\phi_{\tau_1} [\cdot]$ over $\mathbf{C} [\mathcal{G}]$, respectively, where

$$\mathbf{C} [\mathcal{G}] = \left\{ \mathbf{X} \in \mathcal{M}_{3 \times 1}(\mathbb{R}) \mid \sum_{i \in \mathcal{I}} X_i \mathbf{E}_i \in \kappa_{\mathbf{R}} [\mathcal{G}] \right\}.$$

The ϕ -sublevel set of $\phi_{\tau_1} [\cdot]$ at the time instance τ is defined as

$$S_{\tau} [\phi] = \{ \mathbf{X} \in \mathbf{C} [\mathcal{G}] \mid \phi_{\tau_1} [\mathbf{X}] \leq \phi \}.$$

We define the map m_{τ} ,

$$[\phi_{\tau, \text{inf}}, \phi_{\tau, \text{sup}}] \ni \phi \xrightarrow{m_{\tau}} \frac{\text{meas} [S_{\tau} [\phi]]}{\text{meas} [S_{\tau} [\phi_{\max}]]} \in [0, 1],$$

where $\text{meas} [\cdot]$ gives the Lebesgue measure of a set. When m_{τ} has an inverse we define the 95th percentile MPLS at the time instance τ , $\phi^{95} [\tau]$, to be $m_{\tau}^{-1} [0.95]$.

The maximum 95th percentile MPLS, $\phi_{\max}^{95}$, is the supremum of the values attained by $\phi^{95} [\cdot]$ over all time. In our head model for arbitrary loadings it is not possible to determine the supremum of $\phi^{95} [\cdot]$. This is because, since our head model does not include viscous effects and is non-linear, the dynamics in it may not reach a periodic state. Therefore we take $\phi_{\max}^{95}$ in our head model to be the supremum of the values attained by $\phi^{95} [\cdot]$ only over a finite time interval. Specifically, we take $\phi_{\max}^{95}$ to be the supremum of the values taken by $\phi^{95} [\cdot]$ over the time interval $[0, \tau_1 + 10\tau_s)$.

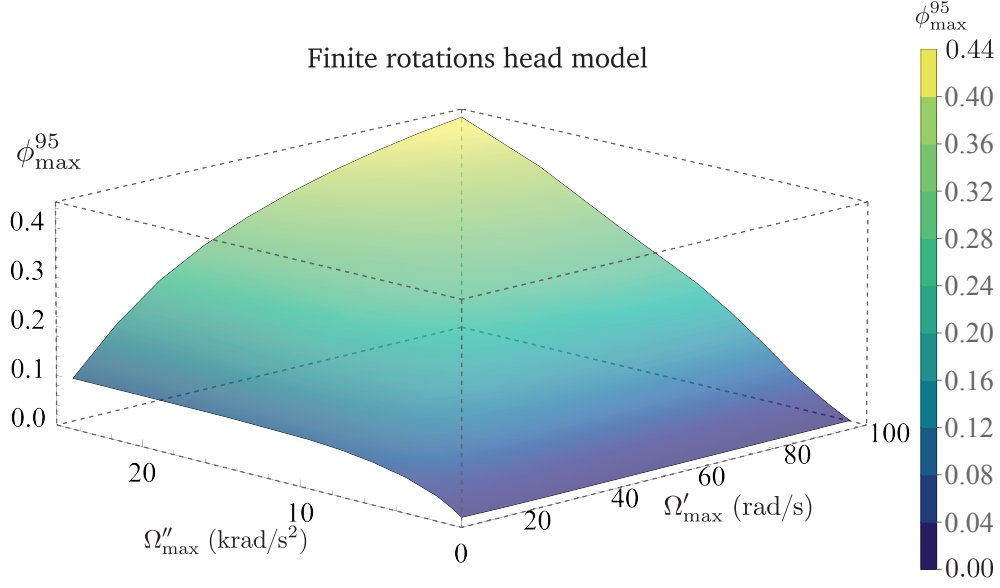


Figure 3.7: The maximum 95th percentile MPLS, $\phi_{\max}^{95}$, predicted by the finite rotations head model. The $\phi_{\max}^{95}$ values shown are from a family of motions; all of which correspond to the material and geometry parameters (3.63), and the loading function $\hat{\omega}[\cdot]$ given in (3.64). The family of motions were generate by varying (τ_1, τ_2) (resp. $(\Omega'_{\max}, \Omega''_{\max})$) from $(33.14 \times 10^{-3}, 100 \times 10^{-3})$ (resp. $(10, 0.5 \times 10^3)$) to $(6.63 \times 10^{-3}, 10 \times 10^{-3})$ (resp. $(100, 25 \times 10^3)$). The quantity $\phi_{\max}^{95}$ is discussed in §3.5.1.2.

3.5.1.3 Comparison with finite element solutions

As we mentioned in §3.5.1, we compared the “maximum 95th percentile MPLS” $\phi_{\max}^{95}$ values from the finite rotations head model with those from a related, finite element based, 2D head model [26].

To make our head model approximate the finite element head model as closely as possible in our

head model we take the non-dimensional head radius r_0 , density ρ_0 , and shear modulus μ ⁵ to be

$$r_0 = 0.06525, \quad (3.63a)$$

$$\rho_0 = 1040, \quad (3.63b)$$

$$\mu = 34000. \quad (3.63c)$$

(It follows from (3.38) that the value of the parameter τ_s for these geometry and material parameters is 0.0114.)

We agree that the assumed shear modulus of 34000 Pa is approximately an order of magnitude higher than most measurements of the behavior of actual brain tissue. The shear modulus of 34000 Pa is chosen by approximating the finite element (FE) based, 2D head model [26] as closely as possible. In FE head model [26], the main constituents of the brain, white and gray matter are modeled using viscoelastic Holzapfel-Gasser-Ogden material model. Both white and gray matter adopt the instantaneous shear modulus μ_0 of 34000 Pa and the long-term shear modulus at equilibrium μ_∞ of 6400 Pa. When not considering the effect of viscosity, the shear modulus of 34000 Pa is chosen for our finite rotation head model.

We take $\hat{\omega} [\cdot]$ to be

$$\hat{\tau} \mapsto \int_0^{\hat{\tau}} \hat{\psi} [\xi] d\xi, \quad (3.64)$$

where

$$\hat{\psi} [\hat{\tau}] := \begin{cases} 1.657 e^{1 - \frac{1}{1 - (2\hat{\tau} - 1)^2}}, & 0 \leq \hat{\tau} \leq 1, \\ 0, & \hat{\tau} > 1. \end{cases} \quad (3.65)$$

⁵Thus, for comparing with the results of the finite element model we took the shear modulus in our model to be 34 kilopascals. This value is much larger than the quasi-statically measured shear modulus values of white and gray matter, which are generally in the range of hundreds of pascals to kilopascals [109]. Our reasoning for choosing this larger value is as follows.

In the finite element model the white and gray matter were modeled as visco-elastic solids. Their shear relaxation function was taken to be $\mu_0 g [\cdot]$ N/m², where

$$g [\tau] = \frac{\mu_\infty}{\mu_0} + \frac{\mu_0 - \mu_\infty}{\mu_0} e^{-\beta \tau}, \quad (3.62)$$

with $\mu_0 = 34000$, $\mu_\infty = 6400$, and $\beta = 700$. The material constant μ_0 N/m² is called the instantaneous shear modulus, μ_∞ N/m² is called the long term shear modulus, and β Hertz is called the decay constant. As per (3.62) the brain's shear modulus at small time scales (compared to the relaxation time scale $1/\beta$ seconds) is given by μ_0 , while at the large time scales it is given by μ_∞ .

We assume that the largest strains and strain rates occur at the earliest time instances of the loading, rather than at the later time instances. Therefore, we wanted the shear modulus in our model to match the shear modulus in the finite element model at the earliest time instances. Hence, we took the shear modulus in our model, μ , to match the instantaneous shear modulus of the finite element model, μ_0 .

The function $\hat{\omega}' [\cdot]$ corresponding to the $\hat{\omega} [\cdot]$ given in (3.64) is the function $\hat{\psi} [\cdot]$ given in (3.65). The graph of the functions (3.64) and (3.65) are sketched in Figs. 3.2(e) and (f), respectively. The loading in our and the finite element head model depends not only on $\hat{\omega} [\cdot]$ (and hence $\hat{\omega}' [\cdot]$) but also on τ_1 and τ_2 .

It can be shown that for the choice of (3.64) for $\hat{\omega} [\cdot]$ the parameters (τ_1, τ_2) depend on $(\Omega'_{\max}, \Omega''_{\max})$, which we will define shortly, as

$$\Omega'_{\max} = \frac{1}{\tau_2}, \quad (3.66a)$$

and

$$\Omega''_{\max} = \frac{1.657}{\tau_1 \tau_2}. \quad (3.66b)$$

The parameters $\Omega'_{\max}$ and $\Omega''_{\max}$ are, respectively, the maximum values of $\Omega' [\cdot]$ and $\Omega'' [\cdot]$.

We considered a range of values for $(\Omega'_{\max}, \Omega''_{\max})$. For each of that $(\Omega'_{\max}, \Omega''_{\max})$ values we computed the corresponding (τ_1, τ_2) value using (3.66). For that (τ_1, τ_2) value and the chosen r_0 , τ_s , $\hat{\omega} [\cdot]$, $\hat{\omega}' [\cdot]$ we computed $\hat{U}_2^C [\cdot, \cdot]$ by following the first six steps given in procedure 1. Using that $\hat{U}_2^C [\cdot, \cdot]$ and (3.60) we then computed the maximum 95th percentile MPLS, $\phi_{\max}^{95}$. We plot the computed $\phi_{\max}^{95}$ as a function of $(\Omega'_{\max}, \Omega''_{\max})$ in Fig. 3.7. We plot the $\phi_{\max}^{95}$ from the finite element calculations as a function of $(\Omega'_{\max}, \Omega''_{\max})$ in Fig. 3.8. In Fig. 3.9 we show both the $\phi_{\max}^{95}$ from our finite rotations head model as well as that from the finite element head model. The insets in Fig. 3.9 show different views of the 3D plot shown in Fig. 3.9.

As can be seen from Fig. 3.9, the $\phi_{\max}^{95}$ values from our head model are comparable to those from the finite element head model. For example, for $(\Omega'_{\max}, \Omega''_{\max}) = (100, 25000)$ our model predicts a value of 0.44 for $\phi_{\max}^{95}$, whereas the finite element head model predicts a value of 0.6. The maximum difference in the predicted values for $\phi_{\max}^{95}$ from our finite rotations head model and the finite element head model is 0.36 (finite rotations head model 0.17; finite element head model 0.53). This occurs when $(\Omega'_{\max}, \Omega''_{\max}) = (100, 8000)$. The dependence of $\phi_{\max}^{95}$ on $(\Omega'_{\max}, \Omega''_{\max})$ in our model is qualitatively similar to the dependence of $\phi_{\max}^{95}$ on $(\Omega'_{\max}, \Omega''_{\max})$ in the finite element head model. As we stated in §3.5.1 we find it remarkable that $\phi_{\max}^{95}$ values from our head model and the finite element head model are as close as they are in Fig. 3.9 considering that the finite element

head model is far more sophisticated than our head model. We give some of the details of the finite element head model in the caption of Fig. 3.8.

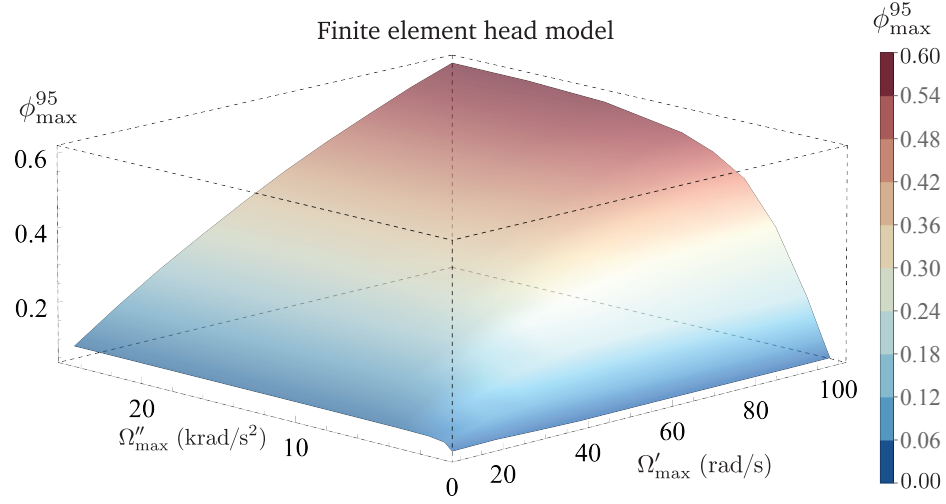


Figure 3.8: The maximum 95th percentile MPLS, ϕ_{max}^{95} , predicted by the 2D finite element head model presented in [26] for a family of loadings. Those loadings are same as the loadings used to generate the family of motions whose ϕ_{max}^{95} values are shown in Fig. 3.7. Some details of how the 2D finite element head model was created are as follows. The model was generated by processing magnetic resonance images (MRI) and diffusion tensor images (DTI) of an adult subject. It included all major anatomical structures, such as the skull, white matter, gray matter, cerebral spinal fluid, ventricles, bridging veins, falx cerebri, and tentorium cerebelli. Different 2D finite element head models were created by taking cross-sections of the head geometry along the sagittal, coronal, and axial planes. Plane strain conditions were assumed in each of those 2D finite element models. The ϕ_{max}^{95} values in this figure and Fig. 3.9 are from the 2D coronal finite element head model. The brain tissue in the 2D coronal finite element head model was modeled as an anisotropic hyper-viscoelastic material using the Holzapfel-Gasser-Ogden strain energy function as described in detail in Wright *et al.* [42]. The cerebral spinal fluid in it was modeled using a Mie-Gruneisen equation of state. The material interfaces in the model were tied so that no tangential sliding or normal separation could occur.

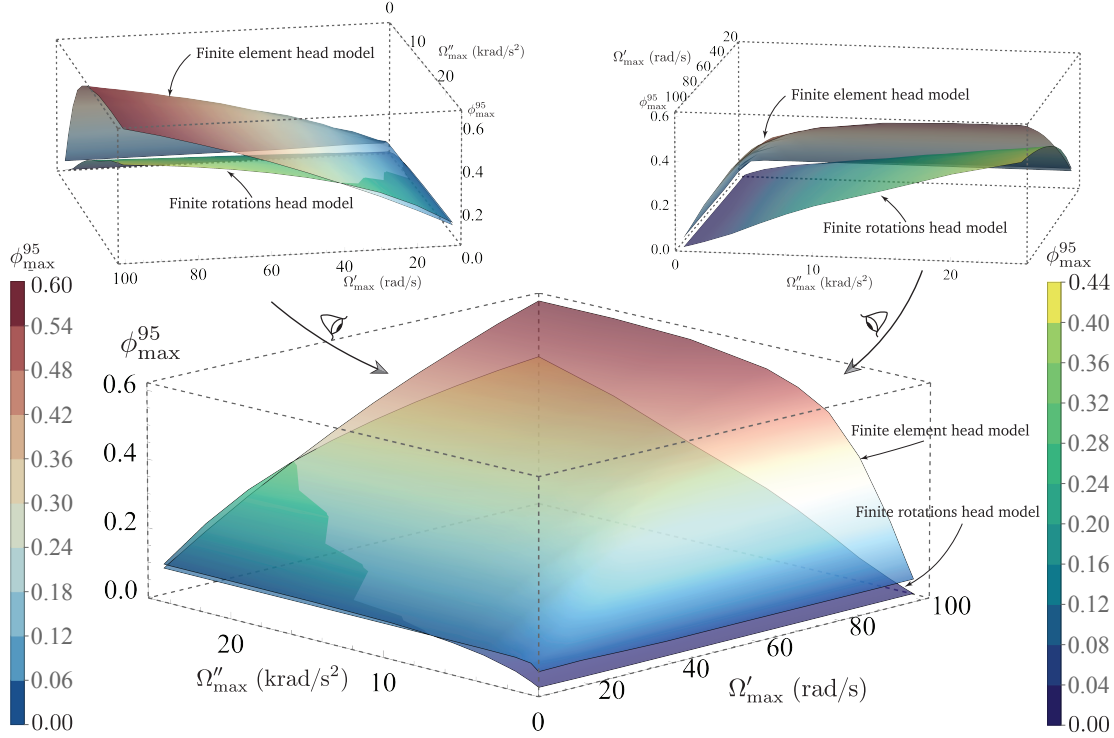


Figure 3.9: Comparison of the maximum 95th percentile MPLS, $\phi_{\max}^{95}$, values from the finite rotations head model and the finite element head model. The $\phi_{\max}^{95}$ values for the finite rotations head model are the ones from Fig. 3.7. The $\phi_{\max}^{95}$ values for the finite element head model are the ones from Fig. 3.8. The insets in the figure show different views of the 3D plot shown in this figure.

3.5.2 Other strain measures

Additionally, we calculated the 95th percentile maximum principal infinitesimal strain (MPIS) and its peak value over time, ϕ_{Hmax}^{95} , as well as the 95th percentile maximum principal Green-Lagrange strain (MPGLS) and its peak value over time, ϕ_{Emax}^{95} . Both the maximum 95th percentile MPIS and MPGLS are defined in a similar way to the maximum 95th percentile MPLS (§3.5.1.1 and §3.5.1.2). The primary difference is that we calculated the maximum eigenvalue of the infinitesimal strain tensor, defined as,

$$\epsilon[\mathbf{X}, \tau] := \frac{1}{2} \left(\mathbf{H}_{\tau}^*[\mathbf{X}] + \mathbf{H}_{\tau}^{*\top}[\mathbf{X}] \right), \quad (3.67)$$

and the Green-Lagrange strain tensor defined in (3.54). It can be shown that the MPIS is

$$\frac{1}{2} \left| \frac{U_2^C[\bar{\mathbf{r}}[\mathbf{X}], \tau]}{\bar{\mathbf{r}}[\mathbf{X}]} - \partial^{(1,0)} U_2^C[\bar{\mathbf{r}}[\mathbf{X}], \tau] \right|. \quad (3.68)$$

and the MPGLS is

$$\begin{aligned} & \frac{1}{4\bar{r}[\mathbf{X}]^2} \left(U_2^C [\bar{r}[\mathbf{X}], \tau]^2 + \bar{r}[\mathbf{X}]^2 \left(\partial^{(1,0)} U_2^C [\bar{r}[\mathbf{X}], \tau] \right)^2 \right. \\ & \left. + \left| U_2^C [\bar{r}[\mathbf{X}], \tau] - \bar{r}[\mathbf{X}] \partial^{(1,0)} U_2^C [\bar{r}[\mathbf{X}], \tau] \right| \sqrt{4\bar{r}[\mathbf{X}]^2 + \left(U_2^C [\bar{r}[\mathbf{X}], \tau] + \bar{r}[\mathbf{X}] \partial^{(1,0)} U_2^C [\bar{r}[\mathbf{X}], \tau] \right)^2} \right). \end{aligned} \quad (3.69)$$

We again calculated ϕ_{Hmax}^{95} and ϕ_{Emax}^{95} using the material and geometry parameters specified in (3.63), and the loading function $\hat{\omega}[\cdot]$ to be the one given in (3.64). We considered the same range of (τ_1, τ_2) (equivalently $(\Omega'_{\text{max}}, \Omega''_{\text{max}})$) values that we used for comparing our head model with the finite element head model in §3.5.1.3. We plot ϕ_{Hmax}^{95} and ϕ_{Emax}^{95} from our finite rotations head model as a function of $(\Omega'_{\text{max}}, \Omega''_{\text{max}})$ in Fig. 3.10 and Fig. 3.11, respectively.

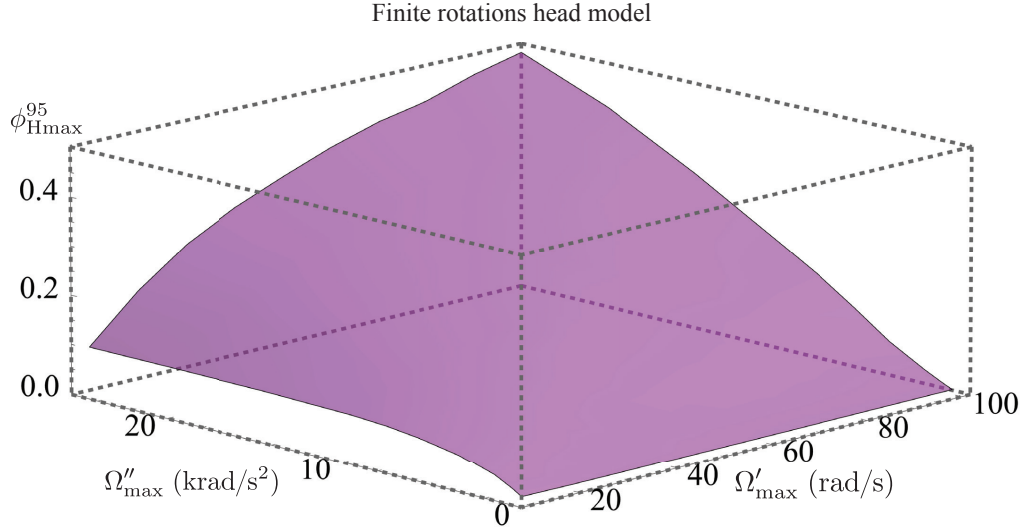


Figure 3.10: The maximum 95th percentile MPIS, ϕ_{Hmax}^{95} , predicted by the finite rotations head model. The ϕ_{Hmax}^{95} values shown are from a family of motions; all of which correspond to the material and geometry parameters (3.63), and the loading function $\hat{\omega}[\cdot]$ given in (3.64). The family of motions were generate by varying (τ_1, τ_2) (resp. $(\Omega'_{\text{max}}, \Omega''_{\text{max}})$) from $(33.14 \times 10^{-3}, 100 \times 10^{-3})$ (resp. $(10, 0.5 \times 10^3)$) to $(6.63 \times 10^{-3}, 10 \times 10^{-3})$ (resp. $(100, 25 \times 10^3)$). The quantity ϕ_{Hmax}^{95} is discussed in §3.5.2.

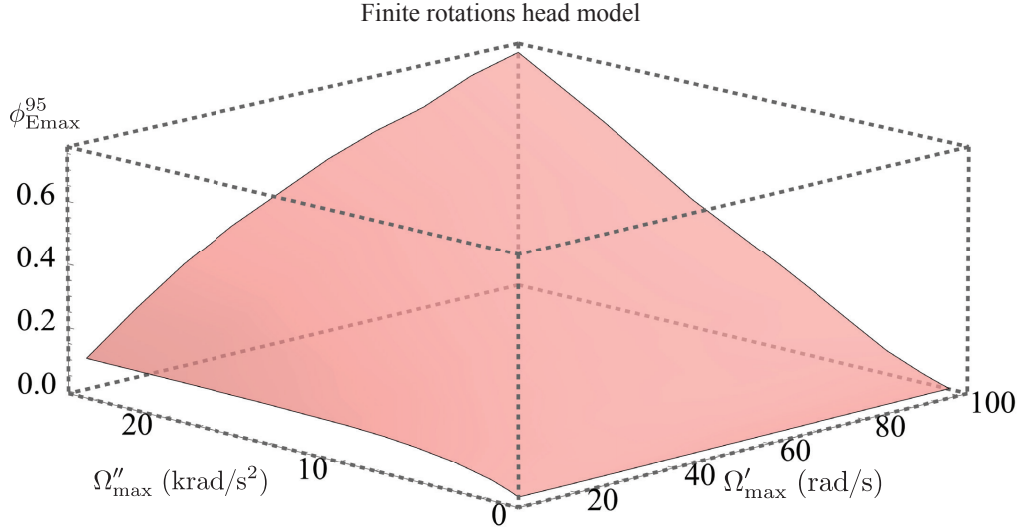


Figure 3.11: The maximum 95th percentile MPGLS, $\phi_{E_{max}}^{95}$, predicted by the finite rotations head model. The $\phi_{E_{max}}^{95}$ values shown are from a family of motions; all of which correspond to the material and geometry parameters (3.63), and the loading function $\hat{\omega}[\cdot]$ given in (3.64). The family of motions were generate by varying (τ_1, τ_2) (resp. $(\Omega'_{max}, \Omega''_{max})$) from $(33.14 \times 10^{-3}, 100 \times 10^{-3})$ (resp. $(10, 0.5 \times 10^3)$) to $(6.63 \times 10^{-3}, 10 \times 10^{-3})$ (resp. $(100, 25 \times 10^3)$). The quantity $\phi_{E_{max}}^{95}$ is discussed in §3.5.2.

3.5.3 Comparison with previous idealized continuum mechanics based head models

As we mentioned in the introduction §3.1, all the previous idealized continuum mechanics based head models that we surveyed take the head's rotations to be small. For that reason, we refer to them also as the *small rotations head models*.

3.5.3.1 Comparison of the governing equations

Recall that we introduced the tangential component of displacement $u_2^C[\cdot, \cdot]$ in (3.34). Introducing the scaled tangential displacement component $\hat{u}_2^C[\cdot, \cdot] : [0, 1] \times \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$ as

$$\hat{u}_2^C[\hat{r}, \hat{\tau}] = u_2^C[\hat{r}r_0, \hat{\tau}\tau_1]/r_0, \quad (3.70)$$

we can write the initial boundary value problem (IBVP) (3.41) in terms of $\hat{u}_2^C [\cdot, \cdot]$ as follows. The governing partial differential equation (PDE) (3.41a) in terms of $\hat{u}_2^C [\cdot, \cdot]$ reads

$$\begin{aligned} \partial^{(2,0)} \hat{u}_2^C [\hat{r}, \hat{\tau}] + \frac{\partial^{(1,0)} \hat{u}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}} - \frac{\hat{u}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}^2} = & \frac{\tau_s^2 \partial^{(0,2)} \hat{u}_2^C [\hat{r}, \hat{\tau}]}{\tau_1^2} \\ & + \frac{\tau_s^2 \tan \circ \hat{\Omega} [\hat{\tau}] \sec \circ \hat{\Omega} [\hat{\tau}]}{2\tau_2^2} \left(-3\hat{r} + \hat{r} \cos \circ 2\hat{\Omega} [\hat{\tau}] \right. \\ & \left. + 4 \sin \circ \hat{\Omega} [\hat{\tau}] \hat{u}_2^C [\hat{r}, \hat{\tau}] \right) \hat{\omega} [\hat{\tau}]^2 \\ & + \frac{\tau_s^2}{\tau_1 \tau_2} \tan \circ \hat{\Omega} [\hat{\tau}] \left(-\hat{r} \sin \circ \hat{\Omega} [\hat{\tau}] + \hat{u}_2^C [\hat{r}, \hat{\tau}] \right) \hat{\omega}' [\hat{\tau}] \\ & + \frac{2\tau_s^2}{\tau_1 \tau_2} \tan \circ \hat{\Omega} [\hat{\tau}] \partial^{(0,1)} \hat{u}_2^C [\hat{r}, \hat{\tau}] \hat{\omega} [\hat{\tau}], \end{aligned} \quad (3.71a)$$

where $\hat{\Omega} [\cdot] : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$, is defined as

$$\hat{\Omega} [\hat{\tau}] = \Omega [\hat{\tau} \tau_1]. \quad (3.71b)$$

The boundary condition (3.41b) reads

$$\hat{u}_2^C [\hat{r} = 1, \hat{\tau}] = \sin \circ \hat{\Omega} [\hat{\tau}], \quad (3.71c)$$

and the initial conditions (3.41c) and (3.41d) read

$$\hat{u}_2^C [\hat{r}, \hat{\tau} = 0] = 0, \quad \partial^{(0,1)} \hat{u}_2^C [\hat{r}, \hat{\tau} = 0] = 0. \quad (3.71d, 3.71e)$$

When the the skull's rotations are small, the governing PDE of the IBVP (3.71a) reduces to

$$\partial^{(2,0)} \hat{u}_2^C [\hat{r}, \hat{\tau}] + \frac{\partial^{(1,0)} \hat{u}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}} - \frac{\hat{u}_2^C [\hat{r}, \hat{\tau}]}{\hat{r}^2} = \frac{\tau_s^2 \partial^{(0,2)} \hat{u}_2^C [\hat{r}, \hat{\tau}]}{\tau_1^2}, \quad (3.72a)$$

and the boundary condition (3.71c) to

$$\hat{u}_2^C [\hat{r} = 1, \hat{\tau}] = \hat{\Omega} [\hat{\tau}]. \quad (3.72b)$$

The initial conditions (3.71d), (3.71e) remain the same. In arriving at (3.72) we ignored terms of $o(\hat{\Omega})$.

Most of the 2D ICM based head models presented to date include viscous effects. To our knowledge all the 2D ICM based head models reported to date reduce to the IBVP (3.72) on ignoring any viscous effects in them. Some of such 2D head models are the ones reported by Ljung [64], Margulies and Thibault [65], Massouros *et al.* [66], and Massouros [68].

Bayly *et al.* [69] studied the dynamics of a viscoelastic cylinder where the loading is prescribed through the motion of the cylinder's boundary. On ignoring the viscous effects the problem studied in [69] too reduces to the problem (3.72).

3.5.3.2 Comparison of a quantitative prediction

In order to check how quantitatively different our finite rotations head model is to the small rotations head models we again compared predictions for $\phi_{\max}^{95}$. For this comparison we again took the material and geometry parameters to be those given by (3.63), and the loading function $\hat{\omega}[\cdot]$ to be the one given in (3.64); and considered the same range of (τ_1, τ_2) (equivalently $(\Omega'_{\max}, \Omega''_{\max})$) values that we used for comparing our head model with the finite element head model in §3.5.1.3.

The procedure for computing $\phi_{\max}^{95}$ in the small rotations head models is almost the same as the procedure given in §3.5.1.1 and §3.5.1.2 for computing $\phi_{\max}^{95}$ in our finite rotations head model. The primary difference is that in the small rotations head models the MPLS is

$$\frac{1}{2} \ln \left(1 + \frac{|u_2^C[\bar{r}[\mathbf{X}], \tau] - \bar{r}[\mathbf{X}] \partial^{(1,0)} u_2^C[\bar{r}[\mathbf{X}], \tau]|}{\bar{r}[\mathbf{X}]} \right), \quad (3.73)$$

where $u_2^C[\cdot, \cdot]$ is the solution of (3.72a) subjected to the boundary condition (3.72b), and the initial conditions (3.71d)–(3.71e). We solve the initial boundary value problem for $u_2^C[\cdot, \cdot]$ in the small rotations head models numerically.

We plot $\phi_{\max}^{95}$ from the small rotations head models as a function of $(\Omega'_{\max}, \Omega''_{\max})$ in Fig. 3.12. In Fig. 3.13 we show both the $\phi_{\max}^{95}$ from the small rotations head models as well as that from our finite rotations head model. The insets in Fig. 3.13 show different views of the 3D plot shown in

Fig. 3.13.

As can be seen in Fig. 3.13 except for when $\Omega'_{\max}$ and $\Omega''_{\max}$ are small the predictions from the small rotations head models are quite different from those from the finite rotations head model. For example, for $(\Omega'_{\max}, \Omega''_{\max}) = (100, 25000)$ the small rotations head model predicts a value of 0.33 for $\phi_{\max}^{95}$, whereas our model predicts a value of 0.44.

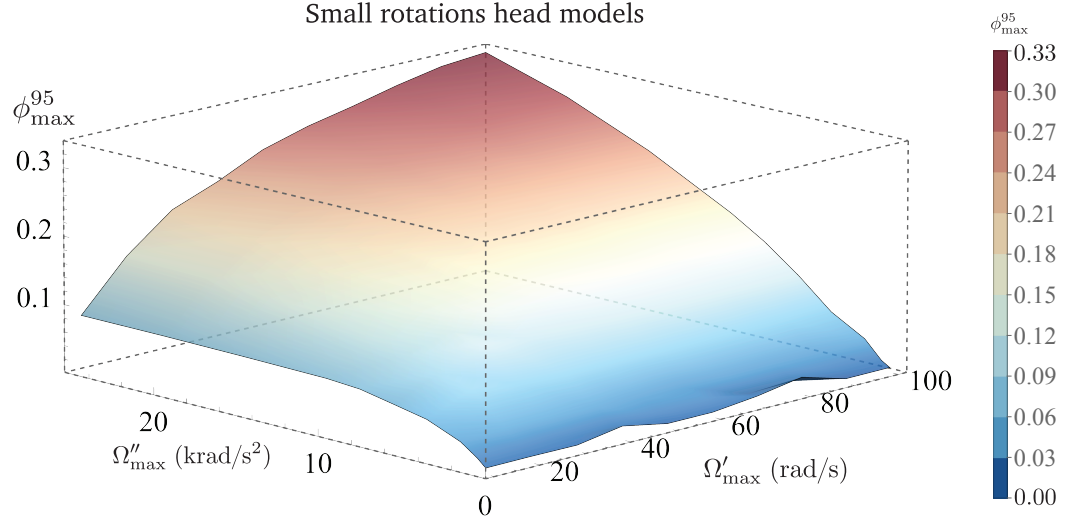


Figure 3.12: The maximum 95th percentile MPLS, $\phi_{\max}^{95}$, predicted by the small rotations head model (see §3.5.3.1 for details). The $\phi_{\max}^{95}$ values shown are from a family of motions, all which correspond to the same material and geometry properties, and loading function; but a range of (τ_1, τ_2) (resp. $(\Omega'_{\max}, \Omega''_{\max})$) values. The material and geometry properties, the loading function, and the range of (τ_1, τ_2) values are the same as those used for generating the motions in Fig. 3.7. The $\phi_{\max}^{95}$ values were computed by generating the family of motions by solving the equations given in §3.5.3.1 and using (3.73).

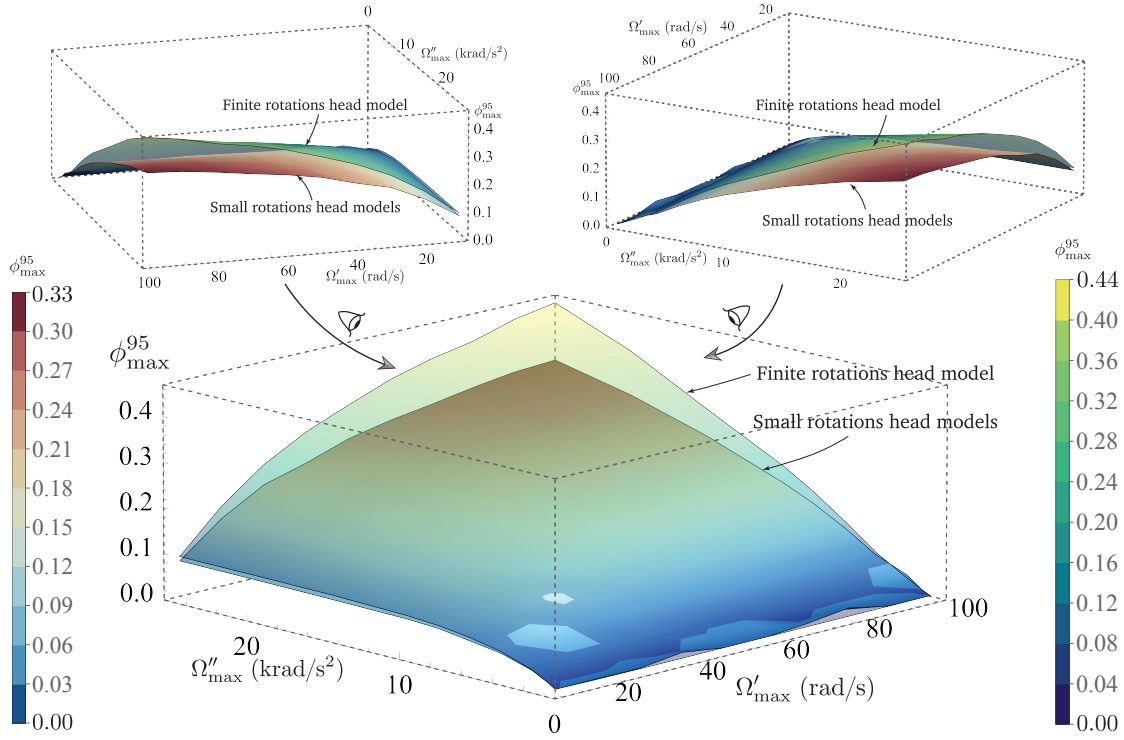


Figure 3.13: Comparison of the maximum 95th percentile MPLS, $\phi_{\max}^{95}$, values from the finite rotations head model and the small rotations head model. The $\phi_{\max}^{95}$ values for the finite rotations head model are the ones from Fig. 3.7. The $\phi_{\max}^{95}$ values for the small rotations head model are the ones from Fig. 3.12. The insets in the figure show different views of the 3D plot shown in this figure.

3.6 Concluding remarks

1. In deriving the finite rotations head model, we have assumed the brain to be homogeneous and isotropic; to undergo small strains; have no viscous effects; and even more dramatically, to undergo 2D deformation. Considering these type of simplifications, we find it remarkable that the ICM head models are capable of providing the kind of first order estimates for the peak strains and strain rates as those shown in Figs. 3.7, 3.8, 3.9, 3.12, and 3.13.
2. We plan on incorporating viscous effects and considering the 3D nature of the brain's deformation shortly in the future. However, despite those augmentations, it is likely that the estimates provided by computational mechanics (CM) based head models will be more accurate than those provided by our, or any other idealized continuum mechanics (ICM) based, head model. As we mentioned in the introduction, the primary advantage of ICM based head models compared to CM based models is that they're much easier and faster to apply for assessing the

injury risk of a mechanically traumatic event.

3. Given the head geometry, brain material property details, and a quantitative representation of the loading, procedure 1 can be used for determining the displacements in the finite rotations head model. Strains and strain rates can then be computed from the displacements by using (3.55) and (3.56), respectively. Though this procedure is far simpler than that involved in using a CM based model, health care and medical professionals interested in mTBI may still find it difficult to apply our procedure. In order to make the finite rotations head model easy to apply, especially by non-engineers, we have built a web application that automatically solves the finite rotations head model and provides the displacements, strains, and strain rates for a given loading input. This web app can be accessed at <http://18.233.10.106:8501/>.

Chapter 4

Extension of the finite rotation, small strain 2D elastic head model to a viscoelastic head model

4.1 Introduction

In Chapter 3, we presented an idealized continuum mechanics (ICM) based, small strain, elastic head model that accounts for the head's finite rotation, which we refer to as elastic ICM head model. In the elastic ICM head model, we have assumed the brain to be homogeneous and isotropic; to undergo small strains; have no viscous effects, see §3.3.1 for details. In this chapter, we extend the elastic ICM head model to a head model counting for the brain's viscosity, which is referred as viscoelastic ICM head model. The presented viscoelastic and elastic ICM head models share the same underlying assumptions (§3.3.1) with the only distinction being that the viscoelastic head model assumes the brain to behave as a linear viscoelastic material.

In the viscoelastic ICM head model, (A.i) we model the brain as a homogeneous, isotropic, incompressible, linear viscoelastic solid, and the skull as a rigid solid. (A.ii) We will be carrying out a 2D analysis. (A.iii) The displacements and deformations in the brain with respect to the skull are taken to be small. (A.iv) The displacements of the brain w.r.t. the skull are taken to be axisymmetric.

(A.v) With regard to geometry, we model the brain's cross-sections perpendicular to the rotation axis as disks. (A.vi) We model the interaction between the brain and the skull by positing that the brain's outer surface is rigidly bonded to the skull's interior surface. (A.vii) The brain and the skull are assumed to be initially at rest or rotating with a constant angular velocity. (A.viii) Initially the brain is assumed to not have any displacements with respect to the skull, and (A.ix) to be stress free.

This chapter is organized as follows. In §4.2, we present the equations that govern the mechanics of the viscoelastic ICM head model. In §4.3, we provide a semi-analytical solution for the governing equations and derive expressions for the strains and strain rates in the viscoelastic ICM head model. In §4.4, we compare the strains from the viscoelastic ICM head model with those from a recently presented finite element head model [26], as well as with those from the elastic ICM head model.

4.2 Analytical model for head motion and brain deformation

In the viscoelastic ICM head model, we model the brain as a linear viscoelastic material and assume that the reference configuration $\kappa_R [\mathcal{B}]$ is stress free. It then follows that as $\mathbf{H}^\star \in \mathcal{M}_{3 \times 3}(\mathbb{R}) \rightarrow \mathbf{o}$, the zero element in $\mathcal{M}_{3 \times 3}(\mathbb{R})$, the 1st Piola-Kirchhoff stress tensors

$$\mathbf{P}^\star [\mathbf{X}, \tau] = \check{\mathbf{P}}^\star [\mathbf{X}, \tau] - p_m [\bar{r} [\mathbf{X}], \tau] \mathbf{I} + o(\mathbf{H}^\star [\mathbf{X}, \tau]), \quad (4.1)$$

where the quantity $p_m [\bar{r} [\mathbf{X}], \tau]$ acts as a Lagrange undetermined multiplier, which can be interpreted as the hydrostatic pressure at the time instance τ at the material particle $\mathbf{X}$. And

$$\check{\mathbf{P}}^\star [\mathbf{X}, \tau] = 2\mu_0 G [\tau] \epsilon [\mathbf{X}, 0] + 2\mu_0 \int_0^\tau G [\tau - \xi] \partial^{(0,1)} \epsilon [\mathbf{X}, \xi] d\xi, \quad (4.2)$$

where $\epsilon [\mathbf{X}, \tau] := 1/2 \left(\mathbf{H}^\star [\mathbf{X}, \tau] + \mathbf{H}^{\star T} [\mathbf{X}, \tau] \right)$. In (4.2), $\mu_0 G [\cdot]$ is the shear relaxation function, defined as

$$G [\tau] = \left(1 - \sum_{i=1}^{n_g} \hat{\mu}_i \left(1 - e^{-\beta_i \tau} \right) \right), \quad (4.3)$$

where μ_0 is defined such that $\mu_0 \text{ N/m}^2$ is $\mathcal{B}$'s instantaneous shear modulus; β_i Hertz is the decay constant; $n_g \in \mathbb{N}$, $\hat{\mu}_i \in \mathbb{R}_{>0}$ are material constants.

Writing $A_\tau [X]$ in (3.28) in terms of $Q[\tau]$ and $U_X [\tau]$ using (3.19); replacing $U_X [\tau]$ in the resulting equation with $U_\tau [X]$ and then writing $U_\tau [X]$ in terms of $U_2^C [\bar{r} [X], \tau]$ and $c_2 [X]$ using (3.25); writing $Q[\tau]$ in the resulting equation in terms of $\Omega[\tau]$ using (3.9); substituting $P^* [X, \tau]$ in (3.28) with the right hand side of (4.1) and writing $\check{P}^* [X, \tau]$ in the resulting equation in terms of $G [\tau]$ and U_2^C using (4.2), (3.22), and (3.25); and writing $c_2 [X]$ in terms of X_1, X_2 in the resulting equation using (3.26) and simplifying; finally assuming H_τ^* to be uniformly small, i.e, in the limit of the deformations of the brain with respect to the skull vanishing uniformly over the brain, we get that

$$\begin{aligned} \mu_0 G [\tau] \left(\mathcal{D} U_2^C [\cdot, 0] \right) [r] + \mu_0 \int_0^\tau G [\tau - \xi] \left(\mathcal{D} \partial^{(0,1)} U_2^C [\cdot, \xi] \right) [r] d\xi = & \rho_0 r \Omega'' [\tau] + \rho_0 \partial^{(0,2)} U_2^C [r, \tau] \\ & - \rho_0 \Omega' [\tau]^2 U_2^C [r, \tau], \end{aligned} \quad (4.4)$$

where we recall that the operator $\mathcal{D}$ is defined such that

$$\left(\mathcal{D} U_2^C [\cdot, \tau] \right) [r] = \partial^{(2,0)} U_2^C [r, \tau] + \frac{\partial^{(1,0)} U_2^C [r, \tau]}{r} - \frac{U_2^C [r, \tau]}{r^2}. \quad (4.5)$$

In the viscoelastic ICM head model, the boundary condition and initial conditions to $U_2^C [\cdot, \cdot]$ remain the same as (3.36) and (3.37), respectively, in the elastic ICM head model.

4.3 Semi-analytical solution

4.3.1 Scaling of the equation

We scale the equation (4.4) in the same way as we did in §3.4 for the elastic ICM head model. The key difference is that we define the time scale τ_s seconds as

$$\tau_s = \frac{r_0}{\sqrt{\mu_0/\rho_0}}, \quad (4.6)$$

where, recall that, μ_0 N/m² is the brain's instantaneous shear modulus in our model (cf. (4.2)).

Additionally, we define the scaled shear relaxation function $\hat{G} [\cdot] : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}$

$$\hat{G} [\hat{\tau}] := G [\hat{\tau} \tau_1] . \quad (4.7)$$

Finally, in terms of $\hat{U}_2^C [\cdot, \cdot]$, $\hat{G} [\cdot]$, $\hat{\omega} [\cdot]$, $\hat{r}$, and $\hat{\tau}$ the equation of motion (4.4) reads

$$\begin{aligned} \hat{G} [\hat{\tau}] \left(\mathcal{D} \hat{U}_2^C [\cdot, 0] \right) [\hat{r}] + \int_0^{\hat{\tau}} \hat{G} [\hat{\tau} - \xi] \left(\mathcal{D} \partial^{(0,1)} \hat{U}_2^C [\cdot, \xi] \right) [\hat{r}] d\xi = \frac{\tau_s^2}{\tau_1^2} \left(\frac{\tau_1}{\tau_2} \hat{r} \hat{\omega}' [\hat{\tau}] + \partial^{(0,2)} \hat{U}_2^C [\hat{r}, \hat{\tau}] - \right. \\ \left. \left(\frac{\tau_1}{\tau_2} \right)^2 \hat{\omega}^2 [\hat{\tau}] \hat{U}_2^C [\hat{r}, \hat{\tau}] \right) . \end{aligned} \quad (4.8)$$

In our problem, we consider instant loading, that is

$$\frac{\tau_1}{\tau_2} \rightarrow 0, \quad (4.9)$$

then the third term of right hand side of (4.8) is much smaller than the first and second term of the right hand side of (4.8). Based on this consideration, we will ignore the third term. Then the equations we will solve is

$$\hat{G} [\hat{\tau}] \left(\mathcal{D} \hat{U}_2^C [\cdot, 0] \right) [\hat{r}] + \int_0^{\hat{\tau}} \hat{G} [\hat{\tau} - \xi] \left(\mathcal{D} \partial^{(0,1)} \hat{U}_2^C [\cdot, \xi] \right) [\hat{r}] d\xi = \frac{\tau_s^2}{\tau_1^2} \left(\frac{\tau_1}{\tau_2} \hat{r} \hat{\omega}' [\hat{\tau}] + \partial^{(0,2)} \hat{U}_2^C [\hat{r}, \hat{\tau}] \right), \quad (4.10)$$

with the boundary condition (3.41b) and initial conditions (3.41c) and (3.41d).

4.3.2 Laplace transform of the equation

In the section we present the Laplace transform of (4.10) with the initial conditions (3.41c) and (3.41d). The Laplace transform of a function $f : \tau \in \mathbb{R}_{\geq 0} \mapsto f [\tau] \in \mathbb{R}$ is defined as

$$(\mathfrak{L}f) [s] = \int_0^\infty f [\tau] e^{-s\tau} d\tau, \quad (4.11)$$

where $s \in \mathbb{C}$. By applying the Laplace transform on (4.10) and simplifying the resulting equations, we get

$$s \left(\mathfrak{L} \hat{G} \right) [s] \left(\partial^{(2,0)} \left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r}, s] + \frac{\partial^{(1,0)} \left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r}, s]}{\hat{r}} - \frac{\left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r}, s]}{\hat{r}^2} \right) = \frac{\tau_s^2}{\tau_1^2} \left(\frac{\tau_1}{\tau_2} \hat{r} s \left(\mathfrak{L} \hat{\omega} \right) [s] + s^2 \left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r}, s] \right). \quad (4.12)$$

In (4.12), the Laplace transform of $\hat{\omega}' [\cdot]$ is derived using the initial condition (3.10) and (3.40). The Laplace transform of the boundary condition (3.41b) is

$$\left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r} = 1, s] = 0. \quad (4.13)$$

4.3.3 Solution using eigenfunction expansions and Sturm-Liouville theory

In this section we solve for $\left(\mathfrak{L} \hat{U}_2^C \right) [\cdot, \cdot]$ using the method of eigenfunction expansion that we used to derive the solution for the elastic ICM head model in §3.4.2. We postulate that $\left(\mathfrak{L} \hat{U}_2^C \right) [\cdot, \cdot]$ can be expressed as,

$$\sum_{n=1}^{\infty} f_n [s] g_n [\hat{r}], \quad (4.14)$$

where $f_n : \mathbb{C} \rightarrow \mathbb{C}$, and the function $g_n [\cdot]$ is defined in (3.43).

It can be shown that the form (4.14) postulated for $\left(\mathfrak{L} \hat{U}_2^C \right) [\cdot, \cdot]$ satisfies the boundary condition (4.13). Next we derive a solution for $f_n [\cdot]$.

Following steps in §3.4.2, replacing the function $(0, 1) \ni \hat{r} \mapsto \hat{r}$ that appears in the first term on the right hand side of (4.12) with $\sum_{n \in \mathbb{N}} p_n g_n [\hat{r}]$, where p_n is calculated using (3.48), and replacing $\left(\mathfrak{L} \hat{U}_2^C \right) [\hat{r}, s]$ in (4.12) with the form given in (4.14), we get that

$$\sum_{n=1}^{\infty} s \left(\mathfrak{L} \hat{G} \right) [s] f_n [s] \left(\mathcal{D} g_n \right) [\hat{r}] = \sum_{n=1}^{\infty} \frac{\tau_s^2}{\tau_1 \tau_2} p_n g_n [\hat{r}] s \left(\mathfrak{L} \hat{\omega} \right) [s] + \sum_{n=1}^{\infty} \frac{\tau_s^2}{\tau_1^2} s^2 f_n [s] g_n [\hat{r}]. \quad (4.15)$$

Again following the steps in §3.4.2, substituting $\left(\mathcal{D} g_n \right) [\hat{r}]$ in equation (3.49) with the right side of (3.51); taking inner product of the resulting equation with $g_m [\cdot]$, $m \in \mathbb{N}$, via (3.46); and

simplifying the resulting equation using (3.47) we get a solution of $f_m [\cdot]$ to be

$$f_m [s] = \frac{-\frac{\tau_s^2}{\tau_1 \tau_2} p_m s (\mathfrak{L}\hat{\omega}) [s]}{s \left(\mathfrak{L}\hat{G} \right) [s] j_{1,m}^2 + \frac{\tau_s^2}{\tau_1^2} s^2}. \quad (4.16)$$

4.3.4 Displacements, strains, and strain rates

We outline a procedure for computing displacements in our head model in procedure 2. Using that $U_2^C [\cdot, \cdot]$, the various strain measures and strain rate measures can be computed following §3.4.3.2 and S3.4.3.3, respectively.

Procedure 2 Computing displacements in the head model

1. Given an individual's head's biometric and other details, e.g., from MRI scans, select values for the characteristic length scale r_0 , the characteristic time scale τ_s , and the shear relaxation modulus constants μ_0 , n_g , β_i , and $\hat{\mu}_i$.
 2. Given a $\Omega [\cdot]$, select the loading times scales τ_1 , τ_2 based on the characteristics of $\Omega [\cdot]$.
 3. Construct non-dimensional scaled loading function $\hat{\omega} [\cdot]$ using the given $\Omega [\cdot]$, the τ_1 and τ_2 from the previous step, and (3.40).
 4. Construct the shear relaxation function $G [\cdot]$ using n_g , β_i , $\hat{\mu}_i$, and (4.3), and then construct non-dimensional scaled shear relaxation function $\hat{G} [\cdot]$ using the given $G [\cdot]$, the τ_1 from the previous step, and (4.7).
 5. Using (3.43) and (3.48) compute p_n , $n \in \mathbb{N}$.
 6. Calculate the Laplace transform $(\mathfrak{L}\hat{\omega}) [\cdot]$ of $\hat{\omega} [\cdot]$ and $(\mathfrak{L}\hat{G}) [\cdot]$ of $\hat{G} [\cdot]$
 7. Using the τ_s , τ_1 , τ_2 , $(\mathfrak{L}\hat{\omega}) [\cdot]$, $(\mathfrak{L}\hat{G}) [\cdot]$, and p_n from the previous steps calculate f_m , $m \in \mathbb{N}$ using (4.16).
 8. Using the calculated f_m and the g_m given by (3.43), construct $\mathfrak{L}\hat{U}_2^C [\cdot, \cdot]$ using (3.42).
 9. Calculating $\hat{U}_2^C [\cdot, \cdot]$ by inverse Laplace transforming $(\mathfrak{L}\hat{U}_2^C) [\cdot, \cdot]$ from the previous step and constructing $U_2^C [\cdot, \cdot]$ using (3.39).
 10. Compute the (non-dimensional version of the) intermediate displacement field, U_τ , using $U_2^C [\cdot, \cdot]$ from the previous step and (3.25).
 11. Compute x_τ using the U_τ from the previous step and (3.12), and the displacement of the material particle X at the time instance τ , as $x_\tau [X] - X$.
-

4.4 Discussion

4.4.1 Comparison with detailed finite element solutions

we compared the “maximum 95th percentile MPLS” $\phi_{\max}^{95}$ values from the viscoelastic ICM head model with those from a related, finite element based, 2D head model [26], which is referred to as “Full FEM head model”. The term “MPLS” stands for maximum principal logarithmic strain. The 95th percentile MPLS at the time instance τ is defined to be the MPLS value such that at time instance τ the MPLS values over 95% of the brain are smaller than it. The “maximum 95th percentile MPLS”, $\phi_{\max}^{95}$, is the supremum of the values of 95th percentile MPLS over all time. For details on the definition of $\phi_{\max}^{95}$, refer to §3.5.1.1 and §3.5.1.2.

To make our head model approximate the Full FEM head model as closely as possible in our head model, we take the non-dimensional head radius r_0 , density ρ_0 , and instantaneous shear modulus μ_0 to be

$$r_0 = 0.06525, \quad (4.17a)$$

$$\rho_0 = 1040, \quad (4.17b)$$

$$\mu_0 = 34000. \quad (4.17c)$$

(It follows from (4.6) that the value of the parameter τ_s for these geometry and material parameters is 0.0114.) And we take one-term shear relaxation function $G [\cdot]$ to be

$$G [\tau] = 1 - \frac{\mu_o - \mu_\infty}{\mu_o} \left(1 - e^{-\beta_1 \tau} \right), \quad (4.18)$$

where $\mu_\infty = 6400$ and $\beta_1 = 700$. In Full FEM head model [26], the main constituents of the brain, white and gray matter are modeled using viscoelastic Holzapfel–Gasser–Ogden material model. Both white and gray matter adopt the instantaneous shear modulus of 34000 Pa, the long-term shear modulus at equilibrium of 6400 Pa, and the decay constant of 700 s^{-1} .

We took the loading function $\hat{\omega} [\cdot]$ to be the one given in (3.64); and considered the same range of (τ_1, τ_2) (equivalently $(\Omega'_{\max}, \Omega''_{\max})$) values that we used for comparing the elastic ICM head model

with the Full FEM head model in §3.5.1.3.

For a (τ_1, τ_2) value and the chosen $r_0, \tau_s, \hat{\omega} [\cdot], G [\cdot]$ we computed $\hat{U}_2^C [\cdot, \cdot]$ by following the first nine steps given in procedure 2. Using that $\hat{U}_2^C [\cdot, \cdot]$ and (3.60) we then computed the maximum 95th percentile MPLS, $\phi_{\max}^{95}$. We plot the computed $\phi_{\max}^{95}$ as a function of $(\Omega'_{\max}, \Omega''_{\max})$ in Fig. 4.1. In Fig. 4.2 we show both the $\phi_{\max}^{95}$ from our viscoelastic ICM head model as well as that from Full FEM head model. The insets in Fig. 4.2 show different views of the 3D plot shown in Fig. 4.2.

As can be seen in Fig. 4.2, the $\phi_{\max}^{95}$ values from our viscoelastic ICM head model are comparable to those from the Full FEA head model. The maximum relative difference in the predicted $\phi_{\max}^{95}$ values from our viscoelastic ICM head model and the Full FEA head model is less than 8%.

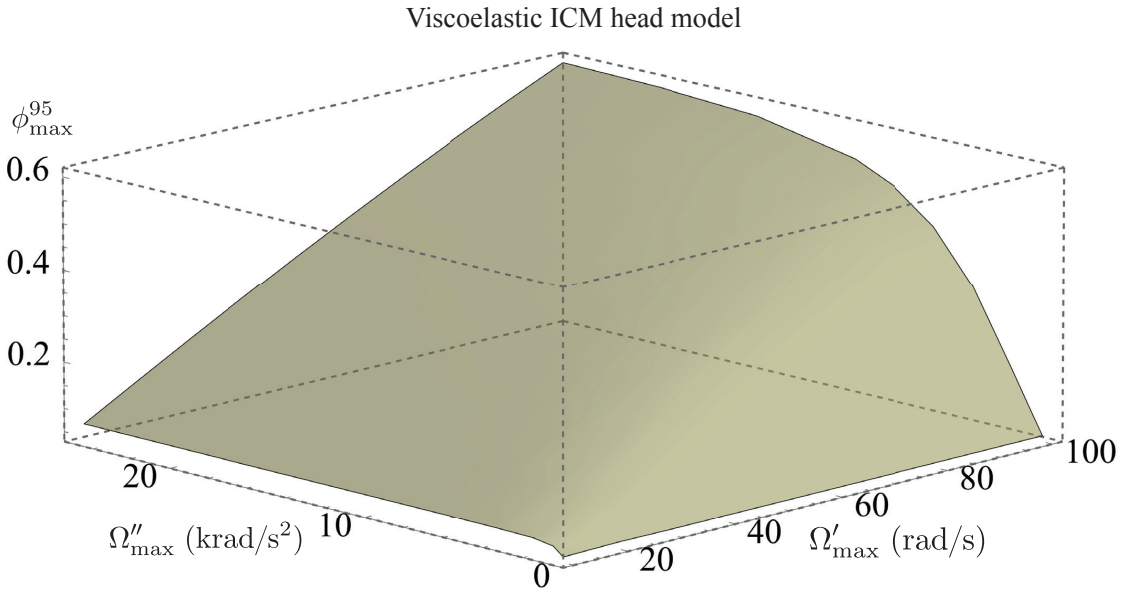


Figure 4.1: The maximum 95th percentile MPLS, $\phi_{\max}^{95}$, predicted by the viscoelastic ICM head model. The $\phi_{\max}^{95}$ values shown are from a family of motions; all of which correspond to the material and geometry parameters (4.17) and (4.18), and the loading function $\hat{\omega} [\cdot]$ given in (3.64). The family of motions were generate by varying (τ_1, τ_2) (resp. $(\Omega'_{\max}, \Omega''_{\max})$) from $(33.14 \times 10^{-3}, 100 \times 10^{-3})$ (resp. $(10, 0.5 \times 10^3)$) to $(6.63 \times 10^{-3}, 10 \times 10^{-3})$ (resp. $(100, 25 \times 10^3)$). The quantity $\phi_{\max}^{95}$ is discussed in §3.5.1.2.

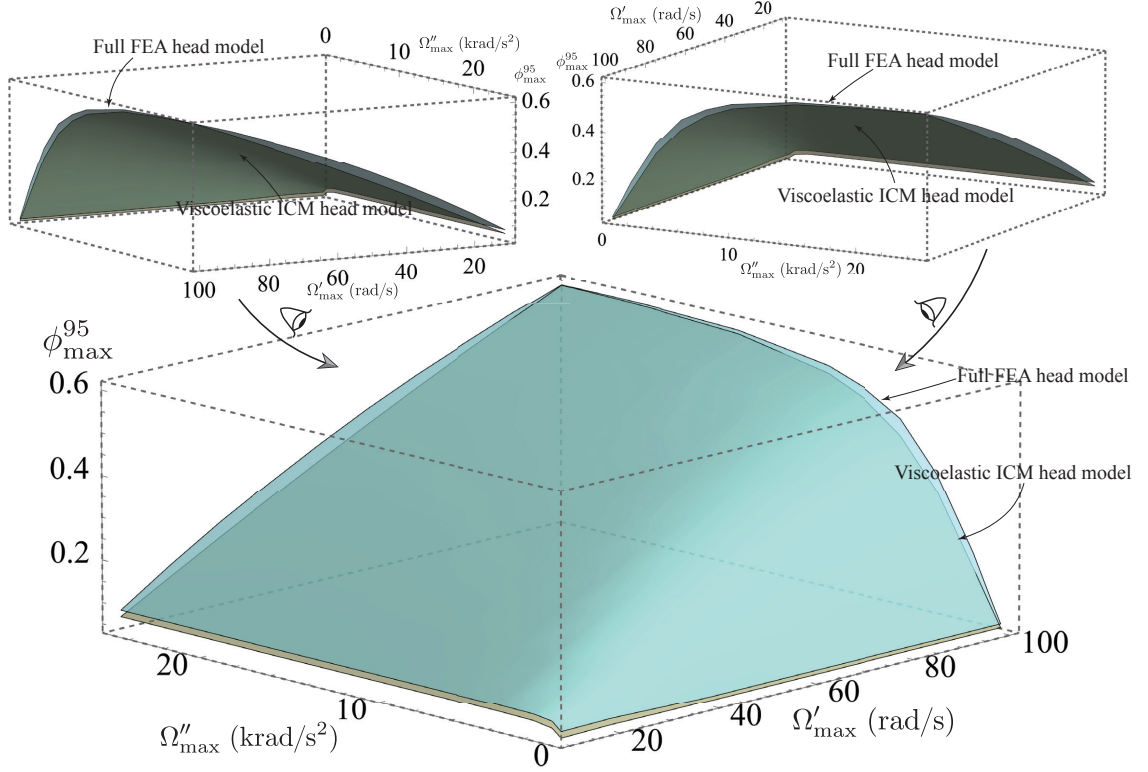


Figure 4.2: Comparison of the maximum 95th percentile MPLS, $\phi_{\max}^{95}$, values from the viscoelastic ICM head model and the Full FEM head model. The $\phi_{\max}^{95}$ values for the viscoelastic ICM head model are the ones from Fig. 4.1. The $\phi_{\max}^{95}$ values for the Full FEM head model are the ones from Fig. 3.8. The insets in the figure show different views of the 3D plot shown in this figure.

4.4.2 Comparison with the elastic ICM head model

As we mentioned in the introduction §4.1, our viscoelastic ICM head model is an extension of the elastic ICM head model (developed in §3) to a head model that counts for brain's viscosity. In order to check how quantitatively different our viscoelastic ICM head model is to the elastic ICM head model, we again compared predictions for $\phi_{\max}^{95}$. For this comparison we again took the material and geometry parameters to be those given by (4.17) and (4.18), and the loading function $\hat{\omega}[\cdot]$ to be the one given in (3.64); and considered the same range of (τ_1, τ_2) (equivalently $(\Omega'_{\max}, \Omega''_{\max})$) values that we used for comparing our viscoelastic ICM head model with the Full FEM head model in §4.4.1.

In Fig. 4.3 we show both the $\phi_{\max}^{95}$ from the Full FEM head model as well as that from our viscoelastic ICM head model. The insets in Fig. 4.3 show different views of the 3D plot shown in Fig. 4.3. As can be seen in Fig. 4.3, except for when $\Omega'_{\max}$ is small and $\Omega''_{\max}$ is large, the predictions

from the viscoelastic ICM head model are quite different from those from the elastic ICM head model. For example, for $(\Omega'_{\max}, \Omega''_{\max}) = (100, 10000)$ the elastic ICM head model predicts a value of 0.22 for $\phi_{\max}^{95}$, whereas the viscoelastic ICM head model predicts a value of 0.55.

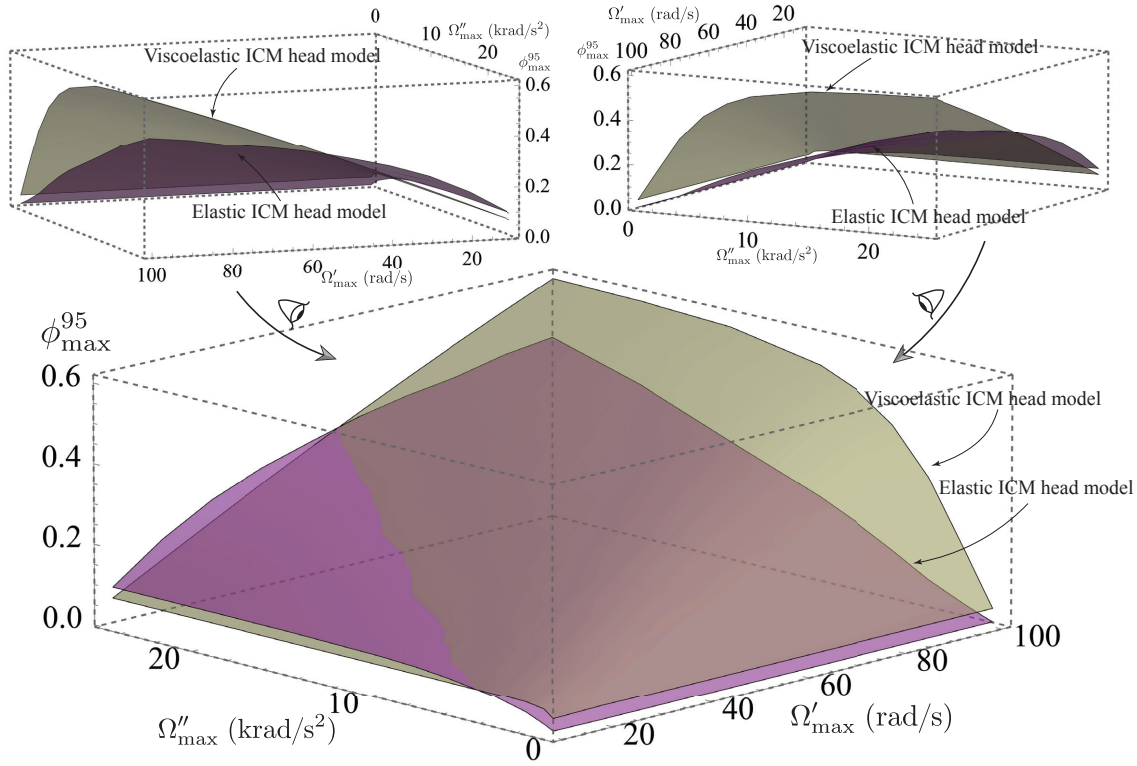


Figure 4.3: Comparison of the maximum 95th percentile MPLS, $\phi_{\max}^{95}$, values from the viscoelastic ICM head model and the elastic ICM head model. The $\phi_{\max}^{95}$ values for the viscoelastic ICM head model are the ones from Fig. 4.1. The $\phi_{\max}^{95}$ values for the elastic ICM head model are the ones from Fig. 3.7. The insets in the figure show different views of the 3D plot shown in this figure.

Chapter 5

A mechanics theory for the exploration of a high-throughput, sterile 3D *in vitro* traumatic brain injury model

Note: A version of this chapter is published in *Biomechanics and Modeling in Mechanobiology*. Data and figures have been used with all co-authors' consent.

Yang Wan, Rafael D. González-Cruz, Diane Hoffman-Kim, and Haneesh Kesari. A mechanics theory for the exploration of a high-throughput, sterile 3D in vitro traumatic brain injury model. Biomechanics and Modeling in Mechanobiology, 23, 1179-1196, 2024.

5.1 Introduction

We propose a new 3D *in vitro* TBI model in which mechanical loads are applied to cortical spheroids, via the aid of centrifugal forces. Cortical spheroids are grown within a 3D soft substrate, i.e., within the cavities (such as microwells) molded into the surface of a soft material (see inset in Fig. 5.1 (a)), such as an agarose hydrogel. Cell culture media bathes the cortical spheroids as well as their soft substrate. We propose to load the cortical spheroids by spinning them along with their soft substrates and their fluid media using a centrifuge (see Fig. 5.1 (a)). The centrifuge's angular velocity can

vary during the experiment. In a frame that rotates with the centrifuge's rotating arm, the cortical spheroid, the soft substrate, and the fluid media experience body forces that push them away from the axis of rotation (Fig. 5.1 (b)). These forces cause the fluid media and the soft substrate to push the cortical spheroid's surfaces that they are respectively in contact with towards each other; thus squeezing the spheroid (Fig. 5.1 (c)). We will experimentally demonstrate in a follow up publication that this centrifugation based method of loading the cortical spheroids can provide the basis for the development of a high-throughput and sterile *in vitro* 3D TBI model.

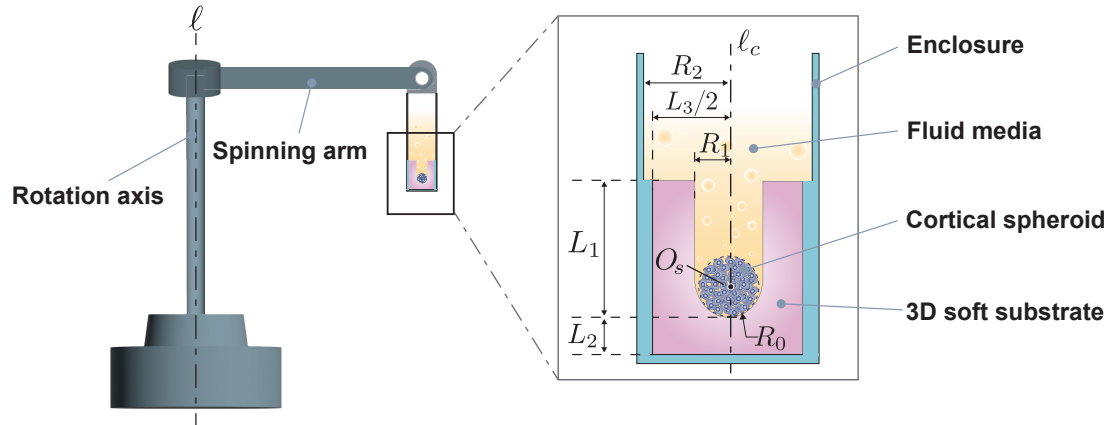
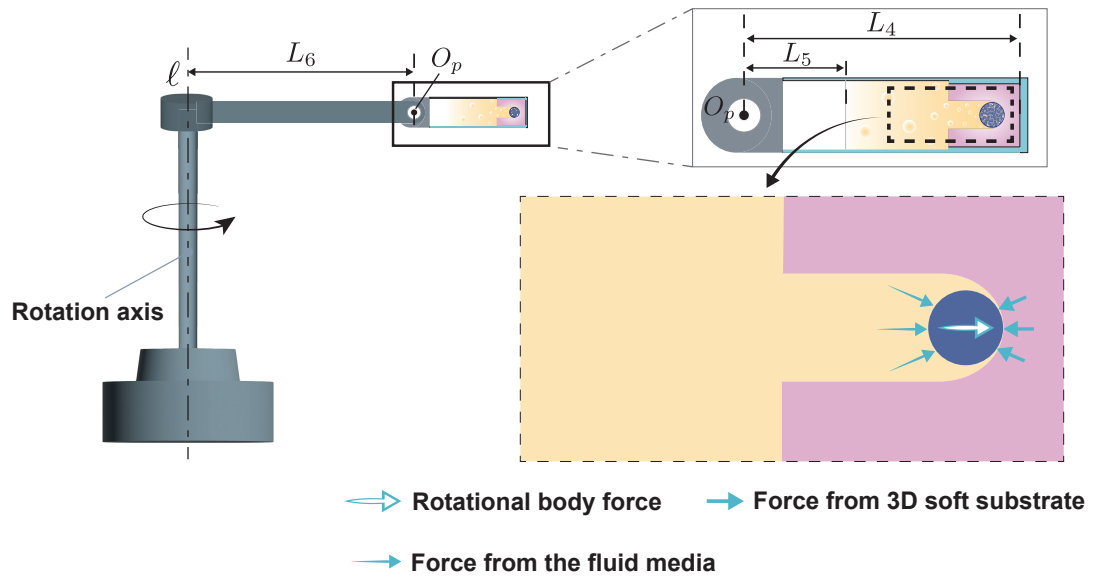
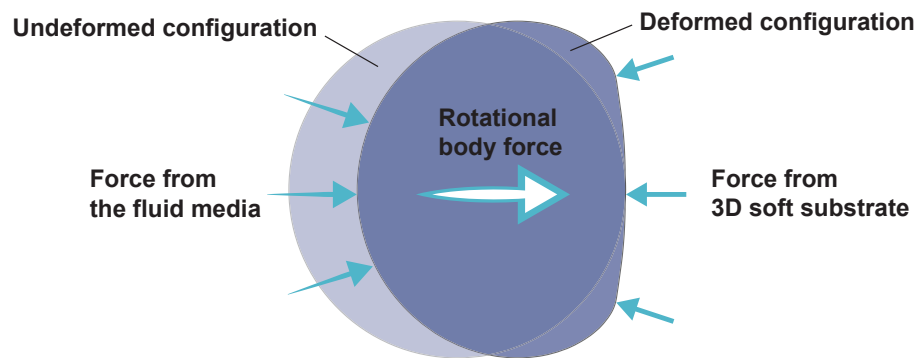
a Centrifuge (not to scale)**b Reference configuration in mechanics model (not to scale)****c Cortical spheroid (to scale)**

Figure 5.1: An illustration of the centrifuge-TBI-model. For ease of exposition, especially when we develop the mechanics theory in §5.4, we only show one cortical spheroid as being spun in (a) and (b). However, in practice several thousands of spheroids can be spun simultaneously.

The above proposed centrifugation based 3D *in vitro* TBI model (centrifuge-TBI-model) does not have the limitations (1)–(3) of the *in vivo* models (§1.2.3), neither does it have the limitation of insufficient sophistication of the 2D *in vitro* models. The model is designed to be high throughput; maintain conditions that are as sterile as those present at the time the spheroids are grown; and allow loading of the spheroids in a reproducible manner.

The centrifuge-TBI-model is of high throughput, since thousands of spheroids can be grown simultaneously, and then spun simultaneously as well¹; with no steps between the two that involve processing the spheroids serially, such as individual pipetting, positioning, or probing. These features are possible for the following reasons. The 3D soft substrate is micromachined to have close to a hundred cavities/microwells on it. A single spheroid (which could be as large as 8000 cells) grows in a cavity via self-assembly given the nonadhesive properties of the soft substrate. It has already been demonstrated that using the 3D soft substrates several thousands of spheroids can be grown simultaneously [93]. A single substrate is usually smaller than a cubic centimeter in size, therefore a lab grade centrifuge will be able to hold over a hundred substrates. Hence, several thousands of spheroids can be spun simultaneously if desired. In addition to the growth and loading operations in the centrifuge-TBI-model each being parallel in nature, another aspect of the model that critically contributes to it being high throughput is that the spheroids are loaded *in situ*, i.e., they are tested in the same substrate as that in which they are grown (compare, e.g., Fig. 5.1 (a) and (b)).

The loading conditions are sterile due to the *in situ* testing, and because the loading on the spheroids is being primarily performed by the same fluid media and the soft substrate used to grow them (e.g., see Figs. 5.1 (c) and (d)).

The mechanical loading, i.e., the force on the spheroids, in the experiment can be easily and robustly tuned via the centrifuge's angular velocity and the volume of the fluid media.

However, as with all models, the proposed model too has some limitations. One of the most significant of those is the same as limitation (4) of the *in vivo* models, which, to reiterate, is the inability to visualize tissue deformation in real time and correlate injury severity to those deformations. In order to address this limitation, in this chapter we consider a special case of

¹In Fig. 5.1 we only show a single cortical spheroid being spun, even though in practice several thousand can be spun simultaneously. We do so since showing a single spheroid makes it easier to use Fig. 5.1 to explain the development of the mechanics theory.

the centrifuge-TBI-model and develop a mechanics theory for determining the cortical spheroids' deformation in it. The special case we consider is the one which the centrifuge's angular velocity is constant as a function of time. The primary assumptions in our mechanics theory are described in §5.2. The mathematical preliminaries necessary for detailing our theory are described in §5.3. We develop the theory in §5.4, and summarize it in §5.5. Results from numerical solutions of our theory for two representative values of centrifuge angular velocity are presented in §5.6.

The proposed centrifuge-TBI-model, in theory, has many advantages compared to other models. The theory developed in this paper applies to a special case of the centrifuge-TBI-model, and it provides an indirect means of determining the deformations. Future studies will work toward a general and direct approach for determining deformations. We believe that our theory is of value since it allows us to explore the centrifuge-TBI-model and its viability and potential for *in vitro* TBI studies.

5.2 Primary assumptions and modeling decisions, and their underlying rationale

In this section we list some of the primary assumptions and modeling decisions that we made for developing our theory to estimate the deformations in the cortical spheroid as it is spun by the centrifuge.

As we already mentioned in §5.1, we restrict ourselves to the case where the centrifuge is being operated at a constant angular velocity of $\omega_{\max}$ rad/s. We made this decision in order to reduce the theory's complexity.

For modeling the deformations of the cortical spheroid, we also model the motion and deformations of the 3D soft substrate containing it, and the fluid media that bathes the spheroid and the substrate.

We ignore acceleration due to gravity in our theory. ²

²This assumption is only valid when the centrifuge's angular velocities and the spheroid's distance from the centrifuge rotation axis at those velocities are such that the centripetal accelerations in and around the vicinity of the spheroid are much larger than the acceleration due to gravity. We assume that this in fact is always the case in the centrifuge-mTBI-model, which our theory is intended to model.

We assume that the deformations and stresses in the spheroid and the substrate's region that is in its vicinity are axi-symmetric.

We assume that the mechanics of the spheroid, the substrate, and the fluid media can be well modeled using continuum theories [107]. Consequently, we model the spheroid and the substrate as homogeneous solids, and the fluid media as a homogeneous fluid. Even more specifically, we model the spheroid as a spherical ball composed of an incompressible neo-Hookean material, the substrate as composed of a compressible neo-Hookean material, and the fluid media as an incompressible Newtonian fluid. The constitutive law for the fluid media is given in §5.4.3 and those for the spheroid and substrate are given in §5.4.4. Additionally, we model the interaction between the spheroid and substrate as non-adhesive frictionless contact. We refer to (i) the cortical spheroid, (ii) the 3D soft substrate, and (iii) the fluid media collectively as the continua.

We assume that in a frame that rotates with the centrifuge's spinning arm, all mechanical fields remain stationary w.r.t. time. (see §5.7 for a discussion of this important assumption).

On account of the previous assumption it follows that the motion of the continua can be described as

$$\mathbf{x}_\tau[\mathbf{X}] = \mathbf{Q}_\tau \mathbf{I}_{\mathbb{R} \rightarrow \mathbb{E}}(\mathbf{X} + \mathbf{U}^\star[\mathbf{X}]) . \quad (5.1)$$

Here $\mathbf{X}$ is the position vector of a continuum material particle $\mathcal{X}$ in a reference configuration. (The particle can belong to the spheroid, the soft substrate, or the fluid media.) We will henceforth be referring to the material particle $\mathcal{X}$ by its reference position vector, $\mathbf{X}$. We call $\mathbf{x}_\tau[\cdot]$ the deformation map and $\mathbf{x}_\tau[\mathbf{X}]$ the material particle $\mathbf{X}$'s current position vector at the time instance τ . We call $\mathbf{U}^\star[\mathbf{X}]$ the intermediate displacements of the material particle $\mathbf{X}$. In general the intermediate displacements in addition to $\mathbf{X}$ also depend on the time instance τ . In our theory the intermediate displacements only depending on $\mathbf{X}$ too is a consequence of our previous assumption. We define the symbols $\mathbf{Q}_\tau$, and $\mathbf{I}_{\mathbb{R} \rightarrow \mathbb{E}}$ appearing in (5.1) in §5.4.1.2. For a mathematically complete and rigorous formulation of (5.1) see [90].

The material field $\mathbf{U}^\star[\cdot]$ is an unknown *a priori*. The strains and the stresses in the continua depend on the values of its gradient, $\{\nabla_{\mathbf{X}}[\mathbf{U}^\star]\}[\cdot]$. In §5.4 we derive the equations whose solution (cf. §5.5 and §5.6) will yield $\mathbf{U}^\star[\cdot]$, and hence the strains and the stresses.

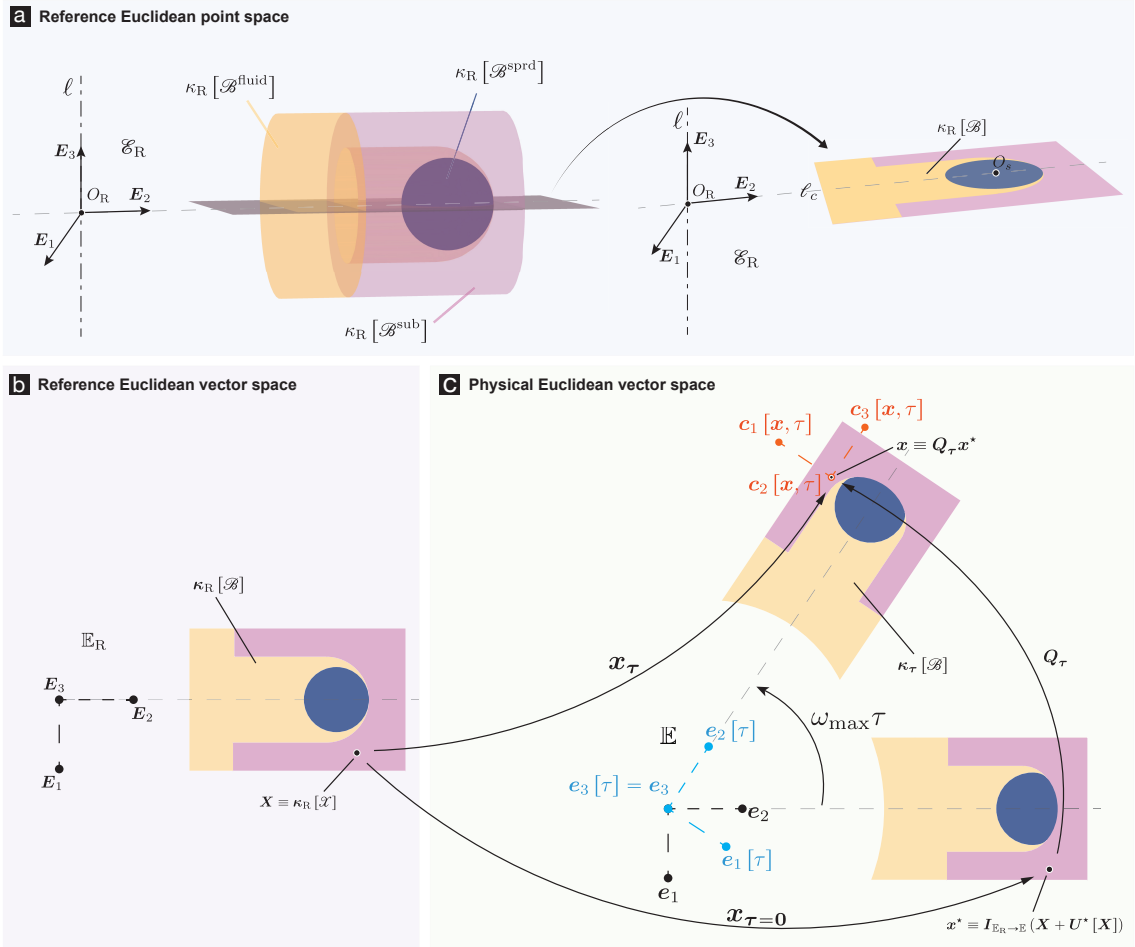


Figure 5.2: An illustration of the various mathematical objects that we use in our mechanics model of spheroid centrifugation. All objects are defined in §5.3 and §5.4.1.

5.3 Mathematical preliminaries

In this section we present the preliminary mechanics and mathematical notions that are needed for the development of our theory. Some of these notions can also be found in [89, §2.1] and [88, §2.1]

5.3.1 Abstract vector spaces in our model

Let $\mathbb{E}_{\mathbf{R}}$ be an oriented Euclidean vector space, i.e., an oriented finite dimensional, real, inner product space, and let the affine point space $\mathcal{E}_{\mathbf{R}}$ have $\mathbb{E}_{\mathbf{R}}$ as its associated vector translation space. We refer to $\mathbb{E}_{\mathbf{R}}$ and $\mathcal{E}_{\mathbf{R}}$ as the reference Euclidean vector and point space, respectively. Let $\mathbb{E}$ and $\mathcal{E}$ be another

pair of Euclidean vector and affine point space, respectively. Our continua (which can either be the spheroid, the 3D soft substrate, or the fluid media) execute their motion in $\mathcal{E}$. For that reason, we refer to $\mathbb{E}$ and $\mathcal{E}$ as the physical Euclidean vector space and point space, respectively. We model each of our continuum bodies, spheroid, soft substrate, and the media using the topological spaces $\mathcal{B}^{\text{sprd}}$, $\mathcal{B}^{\text{sub}}$, and $\mathcal{B}^{\text{fluid}}$, respectively (see Fig. 5.2 (a)).

We call a select continuous, injective map from $\mathcal{B}$ (which can be $\mathcal{B}^{\text{fluid}}$, $\mathcal{B}^{\text{sprd}}$, or $\mathcal{B}^{\text{sub}}$) into $\mathbb{E}_R$ the reference configuration and denoted it as κ_R . The elements of $\mathcal{B}$ are called material particles. We call $\mathbf{X} \equiv \kappa_R [\mathcal{X}]$ the particle $\mathcal{X}$'s reference position vector and $\kappa_R [\mathcal{B}]$ the reference body (see Fig. 5.2 (b)). Taking some arbitrary point $O_R \in \mathcal{E}_R$ to be $\mathcal{E}_R$'s origin (see Fig. 5.2 (a)), to κ_R we associate the map $\kappa_R : \mathcal{B} \rightarrow \mathcal{E}_R$ such that $O_R + \kappa_R [\mathcal{X}] = \kappa_R [\mathcal{X}]$. We call $\mathbf{X} \equiv \kappa_R [\mathcal{X}]$ the particle $\mathcal{X}$'s reference point.

We model time as a one-dimensional normed vector space $\mathbb{T}$ and denote a typical element in it as $\tau = \tau s$, where $\tau \in \mathbb{R}$ and s is a fixed vector which has units of seconds.

5.3.2 Cartesian basis vectors

The sets $(\mathbf{E}_i)_{i \in \mathcal{I}}$ and $(\mathbf{e}_i)_{i \in \mathcal{I}}$, where $\mathcal{I} := (1, 2, 3)$, are orthonormal sets of basis vectors for $\mathbb{E}_R$ and $\mathbb{E}$, respectively. By orthonormal we mean that the inner product between $\mathbf{E}_i$ and $\mathbf{E}_j$, or $\mathbf{e}_i$ and $\mathbf{e}_j$, where $i, j \in \mathcal{I}$, equals δ_{ij} , the Kronecker delta symbol, which equals unity iff $i = j$ and zero otherwise. In our problem, we take $\mathbf{E}_i$ and $\mathbf{e}_i$, $i \in \mathcal{I}$, to have the units of meters. The Cartesian co-ordinates of X which we denote as $\check{\mathbf{X}}[X] = (\check{\mathbf{X}}_i[X])_{i \in \mathcal{I}}$, are components of X w.r.t. $\mathbf{E}_i$, that is $\check{\mathbf{X}}_i[X] = X_i$, where $X_i := X \cdot \mathbf{E}_i$. For simplicity, $\mathbf{X} \equiv (X_1, X_2, X_3)$.

We denote the space of all $m \times n$ real nested ordered sets, where $m, n \in \mathbb{N}$, as $\mathcal{M}_{m \times n}(\mathbb{R})$. Thus $\check{\mathbf{X}}[X] \in \mathcal{M}_{3 \times 1}(\mathbb{R})$. We call the map $\mathcal{E}_R \ni X \mapsto \check{\mathbf{X}}[X] \in \mathcal{M}_{3 \times 1}(\mathbb{R})$ the Cartesian co-ordinate map. Let $(\mathbf{x}_i)_{i \in \mathcal{I}}$ be orthonormal sets of basis vectors for $\mathcal{M}_{3 \times 1}(\mathbb{R})$, or $\mathbb{R}^3$, where $\mathbf{x}_1 := (1, 0, 0)$, $\mathbf{x}_2 := (0, 1, 0)$, and $\mathbf{x}_3 := (0, 0, 1)$. When we refer to $X \in \mathbb{E}_R$, $\mathbf{X} \in \mathcal{M}_{3 \times 1}(\mathbb{R})$, or $X \in \mathcal{E}_R$ as a material particle we in fact mean the material particle $\mathcal{X} \in \mathcal{B}$.

5.3.3 Co-rotational Cartesian basis vectors for $\mathbb{E}$

Let

$$(Q_{ij}[\tau])_{i,j \in I} = \begin{pmatrix} \cos[\omega_{\max}\tau] & -\sin[\omega_{\max}\tau] & 0 \\ \sin[\omega_{\max}\tau] & \cos[\omega_{\max}\tau] & 0 \\ 0 & 0 & 1 \end{pmatrix} := \mathbf{Q}[\tau]. \quad (5.2)$$

The matrix $\mathbf{Q}[\tau]$ belongs to the special orthonormal group $SO(3) \subset \mathcal{M}_{3 \times 3}(\mathbb{R})$, and therefore satisfies the equations

$$\mathbf{Q}^T[\tau] \mathbf{Q}[\tau] = \mathbf{I}_{3 \times 3}, \quad (5.3a)$$

and

$$\mathbf{Q}[\tau] \mathbf{Q}^T[\tau] = \mathbf{I}_{3 \times 3}, \quad (5.3b)$$

where $\mathbf{Q}^T[\tau]$ is the transpose of $\mathbf{Q}[\tau]$, i.e., $\mathbf{Q}^T[\tau] = (\mathbf{Q}[\tau])^T$, and $\mathbf{I}_{3 \times 3} = (\delta_{ij})_{i,j \in I} \in \mathcal{M}_{3 \times 3}(\mathbb{R})$.

Using $\mathbf{Q}[\tau]$ we define the co-rotational set of basis vectors for $\mathbb{E}$, $(\mathbf{e}_i[\tau])_{i \in I}$, as

$$\mathbf{e}_i[\tau] = Q_{ji}[\tau] \mathbf{e}_j. \quad (5.4)$$

Note that $(\mathbf{e}_i[\tau])_{i \in I}$ change with time (see Fig. 5.2 (c)). However at each time instance τ , they form an orthonormal set of vectors and provide a basis for $\mathbb{E}$.

5.3.4 Co-rotational Cartesian co-ordinates

Given $\mathbf{x} \in \mathbb{E}$, let

$$\check{x}_i[\mathbf{x}, \tau] := \mathbf{x} \cdot \mathbf{e}_i[\tau], \quad (5.5)$$

where $(\mathbf{e}_i[\tau])_{i \in I}$ is defined via (5.4). We call $(\check{x}_i[\mathbf{x}, \tau])_{i \in I} =: \check{\mathbf{x}}[\mathbf{x}, \tau]$ the co-rotational Cartesian co-ordinates of $\mathbf{x}$ at the time instance τ .

5.3.5 Co-rotational cylindrical basis vectors for $\mathbb{E}$

Using $\check{\mathbf{x}}[\mathbf{x}, \tau]$ we define the set of co-rotational cylindrical basis vectors $(\mathbf{c}_i[\mathbf{x}, \tau])_{i \in I}$ (see Fig. 5.2 (c)) at the point $\mathbf{x}$ at the time instance τ as

$$(\mathbf{c}_i[\mathbf{x}, \tau])_{i \in I} = \mathbf{R}[\check{\mathbf{x}}[\mathbf{x}, \tau]] (\mathbf{e}_i[\tau])_{i \in I}, \quad (5.6)$$

where $\mathbf{R}[\cdot] : \mathcal{M}_{3 \times 1}(\mathbb{R}) \rightarrow \mathcal{M}_{3 \times 3}(\mathbb{R})$,

$$\mathbf{R}[(x_1, x_2, x_3)] := \frac{1}{\sqrt{x_1^2 + x_3^2}} \begin{pmatrix} x_1 & 0 & x_3 \\ x_3 & 0 & -x_1 \\ 0 & \sqrt{x_1^2 + x_3^2} & 0 \end{pmatrix}. \quad (5.7)$$

5.3.6 Linear maps between vector spaces

Say $\mathbb{W}$ and $\mathbb{U}$ are two arbitrary, oriented Euclidean vector spaces; for instance, they can be $\mathbb{E}_R$ and $\mathbb{E}$. We denote the space of all linear maps (transformations/operators) from $\mathbb{W}$ to $\mathbb{U}$ as $\mathcal{L}(\mathbb{W}, \mathbb{U})$. We denote the norm of a vector $\mathbf{w}_1$ in $\mathbb{W}$ that is induced by $\mathbb{W}$'s inner product, i.e., $(\mathbf{w}_1 \cdot \mathbf{w}_1)^{1/2}$, as $\|\mathbf{w}_1\|$. For $\mathbf{u}_1 \in \mathbb{U}$, the expression $\mathbf{u}_1 \otimes \mathbf{w}_1$ denotes the linear map from $\mathbb{W}$ to $\mathbb{U}$ defined as

$$(\mathbf{u}_1 \otimes \mathbf{w}_1) \mathbf{w}_2 = \mathbf{u}_1 (\mathbf{w}_1 \cdot \mathbf{w}_2), \quad (5.8)$$

where $\mathbf{w}_2 \in \mathbb{W}$. If the sets $(\mathbf{u}_i)_{i \in I}$ and $(\mathbf{w}_i)_{i \in I}$ provide bases for $\mathbb{U}$ and $\mathbb{W}$, respectively, then it can be shown that $\left((\mathbf{u}_i \otimes \mathbf{w}_j)_{j \in I} \right)_{i \in I}$, which we will henceforth abbreviate as $(\mathbf{u}_i \otimes \mathbf{w}_j)_{i, j \in I}$, provides a basis for $\mathcal{L}(\mathbb{W}, \mathbb{U})$. The number T_{ij} , where $i, j \in I$, is called the component of $\mathbf{T} \in \mathcal{L}(\mathbb{W}, \mathbb{U})$ w.r.t. $\mathbf{u}_i \otimes \mathbf{w}_j$ iff $T_{ij} = \mathbf{u}_i \cdot (\mathbf{T} \mathbf{w}_j)$. We call the nested ordered set $(T_{ij})_{i, j \in I}$ the component form of $\mathbf{T}$ w.r.t. $(\mathbf{u}_i \otimes \mathbf{w}_j)_{i, j \in I}$, and denote it as $\mathbf{T}$.

From here on, unless otherwise specified, we will be following the Einstein summation convention. As per this convention a repeated index in a term will imply a sum over that term with the repeated index taking values in I . For example, the expression $X_i \mathbf{E}_i$ represents the sum $\sum_{i \in I} X_i \mathbf{E}_i$. And an unrepeated index in a term will signify a set of 3 terms. For example, the term $\mathbf{E}_i$ represents

the set $\{E_i \mid i \in \mathcal{I}\}$.

The operator $D_j [\cdot]$ is defined such that

$$\{D_j [f]\} [\mathbf{X}] = \frac{\partial f [\mathbf{X}]}{\partial X_j}, \quad (5.9)$$

for $f : \mathbb{R}^3 \rightarrow \mathbb{R}$. We abbreviate $D_j [f]$ as $D_j f$.

5.4 Mechanics Model

5.4.1 Kinematics

5.4.1.1 Stationary and reference configurations

We show our assumed geometries for the continua in the centrifuge-TBI-model in Fig. 5.1 (a) and (b). The configuration shown in (a) is for when the centrifuge is stationary (recall that we have ignored acceleration due to gravity), and the one shown in (b) is the reference configuration in our problem.

In Fig. 5.1 (a), the spheroid lies in a 3D soft substrate; while the 3D soft substrate itself lies in an enclosure connected to the centrifuge's spinning arm. The geometries of the spheroid, the substrate, and the enclosure are all axi-symmetric about the axis ℓ_c shown in Fig. 5.1 (a). Thus, the spheroid is, well, a spherical ball of radius R_0 m. The cavity in the substrate that the spheroid lies in has the shape of a test tube. It is open at the top; it is L_1 m deep; the radius of its circular cross-sections is R_1 m; and its base has a hemispherical shape. The substrate does not completely fill the enclosure. The substrate's cross-sections towards the top are annular disks, while those towards the bottom are circular disks. The inner radii of the annular disks is R_1 m. The outer radii of the annular disks (and the radii of the circular disks) is $L_3/2$ m. The enclosure's cross-sections towards the top as well as those towards the bottom are both annular disks, albeit of different inner radii. The inner radii of the enclosure's annular disks that lie towards the bottom are the same as the outer radii of the substrate's annular disks, namely, $L_3/2$ m. The inner radii of the enclosure's annular disks that lie towards the top are R_2 m.

The spheroid's center O_s lies on the central axis ℓ_c (see Fig. 5.1 (a)), and it rests at the bottom of the cavity, with a single point touching (see Fig. 5.1 (a)). The thickness of the substrate under the spheroid is L_2 m.

We refer to the point at which the centrifuge's spinning arm attaches to the enclosure as O_P (marked in Fig. 5.1 (b)). We take the reference configuration for our problem to be the one shown in Fig. 5.1 (b), which is the same as that in Fig. 5.1 (a) except that the enclosure and the continua have undergone a rigid body rotation about the axis that is perpendicular to the plane spanned by the centrifuge's rotation axis and spinning arm (shown in Fig. 5.1 (a)) and passing through O_P ³.

The base of the substrate is at a distance of L_4 m from the point O_P (shown in Fig. 5.1 (b)). The spheroid and the substrate are bathed in the fluid media. The surface of the fluid media is at a distance of L_5 m from the point O_P (shown in Fig. 5.1 (b)). The length of the spinning arm is L_6 m.

Typical values for all the geometry parameters in the stationary and reference configurations, which are partially based on the measurements reported in [93], are given in Table 5.1.

Table 5.1: Typical values for the geometry parameters R_0 – R_2 , and L_1 – L_6 in the centrifuge-TBI-model. These parameters are defined in Fig. 5.1 (a)–(b). These typical values are based on the spheroid, and 3D soft substrate dimensions reported in [93]; geometry of the enclosure (marked in Fig. 5.1 (a)) in which the 3D soft substrate containing the spheroid is typically held; the amount of fluid media that is typically added to the enclosure, which is around 1 ml [93]; and the dimensions of a typical lab grade centrifuge, such as 5810R Eppendorf [110]. The units of all parameters are *meters*.

Parameter name	R_0	R_1	R_2	L_1	L_2	L_3	L_4	L_5	L_6
Parameter value $\times 10^3$	0.08	0.2	7.96	0.8	1.5	0.8	66	59.61	112.69

5.4.1.2 Deformation mapping

The motion of the continua is given by (5.1). To partially reiterate, the vector $X \in \mathbb{E}_R$ is the reference position vector of the material particle $\mathcal{X}$. The symbol $\mathbf{I}_{\mathbb{E}_R \rightarrow \mathbb{E}}$ denote the identity linear map from $\mathbb{E}_R$ onto $\mathbb{E}$. More explicitly, $\mathbf{I}_{\mathbb{E}_R \rightarrow \mathbb{E}} = \mathbf{e}_i \otimes \mathbf{E}_i$. Without loss of generality, we take that the continua rotate about $\mathbf{e}_3$. Since we restrict ourselves to the case in which the continua rotate at a fixed angular velocity of $\omega_{\max}$ rad/s, the assumption of rotation about $\mathbf{e}_3$ implies that $\mathbf{Q}_\tau = Q_{ij}[\tau] \mathbf{e}_i \otimes \mathbf{e}_j$, where

³Note that in our problem the reference configuration and the configuration when the centrifuge is not spinning are not isomorphic. When the centrifuge is not spinning the central axis and the rotation axis are parallel to each other, where as in the reference configuration the central axis ℓ_c and the rotation axis are perpendicular to each other.

$Q_{ij} [\tau]$ are defined in (5.2). The map $\mathbf{U}^\star : \mathbb{E}_R \rightarrow \mathbb{E}_R$ is the intermediate displacement field of $\mathcal{B}$. The symbol $\mathbf{x}_\tau [X]$ is the material particle X 's position vector in $\mathbb{E}$ at the time instance τ . The set $\kappa_\tau [\mathcal{B}] = \{\mathbf{x}_\tau [X] \in \mathbb{E} \mid X \in \kappa_R [\mathcal{B}]\}$ is called the current body (see Fig. 5.2 (c)).

As per (5.1) the continua's deformations are time invariant in the co-rotational basis $(\mathbf{e}_i [\tau])_{i \in \mathcal{I}}$. The co-rotational basis themselves rotate about the time stationary vector $\mathbf{e}_3$ with the constant angular velocity $\omega_{\max}$ rad/s (see Fig. 5.2 (c)).

5.4.1.3 Displacements components

Expressing $X = X_i \mathbf{E}_i$, and $\mathbf{U}^\star [X] = U_i^\star [\mathbf{X}] \mathbf{E}_i$, and using (5.1) and (5.4) it can be shown that

$$\mathbf{x}_\tau [X] = \bar{x}_i [\mathbf{X}] \mathbf{e}_i [\tau], \quad (5.10a)$$

where

$$\bar{x}_i [\mathbf{X}] = X_i + U_i^\star [\mathbf{X}]. \quad (5.10b)$$

Denoting $(\bar{x}_i [\mathbf{X}])_{i \in \mathcal{I}}$ as $\bar{\mathbf{x}} [\mathbf{X}]$ and $(U_i^\star [\mathbf{X}])_{i \in \mathcal{I}}$ as $\mathbf{U}^\star [\mathbf{X}]$, (5.10b) can equivalently be expressed as

$$\bar{\mathbf{x}} [\mathbf{X}] = \mathbf{X} + \mathbf{U}^\star [\mathbf{X}]. \quad (5.11)$$

Let

$$\kappa_R [\mathcal{B}^{\text{sprd}}] := \{\mathbf{X} \in \mathbb{R}^3 \mid X_i \mathbf{E}_i \in \kappa_R [\mathcal{B}^{\text{sprd}}]\}, \quad (5.12)$$

and

$$\kappa_R [\mathcal{B}^{\text{sub}}] := \{\mathbf{X} \in \mathbb{R}^3 \mid X_i \mathbf{E}_i \in \kappa_R [\mathcal{B}^{\text{sub}}]\}. \quad (5.13)$$

We refer to the restriction of $\bar{\mathbf{x}} [\cdot]$ to $\kappa_R [\mathcal{B}^{\text{sprd}}]$ as $\bar{\mathbf{x}}^{\text{sprd}} [\cdot]$. The maps $\bar{\mathbf{x}}^{\text{sub}} [\cdot]$ and $\bar{\mathbf{x}}^{\text{fluid}} [\cdot]$ are defined similarly.

5.4.1.4 Velocity components

We call $\mathcal{L}(\mathbb{T}, \mathbb{E})$ the physical velocity vector space and denote it as $\mathbb{V}$. It can be shown that the set $(\mathbf{v}_i[\tau])_{i \in I}$, where $\mathbf{v}_i[\tau] \in \mathbb{V}$ and are defined such that $\{\mathbf{v}_i[\tau]\}\boldsymbol{\tau} = \boldsymbol{\tau}\mathbf{e}_i[\tau]$, that is $\mathbf{v}_i[\tau] := \mathbf{e}_i[\tau] \otimes s^*$, where s^* is the dual of s , provides an orthonormal basis for $\mathbb{V}$. The velocity of a material particle $\mathbf{X}$ executing its motion in $\mathbb{E}$ lies in $\mathbb{V}$. The velocity of the material particle $\mathbf{X}$ at the instant τ , which we denote as $\mathbf{V}_\tau[\mathbf{X}]$, equals the value of the Fréchet derivative of the map $\mathbb{T} \ni \tau \mapsto \mathbf{x}_\mathbf{X}[\tau] \in \mathbb{E}$, where $\mathbf{x}_\mathbf{X}[\tau] = \mathbf{x}_\tau[\mathbf{X}]$, at the time instance τ . Thus, it follows from (5.1) that for $\tau \geq 0$

$$\mathbf{V}_\tau[\mathbf{X}] = V_i[\mathbf{X}] \mathbf{v}_i[\tau], \quad (5.14a)$$

where

$$V_i[\mathbf{X}] = W_{ij} \left(X_j + U_j^*[\mathbf{X}] \right), \quad (5.14b)$$

and

$$W_{ij} = \mathcal{Q}'_{kj}[\tau] Q_{ki}[\tau]. \quad (5.14c)$$

From (5.2) and (5.14c) it follows that

$$(W_{ij})_{i,j \in I} = \begin{pmatrix} 0 & -\omega_{\max} & 0 \\ \omega_{\max} & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} =: \mathbf{W}. \quad (5.15)$$

Denoting $(V_i[\mathbf{X}])_{i \in I}$ as $\mathbf{V}[\mathbf{X}]$, (5.14b) can equivalently be written as

$$\mathbf{V}[\mathbf{X}] = \mathbf{W}(\mathbf{X} + \mathbf{U}^*[\mathbf{X}]). \quad (5.16)$$

The velocity of the material particle located at $\mathbf{x} \in \mathbb{E}$ at the time instance τ is defined as

$$\mathbf{v}_\tau [\mathbf{x}] = \mathbf{V}_\tau [\mathbf{x}_\tau^{-1} [\mathbf{x}]] . \quad (5.17)$$

From (5.14a), (5.14b), and (5.10b), the equation (5.17) can be written as

$$\mathbf{v}_\tau [\mathbf{x}] = W_{ij} \check{x}_j [\mathbf{x}, \tau] \mathbf{v}_i [\tau] . \quad (5.18)$$

5.4.1.5 Accelerations

We call $\mathcal{L}(\mathbb{T}, \mathbb{V})$ the physical acceleration vector space and denote it as $\mathbb{A}$. It can be shown that the set $(\mathbf{a}_i [\tau])_{i \in \mathcal{I}}$, where $\mathbf{a}_i [\tau] \in \mathbb{A}$ and are defined such that $\{\mathbf{a}_i [\tau]\} \tau = \tau \mathbf{v}_i [\tau]$, i.e., $\mathbf{a}_i [\tau] = \mathbf{v}_i [\tau] \otimes \mathbf{s}^*$, provides an orthonormal basis for $\mathbb{A}$. The acceleration of a material particle $\mathbf{X}$ executing its motion in $\mathbb{E}$ lies in $\mathbb{A}$. The acceleration of $\mathbf{X}$ at the time instance τ equals the value of the Fréchet derivative of the map $\mathbb{T} \ni \tau \mapsto \mathbf{V}_\mathbf{X}(\tau) \in \mathbb{V}$, where $\mathbf{V}_\mathbf{X}(\tau) = \mathbf{V}_\tau(\mathbf{X})$, at the time instance τ . Thus, it follows from (5.14a) that for $\tau \geq 0$

$$\mathbf{A}_\tau [\mathbf{X}] = A_i [\mathbf{X}] \mathbf{a}_i [\tau] , \quad (5.19a)$$

where

$$A_i [\mathbf{X}] = W_{im} W_{mp} \left(X_p + U_p^* [\mathbf{X}] \right) . \quad (5.19b)$$

Denoting $(A_i [\mathbf{X}])_{i \in \mathcal{I}}$ as $\mathbf{A} [\mathbf{X}]$, (5.19b) can be equivalently be written as

$$\mathbf{A} [\mathbf{X}] = \mathbf{W}^2 (\mathbf{X} + \mathbf{U}^* [\mathbf{X}]) . \quad (5.20)$$

The acceleration of the material particle located at $\mathbf{x} \in \mathbb{E}$ at the time instance τ is defined as

$$\mathbf{a}_\tau [\mathbf{x}] = \mathbf{A}_\tau [\mathbf{x}_\tau^{-1} [\mathbf{x}]] . \quad (5.21)$$

From (5.19a), (5.19b), and (5.10b), the equation (5.21) can be written as

$$\mathbf{a}_\tau [\mathbf{x}] = W_{im} W_{mp} \check{x}_p [\mathbf{x}, \tau] \mathbf{a}_i [\tau]. \quad (5.22)$$

5.4.1.6 Deformation gradient and Strains

The deformation gradient corresponding to the deformation mapping $\mathbf{x}_\tau [\cdot]$, given in (5.1), is

$$\{\nabla_{\mathbf{X}} [\mathbf{x}_\tau]\} [X] =: \mathbf{F}_\tau [X] = F_{ij} [\mathbf{X}] \mathbf{e}_i [\tau] \otimes \mathbf{E}_j, \quad (5.23a)$$

where

$$F_{ij} [\mathbf{X}] := \delta_{ij} + D_j U_i^* [\mathbf{X}]. \quad (5.23b)$$

The right Cauchy-Green deformation tensor corresponding to the deformation gradient $F_{ij} [\mathbf{X}] \mathbf{e}_i [\tau] \otimes \mathbf{E}_j$ is

$$\mathbf{C} [X] = C_{ij} [\mathbf{X}] \mathbf{E}_i \otimes \mathbf{E}_j, \quad (5.24a)$$

where

$$C_{ij} [\mathbf{X}] = F_{mi} [\mathbf{X}] F_{mj} [\mathbf{X}]. \quad (5.24b)$$

We abbreviate $(F_{ij} [\mathbf{X}])_{i,j \in \mathcal{I}}$ and $(C_{ij} [\mathbf{X}])_{i,j \in \mathcal{I}}$ as $\mathbf{F} [\mathbf{X}]$ and $\mathbf{C} [\mathbf{X}]$, respectively.

5.4.2 Equilibrium

5.4.2.1 Cauchy-momentum equations in the reference body

It follows from the principle of balance of linear momentum and our decision to ignore acceleration due to gravity that

$$\{\text{Div } \mathbf{F}_\tau \mathbf{S}\} [X] = \rho_0 \mathbf{A}_\tau [X], \quad (5.25)$$

where $\{\text{Div } \mathbf{F}_\tau \mathbf{S}\} [\cdot]$ is the divergence of the field $X \mapsto \mathbf{F}_\tau [X] \mathbf{S} [X]$. Here $\mathbf{S} [X]$ is the 2nd Piola-Kirchhoff stress tensor at the material particle X .

In component form (5.25) can be written as

$$\{D_j [F_{im}S_{mj}]\} [\mathbf{X}] = \rho_0 A_i [\mathbf{X}], \quad (5.26)$$

where $S_{ij} [\mathbf{X}]$, $i, j \in \mathcal{I}$, are the components of $\mathbf{S} [\mathbf{X}]$ ⁴.

In (5.26) replacing $A_i [\mathbf{X}]$ with the RHS of (5.19b) we get

$$\{D_j [F_{im}S_{mj}]\} [\mathbf{X}] = \rho_0 W_{im} W_{mp} (X_p + U_p^* [\mathbf{X}]), \quad (5.27)$$

where $\rho_0 \text{ kg/m}^3$ is the density of the continua.

Noting from (5.15) that $W_{im} W_{mp} = -\omega_{\max}^2 (\delta_{ip} - \delta_{i3} \delta_{3p})$ in (5.27) we get

$$\{D_j [F_{im}S_{mj}]\} [\mathbf{X}] = -\rho_0 \omega_{\max}^2 (X_i + U_i^* [\mathbf{X}] - \delta_{i3} (X_3 + U_3^* [\mathbf{X}])). \quad (5.28)$$

The domain of (5.28) is either $\kappa_{\mathbf{R}} [\mathcal{B}^{\text{sprd}}]$, or $\kappa_{\mathbf{R}} [\mathcal{B}^{\text{sub}}]$, which were, respectively, defined in (5.12) and (5.13). Irrespective, of whether $\mathbf{X}$ belongs to $\kappa_{\mathbf{R}} [\mathcal{B}^{\text{sprd}}]$ or $\kappa_{\mathbf{R}} [\mathcal{B}^{\text{sub}}]$ the co-ordinates X_1 and X_3 are always less than $L_3/2$, where recall that $L_3/2$ is the outer radii of the substrate, and is 0.4×10^{-3} in our model. The co-ordinate X_2 in the domains, however, varies between 176×10^{-3} and 179×10^{-3} . (Fig. 5.1 can help in understanding how we arrived at these ranges for the different co-ordinates.) Therefore, in (5.28) we ignore X_1 , and X_3 , in comparison to X_2 . The intermediate displacement components $U_i^* [\mathbf{X}]$ are unlikely to be larger than the height of the substrate, which is, roughly, 3 millimeters. Therefore, we also ignore $U_i^* [\mathbf{X}]$ in comparison to X_2 in (5.28). In summary, we approximate (5.28) as

$$\{D_j [F_{im}S_{mj}]\} [\mathbf{X}] = -\rho_0 \omega_{\max}^2 \delta_{i2} \delta_{2j} X_j. \quad (5.29)$$

⁴Here we omit providing the mathematical details of how precisely $S_{ij} [\mathbf{X}]$ and $T_{ij} [\mathbf{x}]$ are, respectively, related to $\mathbf{S} [\mathbf{X}]$ and $\mathbf{T}_{\mathbf{r}} [\mathbf{x}]$. Since doing so will require notions from *exterior algebra* and *differential geometry* that need a significant amount of space to properly explain, and hence would distract from the primary focus of this paper.

5.4.2.2 Cauchy-momentum equations in the current body

It follows from the principle of balance of linear momentum that

$$\{\text{Div } \mathbf{T}_\tau\} [\mathbf{x}] = \rho_\tau [\mathbf{x}] \mathbf{a}_\tau [\mathbf{x}], \quad (5.30)$$

for all $\mathbf{x} \in \kappa_\tau [\mathcal{B}]$, where $\{\text{Div } \mathbf{T}_\tau\} [\cdot]$ is the divergence of the field $\mathbf{x} \mapsto \mathbf{T}_\tau [\mathbf{x}]$. Here $\mathbf{T}_\tau [\mathbf{x}]$ is the Cauchy stress tensor at the current position $\mathbf{x}$ at the time instance τ . And $\rho_\tau [\mathbf{x}] := \rho_0 / \text{Det} [\mathbf{F}_\tau [\mathbf{x}_\tau^{-1} [\mathbf{x}]]]$, where $\text{Det} [\cdot]$ is the determinant operator.

In (5.30) replacing $\mathbf{a}_\tau [\mathbf{x}]$ with the RHS of (5.22) and rewriting (5.30) in component form, we get

$$\{D_j T_{ij}\} [\mathbf{x}] = \bar{\rho}_\tau [\mathbf{x}] W_{im} W_{mp} x_p, \quad (5.31)$$

for all $\mathbf{x} \in \kappa_\tau [\mathcal{B}]$,

$$\kappa_\tau [\mathcal{B}] := \{\mathbf{x} := (x_1, x_2, x_3) \in \mathbb{R}^3 \mid x_i \mathbf{e}_i [\tau] \in \kappa_\tau [\mathcal{B}]\}, \quad (5.32)$$

and $T_{ij} [\mathbf{x}]$, $i, j \in \mathcal{I}$, are the components of $\mathbf{T}_\tau [\mathbf{x}]$ ⁴. Here $\bar{\rho}_\tau [\cdot]$ is defined such that $\bar{\rho}_\tau [\mathbf{x}] = \rho_\tau [x_i \mathbf{e}_i [\tau]]$. Noting from (5.15) that $W_{im} W_{mp} = -\omega_{\max}^2 (\delta_{ip} - \delta_{i3} \delta_{3p})$ in (5.31) we get

$$\{D_j T_{ij}\} [\mathbf{x}] = -\bar{\rho}_\tau [\mathbf{x}] \omega_{\max}^2 (x_i - \delta_{i3} x_3). \quad (5.33)$$

5.4.3 Pressure in the fluid media

We model the fluid media as an incompressible Newtonian fluid. It can be shown that in our problem, the rate of deformation tensor is naught (see §C.1 for details). Consequently, from [107, §22], we can get

$$\mathbf{T} [\mathbf{x}] = -p_s^{\text{fluid}} [\mathbf{x}] \mathbf{I}_{3 \times 3}, \quad (5.34)$$

for all $\mathbf{x} \in \kappa_\tau [\mathcal{B}^{\text{fluid}}]$,

$$\kappa_\tau [\mathcal{B}^{\text{fluid}}] := \{\mathbf{x} \in \mathbb{R}^3 \mid x_i \mathbf{e}_i [\tau] \in \kappa_\tau [\mathcal{B}^{\text{fluid}}]\}, \quad (5.35)$$

and $\mathbf{T}[\mathbf{x}] := (T_{ij}[\mathbf{x}])_{i,j \in I}$ and $p_s^{\text{fluid}}[\cdot]$ is the pressure field.

In (5.33) substituting $\bar{\rho}_\tau[\mathbf{x}]$ as ρ_0 , since we have assumed the fluid media as being incompressible, and $\mathbf{T}[\mathbf{x}]$ as $-p_s^{\text{fluid}}[\mathbf{x}] \mathbf{I}_{3 \times 3}$ from (5.34), we get that

$$\{D_i[p_s^{\text{fluid}}]\}[\mathbf{x}] = \rho_0 \omega_{\max}^2 (x_i - \delta_{i3} x_3), \quad \forall \mathbf{x} \in \kappa_\tau [\mathcal{B}^{\text{fluid}}]. \quad (5.36)$$

It can be shown that in our problem the free surface of the fluid at the time instance τ (marked as $\partial \mathbf{B}_{\tau,5}^{\text{fluid}}$ in Fig. 5.3 (a)) is always part of a cylinder. More specifically, it can be shown that

$$\partial \mathbf{B}_{\tau,5}^{\text{fluid}} = \{\mathbf{x} \in \kappa_\tau [\mathcal{B}^{\text{fluid}}] \mid x_1^2 + x_2^2 = l_7^2\}. \quad (5.37)$$

The parameter l_7 in (5.37) is the distance of the center of $\partial \mathbf{B}_{\tau,5}^{\text{fluid}}$ from the rotation axis (see Fig. 5.3 (a)).

The surface $\partial \mathbf{B}_{\tau,5}^{\text{fluid}}$ experiences the atmospheric pressure p^{atm} Pa, where $p^{\text{atm}} = 1.01325 \times 10^5$. Hence, one of the boundary conditions on $p_s^{\text{fluid}}[\cdot]$ is

$$p_s^{\text{fluid}}[\mathbf{x}] = p^{\text{atm}}, \quad \mathbf{x} \in \partial \mathbf{B}_{\tau,5}^{\text{fluid}}, \quad (5.38)$$

Solving (5.36) with the boundary condition (5.38) we get that

$$p_s^{\text{fluid}}[\mathbf{x}] = \frac{1}{2} \rho_0 \omega_{\max}^2 (x_1^2 + x_2^2 - l_7^2) + p^{\text{atm}}. \quad (5.39)$$

5.4.4 Material constitutive laws

5.4.4.1 Constitutive law for the spheroid

In (5.29) when $\mathbf{X} \in \kappa_{\mathbf{R}} [\mathcal{B}^{\text{sprd}}]$

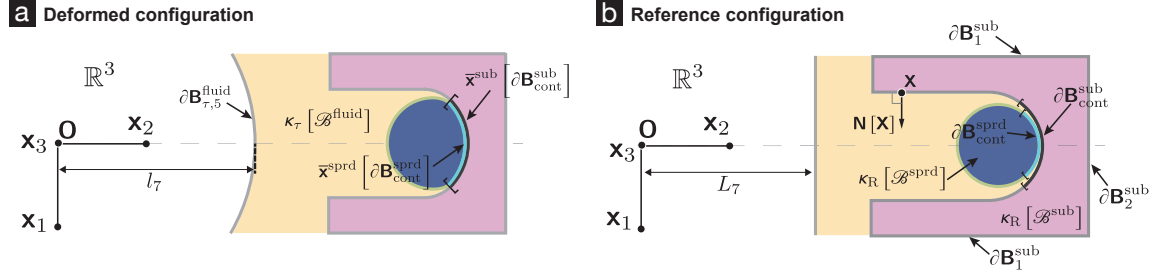


Figure 5.3: An illustration of the cortical spheroid, the fluid media, and the soft substrate surfaces in the deformed (a) and reference (b) configurations.

$$S_{ij}[\mathbf{X}] = \check{S}_{ij}^{\text{sprd}}[\mathbf{C}[\mathbf{X}]] - p_m^{\text{sprd}}[\mathbf{X}] J[\mathbf{C}[\mathbf{X}]] (\mathbf{C}[\mathbf{X}])^{-1}, \quad (5.40a)$$

where $S_{ij}[\mathbf{X}]$ is the i - j^{th} component of $\mathbf{S}[\mathbf{X}]$, the 2nd Piola Kirchhoff stress at the material particle $\mathbf{X}$,

$$J[\mathbf{C}] := \sqrt{\text{Det}[\mathbf{C}]}, \quad (5.40b)$$

$\check{S}_{ij}^{\text{sprd}}[\cdot]$ is the i - j^{th} component of $\check{\mathbf{S}}^{\text{sprd}}[\cdot]$,

$$\check{\mathbf{S}}^{\text{sprd}}[\mathbf{C}] = \mu \text{Det}[\mathbf{C}]^{-\frac{1}{3}} \left(\mathbf{I}_{3 \times 3} - \frac{1}{3} \text{Tr}[\mathbf{C}] \mathbf{C}^{-1} \right), \quad (5.40c)$$

and μ Pa is the shear modulus. Here $\text{Tr}[\cdot]$ is the trace operator. In (5.40a) the quantity $p_m^{\text{sprd}}[\mathbf{X}]$ acts as a Lagrange undetermined multiplier, which can be interpreted as the hydrostatic pressure at the material particle $\mathbf{X}$. Since $p_m^{\text{sprd}}[\cdot]$ is an unknown *a priori*, we solve (5.29) in conjunction with the incompressibility constraint

$$J[\mathbf{C}[\mathbf{X}]] = 1. \quad (5.41)$$

Equations (5.40) are the incompressible neo-Hookean material model from [111, equation 5.50].

5.4.4.2 Constitutive law for the 3D soft substrate

In (5.29) when $\mathbf{X} \in \kappa_R[\mathcal{B}^{\text{sub}}]$

$$S_{ij}[\mathbf{X}] = \check{S}_{ij}^{\text{sub}}[\mathbf{C}[\mathbf{X}]], \quad (5.42a)$$

where $\check{S}_{ij}^{\text{sub}} [\cdot]$ is the i - j th component of $\check{\mathbf{S}}^{\text{sub}} [\cdot]$,

$$\check{\mathbf{S}}^{\text{sub}} [\mathbf{C}] = \mu \text{Det} [\mathbf{C}]^{-1/3} \left(\mathbf{I}_{3 \times 3} - \frac{1}{3} \text{Tr} [\mathbf{C}] \mathbf{C}^{-1} \right) + \left(\lambda + \frac{2}{3} \mu \right) J [\mathbf{C}] (J [\mathbf{C}] - 1) \mathbf{C}^{-1}. \quad (5.42b)$$

Here λ Pa and μ Pa are the Lamé parameters. Equations (5.42) are the compressible neo-Hookean material model from [112, §3.5.5].

5.4.5 Boundary conditions

The solution of (5.29) also requires the use of the following boundary conditions.

Let $\partial \mathbf{B}^{\text{sprd}}$ and $\partial \mathbf{B}^{\text{sub}}$ be the surfaces of the spheroid $\kappa_{\text{R}} [\mathcal{B}^{\text{sprd}}]$ and the substrate $\kappa_{\text{R}} [\mathcal{B}^{\text{sub}}]$, respectively. Let $\partial \mathbf{B}_{\text{cont}}^{\text{sprd}}$ and $\partial \mathbf{B}_{\text{cont}}^{\text{sub}}$ be the surfaces of $\kappa_{\text{R}} [\mathcal{B}^{\text{sprd}}]$ and $\kappa_{\text{R}} [\mathcal{B}^{\text{sub}}]$, respectively, that come into contact with each other (see Fig. 5.3 (b)). They are both unknown *a priori*. The boundary conditions on $\partial \mathbf{B}_{\text{cont}}^{\text{sprd}}$ and $\partial \mathbf{B}_{\text{cont}}^{\text{sub}}$ are that there are no shear tractions on them, and the displacements of the spheroid and substrate on them respectively are such that

$$\bar{\mathbf{x}}^{\text{sprd}} \left[\partial \mathbf{B}_{\text{cont}}^{\text{sprd}} \right] = \bar{\mathbf{x}}^{\text{sub}} \left[\partial \mathbf{B}_{\text{cont}}^{\text{sub}} \right], \quad (5.43)$$

see Fig. 5.3 (a) for an illustration.

Let $\partial \mathbf{B}_1^{\text{sub}}$ and $\partial \mathbf{B}_2^{\text{sub}}$ be the surfaces of $\kappa_{\text{R}} [\mathcal{B}^{\text{sub}}]$ shown in Fig. 5.3 (b). The boundary conditions on $\partial \mathbf{B}^{\text{sprd}} \setminus \partial \mathbf{B}_{\text{cont}}^{\text{sprd}}$ and $\partial \mathbf{B}^{\text{sub}} \setminus \partial \mathbf{B}_{\text{cont}}^{\text{sub}} \setminus \bigcup_{i=1}^2 \partial \mathbf{B}_i^{\text{sub}}$ (see Fig. 5.3 (b)), due to the spheroid's and substrate's, respective, interactions with the fluid media are

$$\left(\mathbf{C} [\mathbf{X}] \mathbf{S} [\mathbf{X}] - p_m^{\text{fluid}} [\mathbf{X}] \mathbf{I}_{3 \times 3} \right) \mathbf{N} [\mathbf{X}] = \mathbf{0}_{3 \times 1}, \quad (5.44a)$$

where $\mathbf{S} [\mathbf{X}] := (S_{ij} [\mathbf{X}])_{i,j \in I}$, $\mathbf{0}_{3 \times 1} \equiv (0, 0, 0)$, and $\mathbf{N} [\mathbf{X}]$ is the unit outward surface normal vector at the location $\mathbf{X}$ (see Fig. 5.3 (b) for example). The field $p_m^{\text{fluid}} [\mathbf{X}]$ in (5.44a) is $p_s^{\text{fluid}} [\bar{\mathbf{x}} [\mathbf{X}]]$, where $p_s^{\text{fluid}} [\cdot]$ is given in (5.39). More concretely,

$$p_m^{\text{fluid}} [\mathbf{X}] = \frac{1}{2} \rho_0 \omega_{\text{max}}^2 \left((X_1 + U_1^* [\mathbf{X}])^2 + (X_2 + U_2^* [\mathbf{X}])^2 - l_7^2 \right) + p^{\text{atm}}. \quad (5.44b)$$

We cannot independently calculate l_7 in our model. Therefore, in (5.44b) we approximate l_7 as L_7 , the distance of the fluid's free surface to the rotation axis under the assumption that none of the continua (the spheroid, the fluid, and the substrate) deform (see Fig. 5.3 (b)). Also, since $U_1^\star$, X_1 , and $U_2^\star$ are much smaller than X_2 , in (5.44b) we approximate $(X_1 + U_1^\star [\mathbf{X}])$ as naught, and $(X_2 + U_2^\star [\mathbf{X}])$ as X_2 . In summary, we compute

$$p_m^{\text{fluid}} [\mathbf{X}] \approx \frac{1}{2} \rho_0 \omega_{\max}^2 (X_2^2 - L_7^2) + p^{\text{atm}}. \quad (5.44c)$$

An additional boundary condition on the substrate is that

$$U_1^\star [\mathbf{X}] X_1 = -U_3^\star [\mathbf{X}] X_3, \quad \forall \mathbf{X} \in \partial \mathbf{B}_1^{\text{sub}} \quad (5.45a)$$

$$U_2^\star [\mathbf{X}] = 0, \quad \forall \mathbf{X} \in \partial \mathbf{B}_2^{\text{sub}}. \quad (5.45b)$$

The boundary condition (5.45a) is a consequence of setting the radial component of the displacement field on $\partial \mathbf{B}_1^{\text{sub}}$ (see Fig. 5.3 (b)) to be naught, which we do to model the constraint from the enclosure (see Fig. 5.1 (a)). We choose the boundary condition (5.45b) to model the fact that the substrate sits in the enclosure (see Fig. 5.1 (a)), which constrains its deformation in the $\mathbf{E}_2$ direction on $\partial \mathbf{B}_2^{\text{sub}}$.

5.5 Coupled boundary value problems

As per our model, the motion of the spheroid and the 3D soft substrate is given by the family of deformation maps $\mathbf{x}_\tau [\cdot]$. This family of deformation maps can be constructed using (5.10) once the displacement field components $U_i^\star [\cdot]$ are known. The restrictions of $U_i^\star [\cdot]$ to $\kappa_R [\mathcal{B}^{\text{sprd}}]$ (resp. $\kappa_R [\mathcal{B}^{\text{sub}}]$) are obtained by solving the partial differential equation (PDE) (5.29) over the region $\kappa_R [\mathcal{B}^{\text{sprd}}]$ (resp. $\kappa_R [\mathcal{B}^{\text{sub}}]$). We refer to the PDE (5.29) posed over the region $\kappa_R [\mathcal{B}^{\text{sprd}}]$ as the spheroid boundary value problem (BVP), and the PDE (5.29) posed over the region $\kappa_R [\mathcal{B}^{\text{sub}}]$ as the substrate BVP. Recall that the functions $\{F_{im} S_{mj}\} [\cdot]$ appearing in (5.29) are defined as

$$\mathbf{X} \mapsto F_{im} [\mathbf{X}] S_{mj} [\mathbf{X}]. \quad (5.46)$$

In the spheroid (resp. substrate) BVP the $F_{im} [\cdot]$ in (5.46) are to be interpreted as the restrictions of the $F_{im} [\cdot]$ defined in (5.23b) to $\kappa_R [\mathcal{B}^{\text{sprd}}]$ (resp. $\kappa_R [\mathcal{B}^{\text{sub}}]$). For the spheroid (resp. substrate) BVP the $S_{ij} [\cdot]$ in (5.46) is given by the function (5.40) (resp. (5.42)). In the spheroid BVP, due to the presence of the Lagrange multiplier (pressure field) $p_m^{\text{sprd}} [\cdot]$ in (5.40), the PDE (5.29) needs to be solved jointly with the incompressibility constraint equation (5.41). The boundary conditions in the spheroid and the substrate BVPs are detailed in §5.4.5.

Note that the contact boundary condition (5.43) is part of both the spheroid and the substrate BVPs. It couples the two BVPs, since it involves displacement components from both BVPs. Therefore, the two BVPs cannot be solved independently. We solve the spheroid and the substrate BVPs simultaneously using finite element techniques.

5.6 Representative numerical solutions of the theory

To get a preliminary understanding into the type of strains and stresses that the cortical spheroids experience in the constant angular velocity centrifuge-TBI-model, and for demonstrating our theory, we compute various strain and stress measures in the cortical spheroids using our theory for some representative values of angular velocity, geometry parameters, and material properties.

Typical lab grade centrifuges are capable of reaching top speeds in the range of 1000–5000 revolution per minute (RPM). Therefore, for the representative calculations we consider angular velocities of 2000 RPM, i.e., 209 radians per second (rad/s); and 4000 RPM, i.e., 419 rad/s.

For the representative calculations we take the 3D soft substrate to be composed of agarose hydrogel, and the cortical spheroids and fluid media to be the ones described in [93]. Consequently, we take the densities of the 3D soft substrate, cortical spheroid, and fluid media to be 1640, 1240, and 980 kg/m³, respectively. Based on the measurements in [113, 114] we take the agarose hydrogel's Lamé parameters to be $\lambda = 4.28571 \times 10^5$ (Pa) and $\mu = 1.07143 \times 10^5$ (Pa). Based on the measurements in [115] we take the shear modulus of the cortical spheroids to be $\mu = 1.33 \times 10^3$ (Pa).

We take the values for the geometry parameters to be the ones given in Table 5.1.

Some strain measures from the representative calculations are shown in Figs. 5.4–5.6, and stress

measures in Figs. 5.7–5.10. The definitions of some of those strain (resp. stress) measures are discussed in §5.6.1 (resp. §5.6.2).

5.6.1 Strains

For the strain measure we use the logarithmic strain tensor $\mathbf{H}_\tau [\mathbf{x}]$. The logarithmic strain tensor is defined as

$$\mathbf{H}_\tau [\mathbf{x}] = \ln \mathbf{V}_\tau [\mathbf{x}], \quad (5.47a)$$

where

$$\mathbf{V}_\tau [\mathbf{x}] \mathbf{V}_\tau [\mathbf{x}] := \mathbf{F}_\tau [\mathbf{x}_\tau^{-1} [\mathbf{x}]] \left(\mathbf{F}_\tau [\mathbf{x}_\tau^{-1} [\mathbf{x}]] \right)^\top. \quad (5.47b)$$

The co-rotational Cartesian components of $\mathbf{H}_\tau [\mathbf{x}]$ are defined as

$$H_{ij} [\mathbf{x}] = \mathbf{e}_i [\tau] \cdot \left\{ \mathbf{H}_\tau [\check{\mathbf{x}} [\mathbf{x}, \tau]] \mathbf{e}_j [\tau] \right\}, \quad (5.48)$$

where $\check{\mathbf{x}} [\mathbf{x}, \tau]$ is the vector in $\mathbb{E}$ such that $\mathbf{x}$ is its set of co-rotational Cartesian co-ordinates at the time instance τ . We denote the matrix $(H_{ij} [\mathbf{x}])_{i,j \in I}$ as $\mathbf{H}^{\{\mathbf{e}_i[\tau]\}} [\mathbf{x}]$.

Let

$$\mathbf{H}^{\{\mathbf{c}_i[\mathbf{x}, \tau]\}} [\mathbf{x}] := \mathbf{R} [\mathbf{x}] \left\{ \mathbf{H}^{\{\mathbf{e}_i[\tau]\}} [\mathbf{x}] \right\} \mathbf{R} [\mathbf{x}]^\top. \quad (5.49)$$

We call $\mathbf{H}^{\{\mathbf{c}_i[\mathbf{x}, \tau]\}} [\mathbf{x}]$, the co-rotational cylindrical components form of $\mathbf{H}_\tau [\mathbf{x}]$. We denote the (1, 1), (2, 2), (3, 3), and (1, 3) components of $\mathbf{H}^{\{\mathbf{c}_i[\mathbf{x}, \tau]\}} [\mathbf{x}]$, respectively, as $H_{rr} [\mathbf{x}]$, $H_{\theta\theta} [\mathbf{x}]$, $H_{zz} [\mathbf{x}]$, and $H_{rz} [\mathbf{x}]$.

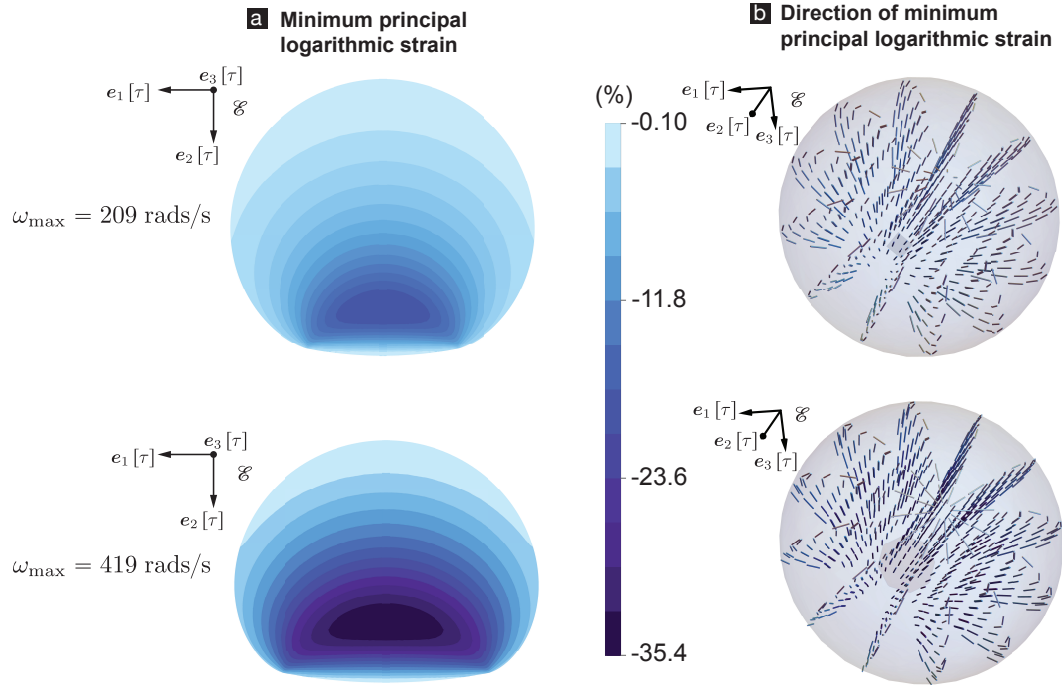


Figure 5.4: Strains in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . (a) Contour plots of the minimum principal value of the logarithmic strain tensor. The minimum principal value is the smallest eigenvalue. The logarithmic strain tensor is defined in §5.6.1. (b) Each line segment shows a section of the fiber associated with the eigenvectors that correspond to the minimum eigenvalue at the location of its midpoint.

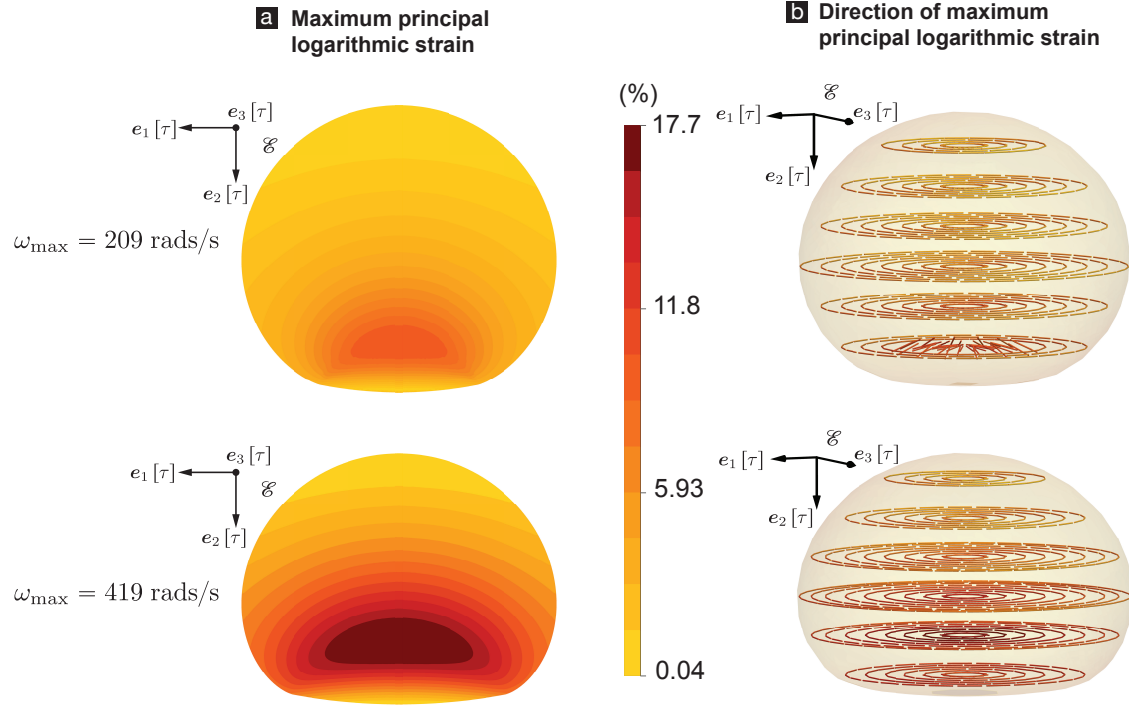


Figure 5.5: Strains in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . (a) Contour plots of the maximum principal value of the logarithmic strain tensor. The maximum principal value is the largest eigenvalue. The logarithmic strain tensor is defined in §5.6.1. (b) Each line segment shows a section of the fiber associated with the eigenvectors that correspond to the maximum eigenvalue at the location of its midpoint.

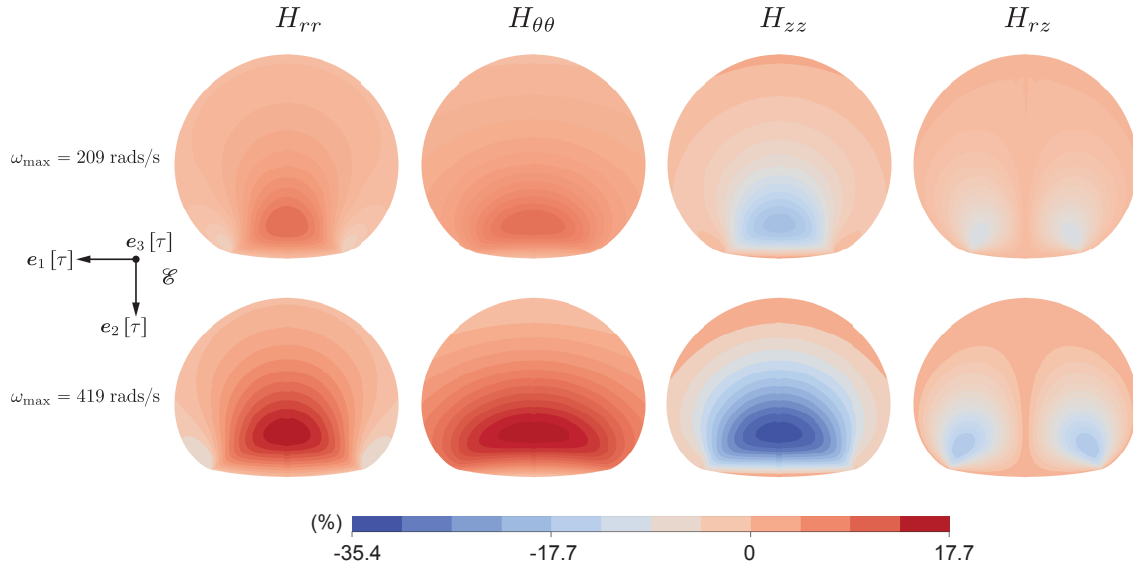


Figure 5.6: Strains in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . The logarithmic strain tensor is defined in §5.6.1. The columns show contour plots of H_{rr} , $H_{\theta\theta}$, H_{zz} , and H_{rz} , respectively, which are the co-rotational cylindrical components of the logarithmic strain tensor. They are defined in §5.6.1. The top row corresponds to the angular velocity 209 rad/s, and the bottom row to 419 rad/s.

5.6.2 Stresses

For the stress measure we use the Cauchy stress tensor. In §5.6.1 we defined the co-rotational Cartesian components of the logarithmic strain tensor, $\mathbf{H}^{\{e_i[\tau]\}}[\mathbf{x}]$, using the logarithmic strain tensor, $\mathbf{H}_\tau[\mathbf{x}]$, and the co-rotational Cartesian basis $(\mathbf{e}_i[\tau])_{i \in \mathcal{I}}$. In the same way we can define the co-rotational Cartesian components of the Cauchy stress tensor, $\mathbf{T}^{\{e_i[\tau]\}}[\mathbf{x}]$, using the Cauchy stress tensor, $\mathbf{T}_\tau[\mathbf{x}]$, and $(\mathbf{e}_i[\tau])_{i \in \mathcal{I}}$. In §5.6.1 we further defined the co-rotational cylindrical components of the logarithmic strain tensor, $\mathbf{H}^{\{c_i[\mathbf{x}, \tau]\}}[\mathbf{x}]$, using $\mathbf{H}^{\{e_i[\tau]\}}[\mathbf{x}]$ and the function $\mathbf{R}[\cdot]$ via (5.49). In the same way we can define the co-rotational cylindrical components of the Cauchy stress tensor, $\mathbf{T}^{\{c_i[\mathbf{x}, \tau]\}}[\mathbf{x}]$, using $\mathbf{T}^{\{e_i[\tau]\}}[\mathbf{x}]$ and $\mathbf{R}[\cdot]$.

We denote the (1, 1), (2, 2), (3, 3), and (1, 3) components of $\mathbf{T}^{\{c_i[\mathbf{x}, \tau]\}}[\mathbf{x}]$, respectively, as $T_{rr}[\mathbf{x}]$, $T_{\theta\theta}[\mathbf{x}]$, $T_{zz}[\mathbf{x}]$, and $T_{rz}[\mathbf{x}]$.

The pressure at the location $\mathbf{x}$ at the time instance τ is defined as negative one third the trace of $\mathbf{T}_\tau[\mathbf{x}]$.

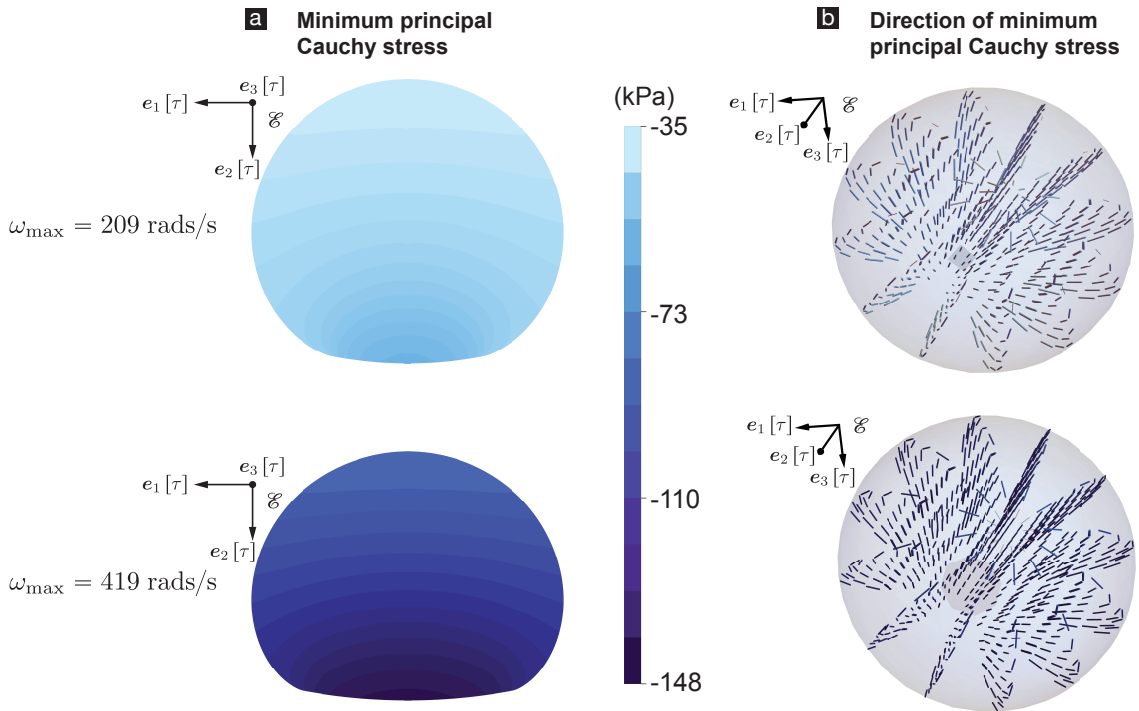


Figure 5.7: Stresses in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . (a) Contour plots of the minimum principal value of the Cauchy stress tensor. The minimum principal value is the smallest eigenvalue. (b) Each line segment shows a section of the fiber associated with the eigenvectors that correspond to the minimum eigenvalue at the location of its midpoint.

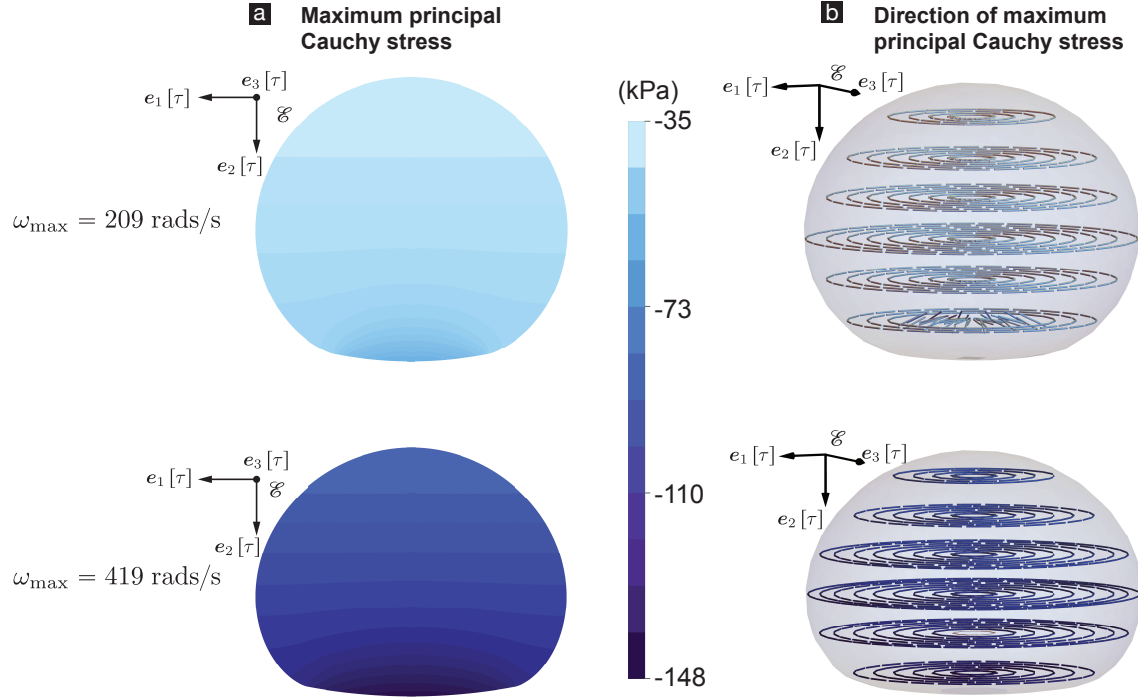


Figure 5.8: Stresses in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . (a) Contour plots of the maximum principal value of the Cauchy stress tensor. The maximum principal value is the largest eigenvalue. (b) Each line segment shows a section of the fiber associated with the eigenvectors that correspond to the maximum eigenvalue at the location of its midpoint.

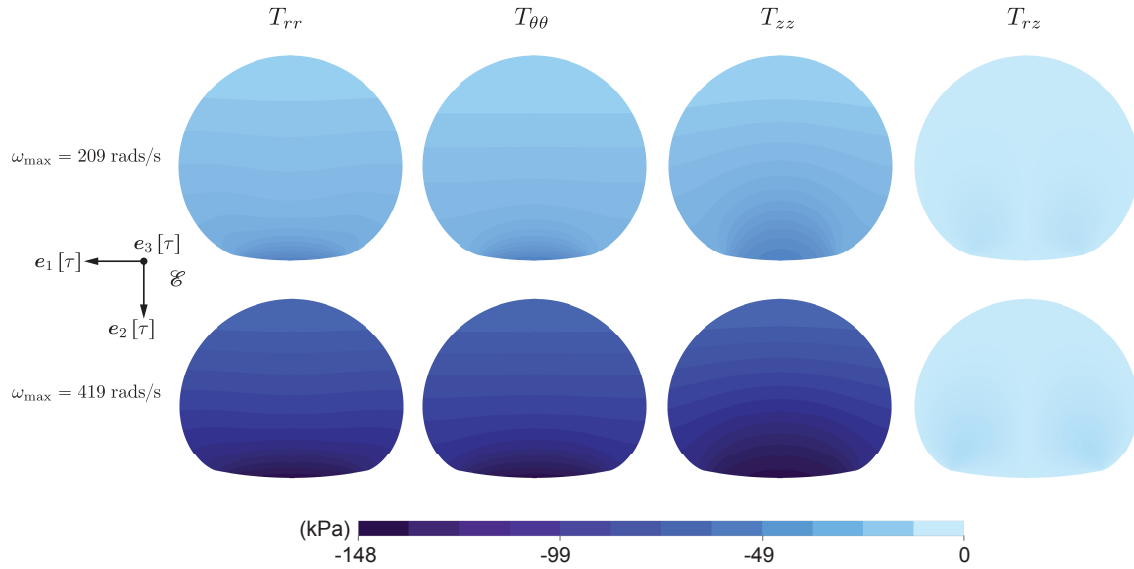


Figure 5.9: Stresses in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . The columns show contour plots of T_{rr} , $T_{\theta\theta}$, T_{zz} , and T_{rz} , respectively, which are the co-rotational cylindrical components of the Cauchy stress tensor (see §5.6.2 for details). The top row corresponds to the angular velocity 209 rad/s, and the bottom row to 419 rad/s.

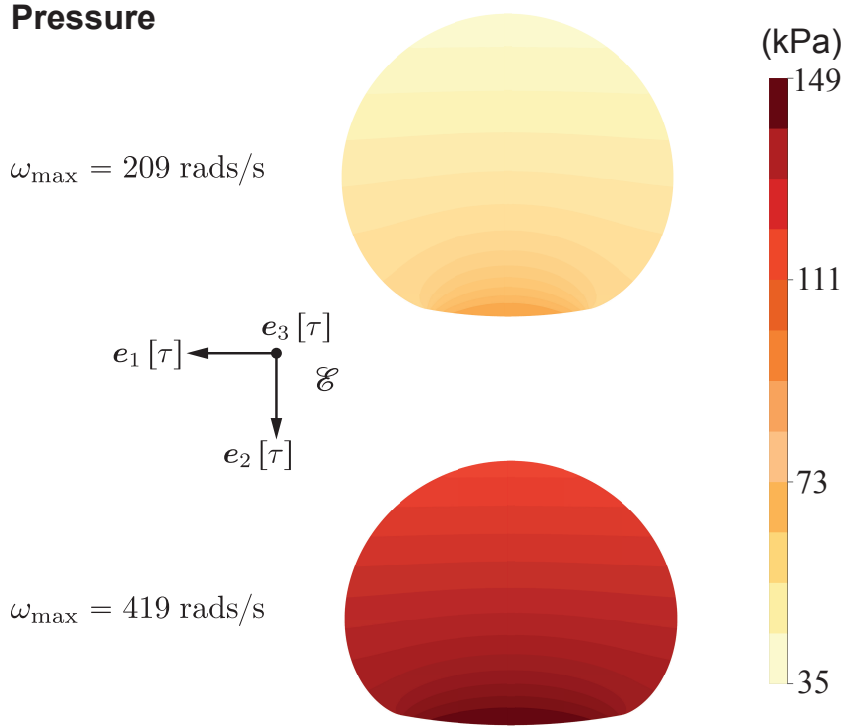


Figure 5.10: Pressures in the spheroid predicted by our theory for the representative values of angular velocity, geometry parameters, and material properties described in §5.6 at an arbitrary time instance τ . The top plot corresponds to the angular velocity 209 rad/s, and the bottom plot to 419 rad/s.

5.7 Concluding remarks

1. In the proposed centrifuge-TBI-model design (Fig. 5.1), the cortical spheroids primarily undergo a type of squeezing deformation (Fig. 5.1 (c)). However, with more sophisticated designs for the 3D soft substrate, it is possible to apply other types of deformations to the cortical spheroids.
2. We have stated earlier (see beginning of §5.2) that we restrict ourselves to the *in vitro* experiment in which the centrifuge's angular velocity is constant as a function of time. However, based on some preliminary experiments and noting that most micro-tissues are viscoelastic in nature, we believe that the final results from our theory will continue to apply with a minor modification even when the angular velocity is not a constant. Say the angular velocity changes with time as dictated by the function $\tau \mapsto \check{\omega}_{\max}[\tau]$. Then we believe that our results modified by replacing $\omega_{\max}$ in them with $\check{\omega}_{\max}[\tau]$ will apply at the time instance τ . A caveat for our

modified results to apply is that the variation of angular velocity with time be of a moderate character. That is, at the least, the derivative of $\tilde{\omega} [\cdot]$ be well defined and bounded. Since, clearly, our modified results will not apply when the angular velocity is changed abruptly, i.e., when $\tilde{\omega} [\cdot]$ is a step function. (Such a step change in angular velocity will agitate the fluid media in addition to creating other complications.) Full 3D solution of the Navier-Stokes equations, in the context of the centrifuge-TBI-model, is needed in order to precisely determine the regime of applicability of our modified results.

3. The developed theory is fairly general. For deriving the numerical results shown in Figs. 5.4–5.10, we modeled the mechanical behavior of the spheroids using the incompressible neo-Hookean constitutive law (5.40). This model does not incorporate any sophisticated material effects such as visco-elasticity, poro-elasticity, plasticity, etc. We chose it not because we believe that the spheroid's deformation does not contain any such sophisticated effects, but rather because currently those effects are not sufficiently characterized in the spheroids; because of which (1) it is harder to justify the computational expense involved in incorporating those effects, and (2) the material constants in the material models incorporating those sophisticated effects will have to be selected in an almost arbitrary fashion. The elastic constants in the incompressible neo-Hookean constitutive law were chosen such that the spheroid's elastic behavior was consistent with the brain white matter's elasticity, which was characterized using magnetic resonance elastography (MRE) [115], and the spheroids' contact deformations, which were characterized using an Atomic Force Microscope [93]. Once the more sophisticated material effects in the spheroid's mechanical deformation become well characterized, new numerical results can be derived in a straightforward manner from the developed theory by simply changing the spheroid's constitutive equation (5.40) to be something different that incorporates those effects. For example, it can be changed to the one developed in [116], [117], and [118], which takes the brain's poro-elastic behavior into account.
4. There are several studies that focus on the deformation of a spherical elastic solid. A number of axi-symmetric solutions can be found in standard elasticity monographs and textbooks, such as [119] and [120]. For axi-symmetric solutions involving contact between two nominally spherical solids see, e.g., [121, 122, 123]. For results involving both non-axi-symmetric

deformation and contact see [124]. In the context of the vast amount of literature available on the deformation of a spherical elastic solid, a distinguishing aspect of the developed theory is that it contains the following four physical effects simultaneously: (a) contact between two deformable bodies, (b) finite deformation of both solids, (c) individual contact between the solids and a surrounding fluid, and (d) body forces on both solids and the fluid due to the rotation of their assembly.

5. From a mechanics and engineering perspective, we do not see anything that curtails one from using the idea of applying mechanical loads via centrifugation in other mechanobiology studies. Especially, *in vitro* traumatic injury studies can be envisioned with other micro-tissues, such as those composed of lung or liver cells.
6. As we mentioned in §5.1, the mechanical loading, i.e., the force on the cortical spheroid, can be easily and robustly tuned via the centrifuge's angular velocity and the volume of the fluid media. As we highlighted in Fig. 5.1 (c), the forces acting on the cortical spheroid consist of the tractions from the fluid media, tractions from the 3D soft substrate, and the body forces due to the rotations. The fact that the tractions on the cortical spheroid from the fluid media depend on the centrifuge's angular velocity and the volume of the fluid media can be seen from (5.44c).

Chapter 6

Conclusions and outlook

In this dissertation, we developed a multiscale, physics-based approach to assess the potential risk of brain injury from mechanical loading. This work enhances our understanding of the relationship between head motion and brain deformation, as well as how brain deformation ultimately leads to injury. Such insights are crucial for improving and developing protective equipment aimed at reducing the incidence of mild traumatic brain injury. The main finds of this dissertation are summarized as follows:

1. We developed a new motion-prediction algorithm, the $\sqrt{A0}$ -algorithm, which is able to determine the complete motion of a rigid body using only data from four tri-axial accelerometers. The algorithm was validated using the virtual accelerometer data generated from the simulation of a rigid ellipsoid impacting an elastic half-space.
2. We compared the predicted acceleration from proposed $\sqrt{A0}$ -algorithm and the AO-algorithm using the virtual data, to which differing amounts of bias and noise type errors have been added. The comparison indicates that the $\sqrt{A0}$ -algorithm is less sensitive to bias type errors compared to the AO-algorithm. When subjected to noise-type errors alone, the performance of the $\sqrt{A0}$ -algorithm is comparable to that of the AO-algorithm.
3. We developed an idealized continuum mechanics (ICM) based, finite rotation, small strain, elastic head model, referred to as the elastic ICM head model. This model was then extended to incorporate brain's viscosity, resulting in what we refer to as the viscoelastic ICM head

model. In deriving the viscoelastic ICM head model, we have assumed the brain to be homogeneous and isotropic; to undergo small strains; and even more dramatically, to undergo 2D deformation. Despite these simplifications, it is remarkable that the viscoelastic ICM head model is capable of providing first-order estimates of peak strains comparable to those predicted by more sophisticated finite element analysis (FEA) head models. The maximum relative difference between the predicted peak strain values from our viscoelastic ICM head model and the full FEA head model is less than 8%.

4. We developed a spheroid-based, *in vitro* TBI model in which the spheroids are mechanically loaded by being spun by a centrifuge. In conjunction with the proposed TBI model, we developed a continuum mechanics-based theory to estimate the strains experienced by the spheroids when spun at a constant angular velocity. The TBI model, as demonstrated in [125], offers high throughput; maintains conditions that are as sterile as those present at the time the spheroids are grown; and enables loading of the spheroids in a reproducible manner.

Despite the advances achieved in this dissertation, several potential limitations remain that could be explored in future research. These areas of exploration are discussed as follows:

1. In this dissertation, we present the validation of $\sqrt{A0}$ -algorithm using the virtual accelerometer data. Additionally, we have developed a wearable sensor system for implementing the algorithm. This system has been successfully tested in a fuselage drop test to determine the acceleration of the center of mass of a Hybrid III (HIII) dummy's head. The results of this application will be published shortly.
2. The only constraint for the $\sqrt{A0}$ -algorithm is that it requires the four tri-axial accelerometers to not lie in the same plane. We have since developed a new algorithm that determines rigid body motion using three tri-axial accelerometers and one tri-axial gyroscope. This new algorithm only requires that the three accelerometers do not lie along the same line. The algorithm, along with its demonstration, will be published shortly.
3. Despite the presence of bias and noise type errors in accelerometer measurements, the accuracy of the predictions from both the $\sqrt{A0}$ -algorithm and AO-algorithm is influenced by

the arrangement of the four accelerometers. There may exist an optimal configuration of accelerometers for both algorithms. In the future, we plan to conduct an error analysis of both the $\sqrt{A0}$ -algorithm and AO-algorithm with respect to the placement of accelerometers. These findings will be critical for designing the optimal positioning of accelerometers in wearable sensor systems.

4. In deriving the elastic and viscoelastic ICM head models, we assumed the brain undergoes small strains and 2D deformation. In the future, we plan to extend these head models to account for 3D deformation and finite strains, providing a more comprehensive representation of brain tissue mechanics.
5. We provided a procedure to calculate the peak strain in both the elastic and viscoelastic ICM head models. While this procedure is significantly simpler than the one used for CM-based head models, there is no closed-form expression to represent the relationship between peak strain, peak angular velocity, and peak angular acceleration, as illustrated by the curved surface in Figs. 3.7 and 4.1. In the future, we plan to derive a simplified formula to approximate this curved surface for more straightforward application.
6. We plan to compare the location within the brain where peak strains occur, as predicted by our elastic and viscoelastic ICM head models, to the predictions made by the more sophisticated FEA head model. This comparison will help evaluate the performance of our models in identifying the regions of peak strain, which are critical to assess potential injury risk.
7. In [125], we demonstrate our proposed TBI model, wherein a group of spheroids were centrifuged to assess cellular injury. Additionally, we experimentally validated the predicted pressure exerted by the media on the spheroid using our mechanics theory. In the future, we plan to design an experiment that utilizes imaging techniques to measure the deformation of the spheroids during centrifugation.

Appendix A

Supplementary Material: Determining rigid body motion from accelerometer data through the square-root of a negative semi-definite tensor

A.1 Definition of the map $\star(\cdot)$

For $n_{sd} = 2$, the map $\star(\cdot) : \mathfrak{so}(\mathbb{R}, 2) \rightarrow \mathbb{R}$ is defined by the equation $\star(\cdot) = (\cdot)_{21}$. The inverse of $\star(\cdot)$ is the map $\star^{-1}(\cdot) : \mathbb{R} \rightarrow \mathfrak{so}(\mathbb{R}, 2)$ defined by the equation

$$\star(\alpha) = \begin{pmatrix} 0 & -\alpha \\ \alpha & 0 \end{pmatrix}. \quad (\text{A.1})$$

For $n_{sd} = 3$, the map $\star(\cdot) : \mathfrak{so}(\mathbb{R}, 3) \rightarrow \mathcal{M}_{3,1}(\mathbb{R})$ is defined by the equation $\star(\cdot) = ((\cdot)_{32}, (\cdot)_{13}, (\cdot)_{21})$. The inverse of $\star(\cdot)$ is the map $\star^{-1}(\cdot) : \mathcal{M}_{3,1}(\mathbb{R}) \rightarrow \mathfrak{so}(\mathbb{R}, 3)$ defined by the equation

$$\star((\alpha_1, \alpha_2, \alpha_3)) = \begin{pmatrix} 0 & -\alpha_3 & \alpha_2 \\ \alpha_3 & 0 & -\alpha_1 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}. \quad (\text{A.2})$$

To make our notation appear less cumbersome we denote $\star^{-1}(\cdot)$ too as $\star(\cdot)$. Whether we mean $\star(\cdot)$ or $\star^{-1}(\cdot)$ will be clear from the argument of $\star(\cdot)$.

A.2 Derivation of (2.3), i.e., proof of the statement that square of $\overline{\mathbf{W}}(\tau)$ is equal to the symmetric part of $\mathbf{P}(\tau)$

The following lemmas can be shown to be equivalent to some of the standard results in the mechanics of rigid solids, see the work in [126, §2.5.2] and [127, §6.4], which treats the rigid body motion in a modern continuum mechanics style; or see the work in [128, §9.4] and [129, §15], which treats the rigid body motion from a perspective of geometric mechanics. However, at a cursory level, due to our notation and formalism, those results might appear to be different from the below lemmas. The differences in notation and formalism are primarily due to the fact that in our work we distinguish between the vector spaces to which the various physical quantities, e.g., the rotation operation, belong to and the (non-dimensional) matrix vector spaces to which the component representations of those quantities belong to. For that reason, we believed that it would be helpful to the reader if we presented the following lemmas using the notation and formalism that we use in the current work.

A.2.1 Skew symmetry of $\overline{\mathbf{W}}(\tau)$

Lemma A.2.1. The matrix $\overline{\mathbf{W}}(\tau)$, defined in (2.18), is skew-symmetric.

Proof. It can be shown using $\overline{\mathbf{W}}(\tau)$'s definition (2.18) and equations (2.9) and (2.6a) that

$$\overline{\mathbf{W}}(\tau) = \mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau). \quad (\text{A.3})$$

Differentiating (2.6a) we get

$$\left(\mathbf{Q}^\top\right)'(\tau) \mathbf{Q}(\tau) + \mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau) = \mathbf{0}. \quad (\text{A.4})$$

Noting that $\left(\mathbf{Q}^\top\right)'(\tau) = (\mathbf{Q}')^\top(\tau)$ we see that the first term on the left hand side of (A.4) is equal to $(\mathbf{Q}')^\top(\tau) \mathbf{Q}(\tau)$, which, in fact, is equal to the transpose of the second term on the left hand side of (A.4). Thus, it follows from (A.4) that $\text{sym}\left(\mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau)\right) = \mathbf{0}$. That is, that $\mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau)$ is skew-symmetric. The result that $\overline{\mathbf{W}}(\tau)$ too is skew-symmetric now immediately follows from (A.3).

A.2.2 Derivation of equation (2.3)

Lemma A.2.2. The symmetric part of $\mathbf{P}(\tau)$ is equal to the square of $\overline{\mathbf{W}}(\tau)$.

Proof. Differentiating (2.6a) twice and rearranging we get that

$$\left(\mathbf{Q}^\top\right)''(\tau) \mathbf{Q}(\tau) + \mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau) = -2 \left(\mathbf{Q}^\top\right)'(\tau) \mathbf{Q}'(\tau). \quad (\text{A.5})$$

The equation (A.5) on noting that the first and second terms on its left hand side are in fact transposes of each other simplifies to

$$\text{sym}\left(\mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau)\right) = -\left(\mathbf{Q}^\top\right)'(\tau) \mathbf{Q}'(\tau). \quad (\text{A.6})$$

Writing the term on the right hand side of (A.6) as $-\left(\mathbf{Q}^\top\right)'(\tau) \mathbf{I} \mathbf{Q}'(\tau)$, and then using (2.6b) and replacing the $\mathbf{I}$ in the resulting equation with $\mathbf{Q}(\tau) \mathbf{Q}^\top(\tau)$, we get

$$\text{sym}\left(\mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau)\right) = -\left(\left(\mathbf{Q}^\top\right)'(\tau) \mathbf{Q}(\tau)\right) \left(\mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau)\right). \quad (\text{A.7})$$

Noting that $\left(\mathbf{Q}^\top\right)'(\tau) = (\mathbf{Q}')^\top(\tau)$ we see that the first factor on the right hand side of (A.7) is equal to $(\mathbf{Q}')^\top(\tau) \mathbf{Q}(\tau)$, which is the transpose of the second factor on the right hand side of (A.7), namely

$\mathbf{Q}^\top(\tau) \mathbf{Q}'(\tau)$. We, however, know from (A.3) that this second factor is equal to $\overline{\mathbf{W}}(\tau)$. Thus, we get from (A.7) that

$$\text{sym} \left(\mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau) \right) = -\overline{\mathbf{W}}^\top(\tau) \overline{\mathbf{W}}(\tau), \quad (\text{A.8a})$$

which simplifies on using Lemma A.2.1 to

$$\text{sym} \left(\mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau) \right) = \overline{\mathbf{W}}^2(\tau). \quad (\text{A.8b})$$

Using (2.16) and replacing the quantity $\mathbf{Q}^\top(\tau) \mathbf{Q}''(\tau)$ appearing on the left hand side of (A.8b) with $\mathbf{P}(\tau)$, we get

$$\text{sym}(\mathbf{P}(\tau)) = \overline{\mathbf{W}}^2(\tau). \quad (\text{A.9})$$

A.3 The matrix $\text{sym}(\mathbf{P}(\tau))$ is negative semidefinite and its negative eigenvalues, if they exist, have even algebraic multiplicities

The entries of $\overline{\mathbf{W}}(\tau)$ and $\text{sym}(\mathbf{P}(\tau))$ are all real numbers. However, in this section we consider $\overline{\mathbf{W}}(\tau)$ and $\text{sym}(\mathbf{P}(\tau))$ to be elements of $\mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{C})$, where $\mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{C})$ is the space of all $n_{\text{sd}} \times n_{\text{sd}}$ matrices whose entries belong to $\mathbb{C}$, the space of complex numbers.

A.3.1 Negative semi-definiteness of $\text{sym}(\mathbf{P}(\tau))$

Lemma A.3.1. The matrix $\text{sym}(\mathbf{P}(\tau))$ is negative semidefinite.

Proof. The matrix $\text{sym}(\mathbf{P}(\tau))$ is self-adjoint since it is equal to its transpose, which, as all of $\text{sym}(\mathbf{P}(\tau))$'s entries are real, is equal to its conjugate-transpose, i.e., to its adjoint. Hence it follows from [130, 7.31] that the matrix $\text{sym}(\mathbf{P}(\tau))$ is negative semidefinite iff the inner product $\langle \text{sym}(\mathbf{P}(\tau)) \mathbf{X}, \mathbf{X} \rangle$, where $\mathbf{X} \in \mathcal{M}_{n_{\text{sd}}, 1}(\mathbb{C})$ but is otherwise arbitrary, is always non-positive.

It follows from (2.3) that

$$\langle \text{sym}(\mathbf{P}(\tau)) \mathbf{X}, \mathbf{X} \rangle = \langle \overline{\mathbf{W}}^2(\tau) \mathbf{X}, \mathbf{X} \rangle. \quad (\text{A.10})$$

Since we know from Lemma A.2.1 that $\overline{\mathbf{W}}(\tau)$ is skew-symmetric, we can write $\overline{\mathbf{W}}^2(\tau)$ on the right hand side of (A.10) as $-\overline{\mathbf{W}}^T(\tau) \overline{\mathbf{W}}(\tau)$. On doing so and using the properties of the inner product we get

$$\langle \text{sym}(\mathbf{P}(\tau)) \mathbf{X}, \mathbf{X} \rangle = -\langle \overline{\mathbf{W}}(\tau) \mathbf{X}, \overline{\mathbf{W}}(\tau) \mathbf{X} \rangle. \quad (\text{A.11})$$

It also follows from the properties of the inner product that $\langle \overline{\mathbf{W}}(\tau) \mathbf{X}, \overline{\mathbf{W}}(\tau) \mathbf{X} \rangle$ is always non-negative. Therefore we get from (A.11) that $\langle \text{sym}(\mathbf{P}(\tau)) \mathbf{X}, \mathbf{X} \rangle$ is always non-positive, or equivalently that $\text{sym}(\mathbf{P}(\tau))$ is negative semidefinite.

A.3.2 Form of the eigenvalues of $\text{sym}(\mathbf{P}(\tau))$

Lemma A.3.2. The matrix $\text{sym}(\mathbf{P}(\tau))$'s negative eigenvalues, if they exist, have even algebraic multiplicities.

Proof. Say $\mathbf{S} \in \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{C})$ then $\mathbf{S}$ is said to be normal when it commutes with its conjugate-transpose $\mathbf{S}^H$, i.e., when $\mathbf{S} \mathbf{S}^H = \mathbf{S}^H \mathbf{S}$. Note that

$$\overline{\mathbf{W}}(\tau) \overline{\mathbf{W}}^T(\tau) = \overline{\mathbf{W}}(\tau) (-\overline{\mathbf{W}}(\tau)) = (-\overline{\mathbf{W}}(\tau)) \overline{\mathbf{W}}(\tau) = \overline{\mathbf{W}}^T(\tau) \overline{\mathbf{W}}(\tau). \quad (\text{A.12})$$

The first and the third equalities in (A.12) follow from the fact that $\overline{\mathbf{W}}(\tau)$ is skew-symmetric (Lemma A.2.1). Since all the entries of $\overline{\mathbf{W}}(\tau)$ are real, the transpose of $\overline{\mathbf{W}}(\tau)$ is equal to its conjugate-transpose. For this reason it follows from (A.12) that $\overline{\mathbf{W}}(\tau) \overline{\mathbf{W}}^H(\tau) = \overline{\mathbf{W}}^H(\tau) \overline{\mathbf{W}}(\tau)$, or equivalently that $\overline{\mathbf{W}}(\tau)$ is normal.

Since $\overline{\mathbf{W}}(\tau)$ is normal it follows from the *Complex Spectral Theorem*, e.g., see [130, 7.24] that there exists an unitary matrix $\mathbf{U}(\tau) \in \mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{C})$ such that $\mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) = \text{diag}(\mu_i(\tau))_{i \in I}$, where $\mu_i(\tau) \in \mathbb{C}$ and $\text{diag}(\mu_i(\tau))_{i \in I}$ is a diagonal matrix whose diagonal entries are $\mu_1(\tau), \mu_2(\tau) \dots, \mu_{n_{\text{sd}}}(\tau)$. That is,

$$\mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) = \begin{pmatrix} \mu_1(\tau) & 0 & \dots & 0 \\ 0 & \mu_2(\tau) & \dots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \mu_{n_{sd}}(\tau) \end{pmatrix}. \quad (\text{A.13})$$

The complex numbers $\mu_i(\tau)$, not necessarily distinct, are the eigenvalues of $\overline{\mathbf{W}}(\tau)$ and the columns of $\mathbf{U}(\tau)$ are the eigenvectors of $\overline{\mathbf{W}}(\tau)$. To be more specific,

$$\overline{\mathbf{W}}(\tau) \mathbf{u}_i(\tau) = \mu_i(\tau) \mathbf{u}_i(\tau) \text{ (no sum over } i), \quad (\text{A.14})$$

where $\mathbf{u}_i(\tau) = \left((\mathbf{U}(\tau))_{ji} \right)_{j \in I}$. Applying the operation of complex-conjugation to both sides of (A.14) and noting that the entries of $\overline{\mathbf{W}}(\tau)$ are all real we get that

$$\overline{\mathbf{W}}(\tau) \mathbf{u}_i^*(\tau) = \mu_i^*(\tau) \mathbf{u}_i^*(\tau) \text{ (no sum over } i), \quad (\text{A.15})$$

where $\mu_i^*(\tau)$ and $\mathbf{u}_i^*(\tau)$ are, respectively, the complex-conjugates of $\mu_i(\tau)$ and $\mathbf{u}_i(\tau)$. It follows from (A.15) that if $\mu_i(\tau)$ is an eigenvalue of $\overline{\mathbf{W}}(\tau)$ then so is $\mu_i^*(\tau)$. Thus $(\mu_i(\tau))_{i \in I}$ has the form $\varsigma(z_1(\tau), z_1^*(\tau), z_2(\tau), z_2^*(\tau), \dots, z_k(\tau), z_k^*(\tau), \alpha_1(\tau), \alpha_2(\tau), \dots, \alpha_l(\tau))$, where $\varsigma(\cdot)$ is the permutation operation, $z_i(\tau) \in \mathbb{C}$ with $\text{Im}(z_i(\tau)) \neq 0$, $z_i^*(\tau)$ is the complex-conjugate of $z_i(\tau)$, $0 \leq k \leq \lfloor n_{sd}/2 \rfloor^1$, $\alpha_i(\tau) \in \mathbb{R}$, and $l = n_{sd} - 2k$. It is not necessary that the complex numbers $z_i(\tau)$ be distinct from one another. The same is the case with the real numbers $\alpha_i(\tau)$.

Taking the square on both sides of (A.13) and using our knowledge about the form of $(\mu_i(\tau))_{i \in I}$ we get that

$$\left(\mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) \right)^2 = \text{diag } \varsigma \left(z_1^2(\tau), z_1^{*2}(\tau), \dots, z_k^2(\tau), z_k^{*2}(\tau), \alpha_1^2(\tau), \dots, \alpha_l^2(\tau) \right). \quad (\text{A.16})$$

The expression on the left hand side of (A.16) can be simplified as $\left(\mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) \right)^2 = \mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) \mathbf{U}^H(\tau) \overline{\mathbf{W}}(\tau) \mathbf{U}(\tau) = \mathbf{U}^H(\tau) \overline{\mathbf{W}}^2(\tau) \mathbf{U}(\tau) = \mathbf{U}^H(\tau) \text{sym}(\mathbf{P}(\tau)) \mathbf{U}(\tau)$, where the second equality follows from the fact that $\mathbf{U}(\tau)$ is an unitary matrix, and the third equality from

¹Here $\lfloor \cdot \rfloor$ is the *floor* function.

(2.3). Thus we can get from (A.16) that

$$\mathbf{U}^H(\tau) \text{sym}(\mathbf{P}(\tau)) \mathbf{U}(\tau) = \text{diag } \varsigma \left(z_1^2(\tau), z_1^{*2}(\tau), \dots, z_k^2(\tau), z_k^{*2}(\tau), \alpha_1^2(\tau), \dots, \alpha_l^2(\tau) \right), \quad (\text{A.17a})$$

which implies that $\mathbf{u}_i(\tau)$ are the eigenvectors of $\text{sym}(\mathbf{P}(\tau))$ as well, but with their corresponding eigenvalues being $\varsigma \left(z_1^2(\tau), z_1^{*2}(\tau), \dots, z_k^2(\tau), z_k^{*2}(\tau), \alpha_1^2(\tau), \dots, \alpha_l^2(\tau) \right)$. We know from Lemma A.3.1 that all of $\text{sym}(\mathbf{P}(\tau))$'s eigenvalues are non-positive. Therefore, each $z_i(\tau)$ must be of the form $\lambda_i(\tau)\sqrt{-1}$, where $\lambda_i(\tau) \in \mathbb{R}$ and $\lambda_i(\tau) \neq 0$, and $\alpha_i(\tau) = 0$. Hence, we get from (A.17a) that

$$\mathbf{U}^H(\tau) \text{sym}(\mathbf{P}(\tau)) \mathbf{U}(\tau) = \text{diag } \varsigma \left(-\lambda_1^2(\tau), -\lambda_1^2(\tau), \dots, -\lambda_k^2(\tau), -\lambda_k^2(\tau), 0, \dots, 0 \right), \quad (\text{A.18})$$

where, to reiterate, $0 \leq k \leq \lfloor n_{\text{sd}}/2 \rfloor$ and $\lambda_i(\tau)$, when they exist, are non-zero and not necessarily distinct. It can be noted from this last assertion that all of $\text{sym}(\mathbf{P}(\tau))$'s negative eigenvalues, specifically those corresponding to $\lambda_i(\tau)$, are of even geometric multiplicities. For the case of symmetric matrices, algebraic and geometric multiplicities are one and the same. Therefore, $\text{sym}(\mathbf{P}(\tau))$'s negative eigenvalues, when they exist, are also of even algebraic multiplicities.

A.4 Calculating $\overline{\mathbf{W}}(\tau)$ as the square-root of $\text{sym}(\mathbf{P}(\tau))$

A.4.1 A spectral decomposition of $\text{sym}(\mathbf{P}(\tau))$

Since $\text{sym}(\mathbf{P}(\tau))$ is a real symmetric matrix it follows from the *Real Spectral Theorem* [130, 7.29] that it can be decomposed as

$$\mathbf{N}(\tau) \mathbf{D}(\tau) \mathbf{N}^T(\tau), \quad (\text{A.19})$$

where $\mathbf{D}(\tau)$ and $\mathbf{N}(\tau)$ belong to $\mathcal{M}_{n_{\text{sd}}, n_{\text{sd}}}(\mathbb{R})$. We first describe $\mathbf{N}(\tau)$ and then $\mathbf{D}(\tau)$, in the next paragraph. The matrix $\mathbf{N}(\tau) := (\mathbf{n}_1(\tau), \dots, \mathbf{n}_{n_{\text{sd}}}(\tau))^T$ where $\mathbf{n}_i(\tau) \in \mathcal{M}_{n_{\text{sd}}, 1}(\mathbb{R})$ are $\text{sym}(\mathbf{P}(\tau))$'s

eigenvectors that are constructed such that $\mathbf{n}_i(\tau) \cdot \mathbf{n}_j(\tau) = \delta_{ij}$, or equivalently

$$\mathbf{N}(\tau) \mathbf{N}^T(\tau) = \mathbf{I}. \quad (\text{A.20})$$

Using (A.18) it can be shown that for $n_{\text{sd}} = 1$, $\mathbf{D}(\tau) = (0)$; for $n_{\text{sd}} = 2$, $\mathbf{D}(\tau) = \text{diag}(0, 0)$ or $\text{diag}(-\lambda_1^2(\tau), -\lambda_1^2(\tau))$, where $\lambda_1(\tau) \neq 0$; and for $n_{\text{sd}} = 3$, $\mathbf{D}(\tau) = \text{diag}(0, 0, 0)$ or $\text{diag}_\zeta(-\lambda_1^2(\tau), -\lambda_1^2(\tau), 0)$. The last two results can be summarized by saying that when $n_{\text{sd}} = 2$,

$$\mathbf{D}(\tau) = \text{diag}\left(-\lambda^2(\tau), -\lambda^2(\tau)\right), \quad (\text{A.21})$$

where $\lambda(\tau) \in \mathbb{R}$, and when $n_{\text{sd}} = 3$, $\mathbf{D}(\tau) = \text{diag}_\zeta(-\lambda(\tau)^2, -\lambda(\tau)^2, 0)$. Without loss of generality, we can choose the order of $\mathbf{n}_i(\tau)$ so that their respective eigenvalues form a non-increasing sequence². Therefore, for concreteness in the case of $n_{\text{sd}} = 3$ we take

$$\mathbf{D}(\tau) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\lambda^2(\tau) & 0 \\ 0 & 0 & -\lambda^2(\tau) \end{pmatrix}. \quad (\text{A.22})$$

A.4.2 Calculation of $\overline{\mathbf{W}}(\tau)$ from $\text{sym}(\mathbf{P}(\tau))$ using (2.3)

Let

$$\mathbf{F}(\tau) := \mathbf{N}^T(\tau) \overline{\mathbf{W}}(\tau) \mathbf{N}(\tau). \quad (\text{A.23})$$

It can be shown using $\mathbf{F}(\tau)$'s definition, equations (A.20), and (2.3), and $\text{sym}(\mathbf{P}(\tau))$'s decomposition that is derived in A.4.1 and summarized in (2.22) that

$$\mathbf{F}^2(\tau) = \mathbf{D}(\tau). \quad (\text{A.24})$$

²The matrix $\overline{\mathbf{W}}(\tau)$ as the square root of $\text{sym}(\mathbf{P}(\tau))$ does not depend on the order of $\mathbf{n}_i(\tau)$. Different orders will lead to the same $\overline{\mathbf{W}}(\tau)$.

Substituting $\mathbf{D}(\tau)$ in (A.24) from (A.21) and (A.22) for then noting from Lemma A.2.1 and $\mathbf{F}(\tau)$'s definition that $\mathbf{F}(\tau)$ is skew-symmetric, it can be shown that for $n_{sd} = 2$ and 3

$$\mathbf{F}(\tau) = \pm \star (\lambda(\tau)) \quad (\text{A.25})$$

and

$$\mathbf{F}(\tau) = \pm \star ((\lambda(\tau), 0, 0)), \quad (\text{A.26})$$

respectively. Equations 2.26 follow from (A.23), (A.25), and (A.26).

Appendix B

Supplementary Material: A finite rotation, small strain 2D elastic head model

B.1 Derivation of $U_1^C = 0$

Following the assumption that $\mathcal{B}$ is incompressible (assumption *A.i.c* in §3.3.1), we have that

$$\text{Det} [F_\tau [X]] = 1, \tag{B.1}$$

where Det is determinant operator. Using (3.21b) and substituting $F_\tau [X]$ as $Q [\tau] F_\tau^\star [X]$ in (B.1); in the resulting equation, using (3.21a) and substituting $F_\tau^\star [X]$ as $I + H_\tau^\star [X]$; and then noting that $\text{Det} [Q [\tau]] = 1$, since $Q [\tau] \in SO(3)$, we get that

$$\text{Det} [I + H_\tau^\star [X]] = 1. \tag{B.2}$$

From our assumption that the displacements and deformations of the brain w.r.t. the skull are uniformly small (assumption A.iii in §3.3.1) and the identity that for any $H \in \mathcal{M}_{3 \times 3}(\mathbb{R})$ as $\epsilon \rightarrow 0$

$$\text{Det}[I + \epsilon H] = 1 + \epsilon \text{Tr}[H] + o(\epsilon), \quad (\text{B.3})$$

we get that up to terms of $o(H_\tau^*[X])$,

$$\text{Tr}[H_\tau^*[X]] = 0. \quad (\text{B.4})$$

Writing H_τ^* in (B.4) in terms of $(U_i^C)_{i \in \mathcal{I}}$ using (3.22), (3.14), and (3.20) we get that

$$r \partial^{(1,0)} U_1^C[r, \tau] + U_1^C[r, \tau] = 0, \quad (\text{B.5})$$

for $r \in (0, r_0)$. It follows from our assumptions A.i.e (skull is a rigid solid), and A.vi (brain is rigidly connected to the skull) that

$$U_1^C[r = r_0, \tau] = 0. \quad (\text{B.6})$$

By solving (B.5) with the boundary condition (B.6) it can be shown that

$$U_1^C[r, \tau] = 0, \quad (\text{B.7})$$

for all admissible r and τ .

Appendix C

Supplementary Material: A mechanics theory for the exploration of a high-throughput, sterile 3D *in vitro* traumatic brain injury model

C.1 Vanishing of the rate of deformation tensor

The rate of deformation tensor is defined as

$$D_{\tau} [\mathbf{x}] = \frac{1}{2} \left(L_{\tau} [\mathbf{x}] + L_{\tau} [\mathbf{x}]^{\top} \right), \quad (\text{C.1})$$

where $L_{\tau} [\mathbf{x}]$ is the spatial velocity gradient, defined by

$$L_{\tau} [\mathbf{x}] = \{ \nabla_{\mathbf{x}} \mathbf{v}_{\tau} \} [\mathbf{x}]. \quad (\text{C.2})$$

From (5.18), (C.2) can be written as

$$L_{\tau} [\mathbf{x}] = W_{ij} [\tau] \mathbf{v}_i [\tau] \otimes \mathbf{e}_j [\tau]. \quad (\text{C.3})$$

From the definition of W_{ij} (5.15), it follows that $W_{ij} = -W_{ji}$. Then from (C.3) and (C.1) we get that

$$\boldsymbol{D}_\tau [\boldsymbol{x}] = \mathbf{0}. \quad (\text{C.4})$$

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